ZF

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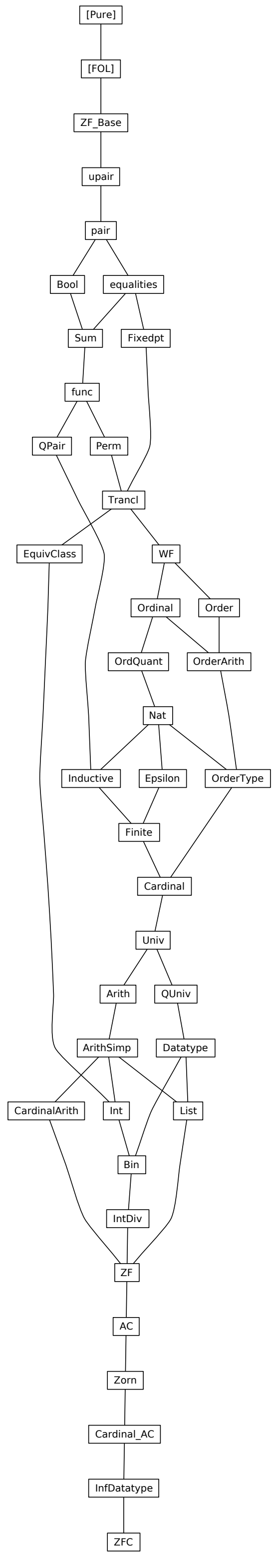
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1 Base of Zermelo-Fraenkel Set Theory

theory *ZF-Base*
imports *FOL*
begin

1.1 Signature

declare [[*eta-contract* = *false*]]

typedecl *i*
instance *i* :: *term* **..**

axiomatization *mem* :: [*i*, *i*] ⇒ *o* (**infixl** ‹∈› *50*) — membership relation
  and *zero* :: *i* (‹*0*›) — the empty set
  and *Pow* :: *i* ⇒ *i* — power sets
  and *Inf* :: *i* — infinite set
  and *Union* :: *i* ⇒ *i* (‹(‹*open-block notation*=‹*prefix* ⋃››⋃ -)› [*90*] *90*)
  and *PrimReplace* :: [*i*, [*i*, *i*] ⇒ *o*] ⇒ *i*

abbreviation *not-mem* :: [*i*, *i*] ⇒ *o* (**infixl** ‹∉› *50*) — negated membership relation
  where *x* ∉ *y* ≡ ¬ (*x* ∈ *y*)

1.2 Bounded Quantifiers

definition *Ball* :: [*i*, *i* ⇒ *o*] ⇒ *o*
  where *Ball*(*A*, *P*) ≡ ∀ *x*. *x*∈*A* ⟶ *P*(*x*)

definition *Bex* :: [*i*, *i* ⇒ *o*] ⇒ *o*
  where *Bex*(*A*, *P*) ≡ ∃ *x*. *x*∈*A* ∧ *P*(*x*)

syntax
  -Ball :: [*pttrn*, *i*, *o*] ⇒ *o* (‹(‹*indent*=*3 notation*=‹*binder* ∀∈››∀ -∈-./ -)› *10*)
  -Bex :: [*pttrn*, *i*, *o*] ⇒ *o* (‹(‹*indent*=*3 notation*=‹*binder* ∃∈››∃ -∈-./ -)› *10*)
syntax-consts
  -Ball ⇌ *Ball* **and**
  -Bex ⇌ *Bex*
translations
  ∀ *x*∈*A*. *P* ⇌ *CONST Ball*(*A*, λ*x*. *P*)
  ∃ *x*∈*A*. *P* ⇌ *CONST Bex*(*A*, λ*x*. *P*)

1.3 Variations on Replacement

definition *Replace* :: [*i*, [*i*, *i*] ⇒ *o*] ⇒ *i*
  where *Replace*(*A*,*P*) ≡ *PrimReplace*(*A*, λ*x y*. (∃!*z*. *P*(*x*,*z*)) ∧ *P*(*x*,*y*))

syntax
  -Replace :: [*pttrn*, *pttrn*, *i*, *o*] ⇒ *i* (‹(‹*indent*=*1 notation*=‹*mixfix relational replacement*››{- ./ - ∈ -, -})›)

syntax-consts
  -Replace ⇌ *Replace*
translations
  $\{y.\ x{\in}A,\ Q\}$ ⇌ *CONST Replace*$(A,\ \lambda x\ y.\ Q)$

definition *RepFun* :: $[i,\ i \Rightarrow i] \Rightarrow i$
  where *RepFun*$(A,f) \equiv \{y\ .\ x{\in}A,\ y{=}f(x)\}$

syntax
  -RepFun :: $[i,\ pttrn,\ i] \Rightarrow i$ (‹(‹*indent=1 notation=*‹*mixfix functional replacement*››{- ./ - ∈ -})› [*51*,*0*,*51*])
syntax-consts
  -RepFun ⇌ *RepFun*
translations
  $\{b.\ x{\in}A\}$ ⇌ *CONST RepFun*$(A,\ \lambda x.\ b)$

definition *Collect* :: $[i,\ i \Rightarrow o] \Rightarrow i$
  where *Collect*$(A,P) \equiv \{y\ .\ x{\in}A,\ x{=}y \wedge P(x)\}$

syntax
  -Collect :: $[pttrn,\ i,\ o] \Rightarrow i$ (‹(‹*indent=1 notation=*‹*mixfix set comprehension*››{- ∈ - ./ -})›)
syntax-consts
  -Collect ⇌ *Collect*
translations
  $\{x{\in}A.\ P\}$ ⇌ *CONST Collect*$(A,\ \lambda x.\ P)$

1.4 General union and intersection

definition *Inter* :: $i \Rightarrow i$ (‹(‹*open-block notation=*‹*prefix* ⋂››⋂ -)› [*90*] *90*)
  where $\bigcap(A) \equiv \{\ x{\in}\bigcup(A)\ .\ \forall y{\in}A.\ x{\in}y\}$

syntax
  -UNION :: $[pttrn,\ i,\ i] \Rightarrow i$ (‹(‹*indent=3 notation=*‹*binder* ⋃∈››⋃ -∈-./ -)› *10*)
  -INTER :: $[pttrn,\ i,\ i] \Rightarrow i$ (‹(‹*indent=3 notation=*‹*binder* ⋂∈››⋂ -∈-./ -)› *10*)
syntax-consts
  -UNION == *Union* **and**
  -INTER == *Inter*
translations
  $\bigcup x{\in}A.\ B$ == *CONST Union*$(\{B.\ x{\in}A\})$
  $\bigcap x{\in}A.\ B$ == *CONST Inter*$(\{B.\ x{\in}A\})$

1.5 Finite sets and binary operations

definition *Upair* :: $[i,\ i] \Rightarrow i$
  where *Upair*$(a,b) \equiv \{y.\ x{\in}Pow(Pow(0)),\ (x{=}0 \wedge y{=}a)\ |\ (x{=}Pow(0) \wedge y{=}b)\}$

definition *Subset* :: $[i, i] \Rightarrow o$ (**infixl** ‹$\subseteq$› *50*) — subset relation
where *subset-def*: $A \subseteq B \equiv \forall x{\in}A.\ x{\in}B$

definition *Diff* :: $[i, i] \Rightarrow i$ (**infixl** ‹$-$› *65*) — set difference
where $A - B \equiv \{\ x{\in}A\ .\ \neg(x{\in}B)\ \}$

definition *Un* :: $[i, i] \Rightarrow i$ (**infixl** ‹$\cup$› *65*) — binary union
where $A \cup B \equiv \bigcup(\mathit{Upair}(A,B))$

definition *Int* :: $[i, i] \Rightarrow i$ (**infixl** ‹$\cap$› *70*) — binary intersection
where $A \cap B \equiv \bigcap(\mathit{Upair}(A,B))$

definition *cons* :: $[i, i] \Rightarrow i$
where $\mathit{cons}(a,A) \equiv \mathit{Upair}(a,a) \cup A$

definition *succ* :: $i \Rightarrow i$
where $\mathit{succ}(i) \equiv \mathit{cons}(i, i)$

nonterminal *is*
syntax
:: $i \Rightarrow is$ (‹-›)
-Enum :: $[i, is] \Rightarrow is$ (‹-,/ -›)
-Finset :: $is \Rightarrow i$ (‹(‹*indent=1 notation=*‹*mixfix set enumeration*››{-})›)
translations
$\{x, xs\}$ == *CONST* $\mathit{cons}(x, \{xs\})$
$\{x\}$ == *CONST* $\mathit{cons}(x, 0)$

1.6 Axioms

axiomatization
where
extension: $A = B \longleftrightarrow A \subseteq B \wedge B \subseteq A$ **and**
Union-iff: $A \in \bigcup(C) \longleftrightarrow (\exists B{\in}C.\ A{\in}B)$ **and**
Pow-iff: $A \in \mathit{Pow}(B) \longleftrightarrow A \subseteq B$ **and**

infinity: $0 \in \mathit{Inf} \wedge (\forall y{\in}\mathit{Inf}.\ \mathit{succ}(y) \in \mathit{Inf})$ **and**

foundation: $A = 0 \vee (\exists x{\in}A.\ \forall y{\in}x.\ y{\notin}A)$ **and**

replacement: $(\forall x{\in}A.\ \forall y\ z.\ P(x,y) \wedge P(x,z) \longrightarrow y = z) \Longrightarrow$
$b \in \mathit{PrimReplace}(A,P) \longleftrightarrow (\exists x{\in}A.\ P(x,b))$

1.7 Definite descriptions – via Replace over the set "1"

definition *The* :: $(i \Rightarrow o) \Rightarrow i$ (**binder** ‹*THE* › *10*)
where *the-def*: $\mathit{The}(P) \equiv \bigcup(\{y\ .\ x \in \{0\}, P(y)\})$

definition *If* :: [*o*, *i*, *i*] ⇒ *i* (‹(‹*notation*=‹*mixfix if then else*››*if* (-)/ *then* (-)/ *else* (-))› [*10*] *10*)
  where *if-def*: *if P then a else b* ≡ *THE z*. *P* ∧ *z*=*a* | ¬*P* ∧ *z*=*b*

abbreviation (*input*)
  old-if :: [*o*, *i*, *i*] ⇒ *i* (‹*if* ′(-,-,-′)›)
  where *if*(*P*,*a*,*b*) ≡ *If*(*P*,*a*,*b*)

1.8 Ordered Pairing

definition *Pair* :: [*i*, *i*] ⇒ *i*
  where *Pair*(*a*,*b*) ≡ {{*a*,*a*}, {*a*,*b*}}

definition *fst* :: *i* ⇒ *i*
  where *fst*(*p*) ≡ *THE a*. ∃ *b*. *p* = *Pair*(*a*, *b*)

definition *snd* :: *i* ⇒ *i*
  where *snd*(*p*) ≡ *THE b*. ∃ *a*. *p* = *Pair*(*a*, *b*)

definition *split* :: [[*i*, *i*] ⇒ ′*a*, *i*] ⇒ ′*a*::{} — for pattern-matching
  where *split*(*c*) ≡ λ*p*. *c*(*fst*(*p*), *snd*(*p*))

nonterminal *tuple-args*
syntax
  :: *i* ⇒ *tuple-args* (‹-›)
  -Tuple-args :: [*i*, *tuple-args*] ⇒ *tuple-args* (‹-,/ -›)
  -Tuple :: [*i*, *tuple-args*] ⇒ *i* (‹(‹*indent*=*1* *notation*=‹*mixfix tuple enumeration*››⟨-,/ -⟩)›)
translations
  ⟨*x*, *y*, *z*⟩ == ⟨*x*, ⟨*y*, *z*⟩⟩
  ⟨*x*, *y*⟩ == *CONST Pair*(*x*, *y*)

nonterminal *patterns*
syntax
  -pattern :: *patterns* ⇒ *pttrn* (‹(‹*open-block notation*=‹*pattern tuple*››⟨-⟩)›)
  :: *pttrn* ⇒ *patterns* (‹-›)
  -patterns :: [*pttrn*, *patterns*] ⇒ *patterns* (‹-,/-›)
syntax-consts
  -pattern -patterns == *split*
translations
  λ⟨*x*,*y*,*zs*⟩.*b* == *CONST split*(λ*x* ⟨*y*,*zs*⟩.*b*)
  λ⟨*x*,*y*⟩.*b* == *CONST split*(λ*x y*. *b*)

definition *Sigma* :: [*i*, *i* ⇒ *i*] ⇒ *i*
  where *Sigma*(*A*,*B*) ≡ ⋃*x*∈*A*. ⋃*y*∈*B*(*x*). {⟨*x*,*y*⟩}

abbreviation *cart-prod* :: [*i*, *i*] ⇒ *i* (**infixr** ‹×› *80*) — Cartesian product

where $A \times B \equiv Sigma(A, \lambda\text{-}.\ B)$

1.9 Relations and Functions

definition *converse* :: $i \Rightarrow i$
where $converse(r) \equiv \{z.\ w{\in}r,\ \exists\, x\ y.\ w{=}\langle x{,}y\rangle \wedge z{=}\langle y{,}x\rangle\}$

definition *domain* :: $i \Rightarrow i$
where $domain(r) \equiv \{x.\ w{\in}r,\ \exists\, y.\ w{=}\langle x{,}y\rangle\}$

definition *range* :: $i \Rightarrow i$
where $range(r) \equiv domain(converse(r))$

definition *field* :: $i \Rightarrow i$
where $field(r) \equiv domain(r) \cup range(r)$

definition *relation* :: $i \Rightarrow o$ — recognizes sets of pairs
where $relation(r) \equiv \forall\, z{\in}r.\ \exists\, x\ y.\ z = \langle x{,}y\rangle$

definition *function* :: $i \Rightarrow o$ — recognizes functions; can have non-pairs
where $function(r) \equiv \forall\, x\ y.\ \langle x{,}y\rangle \in r \longrightarrow (\forall\, y'.\ \langle x{,}y'\rangle \in r \longrightarrow y = y')$

definition *Image* :: $[i, i] \Rightarrow i$ (**infixl** ‹‘‘› *90*) — image
where *image-def*: $r\ \text{‘‘}\ A \equiv \{y \in range(r).\ \exists\, x{\in}A.\ \langle x{,}y\rangle \in r\}$

definition *vimage* :: $[i, i] \Rightarrow i$ (**infixl** ‹-‘‘› *90*) — inverse image
where *vimage-def*: $r\ -\text{‘‘}\ A \equiv converse(r)\text{‘‘}A$

definition *restrict* :: $[i, i] \Rightarrow i$
where $restrict(r{,}A) \equiv \{z \in r.\ \exists\, x{\in}A.\ \exists\, y.\ z = \langle x{,}y\rangle\}$

definition *Lambda* :: $[i, i \Rightarrow i] \Rightarrow i$
where *lam-def*: $Lambda(A{,}b) \equiv \{\langle x{,}b(x)\rangle.\ x{\in}A\}$

definition *apply* :: $[i, i] \Rightarrow i$ (**infixl** ‹‘› *90*) — function application
where $f\text{‘}a \equiv \bigcup(f\text{‘‘}\{a\})$

definition *Pi* :: $[i, i \Rightarrow i] \Rightarrow i$
where $Pi(A{,}B) \equiv \{f{\in}Pow(Sigma(A{,}B)).\ A{\subseteq}domain(f) \wedge function(f)\}$

abbreviation *function-space* :: $[i, i] \Rightarrow i$ (**infixr** ‹->› *60*) — function space
where $A \to B \equiv Pi(A, \lambda\text{-}.\ B)$

syntax
-PROD :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(‹*indent=3 notation=*‹*mixfix* ∏∈››∏ -∈-./ -)› *10*)
-SUM :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(‹*indent=3 notation=*‹*mixfix* ∑∈››∑ -∈-./ -)› *10*)
-lam :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(‹*indent=3 notation=*‹*mixfix* λ∈››λ-∈-./ -)› *10*)
syntax-consts
-PROD == *Pi* **and**
-SUM == *Sigma* **and**
-lam == *Lambda*
translations
∏*x*∈*A*. *B* == *CONST Pi*(*A*, λ*x*. *B*)
∑*x*∈*A*. *B* == *CONST Sigma*(*A*, λ*x*. *B*)
λ*x*∈*A*. *f* == *CONST Lambda*(*A*, λ*x*. *f*)

1.10 ASCII syntax

notation (*ASCII*)
cart-prod (**infixr** ‹∗› *80*) **and**
Int (**infixl** ‹*Int*› *70*) **and**
Un (**infixl** ‹*Un*› *65*) **and**
function-space (**infixr** ‹−>› *60*) **and**
Subset (**infixl** ‹<=› *50*) **and**
mem (**infixl** ‹:› *50*) **and**
not-mem (**infixl** ‹¬:› *50*)

syntax (*ASCII*)
-Ball :: [*pttrn*, *i*, *o*] ⇒ *o* (‹(‹*indent=3 notation=*‹*binder ALL*:››*ALL* -:-./ -)› *10*)
-Bex :: [*pttrn*, *i*, *o*] ⇒ *o* (‹(‹*indent=3 notation=*‹*binder EX*:››*EX* -:-./ -)› *10*)
-Collect :: [*pttrn*, *i*, *o*] ⇒ *i* (‹(‹*indent=1 notation=*‹*mixfix set comprehension*››{-: - ./ -})›)
-Replace :: [*pttrn*, *pttrn*, *i*, *o*] ⇒ *i* (‹(‹*indent=1 notation=*‹*mixfix relational replacement*››{- ./ -: -, -})›)
-RepFun :: [*i*, *pttrn*, *i*] ⇒ *i* (‹(‹*indent=1 notation=*‹*mixfix functional replacement*››{- ./ -: -})› [*51*,*0*,*51*])
-UNION :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(‹*indent=3 notation=*‹*binder UN*:››*UN* -:-./ -)› *10*)
-INTER :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(‹*indent=3 notation=*‹*binder INT*:››*INT* -:-./ -)› *10*)
-PROD :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(‹*indent=3 notation=*‹*binder PROD*:››*PROD* -:-./ -)› *10*)
-SUM :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(‹*indent=3 notation=*‹*binder SUM*:››*SUM* -:-./ -)› *10*)
-lam :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(‹*indent=3 notation=*‹*binder lam*:››*lam* -:-./ -)› *10*)

-Tuple :: [*i*, *tuple-args*] ⇒ *i* (‹(‹*indent=1 notation=*‹*mixfix tuple enumeration*››<-,/ ->)›)
-pattern :: *patterns* ⇒ *pttrn* (‹<->›)

1.11 Substitution

lemma *subst-elem*: $[\![b{\in}A;\ \ a{=}b]\!] \Longrightarrow a{\in}A$
by (*erule ssubst*, *assumption*)

1.12 Bounded universal quantifier

lemma *ballI* [*intro*!]: $[\![\bigwedge x.\ x{\in}A \Longrightarrow P(x)]\!] \Longrightarrow \forall x{\in}A.\ P(x)$
by (*simp add*: *Ball-def*)

lemmas *strip* = *impI allI ballI*

lemma *bspec* [*dest?*]: $[\![\forall x{\in}A.\ P(x);\ \ x{:}\ A]\!] \Longrightarrow P(x)$
by (*simp add*: *Ball-def*)

lemma *rev-ballE* [*elim*]:
$[\![\forall x{\in}A.\ P(x);\ \ x{\notin}A \Longrightarrow Q;\ \ P(x) \Longrightarrow Q]\!] \Longrightarrow Q$
by (*simp add*: *Ball-def*, *blast*)

lemma *ballE*: $[\![\forall x{\in}A.\ P(x);\ \ P(x) \Longrightarrow Q;\ \ x{\notin}A \Longrightarrow Q]\!] \Longrightarrow Q$
by *blast*

lemma *rev-bspec*: $[\![x{:}\ A;\ \ \forall x{\in}A.\ P(x)]\!] \Longrightarrow P(x)$
by (*simp add*: *Ball-def*)

lemma *ball-triv* [*simp*]: $(\forall x{\in}A.\ P) \longleftrightarrow ((\exists x.\ x{\in}A) \longrightarrow P)$
by (*simp add*: *Ball-def*)

lemma *ball-cong* [*cong*]:
$[\![A{=}A';\ \ \bigwedge x.\ x{\in}A' \Longrightarrow P(x) \longleftrightarrow P'(x)]\!] \Longrightarrow (\forall x{\in}A.\ P(x)) \longleftrightarrow (\forall x{\in}A'.\ P'(x))$
by (*simp add*: *Ball-def*)

lemma *atomize-ball*:
$(\bigwedge x.\ x \in A \Longrightarrow P(x)) \equiv$ *Trueprop* $(\forall x{\in}A.\ P(x))$
by (*simp only*: *Ball-def atomize-all atomize-imp*)

lemmas [*symmetric*, *rulify*] = *atomize-ball*
and [*symmetric*, *defn*] = *atomize-ball*

1.13 Bounded existential quantifier

lemma *bexI* [*intro*]: $[\![P(x);\ \ x{:}\ A]\!] \Longrightarrow \exists x{\in}A.\ P(x)$

by (*simp add*: *Bex-def*, *blast*)

lemma *rev-bexI*: $[\![x{\in}A;\ \ P(x)]\!] \Longrightarrow \exists\, x{\in}A.\ P(x)$
by *blast*

lemma *bexCI*: $[\![\forall\, x{\in}A.\ \neg P(x) \Longrightarrow P(a);\ \ a{:}\ A]\!] \Longrightarrow \exists\, x{\in}A.\ P(x)$
by *blast*

lemma *bexE* [*elim*!]: $[\![\exists\, x{\in}A.\ P(x);\ \ \bigwedge x.\ [\![x{\in}A;\ P(x)]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
by (*simp add*: *Bex-def*, *blast*)

lemma *bex-triv* [*simp*]: $(\exists\, x{\in}A.\ P) \longleftrightarrow ((\exists\, x.\ x{\in}A) \wedge P)$
by (*simp add*: *Bex-def*)

lemma *bex-cong* [*cong*]:
$[\![A{=}A';\ \ \bigwedge x.\ x{\in}A' \Longrightarrow P(x) \longleftrightarrow P'(x)]\!]$
$\Longrightarrow (\exists\, x{\in}A.\ P(x)) \longleftrightarrow (\exists\, x{\in}A'.\ P'(x))$
by (*simp add*: *Bex-def cong*: *conj-cong*)

1.14 Rules for subsets

lemma *subsetI* [*intro*!]:
$(\bigwedge x.\ x{\in}A \Longrightarrow x{\in}B) \Longrightarrow A \subseteq B$
by (*simp add*: *subset-def*)

lemma *subsetD* [*elim*]: $[\![A \subseteq B;\ \ c{\in}A]\!] \Longrightarrow c{\in}B$
unfolding *subset-def*
apply (*erule bspec*, *assumption*)
done

lemma *subsetCE* [*elim*]:
$[\![A \subseteq B;\ \ c{\notin}A \Longrightarrow P;\ \ c{\in}B \Longrightarrow P]\!] \Longrightarrow P$
by (*simp add*: *subset-def*, *blast*)

lemma *rev-subsetD*: $[\![c{\in}A;\ A{\subseteq}B]\!] \Longrightarrow c{\in}B$
by *blast*

lemma *contra-subsetD*: $[\![A \subseteq B;\ c \notin B]\!] \Longrightarrow c \notin A$
by *blast*

lemma *rev-contra-subsetD*: $[\![c \notin B;\ \ A \subseteq B]\!] \Longrightarrow c \notin A$
by *blast*

lemma *subset-refl* [*simp*]: $A \subseteq A$
  by *blast*

lemma *subset-trans*: $[\![A \subseteq B;\ \ B \subseteq C]\!] \Longrightarrow A \subseteq C$
  by *blast*

lemma *subset-iff*:
  $A \subseteq B \longleftrightarrow (\forall x.\ x \in A \longrightarrow x \in B)$
  by *auto*

For calculations

declare *subsetD* [*trans*] *rev-subsetD* [*trans*] *subset-trans* [*trans*]

1.15 Rules for equality

lemma *equalityI* [*intro*]: $[\![A \subseteq B;\ \ B \subseteq A]\!] \Longrightarrow A = B$
  by (*rule extension* [*THEN iffD2*], *rule conjI*)

lemma *equality-iffI*: $(\bigwedge x.\ x \in A \longleftrightarrow x \in B) \Longrightarrow A = B$
  by (*rule equalityI*, *blast+*)

lemmas *equalityD1* = *extension* [*THEN iffD1*, *THEN conjunct1*]
lemmas *equalityD2* = *extension* [*THEN iffD1*, *THEN conjunct2*]

lemma *equalityE*: $[\![A = B;\ \ [\![A \subseteq B;\ B \subseteq A]\!] \Longrightarrow P]\!] \ \Longrightarrow\ P$
  by (*blast dest*: *equalityD1 equalityD2*)

lemma *equalityCE*:
  $[\![A = B;\ \ [\![c \in A;\ c \in B]\!] \Longrightarrow P;\ \ [\![c \notin A;\ c \notin B]\!] \Longrightarrow P]\!] \ \Longrightarrow\ P$
  by (*erule equalityE*, *blast*)

lemma *equality-iffD*:
  $A = B \Longrightarrow (\bigwedge x.\ x \in A \longleftrightarrow x \in B)$
  by *auto*

1.16 Rules for Replace – the derived form of replacement

lemma *Replace-iff*:
  $b \in \{y.\ x \in A,\ P(x,y)\} \ \longleftrightarrow\ (\exists x \in A.\ P(x,b) \land (\forall y.\ P(x,y) \longrightarrow y = b))$
  unfolding *Replace-def*
  by (*rule replacement* [*THEN iff-trans*], *blast+*)

lemma *ReplaceI* [*intro*]:
  $[\![P(x,b);\ \ x{:}\ A;\ \ \bigwedge y.\ P(x,y) \Longrightarrow y = b]\!] \Longrightarrow$
  $b \in \{y.\ x \in A,\ P(x,y)\}$
by (*rule Replace-iff* [*THEN iffD2*], *blast*)

lemma *ReplaceE*:
$[\![b \in \{y.\ x{\in}A,\ P(x,y)\};$
$\bigwedge x.\ [\![x\colon A;\ \ P(x,b);\ \ \forall\, y.\ P(x,y) {\longrightarrow} y{=}b]\!] \Longrightarrow R$
$]\!] \Longrightarrow R$
by (*rule Replace-iff* [*THEN iffD1*, *THEN bexE*], *simp*+)

lemma *ReplaceE2* [*elim*!]:
$[\![b \in \{y.\ x{\in}A,\ P(x,y)\};$
$\bigwedge x.\ [\![x\colon A;\ \ P(x,b)]\!] \Longrightarrow R$
$]\!] \Longrightarrow R$
by (*erule ReplaceE*, *blast*)

lemma *Replace-cong* [*cong*]:
$[\![A{=}B;\ \ \bigwedge x\ y.\ x{\in}B \Longrightarrow P(x,y) \longleftrightarrow Q(x,y)]\!] \Longrightarrow \mathit{Replace}(A,P) = \mathit{Replace}(B,Q)$
apply (*rule equality-iffI*)
apply (*simp add*: *Replace-iff*)
done

1.17 Rules for RepFun

lemma *RepFunI*: $a \in A \Longrightarrow f(a) \in \{f(x).\ x{\in}A\}$
by (*simp add*: *RepFun-def Replace-iff*, *blast*)

lemma *RepFun-eqI* [*intro*]: $[\![b{=}f(a);\ \ a \in A]\!] \Longrightarrow b \in \{f(x).\ x{\in}A\}$
by (*blast intro*: *RepFunI*)

lemma *RepFunE* [*elim*!]:
$[\![b \in \{f(x).\ x{\in}A\};$
$\bigwedge x.[\![x{\in}A;\ \ b{=}f(x)]\!] \Longrightarrow P]\!] \Longrightarrow$
P
by (*simp add*: *RepFun-def Replace-iff*, *blast*)

lemma *RepFun-cong* [*cong*]:
$[\![A{=}B;\ \ \bigwedge x.\ x{\in}B \Longrightarrow f(x){=}g(x)]\!] \Longrightarrow \mathit{RepFun}(A,f) = \mathit{RepFun}(B,g)$
by (*simp add*: *RepFun-def*)

lemma *RepFun-iff* [*simp*]: $b \in \{f(x).\ x{\in}A\} \longleftrightarrow (\exists\, x{\in}A.\ b{=}f(x))$
by (*unfold Bex-def*, *blast*)

lemma *triv-RepFun* [*simp*]: $\{x.\ x{\in}A\} = A$
by *blast*

1.18 Rules for Collect – forming a subset by separation

lemma *separation* [*simp*]: $a \in \{x{\in}A.\ P(x)\} \longleftrightarrow a{\in}A \wedge P(a)$
by (*auto simp*: *Collect-def*)

lemma *CollectI* [*intro*!]: $[\![a{\in}A;\ \ P(a)]\!] \Longrightarrow a \in \{x{\in}A.\ P(x)\}$
by *simp*

lemma *CollectE* [*elim*!]: $[\![a \in \{x{\in}A.\ P(x)\};\ \ [\![a{\in}A;\ P(a)]\!] \Longrightarrow R]\!] \Longrightarrow R$
by *simp*

lemma *CollectD1*: $a \in \{x{\in}A.\ P(x)\} \Longrightarrow a{\in}A$ **and** *CollectD2*: $a \in \{x{\in}A.\ P(x)\} \Longrightarrow P(a)$
by *auto*

lemma *Collect-cong* [*cong*]:
$[\![A{=}B;\ \ \bigwedge x.\ x{\in}B \Longrightarrow P(x) \longleftrightarrow Q(x)]\!]$
$\Longrightarrow Collect(A,\ \lambda x.\ P(x)) = Collect(B,\ \lambda x.\ Q(x))$
by (*simp add*: *Collect-def*)

1.19 Rules for Unions

declare *Union-iff* [*simp*]

lemma *UnionI* [*intro*]: $[\![B{:}\ C;\ \ A{:}\ B]\!] \Longrightarrow A{:}\ \bigcup(C)$
by *auto*

lemma *UnionE* [*elim*!]: $[\![A \in \bigcup(C);\ \ \bigwedge B.[\![A{:}\ B;\ \ B{:}\ C]\!] \Longrightarrow R]\!] \Longrightarrow R$
by *auto*

1.20 Rules for Unions of families

lemma *UN-iff* [*simp*]: $b \in (\bigcup x{\in}A.\ B(x)) \longleftrightarrow (\exists x{\in}A.\ b \in B(x))$
by *blast*

lemma *UN-I*: $[\![a{:}\ A;\ \ b{:}\ B(a)]\!] \Longrightarrow b{:}\ (\bigcup x{\in}A.\ B(x))$
by *force*

lemma *UN-E* [*elim*!]:
$[\![b \in (\bigcup x{\in}A.\ B(x));\ \ \bigwedge x.[\![x{:}\ A;\ \ b{:}\ B(x)]\!] \Longrightarrow R]\!] \Longrightarrow R$
by *blast*

lemma *UN-cong*:
$[\![A{=}B;\ \ \bigwedge x.\ x{\in}B \Longrightarrow C(x){=}D(x)]\!] \Longrightarrow (\bigcup x{\in}A.\ C(x)) = (\bigcup x{\in}B.\ D(x))$
by *simp*

1.21 Rules for the empty set

lemma *not-mem-empty* [*simp*]: $a \notin 0$
using *foundation* **by** (*best dest*: *equalityD2*)

lemmas *emptyE* [*elim*!] = *not-mem-empty* [*THEN notE*]

lemma *empty-subsetI* [*simp*]: $0 \subseteq A$
by *blast*

lemma *equals0I*: $[\![\bigwedge y.\ y{\in}A \Longrightarrow False]\!] \Longrightarrow A{=}0$
by *blast*

lemma *equals0D* [*dest*]: $A{=}0 \Longrightarrow a \notin A$
by *blast*

declare *sym* [*THEN equals0D*, *dest*]

lemma *not-emptyI*: $a{\in}A \Longrightarrow A \neq 0$
by *blast*

lemma *not-emptyE*: $[\![A \neq 0;\ \bigwedge x.\ x{\in}A \Longrightarrow R]\!] \Longrightarrow R$
by *blast*

1.22 Rules for Inter

lemma *Inter-iff*: $A \in \bigcap(C) \longleftrightarrow (\forall x{\in}C.\ A{:}\ x) \wedge C{\neq}0$
by (*force simp*: *Inter-def*)

lemma *InterI* [*intro*!]:
$[\![\bigwedge x.\ x{:}\ C \Longrightarrow A{:}\ x;\ C{\neq}0]\!] \Longrightarrow A \in \bigcap(C)$
by (*simp add*: *Inter-iff*)

lemma *InterD* [*elim*, *Pure.elim*]: $[\![A \in \bigcap(C);\ B \in C]\!] \Longrightarrow A \in B$
by (*force simp*: *Inter-def*)

lemma *InterE* [*elim*]:
$[\![A \in \bigcap(C);\ B{\notin}C \Longrightarrow R;\ A{\in}B \Longrightarrow R]\!] \Longrightarrow R$
by (*auto simp*: *Inter-def*)

1.23 Rules for Intersections of families

lemma *INT-iff*: $b \in (\bigcap x{\in}A.\ B(x)) \longleftrightarrow (\forall x{\in}A.\ b \in B(x)) \wedge A{\neq}0$
by (*force simp add*: *Inter-def*)

lemma *INT-I*: $[\![\bigwedge x.\ x{:}\ A \Longrightarrow b{:}\ B(x);\ A{\neq}0]\!] \Longrightarrow b{:}\ (\bigcap x{\in}A.\ B(x))$
by *blast*

lemma *INT-E*: $[\![b \in (\bigcap x{\in}A.\ B(x));\ a{:}\ A]\!] \Longrightarrow b \in B(a)$
by *blast*

lemma *INT-cong*:
$[\![A{=}B;\ \bigwedge x.\ x{\in}B \Longrightarrow C(x){=}D(x)]\!] \Longrightarrow (\bigcap x{\in}A.\ C(x)) = (\bigcap x{\in}B.\ D(x))$
by *simp*

1.24 Rules for Powersets

lemma *PowI*: $A \subseteq B \Longrightarrow A \in Pow(B)$
by (*erule Pow-iff* [*THEN iffD2*])

lemma *PowD*: $A \in Pow(B) \Longrightarrow A{\subseteq}B$
by (*erule Pow-iff* [*THEN iffD1*])

declare *Pow-iff* [*iff*]

lemmas *Pow-bottom* = *empty-subsetI* [*THEN PowI*] — $0 \in Pow(B)$
lemmas *Pow-top* = *subset-refl* [*THEN PowI*] — $A \in Pow(A)$

1.25 Cantor's Theorem: There is no surjection from a set to its powerset.

lemma *cantor*: $\exists S \in Pow(A).\ \forall x{\in}A.\ b(x) \neq S$
by (*best elim!*: *equalityCE del*: *ReplaceI RepFun-eqI*)

end

2 Unordered Pairs

theory *upair*
imports *ZF-Base*
keywords *print-tcset* :: *diag*
begin

ML-file ‹*Tools/typechk.ML*›

2.1 Unordered Pairs: constant *Upair*

lemma *Upair-iff* [*simp*]: $c \in Upair(a,b) \longleftrightarrow (c{=}a \mid c{=}b)$
by (*unfold Upair-def*, *blast*)

lemma *UpairI1*: $a \in Upair(a,b)$
by *simp*

lemma *UpairI2*: $b \in Upair(a,b)$
by *simp*

lemma *UpairE*: $[\![a \in Upair(b,c);\ a{=}b \Longrightarrow P;\ a{=}c \Longrightarrow P]\!] \Longrightarrow P$
by (*simp*, *blast*)

2.2 Rules for Binary Union, Defined via *Upair*

lemma *Un-iff* [*simp*]: $c \in A \cup B \longleftrightarrow (c \in A \mid c \in B)$
apply (*simp add*: *Un-def*)
apply (*blast intro*: *UpairI1 UpairI2 elim*: *UpairE*)
done

lemma *UnI1*: $c \in A \Longrightarrow c \in A \cup B$
by *simp*

lemma *UnI2*: $c \in B \Longrightarrow c \in A \cup B$
by *simp*

declare *UnI1* [*elim?*] *UnI2* [*elim?*]

lemma *UnE* [*elim*!]: $[\![c \in A \cup B;\ c \in A \Longrightarrow P;\ c \in B \Longrightarrow P]\!] \Longrightarrow P$
by (*simp*, *blast*)

lemma *UnE'*: $[\![c \in A \cup B;\ c \in A \Longrightarrow P;\ [\![c \in B;\ c \notin A]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*simp*, *blast*)

lemma *UnCI* [*intro*!]: $(c \notin B \Longrightarrow c \in A) \Longrightarrow c \in A \cup B$
by (*simp*, *blast*)

2.3 Rules for Binary Intersection, Defined via *Upair*

lemma *Int-iff* [*simp*]: $c \in A \cap B \longleftrightarrow (c \in A \wedge c \in B)$
unfolding *Int-def*
apply (*blast intro*: *UpairI1 UpairI2 elim*: *UpairE*)
done

lemma *IntI* [*intro*!]: $[\![c \in A;\ c \in B]\!] \Longrightarrow c \in A \cap B$
by *simp*

lemma *IntD1*: $c \in A \cap B \Longrightarrow c \in A$
by *simp*

lemma *IntD2*: $c \in A \cap B \Longrightarrow c \in B$
by *simp*

lemma *IntE* [*elim*!]: $[\![c \in A \cap B;\ [\![c \in A;\ c \in B]\!] \Longrightarrow P]\!] \Longrightarrow P$
by *simp*

2.4 Rules for Set Difference, Defined via *Upair*

lemma *Diff-iff* [*simp*]: $c \in A-B \longleftrightarrow (c \in A \wedge c \notin B)$
by (*unfold Diff-def*, *blast*)

lemma *DiffI* [*intro*!]: $[\![c \in A;\ \ c \notin B]\!] \Longrightarrow c \in A - B$
by *simp*

lemma *DiffD1*: $c \in A - B \Longrightarrow c \in A$
by *simp*

lemma *DiffD2*: $c \in A - B \Longrightarrow c \notin B$
by *simp*

lemma *DiffE* [*elim*!]: $[\![c \in A - B;\ \ [\![c \in A;\ c \notin B]\!] \Longrightarrow P]\!] \Longrightarrow P$
by *simp*

2.5 Rules for *cons*

lemma *cons-iff* [*simp*]: $a \in cons(b,A) \longleftrightarrow (a=b \mid a \in A)$
  unfolding *cons-def*
apply (*blast intro*: *UpairI1 UpairI2 elim*: *UpairE*)
done

lemma *consI1* [*simp*,*TC*]: $a \in cons(a,B)$
by *simp*

lemma *consI2*: $a \in B \Longrightarrow a \in cons(b,B)$
by *simp*

lemma *consE* [*elim*!]: $[\![a \in cons(b,A);\ \ a=b \Longrightarrow P;\ \ a \in A \Longrightarrow P]\!] \Longrightarrow P$
by (*simp*, *blast*)

lemma *consE'*:
  $[\![a \in cons(b,A);\ \ a=b \Longrightarrow P;\ \ [\![a \in A;\ \ a \neq b]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*simp*, *blast*)

lemma *consCI* [*intro*!]: $(a \notin B \Longrightarrow a=b) \Longrightarrow a \in cons(b,B)$
by (*simp*, *blast*)

lemma *cons-not-0* [*simp*]: $cons(a,B) \neq 0$
by (*blast elim*: *equalityE*)

lemmas *cons-neq-0* = *cons-not-0* [*THEN notE*]

declare *cons-not-0* [*THEN not-sym*, *simp*]

2.6 Singletons

lemma *singleton-iff*: $a \in \{b\} \longleftrightarrow a=b$
by *simp*

lemma *singletonI* [*intro*!]: $a \in \{a\}$
by (*rule consI1*)

lemmas *singletonE* = *singleton-iff* [*THEN iffD1*, *elim-format*, *elim*!]

2.7 Descriptions

lemma *the-equality* [*intro*]:
$[\![P(a); \bigwedge x. P(x) \Longrightarrow x{=}a]\!] \Longrightarrow (THE\ x.\ P(x)) = a$
unfolding *the-def*
apply (*fast dest*: *subst*)
done

lemma *the-equality2*: $[\![\exists!x. P(x); P(a)]\!] \Longrightarrow (THE\ x.\ P(x)) = a$
by *blast*

lemma *theI*: $\exists!x. P(x) \Longrightarrow P(THE\ x.\ P(x))$
apply (*erule ex1E*)
apply (*subst the-equality*)
apply (*blast*+)
done

lemma *the-0*: $\neg (\exists!x. P(x)) \Longrightarrow (THE\ x.\ P(x)){=}0$
unfolding *the-def*
apply (*blast elim*!: *ReplaceE*)
done

lemma *theI2*:
assumes *p1*: $\neg Q(0) \Longrightarrow \exists!x. P(x)$
and *p2*: $\bigwedge x. P(x) \Longrightarrow Q(x)$
shows $Q(THE\ x.\ P(x))$
apply (*rule classical*)
apply (*rule p2*)
apply (*rule theI*)
apply (*rule classical*)
apply (*rule p1*)
apply (*erule the-0* [*THEN subst*], *assumption*)
done

lemma *the-eq-trivial* [*simp*]: $(THE\ x.\ x = a) = a$
by *blast*

lemma *the-eq-trivial2* [*simp*]: $(THE\ x.\ a = x) = a$

by *blast*

2.8 Conditional Terms: *if−then−else*

lemma *if-true* [*simp*]: (*if True then a else b*) = *a*
by (*unfold if-def*, *blast*)

lemma *if-false* [*simp*]: (*if False then a else b*) = *b*
by (*unfold if-def*, *blast*)

lemma *if-cong*:
$[\![P \longleftrightarrow Q;\ \ Q \Longrightarrow a{=}c;\ \ \neg Q \Longrightarrow b{=}d]\!]$
$\Longrightarrow$ (*if P then a else b*) = (*if Q then c else d*)
by (*simp add*: *if-def cong add*: *conj-cong*)

lemma *if-weak-cong*: $P \longleftrightarrow Q \Longrightarrow$ (*if P then x else y*) = (*if Q then x else y*)
by *simp*

lemma *if-P*: $P \Longrightarrow$ (*if P then a else b*) = *a*
by (*unfold if-def*, *blast*)

lemma *if-not-P*: $\neg P \Longrightarrow$ (*if P then a else b*) = *b*
by (*unfold if-def*, *blast*)

lemma *split-if* [*split*]:
P(*if Q then x else y*) $\longleftrightarrow ((Q \longrightarrow P(x)) \wedge (\neg Q \longrightarrow P(y)))$
by (*case-tac Q*, *simp-all*)

lemmas *split-if-eq1* = *split-if* [*of* $\lambda x.\ x = b$] **for** *b*
lemmas *split-if-eq2* = *split-if* [*of* $\lambda x.\ a = x$] **for** *a*

lemmas *split-if-mem1* = *split-if* [*of* $\lambda x.\ x \in b$] **for** *b*
lemmas *split-if-mem2* = *split-if* [*of* $\lambda x.\ a \in x$] **for** *a*

lemmas *split-ifs* = *split-if-eq1 split-if-eq2 split-if-mem1 split-if-mem2*

lemma *if-iff*: *a*: (*if P then x else y*) $\longleftrightarrow P \wedge a \in x \mid \neg P \wedge a \in y$
by *simp*

lemma *if-type* [*TC*]:
$[\![P \Longrightarrow a \in A;\ \ \neg P \Longrightarrow b \in A]\!] \Longrightarrow$ (*if P then a else b*): *A*
by *simp*

lemma *split-if-asm*: $P(\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y) \longleftrightarrow (\neg((Q \wedge \neg P(x)) \mid (\neg Q \wedge \neg P(y))))$
by *simp*

lemmas *if-splits* = *split-if split-if-asm*

2.9 Consequences of Foundation

lemma *mem-asym*: $[\![a \in b;\ \neg P \Longrightarrow b \in a]\!] \Longrightarrow P$
apply (*rule classical*)
apply (*rule-tac A1* = $\{a,b\}$ **in** *foundation* [*THEN disjE*])
apply (*blast elim*!: *equalityE*)+
done

lemma *mem-irrefl*: $a \in a \Longrightarrow P$
by (*blast intro*: *mem-asym*)

lemma *mem-not-refl*: $a \notin a$
apply (*rule notI*)
apply (*erule mem-irrefl*)
done

lemma *mem-imp-not-eq*: $a \in A \Longrightarrow a \neq A$
by (*blast elim*!: *mem-irrefl*)

lemma *eq-imp-not-mem*: $a=A \Longrightarrow a \notin A$
by (*blast intro*: *elim*: *mem-irrefl*)

2.10 Rules for Successor

lemma *succ-iff*: $i \in \mathit{succ}(j) \longleftrightarrow i=j \mid i \in j$
by (*unfold succ-def*, *blast*)

lemma *succI1* [*simp*]: $i \in \mathit{succ}(i)$
by (*simp add*: *succ-iff*)

lemma *succI2*: $i \in j \Longrightarrow i \in \mathit{succ}(j)$
by (*simp add*: *succ-iff*)

lemma *succE* [*elim*!]:
$[\![i \in \mathit{succ}(j);\ i=j \Longrightarrow P;\ i \in j \Longrightarrow P]\!] \Longrightarrow P$
apply (*simp add*: *succ-iff*, *blast*)
done

lemma *succCI* [*intro*!]: $(i \notin j \Longrightarrow i=j) \Longrightarrow i \in succ(j)$
by (*simp add*: *succ-iff*, *blast*)

lemma *succ-not-0* [*simp*]: $succ(n) \neq 0$
by (*blast elim*!: *equalityE*)

lemmas *succ-neq-0* = *succ-not-0* [*THEN notE*, *elim*!]

declare *succ-not-0* [*THEN not-sym*, *simp*]
declare *sym* [*THEN succ-neq-0*, *elim*!]

lemmas *succ-subsetD* = *succI1* [*THEN* [*2*] *subsetD*]

lemmas *succ-neq-self* = *succI1* [*THEN mem-imp-not-eq*, *THEN not-sym*]

lemma *succ-inject-iff* [*simp*]: $succ(m) = succ(n) \longleftrightarrow m=n$
by (*blast elim*: *mem-asym elim*!: *equalityE*)

lemmas *succ-inject* = *succ-inject-iff* [*THEN iffD1*, *dest*!]

2.11 Miniscoping of the Bounded Universal Quantifier

lemma *ball-simps1*:
$(\forall x\in A.\ P(x) \land Q) \longleftrightarrow (\forall x\in A.\ P(x)) \land (A=0 \mid Q)$
$(\forall x\in A.\ P(x) \mid Q) \longleftrightarrow ((\forall x\in A.\ P(x)) \mid Q)$
$(\forall x\in A.\ P(x) \longrightarrow Q) \longleftrightarrow ((\exists x\in A.\ P(x)) \longrightarrow Q)$
$(\neg(\forall x\in A.\ P(x))) \longleftrightarrow (\exists x\in A.\ \neg P(x))$
$(\forall x\in 0.P(x)) \longleftrightarrow True$
$(\forall x\in succ(i).P(x)) \longleftrightarrow P(i) \land (\forall x\in i.\ P(x))$
$(\forall x\in cons(a,B).P(x)) \longleftrightarrow P(a) \land (\forall x\in B.\ P(x))$
$(\forall x\in RepFun(A,f).\ P(x)) \longleftrightarrow (\forall y\in A.\ P(f(y)))$
$(\forall x\in\bigcup(A).P(x)) \longleftrightarrow (\forall y\in A.\ \forall x\in y.\ P(x))$
by *blast*+

lemma *ball-simps2*:
$(\forall x\in A.\ P \land Q(x)) \longleftrightarrow (A=0 \mid P) \land (\forall x\in A.\ Q(x))$
$(\forall x\in A.\ P \mid Q(x)) \longleftrightarrow (P \mid (\forall x\in A.\ Q(x)))$
$(\forall x\in A.\ P \longrightarrow Q(x)) \longleftrightarrow (P \longrightarrow (\forall x\in A.\ Q(x)))$
by *blast*+

lemma *ball-simps3*:
$(\forall x\in Collect(A,Q).P(x)) \longleftrightarrow (\forall x\in A.\ Q(x) \longrightarrow P(x))$
by *blast*+

lemmas *ball-simps* [*simp*] = *ball-simps1 ball-simps2 ball-simps3*

lemma *ball-conj-distrib*:
$(\forall x{\in}A.\ P(x) \wedge Q(x)) \longleftrightarrow ((\forall x{\in}A.\ P(x)) \wedge (\forall x{\in}A.\ Q(x)))$
by *blast*

2.12 Miniscoping of the Bounded Existential Quantifier

lemma *bex-simps1*:
$(\exists x{\in}A.\ P(x) \wedge Q) \longleftrightarrow ((\exists x{\in}A.\ P(x)) \wedge Q)$
$(\exists x{\in}A.\ P(x) \mid Q) \longleftrightarrow (\exists x{\in}A.\ P(x)) \mid (A{\neq}0 \wedge Q)$
$(\exists x{\in}A.\ P(x) \longrightarrow Q) \longleftrightarrow ((\forall x{\in}A.\ P(x)) \longrightarrow (A{\neq}0 \wedge Q))$
$(\exists x{\in}0.P(x)) \longleftrightarrow False$
$(\exists x{\in}succ(i).P(x)) \longleftrightarrow P(i) \mid (\exists x{\in}i.\ P(x))$
$(\exists x{\in}cons(a,B).P(x)) \longleftrightarrow P(a) \mid (\exists x{\in}B.\ P(x))$
$(\exists x{\in}RepFun(A,f).\ P(x)) \longleftrightarrow (\exists y{\in}A.\ P(f(y)))$
$(\exists x{\in}\bigcup(A).P(x)) \longleftrightarrow (\exists y{\in}A.\ \exists x{\in}y.\ \ P(x))$
$(\neg(\exists x{\in}A.\ P(x))) \longleftrightarrow (\forall x{\in}A.\ \neg P(x))$
by *blast+*

lemma *bex-simps2*:
$(\exists x{\in}A.\ P \wedge Q(x)) \longleftrightarrow (P \wedge (\exists x{\in}A.\ Q(x)))$
$(\exists x{\in}A.\ P \mid Q(x)) \longleftrightarrow (A{\neq}0 \wedge P) \mid (\exists x{\in}A.\ Q(x))$
$(\exists x{\in}A.\ P \longrightarrow Q(x)) \longleftrightarrow ((A{=}0 \mid P) \longrightarrow (\exists x{\in}A.\ Q(x)))$
by *blast+*

lemma *bex-simps3*:
$(\exists x{\in}Collect(A,Q).P(x)) \longleftrightarrow (\exists x{\in}A.\ Q(x) \wedge P(x))$
by *blast*

lemmas *bex-simps* [*simp*] = *bex-simps1 bex-simps2 bex-simps3*

lemma *bex-disj-distrib*:
$(\exists x{\in}A.\ P(x) \mid Q(x)) \longleftrightarrow ((\exists x{\in}A.\ P(x)) \mid (\exists x{\in}A.\ Q(x)))$
by *blast*

lemma *bex-triv-one-point1* [*simp*]: $(\exists x{\in}A.\ x{=}a) \longleftrightarrow (a \in A)$
by *blast*

lemma *bex-triv-one-point2* [*simp*]: $(\exists x{\in}A.\ a{=}x) \longleftrightarrow (a \in A)$
by *blast*

lemma *bex-one-point1* [*simp*]: $(\exists x{\in}A.\ x{=}a \wedge P(x)) \longleftrightarrow (a \in A \wedge P(a))$
by *blast*

lemma *bex-one-point2* [*simp*]: $(\exists x{\in}A.\ a{=}x \wedge P(x)) \longleftrightarrow (a \in A \wedge P(a))$
by *blast*

lemma *ball-one-point1* [*simp*]: $(\forall x{\in}A.\ x{=}a \longrightarrow P(x)) \longleftrightarrow (a \in A \longrightarrow P(a))$
by *blast*

lemma *ball-one-point2* [*simp*]: $(\forall x{\in}A.\ a{=}x \longrightarrow P(x)) \longleftrightarrow (a \in A \longrightarrow P(a))$
by *blast*

2.13 Miniscoping of the Replacement Operator

These cover both *Replace* and *Collect*

lemma *Rep-simps* [*simp*]:
$\{x.\ y \in 0,\ R(x,y)\} = 0$
$\{x \in 0.\ P(x)\} = 0$
$\{x \in A.\ Q\} = (\textit{if } Q \textit{ then } A \textit{ else } 0)$
$RepFun(0,f) = 0$
$RepFun(succ(i),f) = cons(f(i),\ RepFun(i,f))$
$RepFun(cons(a,B),f) = cons(f(a),\ RepFun(B,f))$
by (*simp-all*, *blast*+)

2.14 Miniscoping of Unions

lemma *UN-simps1*:
$(\bigcup x{\in}C.\ cons(a,\ B(x))) = (\textit{if } C{=}0 \textit{ then } 0 \textit{ else } cons(a,\ \bigcup x{\in}C.\ B(x)))$
$(\bigcup x{\in}C.\ A(x) \cup B') \quad = (\textit{if } C{=}0 \textit{ then } 0 \textit{ else } (\bigcup x{\in}C.\ A(x)) \cup B')$
$(\bigcup x{\in}C.\ A' \cup B(x)) \quad = (\textit{if } C{=}0 \textit{ then } 0 \textit{ else } A' \cup (\bigcup x{\in}C.\ B(x)))$
$(\bigcup x{\in}C.\ A(x) \cap B') \ = ((\bigcup x{\in}C.\ A(x)) \cap B')$
$(\bigcup x{\in}C.\ A' \cap B(x)) \ = (A' \cap (\bigcup x{\in}C.\ B(x)))$
$(\bigcup x{\in}C.\ A(x) - B') \quad = ((\bigcup x{\in}C.\ A(x)) - B')$
$(\bigcup x{\in}C.\ A' - B(x)) \quad = (\textit{if } C{=}0 \textit{ then } 0 \textit{ else } A' - (\bigcap x{\in}C.\ B(x)))$
apply (*simp-all add*: *Inter-def*)
apply (*blast intro*!: *equalityI*)+
done

lemma *UN-simps2*:
$(\bigcup x{\in}\bigcup(A).\ B(x)) = (\bigcup y{\in}A.\ \bigcup x{\in}y.\ B(x))$
$(\bigcup z{\in}(\bigcup x{\in}A.\ B(x)).\ C(z)) = (\bigcup x{\in}A.\ \bigcup z{\in}B(x).\ C(z))$
$(\bigcup x{\in}RepFun(A,f).\ B(x)) \quad = (\bigcup a{\in}A.\ B(f(a)))$
by *blast*+

lemmas *UN-simps* [*simp*] = *UN-simps1 UN-simps2*

Opposite of miniscoping: pull the operator out

lemma *UN-extend-simps1*:
$(\bigcup x{\in}C.\ A(x)) \cup B \quad = (\textit{if } C{=}0 \textit{ then } B \textit{ else } (\bigcup x{\in}C.\ A(x) \cup B))$
$((\bigcup x{\in}C.\ A(x)) \cap B) = (\bigcup x{\in}C.\ A(x) \cap B)$
$((\bigcup x{\in}C.\ A(x)) - B) = (\bigcup x{\in}C.\ A(x) - B)$
apply *simp-all*
apply *blast*+
done

lemma *UN-extend-simps2*:
  $cons(a, \bigcup x\in C.\ B(x)) = (if\ C=0\ then\ \{a\}\ else\ (\bigcup x\in C.\ cons(a, B(x))))$
  $A \cup (\bigcup x\in C.\ B(x)) = (if\ C=0\ then\ A\ else\ (\bigcup x\in C.\ A \cup B(x)))$
  $(A \cap (\bigcup x\in C.\ B(x))) = (\bigcup x\in C.\ A \cap B(x))$
  $A - (\bigcap x\in C.\ B(x)) = (if\ C=0\ then\ A\ else\ (\bigcup x\in C.\ A - B(x)))$
  $(\bigcup y\in A.\ \bigcup x\in y.\ B(x)) = (\bigcup x\in\bigcup(A).\ B(x))$
  $(\bigcup a\in A.\ B(f(a))) = (\bigcup x\in RepFun(A,f).\ B(x))$
apply (*simp-all add*: *Inter-def*)
apply (*blast intro*!: *equalityI*)+
done

lemma *UN-UN-extend*:
  $(\bigcup x\in A.\ \bigcup z\in B(x).\ C(z)) = (\bigcup z\in(\bigcup x\in A.\ B(x)).\ C(z))$
by *blast*

lemmas *UN-extend-simps* = *UN-extend-simps1 UN-extend-simps2 UN-UN-extend*

2.15 Miniscoping of Intersections

lemma *INT-simps1*:
  $(\bigcap x\in C.\ A(x) \cap B) = (\bigcap x\in C.\ A(x)) \cap B$
  $(\bigcap x\in C.\ A(x) - B) = (\bigcap x\in C.\ A(x)) - B$
  $(\bigcap x\in C.\ A(x) \cup B) = (if\ C=0\ then\ 0\ else\ (\bigcap x\in C.\ A(x)) \cup B)$
by (*simp-all add*: *Inter-def*, *blast*+)

lemma *INT-simps2*:
  $(\bigcap x\in C.\ A \cap B(x)) = A \cap (\bigcap x\in C.\ B(x))$
  $(\bigcap x\in C.\ A - B(x)) = (if\ C=0\ then\ 0\ else\ A - (\bigcup x\in C.\ B(x)))$
  $(\bigcap x\in C.\ cons(a, B(x))) = (if\ C=0\ then\ 0\ else\ cons(a, \bigcap x\in C.\ B(x)))$
  $(\bigcap x\in C.\ A \cup B(x)) = (if\ C=0\ then\ 0\ else\ A \cup (\bigcap x\in C.\ B(x)))$
apply (*simp-all add*: *Inter-def*)
apply (*blast intro*!: *equalityI*)+
done

lemmas *INT-simps* [*simp*] = *INT-simps1 INT-simps2*

Opposite of miniscoping: pull the operator out

lemma *INT-extend-simps1*:
  $(\bigcap x\in C.\ A(x)) \cap B = (\bigcap x\in C.\ A(x) \cap B)$
  $(\bigcap x\in C.\ A(x)) - B = (\bigcap x\in C.\ A(x) - B)$
  $(\bigcap x\in C.\ A(x)) \cup B = (if\ C=0\ then\ B\ else\ (\bigcap x\in C.\ A(x) \cup B))$
apply (*simp-all add*: *Inter-def*, *blast*+)
done

lemma *INT-extend-simps2*:
  $A \cap (\bigcap x\in C.\ B(x)) = (\bigcap x\in C.\ A \cap B(x))$
  $A - (\bigcup x\in C.\ B(x)) = (if\ C=0\ then\ A\ else\ (\bigcap x\in C.\ A - B(x)))$
  $cons(a, \bigcap x\in C.\ B(x)) = (if\ C=0\ then\ \{a\}\ else\ (\bigcap x\in C.\ cons(a, B(x))))$
  $A \cup (\bigcap x\in C.\ B(x)) = (if\ C=0\ then\ A\ else\ (\bigcap x\in C.\ A \cup B(x)))$

apply (*simp-all add*: *Inter-def*)
apply (*blast intro*!: *equalityI*)+
done

lemmas *INT-extend-simps* = *INT-extend-simps1 INT-extend-simps2*

2.16 Other simprules

lemma *misc-simps* [*simp*]:
  $0 \cup A = A$
  $A \cup 0 = A$
  $0 \cap A = 0$
  $A \cap 0 = 0$
  $0 - A = 0$
  $A - 0 = A$
  $\bigcup(0) = 0$
  $\bigcup(cons(b,A)) = b \cup \bigcup(A)$
  $\bigcap(\{b\}) = b$
by *blast*+

end

3 Ordered Pairs

theory *pair* **imports** *upair*
begin

ML-file ‹*simpdata.ML*›

setup ‹
  map-theory-simpset
    (*Simplifier.set-mksimps* (*fn ctxt* => *map mk-eq o ZF-atomize o Variable.gen-all ctxt*)
      #> *Simplifier.add-cong* @{*thm if-weak-cong*})
›

ML ‹*val ZF-ss* = *simpset-of* ***context***›

simproc-setup *defined-Bex* ($\exists x \in A.\ P(x) \wedge Q(x)$) = ‹
  K (*Quantifier1.rearrange-Bex* (*fn ctxt* => *unfold-tac ctxt* @{*thms Bex-def*}))
›

simproc-setup *defined-Ball* ($\forall x \in A.\ P(x) \longrightarrow Q(x)$) = ‹
  K (*Quantifier1.rearrange-Ball* (*fn ctxt* => *unfold-tac ctxt* @{*thms Ball-def*}))
›

lemma *singleton-eq-iff* [*iff*]: $\{a\} = \{b\} \longleftrightarrow a=b$
by (*rule extension* [*THEN iff-trans*], *blast*)

lemma *doubleton-eq-iff*: $\{a,b\} = \{c,d\} \longleftrightarrow (a=c \wedge b=d) \mid (a=d \wedge b=c)$
by (*rule extension* [*THEN iff-trans*], *blast*)

lemma *Pair-iff* [*simp*]: $\langle a,b\rangle = \langle c,d\rangle \longleftrightarrow a=c \wedge b=d$
by (*simp add*: *Pair-def doubleton-eq-iff*, *blast*)

lemmas *Pair-inject* = *Pair-iff* [*THEN iffD1*, *THEN conjE*, *elim*!]

lemmas *Pair-inject1* = *Pair-iff* [*THEN iffD1*, *THEN conjunct1*]
lemmas *Pair-inject2* = *Pair-iff* [*THEN iffD1*, *THEN conjunct2*]

lemma *Pair-not-0*: $\langle a,b\rangle \neq 0$
  unfolding *Pair-def*
apply (*blast elim*: *equalityE*)
done

lemmas *Pair-neq-0* = *Pair-not-0* [*THEN notE*, *elim*!]

declare *sym* [*THEN Pair-neq-0*, *elim*!]

lemma *Pair-neq-fst*: $\langle a,b\rangle=a \Longrightarrow P$
proof (*unfold Pair-def*)
  assume *eq*: $\{\{a, a\}, \{a, b\}\} = a$
  have $\{a, a\} \in \{\{a, a\}, \{a, b\}\}$ **by** (*rule consI1*)
  hence $\{a, a\} \in a$ **by** (*simp add*: *eq*)
  moreover have $a \in \{a, a\}$ **by** (*rule consI1*)
  ultimately show *P* **by** (*rule mem-asym*)
qed

lemma *Pair-neq-snd*: $\langle a,b\rangle=b \Longrightarrow P$
proof (*unfold Pair-def*)
  assume *eq*: $\{\{a, a\}, \{a, b\}\} = b$
  have $\{a, b\} \in \{\{a, a\}, \{a, b\}\}$ **by** *blast*
  hence $\{a, b\} \in b$ **by** (*simp add*: *eq*)
  moreover have $b \in \{a, b\}$ **by** *blast*
  ultimately show *P* **by** (*rule mem-asym*)
qed

3.1 Sigma: Disjoint Union of a Family of Sets

Generalizes Cartesian product

lemma *Sigma-iff* [*simp*]: $\langle a,b\rangle$: $Sigma(A,B) \longleftrightarrow a \in A \wedge b \in B(a)$
by (*simp add*: *Sigma-def*)

lemma *SigmaI* [*TC*,*intro*!]: $[\![a \in A;\ \ b \in B(a)]\!] \Longrightarrow \langle a,b\rangle \in Sigma(A,B)$
by *simp*

lemmas *SigmaD1* = *Sigma-iff* [*THEN iffD1*, *THEN conjunct1*]
lemmas *SigmaD2* = *Sigma-iff* [*THEN iffD1*, *THEN conjunct2*]

lemma *SigmaE* [*elim*!]:
$[\![c \in Sigma(A,B);$
$\bigwedge x\ y.[\![x \in A;\ \ y \in B(x);\ \ c=\langle x,y\rangle]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
by (*unfold Sigma-def*, *blast*)

lemma *SigmaE2* [*elim*!]:
$[\![\langle a,b\rangle \in Sigma(A,B);$
$[\![a \in A;\ \ b \in B(a)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
by (*unfold Sigma-def*, *blast*)

lemma *Sigma-cong*:
$[\![A=A';\ \ \bigwedge x.\ x \in A' \Longrightarrow B(x)=B'(x)]\!] \Longrightarrow$
$Sigma(A,B) = Sigma(A',B')$
by (*simp add*: *Sigma-def*)

lemma *Sigma-empty1* [*simp*]: $Sigma(0,B) = 0$
by *blast*

lemma *Sigma-empty2* [*simp*]: $A*0 = 0$
by *blast*

lemma *Sigma-empty-iff*: $A*B=0 \longleftrightarrow A=0\ |\ B=0$
by *blast*

3.2 Projections *fst* and *snd*

lemma *fst-conv* [*simp*]: $fst(\langle a,b\rangle) = a$
by (*simp add*: *fst-def*)

lemma *snd-conv* [*simp*]: $snd(\langle a,b\rangle) = b$
by (*simp add*: *snd-def*)

lemma *fst-type* [*TC*]: $p \in Sigma(A,B) \Longrightarrow fst(p) \in A$
by *auto*

lemma *snd-type* [*TC*]: $p \in Sigma(A,B) \Longrightarrow snd(p) \in B(fst(p))$
by *auto*

lemma *Pair-fst-snd-eq*: $a \in Sigma(A,B) \Longrightarrow <fst(a),snd(a)> = a$
by *auto*

3.3 The Eliminator, *split*

lemma *split* [*simp*]: $split(\lambda x\ y.\ c(x,y),\ \langle a,b\rangle) \equiv c(a,b)$
by (*simp add*: *split-def*)

lemma *split-type* [*TC*]:
$[\![p \in Sigma(A,B);$
$\bigwedge x\ y.[\![x \in A;\ y \in B(x)]\!] \Longrightarrow c(x,y):C(\langle x,y\rangle)$
$]\!] \Longrightarrow split(\lambda x\ y.\ c(x,y),\ p) \in C(p)$
by (*erule SigmaE*, *auto*)

lemma *expand-split*:
$u \in A*B \Longrightarrow$
$R(split(c,u)) \longleftrightarrow (\forall x{\in}A.\ \forall y{\in}B.\ u = \langle x,y\rangle \longrightarrow R(c(x,y)))$
by (*auto simp add*: *split-def*)

3.4 A version of *split* for Formulae: Result Type *o*

lemma *splitI*: $R(a,b) \Longrightarrow split(R,\ \langle a,b\rangle)$
by (*simp add*: *split-def*)

lemma *splitE*:
$[\![split(R,z);\ \ z \in Sigma(A,B);$
$\bigwedge x\ y.\ [\![z = \langle x,y\rangle;\ \ R(x,y)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
by (*auto simp add*: *split-def*)

lemma *splitD*: $split(R,\langle a,b\rangle) \Longrightarrow R(a,b)$
by (*simp add*: *split-def*)

Complex rules for Sigma.

lemma *split-paired-Bex-Sigma* [*simp*]:
$(\exists z \in Sigma(A,B).\ P(z)) \longleftrightarrow (\exists x \in A.\ \exists y \in B(x).\ P(\langle x,y\rangle))$
by *blast*

lemma *split-paired-Ball-Sigma* [*simp*]:
$(\forall z \in Sigma(A,B).\ P(z)) \longleftrightarrow (\forall x \in A.\ \forall y \in B(x).\ P(\langle x,y\rangle))$
by *blast*

end

4 Basic Equalities and Inclusions

theory *equalities* **imports** *pair* **begin**

These cover union, intersection, converse, domain, range, etc. Philippe de Groote proved many of the inclusions.

lemma *in-mono*: $A{\subseteq}B \Longrightarrow x{\in}A \longrightarrow x{\in}B$

by *blast*

lemma *the-eq-0* [*simp*]: (*THE x. False*) = *0*
by (*blast intro*: *the-0*)

4.1 Bounded Quantifiers

The following are not added to the default simpset because (a) they duplicate the body and (b) there are no similar rules for *Int*.

lemma *ball-Un*: $(\forall x \in A \cup B.\ P(x)) \longleftrightarrow (\forall x \in A.\ P(x)) \land (\forall x \in B.\ P(x))$
by *blast*

lemma *bex-Un*: $(\exists x \in A \cup B.\ P(x)) \longleftrightarrow (\exists x \in A.\ P(x)) \mid (\exists x \in B.\ P(x))$
by *blast*

lemma *ball-UN*: $(\forall z \in (\bigcup x{\in}A.\ B(x)).\ P(z)) \longleftrightarrow (\forall x{\in}A.\ \forall z \in B(x).\ P(z))$
by *blast*

lemma *bex-UN*: $(\exists z \in (\bigcup x{\in}A.\ B(x)).\ P(z)) \longleftrightarrow (\exists x{\in}A.\ \exists z{\in}B(x).\ P(z))$
by *blast*

4.2 Converse of a Relation

lemma *converse-iff* [*simp*]: $\langle a,b\rangle \in converse(r) \longleftrightarrow \langle b,a\rangle \in r$
by (*unfold converse-def*, *blast*)

lemma *converseI* [*intro*!]: $\langle a,b\rangle \in r \Longrightarrow \langle b,a\rangle \in converse(r)$
by (*unfold converse-def*, *blast*)

lemma *converseD*: $\langle a,b\rangle \in converse(r) \Longrightarrow \langle b,a\rangle \in r$
by (*unfold converse-def*, *blast*)

lemma *converseE* [*elim*!]:
$[\![yx \in converse(r);$
$\bigwedge x\ y.\ [\![yx=\langle y,x\rangle;\ \ \langle x,y\rangle \in r]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
by (*unfold converse-def*, *blast*)

lemma *converse-converse*: $r \subseteq Sigma(A,B) \Longrightarrow converse(converse(r)) = r$
by *blast*

lemma *converse-type*: $r \subseteq A*B \Longrightarrow converse(r) \subseteq B*A$
by *blast*

lemma *converse-prod* [*simp*]: $converse(A*B) = B*A$
by *blast*

lemma *converse-empty* [*simp*]: *converse*(*0*) = *0*

by *blast*

lemma *converse-subset-iff*:
$A \subseteq Sigma(X,Y) \Longrightarrow converse(A) \subseteq converse(B) \longleftrightarrow A \subseteq B$
by *blast*

4.3 Finite Set Constructions Using *cons*

lemma *cons-subsetI*: $[\![a{\in}C;\ B{\subseteq}C]\!] \Longrightarrow cons(a,B) \subseteq C$
by *blast*

lemma *subset-consI*: $B \subseteq cons(a,B)$
by *blast*

lemma *cons-subset-iff* [*iff*]: $cons(a,B){\subseteq}C \longleftrightarrow a{\in}C \wedge B{\subseteq}C$
by *blast*

lemmas *cons-subsetE* = *cons-subset-iff* [*THEN iffD1*, *THEN conjE*]

lemma *subset-empty-iff*: $A{\subseteq}0 \longleftrightarrow A{=}0$
by *blast*

lemma *subset-cons-iff*: $C{\subseteq}cons(a,B) \longleftrightarrow C{\subseteq}B \mid (a{\in}C \wedge C-\{a\} \subseteq B)$
by *blast*

lemma *cons-eq*: $\{a\} \cup B = cons(a,B)$
by *blast*

lemma *cons-commute*: $cons(a,\ cons(b,\ C)) = cons(b,\ cons(a,\ C))$
by *blast*

lemma *cons-absorb*: $a\colon B \Longrightarrow cons(a,B) = B$
by *blast*

lemma *cons-Diff*: $a\colon B \Longrightarrow cons(a,\ B-\{a\}) = B$
by *blast*

lemma *Diff-cons-eq*: $cons(a,B) - C = (if\ a{\in}C\ then\ B-C\ else\ cons(a,B-C))$
by *auto*

lemma *equal-singleton*: $[\![a\colon C;\ \bigwedge y.\ y \in C \Longrightarrow y{=}b]\!] \Longrightarrow C = \{b\}$
by *blast*

lemma [*simp*]: $cons(a,cons(a,B)) = cons(a,B)$
by *blast*

lemma *singleton-subsetI*: $a{\in}C \Longrightarrow \{a\} \subseteq C$
by *blast*

lemma *singleton-subsetD*: $\{a\} \subseteq C \Longrightarrow a{\in}C$
by *blast*

lemma *subset-succI*: $i \subseteq succ(i)$
by *blast*

lemma *succ-subsetI*: $[\![i{\in}j;\ i{\subseteq}j]\!] \Longrightarrow succ(i){\subseteq}j$
by (*unfold succ-def*, *blast*)

lemma *succ-subsetE*:
$[\![succ(i) \subseteq j;\ [\![i{\in}j;\ i{\subseteq}j]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*unfold succ-def*, *blast*)

lemma *succ-subset-iff*: $succ(a) \subseteq B \longleftrightarrow (a \subseteq B \wedge a \in B)$
by (*unfold succ-def*, *blast*)

4.4 Binary Intersection

lemma *Int-subset-iff*: $C \subseteq A \cap B \longleftrightarrow C \subseteq A \wedge C \subseteq B$
by *blast*

lemma *Int-lower1*: $A \cap B \subseteq A$
by *blast*

lemma *Int-lower2*: $A \cap B \subseteq B$
by *blast*

lemma *Int-greatest*: $[\![C{\subseteq}A;\ C{\subseteq}B]\!] \Longrightarrow C \subseteq A \cap B$
by *blast*

lemma *Int-cons*: $cons(a,B) \cap C \subseteq cons(a, B \cap C)$
by *blast*

lemma *Int-absorb* [*simp*]: $A \cap A = A$
by *blast*

lemma *Int-left-absorb*: $A \cap (A \cap B) = A \cap B$
by *blast*

lemma *Int-commute*: $A \cap B = B \cap A$
by *blast*

lemma *Int-left-commute*: $A \cap (B \cap C) = B \cap (A \cap C)$
by *blast*

lemma *Int-assoc*: $(A \cap B) \cap C = A \cap (B \cap C)$
by *blast*

lemmas *Int-ac*= *Int-assoc Int-left-absorb Int-commute Int-left-commute*

lemma *Int-absorb1*: $B \subseteq A \Longrightarrow A \cap B = B$
by *blast*

lemma *Int-absorb2*: $A \subseteq B \Longrightarrow A \cap B = A$
by *blast*

lemma *Int-Un-distrib*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
by *blast*

lemma *Int-Un-distrib2*: $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
by *blast*

lemma *subset-Int-iff*: $A \subseteq B \longleftrightarrow A \cap B = A$
by (*blast elim*!: *equalityE*)

lemma *subset-Int-iff2*: $A \subseteq B \longleftrightarrow B \cap A = A$
by (*blast elim*!: *equalityE*)

lemma *Int-Diff-eq*: $C \subseteq A \Longrightarrow (A-B) \cap C = C-B$
by *blast*

lemma *Int-cons-left*:
$cons(a,A) \cap B = (if\ a \in B\ then\ cons(a,\ A \cap B)\ else\ A \cap B)$
by *auto*

lemma *Int-cons-right*:
$A \cap cons(a,\ B) = (if\ a \in A\ then\ cons(a,\ A \cap B)\ else\ A \cap B)$
by *auto*

lemma *cons-Int-distrib*: $cons(x,\ A \cap B) = cons(x,\ A) \cap cons(x,\ B)$
by *auto*

4.5 Binary Union

lemma *Un-subset-iff*: $A \cup B \subseteq C \longleftrightarrow A \subseteq C \land B \subseteq C$
by *blast*

lemma *Un-upper1*: $A \subseteq A \cup B$
by *blast*

lemma *Un-upper2*: $B \subseteq A \cup B$
by *blast*

lemma *Un-least*: $[\![A \subseteq C;\ \ B \subseteq C]\!] \Longrightarrow A \cup B \subseteq C$
by *blast*

lemma *Un-cons*: $cons(a,B) \cup C = cons(a,\ B \cup C)$
by *blast*

lemma *Un-absorb* [*simp*]: $A \cup A = A$
by *blast*

lemma *Un-left-absorb*: $A \cup (A \cup B) = A \cup B$
by *blast*

lemma *Un-commute*: $A \cup B = B \cup A$
by *blast*

lemma *Un-left-commute*: $A \cup (B \cup C) = B \cup (A \cup C)$
by *blast*

lemma *Un-assoc*: $(A \cup B) \cup C\ =\ A \cup (B \cup C)$
by *blast*

lemmas *Un-ac* = *Un-assoc Un-left-absorb Un-commute Un-left-commute*

lemma *Un-absorb1*: $A \subseteq B \Longrightarrow A \cup B = B$
  by *blast*

lemma *Un-absorb2*: $B \subseteq A \Longrightarrow A \cup B = A$
  by *blast*

lemma *Un-Int-distrib*: $(A \cap B) \cup C\ =\ (A \cup C) \cap (B \cup C)$
by *blast*

lemma *subset-Un-iff*: $A \subseteq B \longleftrightarrow A \cup B = B$
by (*blast elim*!: *equalityE*)

lemma *subset-Un-iff2*: $A \subseteq B \longleftrightarrow B \cup A = B$
by (*blast elim*!: *equalityE*)

lemma *Un-empty* [*iff*]: $(A \cup B = 0) \longleftrightarrow (A = 0 \wedge B = 0)$
by *blast*

lemma *Un-eq-Union*: $A \cup B = \bigcup(\{A,\ B\})$
by *blast*

4.6 Set Difference

lemma *Diff-subset*: $A-B \subseteq A$
by *blast*

lemma *Diff-contains*: $[\![C\subseteq A;\ \ C \cap B = 0]\!] \Longrightarrow C \subseteq A-B$
by *blast*

lemma *subset-Diff-cons-iff*: $B \subseteq A - cons(c,C) \longleftrightarrow B\subseteq A-C \wedge c \notin B$
by *blast*

lemma *Diff-cancel*: $A - A = 0$
by *blast*

lemma *Diff-triv*: $A \cap B = 0 \Longrightarrow A - B = A$
by *blast*

lemma *empty-Diff* [*simp*]: $0 - A = 0$
by *blast*

lemma *Diff-0* [*simp*]: $A - 0 = A$
by *blast*

lemma *Diff-eq-0-iff*: $A - B = 0 \longleftrightarrow A \subseteq B$
by (*blast elim*: *equalityE*)

lemma *Diff-cons*: $A - cons(a,B) = A - B - \{a\}$
by *blast*

lemma *Diff-cons2*: $A - cons(a,B) = A - \{a\} - B$
by *blast*

lemma *Diff-disjoint*: $A \cap (B-A) = 0$
by *blast*

lemma *Diff-partition*: $A\subseteq B \Longrightarrow A \cup (B-A) = B$
by *blast*

lemma *subset-Un-Diff*: $A \subseteq B \cup (A - B)$
by *blast*

lemma *double-complement*: $[\![A\subseteq B;\ B\subseteq C]\!] \Longrightarrow B-(C-A) = A$
by *blast*

lemma *double-complement-Un*: $(A \cup B) - (B-A) = A$
by *blast*

lemma *Un-Int-crazy*:

$(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$
apply *blast*
done

lemma *Diff-Un*: $A - (B \cup C) = (A-B) \cap (A-C)$
by *blast*

lemma *Diff-Int*: $A - (B \cap C) = (A-B) \cup (A-C)$
by *blast*

lemma *Un-Diff*: $(A \cup B) - C = (A - C) \cup (B - C)$
by *blast*

lemma *Int-Diff*: $(A \cap B) - C = A \cap (B - C)$
by *blast*

lemma *Diff-Int-distrib*: $C \cap (A-B) = (C \cap A) - (C \cap B)$
by *blast*

lemma *Diff-Int-distrib2*: $(A-B) \cap C = (A \cap C) - (B \cap C)$
by *blast*

lemma *Un-Int-assoc-iff*: $(A \cap B) \cup C = A \cap (B \cup C) \longleftrightarrow C \subseteq A$
by (*blast elim*!: *equalityE*)

4.7 Big Union and Intersection

lemma *Union-subset-iff*: $\bigcup(A) \subseteq C \longleftrightarrow (\forall x \in A.\ x \subseteq C)$
by *blast*

lemma *Union-upper*: $B \in A \Longrightarrow B \subseteq \bigcup(A)$
by *blast*

lemma *Union-least*: $[\![\bigwedge x.\ x \in A \Longrightarrow x \subseteq C]\!] \Longrightarrow \bigcup(A) \subseteq C$
by *blast*

lemma *Union-cons* [*simp*]: $\bigcup(cons(a,B)) = a \cup \bigcup(B)$
by *blast*

lemma *Union-Un-distrib*: $\bigcup(A \cup B) = \bigcup(A) \cup \bigcup(B)$
by *blast*

lemma *Union-Int-subset*: $\bigcup(A \cap B) \subseteq \bigcup(A) \cap \bigcup(B)$
by *blast*

lemma *Union-disjoint*: $\bigcup(C) \cap A = 0 \longleftrightarrow (\forall B \in C.\ B \cap A = 0)$
by (*blast elim*!: *equalityE*)

lemma *Union-empty-iff*: $\bigcup(A) = 0 \longleftrightarrow (\forall B{\in}A.\ B{=}0)$
by *blast*

lemma *Int-Union2*: $\bigcup(B) \cap A = (\bigcup C{\in}B.\ C \cap A)$
by *blast*

lemma *Inter-subset-iff*: $A{\neq}0 \implies C \subseteq \bigcap(A) \longleftrightarrow (\forall x{\in}A.\ C \subseteq x)$
by *blast*

lemma *Inter-lower*: $B{\in}A \Longrightarrow \bigcap(A) \subseteq B$
by *blast*

lemma *Inter-greatest*: $[\![A{\neq}0;\ \ \bigwedge x.\ x{\in}A \Longrightarrow C{\subseteq}x]\!] \Longrightarrow C \subseteq \bigcap(A)$
by *blast*

lemma *INT-lower*: $x{\in}A \Longrightarrow (\bigcap x{\in}A.\ B(x)) \subseteq B(x)$
by *blast*

lemma *INT-greatest*: $[\![A{\neq}0;\ \ \bigwedge x.\ x{\in}A \Longrightarrow C{\subseteq}B(x)]\!] \Longrightarrow C \subseteq (\bigcap x{\in}A.\ B(x))$
by *force*

lemma *Inter-0* [*simp*]: $\bigcap(0) = 0$
by (*unfold Inter-def*, *blast*)

lemma *Inter-Un-subset*:
$[\![z{\in}A;\ z{\in}B]\!] \Longrightarrow \bigcap(A) \cup \bigcap(B) \subseteq \bigcap(A \cap B)$
by *blast*

lemma *Inter-Un-distrib*:
$[\![A{\neq}0;\ \ B{\neq}0]\!] \Longrightarrow \bigcap(A \cup B) = \bigcap(A) \cap \bigcap(B)$
by *blast*

lemma *Union-singleton*: $\bigcup(\{b\}) = b$
by *blast*

lemma *Inter-singleton*: $\bigcap(\{b\}) = b$
by *blast*

lemma *Inter-cons* [*simp*]:
$\bigcap(cons(a,B)) = (\textit{if } B{=}0 \textit{ then } a \textit{ else } a \cap \bigcap(B))$
by *force*

4.8 Unions and Intersections of Families

lemma *subset-UN-iff-eq*: $A \subseteq (\bigcup i{\in}I.\ B(i)) \longleftrightarrow A = (\bigcup i{\in}I.\ A \cap B(i))$
by (*blast elim*!: *equalityE*)

lemma *UN-subset-iff*: $(\bigcup x{\in}A.\ B(x)) \subseteq C \longleftrightarrow (\forall x{\in}A.\ B(x) \subseteq C)$
by *blast*

lemma *UN-upper*: $x{\in}A \Longrightarrow B(x) \subseteq (\bigcup x{\in}A.\ B(x))$
by (*erule RepFunI* [*THEN Union-upper*])

lemma *UN-least*: $[\![\bigwedge x.\ x{\in}A \Longrightarrow B(x){\subseteq}C]\!] \Longrightarrow (\bigcup x{\in}A.\ B(x)) \subseteq C$
by *blast*

lemma *Union-eq-UN*: $\bigcup(A) = (\bigcup x{\in}A.\ x)$
by *blast*

lemma *Inter-eq-INT*: $\bigcap(A) = (\bigcap x{\in}A.\ x)$
by (*unfold Inter-def*, *blast*)

lemma *UN-0* [*simp*]: $(\bigcup i{\in}0.\ A(i)) = 0$
by *blast*

lemma *UN-singleton*: $(\bigcup x{\in}A.\ \{x\}) = A$
by *blast*

lemma *UN-Un*: $(\bigcup i{\in}\ A \cup B.\ C(i)) = (\bigcup i{\in}\ A.\ C(i)) \cup (\bigcup i{\in}B.\ C(i))$
by *blast*

lemma *INT-Un*: $(\bigcap i{\in}I \cup J.\ A(i)) =$
$(\textit{if } I{=}0 \textit{ then } \bigcap j{\in}J.\ A(j)$
$\textit{else if } J{=}0 \textit{ then } \bigcap i{\in}I.\ A(i)$
$\textit{else } ((\bigcap i{\in}I.\ A(i)) \cap\ (\bigcap j{\in}J.\ A(j))))$
by (*simp*, *blast intro*!: *equalityI*)

lemma *UN-UN-flatten*: $(\bigcup x \in (\bigcup y{\in}A.\ B(y)).\ C(x)) = (\bigcup y{\in}A.\ \bigcup x{\in}\ B(y).\ C(x))$
by *blast*

lemma *Int-UN-distrib*: $B \cap (\bigcup i{\in}I.\ A(i)) = (\bigcup i{\in}I.\ B \cap A(i))$
by *blast*

lemma *Un-INT-distrib*: $I{\neq}0 \Longrightarrow B \cup (\bigcap i{\in}I.\ A(i)) = (\bigcap i{\in}I.\ B \cup A(i))$
by *auto*

lemma *Int-UN-distrib2*:
$(\bigcup i{\in}I.\ A(i)) \cap (\bigcup j{\in}J.\ B(j)) = (\bigcup i{\in}I.\ \bigcup j{\in}J.\ A(i) \cap B(j))$
by *blast*

lemma *Un-INT-distrib2*: $[\![I{\neq}0;\ \ J{\neq}0]\!] \Longrightarrow$

$(\bigcap i{\in}I.\ A(i)) \cup (\bigcap j{\in}J.\ B(j)) = (\bigcap i{\in}I.\ \bigcap j{\in}J.\ A(i) \cup B(j))$
by *auto*

lemma *UN-constant* [*simp*]: $(\bigcup y{\in}A.\ c) = (\mathit{if}\ A{=}0\ \mathit{then}\ 0\ \mathit{else}\ c)$
by *force*

lemma *INT-constant* [*simp*]: $(\bigcap y{\in}A.\ c) = (\mathit{if}\ A{=}0\ \mathit{then}\ 0\ \mathit{else}\ c)$
by *force*

lemma *UN-RepFun* [*simp*]: $(\bigcup y{\in}\ RepFun(A,f).\ B(y)) = (\bigcup x{\in}A.\ B(f(x)))$
by *blast*

lemma *INT-RepFun* [*simp*]: $(\bigcap x{\in}RepFun(A,f).\ B(x)) \quad = (\bigcap a{\in}A.\ B(f(a)))$
by (*auto simp add*: *Inter-def*)

lemma *INT-Union-eq*:
$0 \notin A \Longrightarrow (\bigcap x{\in}\ \bigcup(A).\ B(x)) = (\bigcap y{\in}A.\ \bigcap x{\in}y.\ B(x))$
apply (*subgoal-tac* $\forall x{\in}A.\ x{\neq}0$)
prefer *2* **apply** *blast*
apply (*force simp add*: *Inter-def ball-conj-distrib*)
done

lemma *INT-UN-eq*:
$(\forall x{\in}A.\ B(x) \neq 0)$
$\Longrightarrow (\bigcap z{\in}\ (\bigcup x{\in}A.\ B(x)).\ C(z)) = (\bigcap x{\in}A.\ \bigcap z{\in}\ B(x).\ C(z))$
apply (*subst INT-Union-eq*, *blast*)
apply (*simp add*: *Inter-def*)
done

lemma *UN-Un-distrib*:
$(\bigcup i{\in}I.\ A(i) \cup B(i)) = (\bigcup i{\in}I.\ A(i)) \ \cup\ (\bigcup i{\in}I.\ B(i))$
by *blast*

lemma *INT-Int-distrib*:
$I{\neq}0 \Longrightarrow (\bigcap i{\in}I.\ A(i) \cap B(i)) = (\bigcap i{\in}I.\ A(i)) \cap (\bigcap i{\in}I.\ B(i))$
by (*blast elim*!: *not-emptyE*)

lemma *UN-Int-subset*:
$(\bigcup z{\in}I \cap J.\ A(z)) \subseteq (\bigcup z{\in}I.\ A(z)) \cap (\bigcup z{\in}J.\ A(z))$
by *blast*

lemma *Diff-UN*: $I{\neq}0 \Longrightarrow B - (\bigcup i{\in}I.\ A(i)) = (\bigcap i{\in}I.\ B - A(i))$
by (*blast elim*!: *not-emptyE*)

lemma *Diff-INT*: $I \neq 0 \Longrightarrow B - (\bigcap i \in I.\ A(i)) = (\bigcup i \in I.\ B - A(i))$
by (*blast elim*!: *not-emptyE*)

lemma *Sigma-cons1*: $Sigma(cons(a,B),\ C) = (\{a\} * C(a)) \cup Sigma(B,C)$
by *blast*

lemma *Sigma-cons2*: $A * cons(b,B) = A * \{b\} \cup A * B$
by *blast*

lemma *Sigma-succ1*: $Sigma(succ(A),\ B) = (\{A\} * B(A)) \cup Sigma(A,B)$
by *blast*

lemma *Sigma-succ2*: $A * succ(B) = A * \{B\} \cup A * B$
by *blast*

lemma *SUM-UN-distrib1*:
$(\sum x \in (\bigcup y \in A.\ C(y)).\ B(x)) = (\bigcup y \in A.\ \sum x \in C(y).\ B(x))$
by *blast*

lemma *SUM-UN-distrib2*:
$(\sum i \in I.\ \bigcup j \in J.\ C(i,j)) = (\bigcup j \in J.\ \sum i \in I.\ C(i,j))$
by *blast*

lemma *SUM-Un-distrib1*:
$(\sum i \in I \cup J.\ C(i)) = (\sum i \in I.\ C(i)) \cup (\sum j \in J.\ C(j))$
by *blast*

lemma *SUM-Un-distrib2*:
$(\sum i \in I.\ A(i) \cup B(i)) = (\sum i \in I.\ A(i)) \cup (\sum i \in I.\ B(i))$
by *blast*

lemma *prod-Un-distrib2*: $I * (A \cup B) = I * A \cup I * B$
by (*rule SUM-Un-distrib2*)

lemma *SUM-Int-distrib1*:
$(\sum i \in I \cap J.\ C(i)) = (\sum i \in I.\ C(i)) \cap (\sum j \in J.\ C(j))$
by *blast*

lemma *SUM-Int-distrib2*:
$(\sum i \in I.\ A(i) \cap B(i)) = (\sum i \in I.\ A(i)) \cap (\sum i \in I.\ B(i))$
by *blast*

lemma *prod-Int-distrib2*: $I * (A \cap B) = I{*}A \cap I{*}B$
by (*rule SUM-Int-distrib2*)

lemma *SUM-eq-UN*: $(\sum i{\in}I.\ A(i)) = (\bigcup i{\in}I.\ \{i\} * A(i))$
by *blast*

lemma *times-subset-iff*:
$(A'{*}B' \subseteq A{*}B) \longleftrightarrow (A' = 0 \mid B' = 0 \mid (A'{\subseteq}A) \land (B'{\subseteq}B))$
by *blast*

lemma *Int-Sigma-eq*:
$(\sum x \in A'.\ B'(x)) \cap (\sum x \in A.\ B(x)) = (\sum x \in A' \cap A.\ B'(x) \cap B(x))$
by *blast*

lemma *domain-iff*: a: $domain(r) \longleftrightarrow (\exists\, y.\ \langle a,y\rangle \in r)$
by (*unfold domain-def*, *blast*)

lemma *domainI* [*intro*]: $\langle a,b\rangle \in r \Longrightarrow a$: $domain(r)$
by (*unfold domain-def*, *blast*)

lemma *domainE* [*elim!*]:
$[\![a \in domain(r);\ \bigwedge y.\ \langle a,y\rangle \in r \Longrightarrow P]\!] \Longrightarrow P$
by (*unfold domain-def*, *blast*)

lemma *domain-subset*: $domain(Sigma(A,B)) \subseteq A$
by *blast*

lemma *domain-of-prod*: $b{\in}B \Longrightarrow domain(A{*}B) = A$
by *blast*

lemma *domain-0* [*simp*]: $domain(0) = 0$
by *blast*

lemma *domain-cons* [*simp*]: $domain(cons(\langle a,b\rangle,r)) = cons(a,\ domain(r))$
by *blast*

lemma *domain-Un-eq* [*simp*]: $domain(A \cup B) = domain(A) \cup domain(B)$
by *blast*

lemma *domain-Int-subset*: $domain(A \cap B) \subseteq domain(A) \cap domain(B)$
by *blast*

lemma *domain-Diff-subset*: $domain(A) - domain(B) \subseteq domain(A - B)$
by *blast*

lemma *domain-UN*: $domain(\bigcup x{\in}A.\ B(x)) = (\bigcup x{\in}A.\ domain(B(x)))$

by *blast*

lemma *domain-Union*: *domain*($\bigcup$(*A*)) = ($\bigcup$*x*∈*A*. *domain*(*x*))
by *blast*

lemma *rangeI* [*intro*]: ⟨*a*,*b*⟩∈ *r* ⟹ *b* ∈ *range*(*r*)
unfolding *range-def*
apply (*erule converseI* [*THEN domainI*])
done

lemma *rangeE* [*elim*!]: ⟦*b* ∈ *range*(*r*); ⋀*x*. ⟨*x*,*b*⟩∈ *r* ⟹ *P*⟧ ⟹ *P*
by (*unfold range-def*, *blast*)

lemma *range-subset*: *range*(*A*∗*B*) ⊆ *B*
unfolding *range-def*
apply (*subst converse-prod*)
apply (*rule domain-subset*)
done

lemma *range-of-prod*: *a*∈*A* ⟹ *range*(*A*∗*B*) = *B*
by *blast*

lemma *range-0* [*simp*]: *range*(*0*) = *0*
by *blast*

lemma *range-cons* [*simp*]: *range*(*cons*(⟨*a*,*b*⟩,*r*)) = *cons*(*b*, *range*(*r*))
by *blast*

lemma *range-Un-eq* [*simp*]: *range*(*A* ∪ *B*) = *range*(*A*) ∪ *range*(*B*)
by *blast*

lemma *range-Int-subset*: *range*(*A* ∩ *B*) ⊆ *range*(*A*) ∩ *range*(*B*)
by *blast*

lemma *range-Diff-subset*: *range*(*A*) − *range*(*B*) ⊆ *range*(*A* − *B*)
by *blast*

lemma *domain-converse* [*simp*]: *domain*(*converse*(*r*)) = *range*(*r*)
by *blast*

lemma *range-converse* [*simp*]: *range*(*converse*(*r*)) = *domain*(*r*)
by *blast*

lemma *fieldI1*: $\langle a,b\rangle\in r \Longrightarrow a \in field(r)$
by (*unfold field-def*, *blast*)

lemma *fieldI2*: $\langle a,b\rangle\in r \Longrightarrow b \in field(r)$
by (*unfold field-def*, *blast*)

lemma *fieldCI* [*intro*]:
$(\neg \langle c,a\rangle\in r \Longrightarrow \langle a,b\rangle\in r) \Longrightarrow a \in field(r)$
apply (*unfold field-def*, *blast*)
done

lemma *fieldE* [*elim!*]:
$[\![a \in field(r);$
$\bigwedge x.\ \langle a,x\rangle\in r \Longrightarrow P;$
$\bigwedge x.\ \langle x,a\rangle\in r \Longrightarrow P]\!] \Longrightarrow P$
by (*unfold field-def*, *blast*)

lemma *field-subset*: $field(A*B) \subseteq A \cup B$
by *blast*

lemma *domain-subset-field*: $domain(r) \subseteq field(r)$
unfolding *field-def*
apply (*rule Un-upper1*)
done

lemma *range-subset-field*: $range(r) \subseteq field(r)$
unfolding *field-def*
apply (*rule Un-upper2*)
done

lemma *domain-times-range*: $r \subseteq Sigma(A,B) \Longrightarrow r \subseteq domain(r)*range(r)$
by *blast*

lemma *field-times-field*: $r \subseteq Sigma(A,B) \Longrightarrow r \subseteq field(r)*field(r)$
by *blast*

lemma *relation-field-times-field*: $relation(r) \Longrightarrow r \subseteq field(r)*field(r)$
by (*simp add*: *relation-def*, *blast*)

lemma *field-of-prod*: $field(A*A) = A$
by *blast*

lemma *field-0* [*simp*]: $field(0) = 0$
by *blast*

lemma *field-cons* [*simp*]: $field(cons(\langle a,b\rangle,r)) = cons(a,\ cons(b,\ field(r)))$
by *blast*

lemma *field-Un-eq* [*simp*]: $field(A \cup B) = field(A) \cup field(B)$

by *blast*

lemma *field-Int-subset*: $field(A \cap B) \subseteq field(A) \cap field(B)$
by *blast*

lemma *field-Diff-subset*: $field(A) - field(B) \subseteq field(A - B)$
by *blast*

lemma *field-converse* [*simp*]: $field(converse(r)) = field(r)$
by *blast*

lemma *rel-Union*: $(\forall x\in S.\ \exists A\ B.\ x \subseteq A*B) \Longrightarrow$
$\bigcup(S) \subseteq domain(\bigcup(S)) * range(\bigcup(S))$
by *blast*

lemma *rel-Un*: $[\![r \subseteq A*B;\ \ s \subseteq C*D]\!] \Longrightarrow (r \cup s) \subseteq (A \cup C) * (B \cup D)$
by *blast*

lemma *domain-Diff-eq*: $[\![\langle a,c\rangle \in r;\ c\neq b]\!] \Longrightarrow domain(r-\{\langle a,b\rangle\}) = domain(r)$
by *blast*

lemma *range-Diff-eq*: $[\![\langle c,b\rangle \in r;\ c\neq a]\!] \Longrightarrow range(r-\{\langle a,b\rangle\}) = range(r)$
by *blast*

4.9 Image of a Set under a Function or Relation

lemma *image-iff*: $b \in r``A \longleftrightarrow (\exists x\in A.\ \langle x,b\rangle\in r)$
by (*unfold image-def*, *blast*)

lemma *image-singleton-iff*: $b \in r``\{a\} \longleftrightarrow \langle a,b\rangle\in r$
by (*rule image-iff* [*THEN iff-trans*], *blast*)

lemma *imageI* [*intro*]: $[\![\langle a,b\rangle\in r;\ \ a\in A]\!] \Longrightarrow b \in r``A$
by (*unfold image-def*, *blast*)

lemma *imageE* [*elim*!]:
$[\![b\colon r``A;\ \ \bigwedge x.[\![\langle x,b\rangle\in r;\ \ x\in A]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*unfold image-def*, *blast*)

lemma *image-subset*: $r \subseteq A*B \Longrightarrow r``C \subseteq B$
by *blast*

lemma *image-0* [*simp*]: $r``0 = 0$
by *blast*

lemma *image-Un* [*simp*]: $r``(A \cup B) = (r``A) \cup (r``B)$
by *blast*

lemma *image-UN*: r `` $(\bigcup x{\in}A.\ B(x)) = (\bigcup x{\in}A.\ r$ `` $B(x))$
by *blast*

lemma *Collect-image-eq*:
$\{z \in Sigma(A,B).\ P(z)\}$ `` $C = (\bigcup x \in A.\ \{y \in B(x).\ x \in C \wedge P(\langle x,y\rangle)\})$
by *blast*

lemma *image-Int-subset*: r``$(A \cap B) \subseteq (r$``$A) \cap (r$``$B)$
by *blast*

lemma *image-Int-square-subset*: $(r \cap A{*}A)$``$B \subseteq (r$``$B) \cap A$
by *blast*

lemma *image-Int-square*: $B{\subseteq}A \Longrightarrow (r \cap A{*}A)$``$B = (r$``$B) \cap A$
by *blast*

lemma *image-0-left* [*simp*]: 0``$A = 0$
by *blast*

lemma *image-Un-left*: $(r \cup s)$``$A = (r$``$A) \cup (s$``$A)$
by *blast*

lemma *image-Int-subset-left*: $(r \cap s)$``$A \subseteq (r$``$A) \cap (s$``$A)$
by *blast*

4.10 Inverse Image of a Set under a Function or Relation

lemma *vimage-iff*:
$a \in r-$``$B \longleftrightarrow (\exists\, y{\in}B.\ \langle a,y\rangle{\in}r)$
by (*unfold vimage-def image-def converse-def*, *blast*)

lemma *vimage-singleton-iff*: $a \in r-$``$\{b\} \longleftrightarrow \langle a,b\rangle{\in}r$
by (*rule vimage-iff* [*THEN iff-trans*], *blast*)

lemma *vimageI* [*intro*]: $[\![\langle a,b\rangle{\in}\ r;\ \ b{\in}B]\!] \Longrightarrow a \in r-$``$B$
by (*unfold vimage-def*, *blast*)

lemma *vimageE* [*elim*!]:
$[\![a\!: r-$``$B;\ \ \bigwedge x.[\![\langle a,x\rangle{\in}\ r;\ \ x{\in}B]\!] \Longrightarrow P]\!] \Longrightarrow P$
apply (*unfold vimage-def*, *blast*)
done

lemma *vimage-subset*: $r \subseteq A{*}B \Longrightarrow r-$``$C \subseteq A$
unfolding *vimage-def*
apply (*erule converse-type* [*THEN image-subset*])
done

lemma *vimage-0* [*simp*]: $r-``0 = 0$
by *blast*

lemma *vimage-Un* [*simp*]: $r-``(A \cup B) = (r-``A) \cup (r-``B)$
by *blast*

lemma *vimage-Int-subset*: $r-``(A \cap B) \subseteq (r-``A) \cap (r-``B)$
by *blast*

lemma *vimage-eq-UN*: $f -``B = (\bigcup y\in B. f-``\{y\})$
by *blast*

lemma *function-vimage-Int*:
$function(f) \Longrightarrow f-``(A \cap B) = (f-``A) \cap (f-``B)$
by (*unfold function-def*, *blast*)

lemma *function-vimage-Diff*: $function(f) \Longrightarrow f-``(A-B) = (f-``A) - (f-``B)$
by (*unfold function-def*, *blast*)

lemma *function-image-vimage*: $function(f) \Longrightarrow f `` (f-`` A) \subseteq A$
by (*unfold function-def*, *blast*)

lemma *vimage-Int-square-subset*: $(r \cap A*A)-``B \subseteq (r-``B) \cap A$
by *blast*

lemma *vimage-Int-square*: $B\subseteq A \Longrightarrow (r \cap A*A)-``B = (r-``B) \cap A$
by *blast*

lemma *vimage-0-left* [*simp*]: $0-``A = 0$
by *blast*

lemma *vimage-Un-left*: $(r \cup s)-``A = (r-``A) \cup (s-``A)$
by *blast*

lemma *vimage-Int-subset-left*: $(r \cap s)-``A \subseteq (r-``A) \cap (s-``A)$
by *blast*

lemma *converse-Un* [*simp*]: $converse(A \cup B) = converse(A) \cup converse(B)$
by *blast*

lemma *converse-Int* [*simp*]: $converse(A \cap B) = converse(A) \cap converse(B)$

by *blast*

lemma *converse-Diff* [*simp*]: $converse(A - B) = converse(A) - converse(B)$
by *blast*

lemma *converse-UN* [*simp*]: $converse(\bigcup x{\in}A.\ B(x)) = (\bigcup x{\in}A.\ converse(B(x)))$
by *blast*

lemma *converse-INT* [*simp*]:
$converse(\bigcap x{\in}A.\ B(x)) = (\bigcap x{\in}A.\ converse(B(x)))$
apply (*unfold Inter-def*, *blast*)
done

4.11 Powerset Operator

lemma *Pow-0* [*simp*]: $Pow(0) = \{0\}$
by *blast*

lemma *Pow-insert*: $Pow\ (cons(a,A)) = Pow(A) \cup \{cons(a,X) \ .\ X: Pow(A)\}$
apply (*rule equalityI*, *safe*)
apply (*erule swap*)
apply (*rule-tac* $a = x{-}\{a\}$ **in** *RepFun-eqI*, *auto*)
done

lemma *Un-Pow-subset*: $Pow(A) \cup Pow(B) \subseteq Pow(A \cup B)$
by *blast*

lemma *UN-Pow-subset*: $(\bigcup x{\in}A.\ Pow(B(x))) \subseteq Pow(\bigcup x{\in}A.\ B(x))$
by *blast*

lemma *subset-Pow-Union*: $A \subseteq Pow(\bigcup(A))$
by *blast*

lemma *Union-Pow-eq* [*simp*]: $\bigcup(Pow(A)) = A$
by *blast*

lemma *Union-Pow-iff*: $\bigcup(A) \in Pow(B) \longleftrightarrow A \in Pow(Pow(B))$
by *blast*

lemma *Pow-Int-eq* [*simp*]: $Pow(A \cap B) = Pow(A) \cap Pow(B)$
by *blast*

lemma *Pow-INT-eq*: $A{\neq}0 \Longrightarrow Pow(\bigcap x{\in}A.\ B(x)) = (\bigcap x{\in}A.\ Pow(B(x)))$
by (*blast elim*!: *not-emptyE*)

4.12 RepFun

lemma *RepFun-subset*: $[\![\bigwedge x.\ x{\in}A \Longrightarrow f(x) \in B]\!] \Longrightarrow \{f(x).\ x{\in}A\} \subseteq B$
by *blast*

lemma *RepFun-eq-0-iff* [*simp*]: $\{f(x).x{\in}A\}=0 \longleftrightarrow A=0$
by *blast*

lemma *RepFun-constant* [*simp*]: $\{c.\ x{\in}A\} = (\mathit{if}\ A=0\ \mathit{then}\ 0\ \mathit{else}\ \{c\})$
by *force*

4.13 Collect

lemma *Collect-subset*: $\mathit{Collect}(A,P) \subseteq A$
by *blast*

lemma *Collect-Un*: $\mathit{Collect}(A \cup B,\ P) = \mathit{Collect}(A,P) \cup \mathit{Collect}(B,P)$
by *blast*

lemma *Collect-Int*: $\mathit{Collect}(A \cap B,\ P) = \mathit{Collect}(A,P) \cap \mathit{Collect}(B,P)$
by *blast*

lemma *Collect-Diff*: $\mathit{Collect}(A - B,\ P) = \mathit{Collect}(A,P) - \mathit{Collect}(B,P)$
by *blast*

lemma *Collect-cons*: $\{x{\in}\mathit{cons}(a,B).\ P(x)\} =$
$(\mathit{if}\ P(a)\ \mathit{then}\ \mathit{cons}(a,\ \{x{\in}B.\ P(x)\})\ \mathit{else}\ \{x{\in}B.\ P(x)\})$
by (*simp*, *blast*)

lemma *Int-Collect-self-eq*: $A \cap \mathit{Collect}(A,P) = \mathit{Collect}(A,P)$
by *blast*

lemma *Collect-Collect-eq* [*simp*]:
$\mathit{Collect}(\mathit{Collect}(A,P),\ Q) = \mathit{Collect}(A,\ \lambda x.\ P(x) \wedge Q(x))$
by *blast*

lemma *Collect-Int-Collect-eq*:
$\mathit{Collect}(A,P) \cap \mathit{Collect}(A,Q) = \mathit{Collect}(A,\ \lambda x.\ P(x) \wedge Q(x))$
by *blast*

lemma *Collect-Union-eq* [*simp*]:
$\mathit{Collect}(\bigcup x{\in}A.\ B(x),\ P) = (\bigcup x{\in}A.\ \mathit{Collect}(B(x),\ P))$
by *blast*

lemma *Collect-Int-left*: $\{x{\in}A.\ P(x)\} \cap B = \{x \in A \cap B.\ P(x)\}$
by *blast*

lemma *Collect-Int-right*: $A \cap \{x{\in}B.\ P(x)\} = \{x \in A \cap B.\ P(x)\}$
by *blast*

lemma *Collect-disj-eq*: $\{x{\in}A.\ P(x)\ |\ Q(x)\} = \mathit{Collect}(A,\ P) \cup \mathit{Collect}(A,\ Q)$
by *blast*

lemma *Collect-conj-eq*: $\{x{\in}A.\ P(x) \wedge Q(x)\} = Collect(A,\ P) \cap Collect(A,\ Q)$
by *blast*

lemmas *subset-SIs* = *subset-refl cons-subsetI subset-consI*
Union-least UN-least Un-least
Inter-greatest Int-greatest RepFun-subset
Un-upper1 Un-upper2 Int-lower1 Int-lower2

ML ‹
```
val subset-cs =
  claset-of (context
    delrules [@{thm subsetI}, @{thm subsetCE}]
    addSIs @{thms subset-SIs}
    addIs  [@{thm Union-upper}, @{thm Inter-lower}]
    addSEs [@{thm cons-subsetE}]);

val ZF-cs = claset-of (context delrules [@{thm equalityI}]);
```
›

end

5 Least and Greatest Fixed Points; the Knaster-Tarski Theorem

theory *Fixedpt* **imports** *equalities* **begin**

definition

bnd-mono :: $[i,i{\Rightarrow}i]{\Rightarrow}o$ **where**
$bnd\text{-}mono(D,h) \equiv h(D){<=}D \wedge (\forall\ W\ X.\ W{<=}X \longrightarrow X{<=}D \longrightarrow h(W) \subseteq h(X))$

definition
lfp :: $[i,i{\Rightarrow}i]{\Rightarrow}i$ **where**
$lfp(D,h) \equiv \bigcap(\{X{:}\ Pow(D).\ h(X) \subseteq X\})$

definition
gfp :: $[i,i{\Rightarrow}i]{\Rightarrow}i$ **where**
$gfp(D,h) \equiv \bigcup(\{X{:}\ Pow(D).\ X \subseteq h(X)\})$

The theorem is proved in the lattice of subsets of D, namely $Pow(D)$, with Inter as the greatest lower bound.

5.1 Monotone Operators

lemma *bnd-monoI*:
$[\![h(D){<=}D;$
$\bigwedge W\ X.\ [\![W{<=}D;\ \ X{<=}D;\ \ W{<=}X]\!] \Longrightarrow h(W) \subseteq h(X)$

$\rrbracket \Longrightarrow$ *bnd-mono*(D,h)
by (*unfold bnd-mono-def*, *clarify*, *blast*)

lemma *bnd-monoD1*: *bnd-mono*$(D,h) \Longrightarrow h(D) \subseteq D$
  unfolding *bnd-mono-def*
apply (*erule conjunct1*)
done

lemma *bnd-monoD2*: $\llbracket$*bnd-mono*(D,h); $W<=X$; $X<=D\rrbracket \Longrightarrow h(W) \subseteq h(X)$
by (*unfold bnd-mono-def*, *blast*)

lemma *bnd-mono-subset*:
    $\llbracket$*bnd-mono*(D,h); $X<=D\rrbracket \Longrightarrow h(X) \subseteq D$
by (*unfold bnd-mono-def*, *clarify*, *blast*)

lemma *bnd-mono-Un*:
    $\llbracket$*bnd-mono*(D,h); $A \subseteq D$; $B \subseteq D\rrbracket \Longrightarrow h(A) \cup h(B) \subseteq h(A \cup B)$
  unfolding *bnd-mono-def*
apply (*rule Un-least*, *blast+*)
done

lemma *bnd-mono-UN*:
    $\llbracket$*bnd-mono*(D,h); $\forall i{\in}I.\ A(i) \subseteq D\rrbracket$
     $\Longrightarrow (\bigcup i{\in}I.\ h(A(i))) \subseteq h((\bigcup i{\in}I.\ A(i)))$
  unfolding *bnd-mono-def*
apply (*rule UN-least*)
apply (*elim conjE*)
apply (*drule-tac x*$=A(i)$ **in** *spec*)
apply (*drule-tac x*$=(\bigcup i{\in}I.\ A(i))$ **in** *spec*)
apply *blast*
done

lemma *bnd-mono-Int*:
    $\llbracket$*bnd-mono*(D,h); $A \subseteq D$; $B \subseteq D\rrbracket \Longrightarrow h(A \cap B) \subseteq h(A) \cap h(B)$
apply (*rule Int-greatest*)
apply (*erule bnd-monoD2*, *rule Int-lower1*, *assumption*)
apply (*erule bnd-monoD2*, *rule Int-lower2*, *assumption*)
done

5.2 Proof of Knaster-Tarski Theorem using *lfp*

lemma *lfp-lowerbound*:
    $\llbracket h(A) \subseteq A$; $A<=D\rrbracket \Longrightarrow$ *lfp*$(D,h) \subseteq A$
by (*unfold lfp-def*, *blast*)

lemma *lfp-subset*: *lfp*$(D,h) \subseteq D$

by (*unfold lfp-def Inter-def*, *blast*)

lemma *def-lfp-subset*: $A \equiv lfp(D,h) \Longrightarrow A \subseteq D$
apply *simp*
apply (*rule lfp-subset*)
done

lemma *lfp-greatest*:
$[\![h(D) \subseteq D; \ \bigwedge X. \ [\![h(X) \subseteq X; \ X<=D]\!] \Longrightarrow A<=X]\!] \Longrightarrow A \subseteq lfp(D,h)$
by (*unfold lfp-def*, *blast*)

lemma *lfp-lemma1*:
$[\![bnd\text{-}mono(D,h); \ h(A)<=A; \ A<=D]\!] \Longrightarrow h(lfp(D,h)) \subseteq A$
apply (*erule bnd-monoD2* [*THEN subset-trans*])
apply (*rule lfp-lowerbound*, *assumption*+)
done

lemma *lfp-lemma2*: $bnd\text{-}mono(D,h) \Longrightarrow h(lfp(D,h)) \subseteq lfp(D,h)$
apply (*rule bnd-monoD1* [*THEN lfp-greatest*])
apply (*rule-tac* [*2*] *lfp-lemma1*)
apply (*assumption*+)
done

lemma *lfp-lemma3*:
$bnd\text{-}mono(D,h) \Longrightarrow lfp(D,h) \subseteq h(lfp(D,h))$
apply (*rule lfp-lowerbound*)
apply (*rule bnd-monoD2*, *assumption*)
apply (*rule lfp-lemma2*, *assumption*)
apply (*erule-tac* [*2*] *bnd-mono-subset*)
apply (*rule lfp-subset*)+
done

lemma *lfp-unfold*: $bnd\text{-}mono(D,h) \Longrightarrow lfp(D,h) = h(lfp(D,h))$
apply (*rule equalityI*)
apply (*erule lfp-lemma3*)
apply (*erule lfp-lemma2*)
done

lemma *def-lfp-unfold*:
$[\![A \equiv lfp(D,h); \ bnd\text{-}mono(D,h)]\!] \Longrightarrow A = h(A)$
apply *simp*
apply (*erule lfp-unfold*)
done

5.3 General Induction Rule for Least Fixedpoints

lemma *Collect-is-pre-fixedpt*:

⟦*bnd-mono*(*D*,*h*); ⋀*x*. *x* ∈ *h*(*Collect*(*lfp*(*D*,*h*),*P*)) ⟹ *P*(*x*)⟧
⟹ *h*(*Collect*(*lfp*(*D*,*h*),*P*)) ⊆ *Collect*(*lfp*(*D*,*h*),*P*)
by (*blast intro*: *lfp-lemma2* [*THEN subsetD*] *bnd-monoD2* [*THEN subsetD*]
lfp-subset [*THEN subsetD*])

lemma *induct*:
⟦*bnd-mono*(*D*,*h*); *a* ∈ *lfp*(*D*,*h*);
⋀*x*. *x* ∈ *h*(*Collect*(*lfp*(*D*,*h*),*P*)) ⟹ *P*(*x*)
⟧ ⟹ *P*(*a*)
apply (*rule Collect-is-pre-fixedpt*
[*THEN lfp-lowerbound*, *THEN subsetD*, *THEN CollectD2*])
apply (*rule-tac* [*3*] *lfp-subset* [*THEN Collect-subset* [*THEN subset-trans*]],
blast+)
done

lemma *def-induct*:
⟦*A* ≡ *lfp*(*D*,*h*); *bnd-mono*(*D*,*h*); *a*:*A*;
⋀*x*. *x* ∈ *h*(*Collect*(*A*,*P*)) ⟹ *P*(*x*)
⟧ ⟹ *P*(*a*)
by (*rule induct*, *blast*+)

lemma *lfp-Int-lowerbound*:
⟦*h*(*D* ∩ *A*) ⊆ *A*; *bnd-mono*(*D*,*h*)⟧ ⟹ *lfp*(*D*,*h*) ⊆ *A*
apply (*rule lfp-lowerbound* [*THEN subset-trans*])
apply (*erule bnd-mono-subset* [*THEN Int-greatest*], *blast*+)
done

lemma *lfp-mono*:
assumes *hmono*: *bnd-mono*(*D*,*h*)
and *imono*: *bnd-mono*(*E*,*i*)
and *subhi*: ⋀*X*. *X*<=*D* ⟹ *h*(*X*) ⊆ *i*(*X*)
shows *lfp*(*D*,*h*) ⊆ *lfp*(*E*,*i*)
apply (*rule bnd-monoD1* [*THEN lfp-greatest*])
apply (*rule imono*)
apply (*rule hmono* [*THEN* [*2*] *lfp-Int-lowerbound*])
apply (*rule Int-lower1* [*THEN subhi*, *THEN subset-trans*])
apply (*rule imono* [*THEN bnd-monoD2*, *THEN subset-trans*], *auto*)
done

lemma *lfp-mono2*:
⟦*i*(*D*) ⊆ *D*; ⋀*X*. *X*<=*D* ⟹ *h*(*X*) ⊆ *i*(*X*)⟧ ⟹ *lfp*(*D*,*h*) ⊆ *lfp*(*D*,*i*)
apply (*rule lfp-greatest*, *assumption*)
apply (*rule lfp-lowerbound*, *blast*, *assumption*)
done

lemma *lfp-cong*:
$[\![D=D'; \bigwedge X.\ X \subseteq D' \Longrightarrow h(X) = h'(X)]\!] \Longrightarrow lfp(D,h) = lfp(D',h')$
apply (*simp add*: *lfp-def*)
apply (*rule-tac t=Inter* **in** *subst-context*)
apply (*rule Collect-cong*, *simp-all*)
done

5.4 Proof of Knaster-Tarski Theorem using *gfp*

lemma *gfp-upperbound*: $[\![A \subseteq h(A);\ A<=D]\!] \Longrightarrow A \subseteq gfp(D,h)$
unfolding *gfp-def*
apply (*rule PowI* [*THEN CollectI*, *THEN Union-upper*])
apply (*assumption+*)
done

lemma *gfp-subset*: $gfp(D,h) \subseteq D$
by (*unfold gfp-def*, *blast*)

lemma *def-gfp-subset*: $A \equiv gfp(D,h) \Longrightarrow A \subseteq D$
apply *simp*
apply (*rule gfp-subset*)
done

lemma *gfp-least*:
$[\![bnd\text{-}mono(D,h);\ \bigwedge X.\ [\![X \subseteq h(X);\ X<=D]\!] \Longrightarrow X<=A]\!] \Longrightarrow$
$gfp(D,h) \subseteq A$
unfolding *gfp-def*
apply (*blast dest*: *bnd-monoD1*)
done

lemma *gfp-lemma1*:
$[\![bnd\text{-}mono(D,h);\ A<=h(A);\ A<=D]\!] \Longrightarrow A \subseteq h(gfp(D,h))$
apply (*rule subset-trans*, *assumption*)
apply (*erule bnd-monoD2*)
apply (*rule-tac* [*2*] *gfp-subset*)
apply (*simp add*: *gfp-upperbound*)
done

lemma *gfp-lemma2*: $bnd\text{-}mono(D,h) \Longrightarrow gfp(D,h) \subseteq h(gfp(D,h))$
apply (*rule gfp-least*)
apply (*rule-tac* [*2*] *gfp-lemma1*)
apply (*assumption+*)
done

lemma *gfp-lemma3*:
$bnd\text{-}mono(D,h) \Longrightarrow h(gfp(D,h)) \subseteq gfp(D,h)$
apply (*rule gfp-upperbound*)

apply (*rule bnd-monoD2*, *assumption*)
apply (*rule gfp-lemma2*, *assumption*)
apply (*erule bnd-mono-subset*, *rule gfp-subset*)+
done

lemma *gfp-unfold*: *bnd-mono*(*D*,*h*) $\Longrightarrow$ *gfp*(*D*,*h*) = *h*(*gfp*(*D*,*h*))
apply (*rule equalityI*)
apply (*erule gfp-lemma2*)
apply (*erule gfp-lemma3*)
done

lemma *def-gfp-unfold*:
  $[\![$*A*≡*gfp*(*D*,*h*); *bnd-mono*(*D*,*h*)$]\!]$ $\Longrightarrow$ *A* = *h*(*A*)
apply *simp*
apply (*erule gfp-unfold*)
done

5.5 Coinduction Rules for Greatest Fixed Points

lemma *weak-coinduct*: $[\![$*a*: *X*; *X* ⊆ *h*(*X*); *X* ⊆ *D*$]\!]$ $\Longrightarrow$ *a* ∈ *gfp*(*D*,*h*)
by (*blast intro*: *gfp-upperbound* [*THEN subsetD*])

lemma *coinduct-lemma*:
  $[\![$*X* ⊆ *h*(*X* ∪ *gfp*(*D*,*h*)); *X* ⊆ *D*; *bnd-mono*(*D*,*h*)$]\!]$ $\Longrightarrow$
   X ∪ *gfp*(*D*,*h*) ⊆ *h*(*X* ∪ *gfp*(*D*,*h*))
apply (*erule Un-least*)
apply (*rule gfp-lemma2* [*THEN subset-trans*], *assumption*)
apply (*rule Un-upper2* [*THEN subset-trans*])
apply (*rule bnd-mono-Un*, *assumption*+)
apply (*rule gfp-subset*)
done

lemma *coinduct*:
  $[\![$*bnd-mono*(*D*,*h*); *a*: *X*; *X* ⊆ *h*(*X* ∪ *gfp*(*D*,*h*)); *X* ⊆ *D*$]\!]$
   $\Longrightarrow$ *a* ∈ *gfp*(*D*,*h*)
apply (*rule weak-coinduct*)
apply (*erule-tac* [*2*] *coinduct-lemma*)
apply (*simp-all add*: *gfp-subset Un-subset-iff*)
done

lemma *def-coinduct*:
  $[\![$*A* ≡ *gfp*(*D*,*h*); *bnd-mono*(*D*,*h*); *a*: *X*; *X* ⊆ *h*(*X* ∪ *A*); *X* ⊆ *D*$]\!]$ $\Longrightarrow$
   a ∈ *A*
apply *simp*
apply (*rule coinduct*, *assumption*+)
done

lemma *def-Collect-coinduct*:
  $[\![A \equiv gfp(D, \lambda w.\ Collect(D,P(w)));\ \ bnd\text{-}mono(D, \lambda w.\ Collect(D,P(w)));$
    $a: X;\ \ X \subseteq D;\ \ \bigwedge z.\ z: X \Longrightarrow P(X \cup A,\ z)]\!] \Longrightarrow$
  $a \in A$
apply (*rule def-coinduct*, *assumption*+, *blast*+)
done

lemma *gfp-mono*:
  $[\![bnd\text{-}mono(D,h);\ \ D \subseteq E;$
    $\bigwedge X.\ X{<=}D \Longrightarrow h(X) \subseteq i(X)]\!] \Longrightarrow gfp(D,h) \subseteq gfp(E,i)$
apply (*rule gfp-upperbound*)
apply (*rule gfp-lemma2* [*THEN subset-trans*], *assumption*)
apply (*blast del*: *subsetI intro*: *gfp-subset*)
apply (*blast del*: *subsetI intro*: *subset-trans gfp-subset*)
done

end

6 Booleans in Zermelo-Fraenkel Set Theory

theory *Bool* **imports** *pair* **begin**

abbreviation
  one (‹*1*›) **where**
  $1 \equiv succ(0)$

abbreviation
  two (‹*2*›) **where**
  $2 \equiv succ(1)$

2 is equal to bool, but is used as a number rather than a type.

definition $bool \equiv \{0,1\}$

definition $cond(b,c,d) \equiv if(b{=}1,c,d)$

definition $not(b) \equiv cond(b,0,1)$

definition
  and :: $[i,i]\Rightarrow i$ (**infixl** ‹*and*› *70*) **where**
    $a\ and\ b \equiv cond(a,b,0)$

definition
  or :: $[i,i]\Rightarrow i$ (**infixl** ‹*or*› *65*) **where**
    $a\ or\ b \equiv cond(a,1,b)$

definition

xor :: [*i*,*i*]⇒*i* (**infixl** ‹*xor*› *65*) **where**
a xor b ≡ *cond*(*a*,*not*(*b*),*b*)

lemmas *bool-defs* = *bool-def cond-def*

lemma *singleton-0*: {*0*} = *1*
by (*simp add*: *succ-def*)

lemma *bool-1I* [*simp*,*TC*]: *1* ∈ *bool*
by (*simp add*: *bool-defs*)

lemma *bool-0I* [*simp*,*TC*]: *0* ∈ *bool*
by (*simp add*: *bool-defs*)

lemma *one-not-0*: *1*≠*0*
by (*simp add*: *bool-defs*)

lemmas *one-neq-0* = *one-not-0* [*THEN notE*]

lemma *boolE*:
⟦*c*: *bool*; *c*=*1* ⟹ *P*; *c*=*0* ⟹ *P*⟧ ⟹ *P*
by (*simp add*: *bool-defs*, *blast*)

lemma *cond-1* [*simp*]: *cond*(*1*,*c*,*d*) = *c*
by (*simp add*: *bool-defs*)

lemma *cond-0* [*simp*]: *cond*(*0*,*c*,*d*) = *d*
by (*simp add*: *bool-defs*)

lemma *cond-type* [*TC*]: ⟦*b*: *bool*; *c*: *A*(*1*); *d*: *A*(*0*)⟧ ⟹ *cond*(*b*,*c*,*d*): *A*(*b*)
by (*simp add*: *bool-defs*, *blast*)

lemma *cond-simple-type*: ⟦*b*: *bool*; *c*: *A*; *d*: *A*⟧ ⟹ *cond*(*b*,*c*,*d*): *A*
by (*simp add*: *bool-defs*)

lemma *def-cond-1*: ⟦⋀*b*. *j*(*b*)≡*cond*(*b*,*c*,*d*)⟧ ⟹ *j*(*1*) = *c*
by *simp*

lemma *def-cond-0*: ⟦⋀*b*. *j*(*b*)≡*cond*(*b*,*c*,*d*)⟧ ⟹ *j*(*0*) = *d*
by *simp*

lemmas *not-1* = *not-def* [*THEN def-cond-1*, *simp*]
lemmas *not-0* = *not-def* [*THEN def-cond-0*, *simp*]

lemmas *and-1* = *and-def* [*THEN def-cond-1*, *simp*]
lemmas *and-0* = *and-def* [*THEN def-cond-0*, *simp*]

lemmas *or-1* = *or-def* [*THEN def-cond-1*, *simp*]
lemmas *or-0* = *or-def* [*THEN def-cond-0*, *simp*]

lemmas *xor-1* = *xor-def* [*THEN def-cond-1*, *simp*]
lemmas *xor-0* = *xor-def* [*THEN def-cond-0*, *simp*]

lemma *not-type* [*TC*]: *a*:*bool* ⟹ *not*(*a*) ∈ *bool*
by (*simp add*: *not-def*)

lemma *and-type* [*TC*]: ⟦*a*:*bool*; *b*:*bool*⟧ ⟹ *a and b* ∈ *bool*
by (*simp add*: *and-def*)

lemma *or-type* [*TC*]: ⟦*a*:*bool*; *b*:*bool*⟧ ⟹ *a or b* ∈ *bool*
by (*simp add*: *or-def*)

lemma *xor-type* [*TC*]: ⟦*a*:*bool*; *b*:*bool*⟧ ⟹ *a xor b* ∈ *bool*
by (*simp add*: *xor-def*)

lemmas *bool-typechecks* = *bool-1I bool-0I cond-type not-type and-type or-type xor-type*

6.1 Laws About 'not'

lemma *not-not* [*simp*]: *a*:*bool* ⟹ *not*(*not*(*a*)) = *a*
by (*elim boolE*, *auto*)

lemma *not-and* [*simp*]: *a*:*bool* ⟹ *not*(*a and b*) = *not*(*a*) *or not*(*b*)
by (*elim boolE*, *auto*)

lemma *not-or* [*simp*]: *a*:*bool* ⟹ *not*(*a or b*) = *not*(*a*) *and not*(*b*)
by (*elim boolE*, *auto*)

6.2 Laws About 'and'

lemma *and-absorb* [*simp*]: *a*: *bool* ⟹ *a and a* = *a*
by (*elim boolE*, *auto*)

lemma *and-commute*: ⟦*a*: *bool*; *b*:*bool*⟧ ⟹ *a and b* = *b and a*
by (*elim boolE*, *auto*)

lemma *and-assoc*: *a*: *bool* ⟹ (*a and b*) *and c* = *a and* (*b and c*)
by (*elim boolE*, *auto*)

lemma *and-or-distrib*: $[\![a: bool; b:bool; c:bool]\!] \Longrightarrow$
$(a\ or\ b)\ and\ c\ =\ (a\ and\ c)\ or\ (b\ and\ c)$
by (*elim boolE*, *auto*)

6.3 Laws About 'or'

lemma *or-absorb* [*simp*]: $a: bool \Longrightarrow a\ or\ a = a$
by (*elim boolE*, *auto*)

lemma *or-commute*: $[\![a: bool; b:bool]\!] \Longrightarrow a\ or\ b = b\ or\ a$
by (*elim boolE*, *auto*)

lemma *or-assoc*: $a: bool \Longrightarrow (a\ or\ b)\ or\ c\ =\ a\ or\ (b\ or\ c)$
by (*elim boolE*, *auto*)

lemma *or-and-distrib*: $[\![a: bool; b: bool; c: bool]\!] \Longrightarrow$
$(a\ and\ b)\ or\ c\ =\ (a\ or\ c)\ and\ (b\ or\ c)$
by (*elim boolE*, *auto*)

definition
bool-of-o :: $o \Rightarrow i$ **where**
bool-of-o$(P) \equiv (if\ P\ then\ 1\ else\ 0)$

lemma [*simp*]: *bool-of-o*$(True) = 1$
by (*simp add*: *bool-of-o-def*)

lemma [*simp*]: *bool-of-o*$(False) = 0$
by (*simp add*: *bool-of-o-def*)

lemma [*simp*,*TC*]: *bool-of-o*$(P) \in bool$
by (*simp add*: *bool-of-o-def*)

lemma [*simp*]: $($*bool-of-o*$(P) = 1) \longleftrightarrow P$
by (*simp add*: *bool-of-o-def*)

lemma [*simp*]: $($*bool-of-o*$(P) = 0) \longleftrightarrow \neg P$
by (*simp add*: *bool-of-o-def*)

end

7 Disjoint Sums

theory *Sum* **imports** *Bool equalities* **begin**

And the "Part" primitive for simultaneous recursive type definitions

definition *sum* :: $[i,i] \Rightarrow i$ (**infixr** ‹+› *65*) **where**
$A+B \equiv \{0\}*A \cup \{1\}*B$

definition *Inl* :: $i \Rightarrow i$ **where**
  $Inl(a) \equiv \langle 0,a \rangle$

definition *Inr* :: $i \Rightarrow i$ **where**
  $Inr(b) \equiv \langle 1,b \rangle$

definition *case* :: $[i \Rightarrow i,\ i \Rightarrow i,\ i] \Rightarrow i$ **where**
  $case(c,d) \equiv (\lambda\langle y,z\rangle.\ cond(y,\ d(z),\ c(z)))$

definition *Part* :: $[i,i \Rightarrow i] \Rightarrow i$ **where**
  $Part(A,h) \equiv \{x \in A.\ \exists z.\ x = h(z)\}$

7.1 Rules for the *Part* Primitive

lemma *Part-iff*:
  $a \in Part(A,h) \longleftrightarrow a \in A \wedge (\exists y.\ a=h(y))$
 unfolding *Part-def*
apply (*rule separation*)
done

lemma *Part-eqI* [*intro*]:
  $[\![a \in A;\ \ a=h(b)]\!] \Longrightarrow a \in Part(A,h)$
by (*unfold Part-def*, *blast*)

lemmas *PartI* = *refl* [*THEN* [*2*] *Part-eqI*]

lemma *PartE* [*elim*!]:
  $[\![a \in Part(A,h);\ \ \bigwedge z.\ [\![a \in A;\ \ a=h(z)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
apply (*unfold Part-def*, *blast*)
done

lemma *Part-subset*: $Part(A,h) \subseteq A$
 unfolding *Part-def*
apply (*rule Collect-subset*)
done

7.2 Rules for Disjoint Sums

lemmas *sum-defs* = *sum-def Inl-def Inr-def case-def*

lemma *Sigma-bool*: $Sigma(bool,C) = C(0) + C(1)$
by (*unfold bool-def sum-def*, *blast*)

lemma *InlI* [*intro*!,*simp*,*TC*]: $a \in A \Longrightarrow Inl(a) \in A+B$
by (*unfold sum-defs*, *blast*)

lemma *InrI* [*intro*!,*simp*,*TC*]: $b \in B \Longrightarrow Inr(b) \in A+B$
by (*unfold sum-defs*, *blast*)

lemma *sumE* [*elim*!]:
$[\![u \in A+B;$
$\bigwedge x.\ [\![x \in A;\ \ u=Inl(x)]\!] \Longrightarrow P;$
$\bigwedge y.\ [\![y \in B;\ \ u=Inr(y)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
by (*unfold sum-defs*, *blast*)

lemma *Inl-iff* [*iff*]: $Inl(a)=Inl(b) \longleftrightarrow a=b$
by (*simp add*: *sum-defs*)

lemma *Inr-iff* [*iff*]: $Inr(a)=Inr(b) \longleftrightarrow a=b$
by (*simp add*: *sum-defs*)

lemma *Inl-Inr-iff* [*simp*]: $Inl(a)=Inr(b) \longleftrightarrow False$
by (*simp add*: *sum-defs*)

lemma *Inr-Inl-iff* [*simp*]: $Inr(b)=Inl(a) \longleftrightarrow False$
by (*simp add*: *sum-defs*)

lemma *sum-empty* [*simp*]: $0+0 = 0$
by (*simp add*: *sum-defs*)

lemmas *Inl-inject* = *Inl-iff* [*THEN iffD1*]
lemmas *Inr-inject* = *Inr-iff* [*THEN iffD1*]
lemmas *Inl-neq-Inr* = *Inl-Inr-iff* [*THEN iffD1*, *THEN FalseE*, *elim*!]
lemmas *Inr-neq-Inl* = *Inr-Inl-iff* [*THEN iffD1*, *THEN FalseE*, *elim*!]

lemma *InlD*: $Inl(a): A+B \Longrightarrow a \in A$
by *blast*

lemma *InrD*: $Inr(b): A+B \Longrightarrow b \in B$
by *blast*

lemma *sum-iff*: $u \in A+B \longleftrightarrow (\exists x.\ x \in A \land u=Inl(x)) \mid (\exists y.\ y \in B \land u=Inr(y))$
by *blast*

lemma *Inl-in-sum-iff* [*simp*]: $(Inl(x) \in A+B) \longleftrightarrow (x \in A)$
by *auto*

lemma *Inr-in-sum-iff* [*simp*]: $(Inr(y) \in A{+}B) \longleftrightarrow (y \in B)$
by *auto*

lemma *sum-subset-iff*: $A{+}B \subseteq C{+}D \longleftrightarrow A{<=}C \wedge B{<=}D$
by *blast*

lemma *sum-equal-iff*: $A{+}B = C{+}D \longleftrightarrow A{=}C \wedge B{=}D$
by (*simp add*: *extension sum-subset-iff*, *blast*)

lemma *sum-eq-2-times*: $A{+}A = 2{*}A$
by (*simp add*: *sum-def*, *blast*)

7.3 The Eliminator: *case*

lemma *case-Inl* [*simp*]: $case(c,\ d,\ Inl(a)) = c(a)$
by (*simp add*: *sum-defs*)

lemma *case-Inr* [*simp*]: $case(c,\ d,\ Inr(b)) = d(b)$
by (*simp add*: *sum-defs*)

lemma *case-type* [*TC*]:
$[\![u \in A{+}B;$
$\quad \bigwedge x.\ x \in A \Longrightarrow c(x): C(Inl(x));$
$\quad \bigwedge y.\ y \in B \Longrightarrow d(y): C(Inr(y))$
$]\!] \Longrightarrow case(c,d,u) \in C(u)$
by *auto*

lemma *expand-case*: $u \in A{+}B \Longrightarrow$
$\quad R(case(c,d,u)) \longleftrightarrow$
$\quad ((\forall x{\in}A.\ u = Inl(x) \longrightarrow R(c(x))) \wedge$
$\quad (\forall y{\in}B.\ u = Inr(y) \longrightarrow R(d(y))))$
by *auto*

lemma *case-cong*:
$[\![z \in A{+}B;$
$\quad \bigwedge x.\ x \in A \Longrightarrow c(x){=}c'(x);$
$\quad \bigwedge y.\ y \in B \Longrightarrow d(y){=}d'(y)$
$]\!] \Longrightarrow case(c,d,z) = case(c',d',z)$
by *auto*

lemma *case-case*: $z \in A{+}B \Longrightarrow$
$\quad case(c,\ d,\ case(\lambda x.\ Inl(c'(x)),\ \lambda y.\ Inr(d'(y)),\ z)) =$
$\quad case(\lambda x.\ c(c'(x)),\ \lambda y.\ d(d'(y)),\ z)$
by *auto*

7.4 More Rules for $Part(A,\ h)$

lemma *Part-mono*: $A{<=}B \Longrightarrow Part(A,h){<=}Part(B,h)$
by *blast*

lemma *Part-Collect*: $Part(Collect(A,P),\ h) = Collect(Part(A,h),\ P)$
by *blast*

lemmas *Part-CollectE* =
  Part-Collect [*THEN equalityD1*, *THEN subsetD*, *THEN CollectE*]

lemma *Part-Inl*: $Part(A+B,Inl) = \{Inl(x).\ x \in A\}$
by *blast*

lemma *Part-Inr*: $Part(A+B,Inr) = \{Inr(y).\ y \in B\}$
by *blast*

lemma *PartD1*: $a \in Part(A,h) \Longrightarrow a \in A$
by (*simp add*: *Part-def*)

lemma *Part-id*: $Part(A,\lambda x.\ x) = A$
by *blast*

lemma *Part-Inr2*: $Part(A+B,\ \lambda x.\ Inr(h(x))) = \{Inr(y).\ y \in Part(B,h)\}$
by *blast*

lemma *Part-sum-equality*: $C \subseteq A+B \Longrightarrow Part(C,Inl) \cup Part(C,Inr) = C$
by *blast*

end

8 Functions, Function Spaces, Lambda-Abstraction

theory *func* **imports** *equalities Sum* **begin**

8.1 The Pi Operator: Dependent Function Space

lemma *subset-Sigma-imp-relation*: $r \subseteq Sigma(A,B) \Longrightarrow relation(r)$
by (*simp add*: *relation-def*, *blast*)

lemma *relation-converse-converse* [*simp*]:
  $relation(r) \Longrightarrow converse(converse(r)) = r$
by (*simp add*: *relation-def*, *blast*)

lemma *relation-restrict* [*simp*]: $relation(restrict(r,A))$
by (*simp add*: *restrict-def relation-def*, *blast*)

lemma *Pi-iff*:
  $f \in Pi(A,B) \longleftrightarrow function(f) \land f{<=}Sigma(A,B) \land A{<=}domain(f)$
by (*unfold Pi-def*, *blast*)

lemma *Pi-iff-old*:
  $f \in Pi(A,B) \longleftrightarrow f{<=}Sigma(A,B) \land (\forall x{\in}A.\ \exists!y.\ \langle x,y\rangle{:}\ f)$

by (*unfold Pi-def function-def*, *blast*)

lemma *fun-is-function*: $f \in Pi(A,B) \Longrightarrow function(f)$
by (*simp only*: *Pi-iff*)

lemma *function-imp-Pi*:
$[\![function(f); relation(f)]\!] \Longrightarrow f \in domain(f) -> range(f)$
by (*simp add*: *Pi-iff relation-def*, *blast*)

lemma *functionI*:
$[\![\bigwedge x\ y\ y'.\ [\![\langle x,y\rangle{:}r;\ <x,y'>{:}r]\!] \Longrightarrow y=y']\!] \Longrightarrow function(r)$
by (*simp add*: *function-def*, *blast*)

lemma *fun-is-rel*: $f \in Pi(A,B) \Longrightarrow f \subseteq Sigma(A,B)$
by (*unfold Pi-def*, *blast*)

lemma *Pi-cong*:
$[\![A=A';\ \bigwedge x.\ x \in A' \Longrightarrow B(x)=B'(x)]\!] \Longrightarrow Pi(A,B) = Pi(A',B')$
by (*simp add*: *Pi-def cong add*: *Sigma-cong*)

lemma *fun-weaken-type*: $[\![f \in A->B;\ B<=D]\!] \Longrightarrow f \in A->D$
by (*unfold Pi-def*, *best*)

8.2 Function Application

lemma *apply-equality2*: $[\![\langle a,b\rangle{:}\ f;\ \langle a,c\rangle{:}\ f;\ f \in Pi(A,B)]\!] \Longrightarrow b=c$
by (*unfold Pi-def function-def*, *blast*)

lemma *function-apply-equality*: $[\![\langle a,b\rangle{:}\ f;\ function(f)]\!] \Longrightarrow f`a = b$
by (*unfold apply-def function-def*, *blast*)

lemma *apply-equality*: $[\![\langle a,b\rangle{:}\ f;\ f \in Pi(A,B)]\!] \Longrightarrow f`a = b$
unfolding *Pi-def*
apply (*blast intro*: *function-apply-equality*)
done

lemma *apply-0*: $a \notin domain(f) \Longrightarrow f`a = 0$
by (*unfold apply-def*, *blast*)

lemma *Pi-memberD*: $[\![f \in Pi(A,B);\ c \in f]\!] \Longrightarrow \exists x{\in}A.\ c = <x,f`x>$
apply (*frule fun-is-rel*)
apply (*blast dest*: *apply-equality*)
done

lemma *function-apply-Pair*: $[\![function(f);\;\; a \in domain(f)]\!] \Longrightarrow <a,f`a>: f$
apply (*simp add*: *function-def*, *clarify*)
apply (*subgoal-tac* $f`a = y$, *blast*)
apply (*simp add*: *apply-def*, *blast*)
done

lemma *apply-Pair*: $[\![f \in Pi(A,B);\;\; a \in A]\!] \Longrightarrow <a,f`a>: f$
apply (*simp add*: *Pi-iff*)
apply (*blast intro*: *function-apply-Pair*)
done

lemma *apply-type* [*TC*]: $[\![f \in Pi(A,B);\;\; a \in A]\!] \Longrightarrow f`a \in B(a)$
by (*blast intro*: *apply-Pair dest*: *fun-is-rel*)

lemma *apply-funtype*: $[\![f \in A{-}{>}B;\;\; a \in A]\!] \Longrightarrow f`a \in B$
by (*blast dest*: *apply-type*)

lemma *apply-iff*: $f \in Pi(A,B) \Longrightarrow \langle a,b\rangle: f \longleftrightarrow a \in A \wedge f`a = b$
apply (*frule fun-is-rel*)
apply (*blast intro*!: *apply-Pair apply-equality*)
done

lemma *Pi-type*: $[\![f \in Pi(A,C);\;\; \bigwedge x.\ x \in A \Longrightarrow f`x \in B(x)]\!] \Longrightarrow f \in Pi(A,B)$
apply (*simp only*: *Pi-iff*)
apply (*blast dest*: *function-apply-equality*)
done

lemma *Pi-Collect-iff*:
$(f \in Pi(A,\ \lambda x.\ \{y \in B(x).\ P(x,y)\}))$
$\longleftrightarrow\ f \in Pi(A,B) \wedge (\forall x{\in}A.\ P(x,\ f`x))$
by (*blast intro*: *Pi-type dest*: *apply-type*)

lemma *Pi-weaken-type*:
$[\![f \in Pi(A,B);\;\; \bigwedge x.\ x \in A \Longrightarrow B(x){<=}C(x)]\!] \Longrightarrow f \in Pi(A,C)$
by (*blast intro*: *Pi-type dest*: *apply-type*)

lemma *domain-type*: $[\![\langle a,b\rangle \in f;\;\; f \in Pi(A,B)]\!] \Longrightarrow a \in A$
by (*blast dest*: *fun-is-rel*)

lemma *range-type*: $[\![\langle a,b\rangle \in f;\;\; f \in Pi(A,B)]\!] \Longrightarrow b \in B(a)$
by (*blast dest*: *fun-is-rel*)

lemma *Pair-mem-PiD*: $[\![\langle a,b\rangle : f;\ \ f \in Pi(A,B)]\!] \Longrightarrow a \in A \wedge b \in B(a) \wedge f`a = b$
by (*blast intro*: *domain-type range-type apply-equality*)

8.3 Lambda Abstraction

lemma *lamI*: $a \in A \Longrightarrow <a,b(a)> \in (\lambda x{\in}A.\ b(x))$
  unfolding *lam-def*
apply (*erule RepFunI*)
done

lemma *lamE*:
    $[\![p: (\lambda x{\in}A.\ b(x));\ \ \bigwedge x.[\![x \in A;\ p{=}<x,b(x)>]\!] \Longrightarrow P$
$]\!] \Longrightarrow\ P$
by (*simp add*: *lam-def*, *blast*)

lemma *lamD*: $[\![\langle a,c\rangle: (\lambda x{\in}A.\ b(x))]\!] \Longrightarrow c = b(a)$
by (*simp add*: *lam-def*)

lemma *lam-type* [*TC*]:
    $[\![\bigwedge x.\ x \in A \Longrightarrow b(x): B(x)]\!] \Longrightarrow (\lambda x{\in}A.\ b(x)) \in Pi(A,B)$
by (*simp add*: *lam-def Pi-def function-def*, *blast*)

lemma *lam-funtype*: $(\lambda x{\in}A.\ b(x)) \in A \ {-}{>}\ \{b(x).\ x \in A\}$
by (*blast intro*: *lam-type*)

lemma *function-lam*: *function* $(\lambda x{\in}A.\ b(x))$
by (*simp add*: *function-def lam-def*)

lemma *relation-lam*: *relation* $(\lambda x{\in}A.\ b(x))$
by (*simp add*: *relation-def lam-def*)

lemma *beta-if* [*simp*]: $(\lambda x{\in}A.\ b(x))\ `\ a = (\textit{if}\ a \in A\ \textit{then}\ b(a)\ \textit{else}\ 0)$
by (*simp add*: *apply-def lam-def*, *blast*)

lemma *beta*: $a \in A \Longrightarrow (\lambda x{\in}A.\ b(x))\ `\ a = b(a)$
by (*simp add*: *apply-def lam-def*, *blast*)

lemma *lam-empty* [*simp*]: $(\lambda x{\in}0.\ b(x)) = 0$
by (*simp add*: *lam-def*)

lemma *domain-lam* [*simp*]: $domain(Lambda(A,b)) = A$
by (*simp add*: *lam-def*, *blast*)

lemma *lam-cong* [*cong*]:
    $[\![A{=}A';\ \ \bigwedge x.\ x \in A' \Longrightarrow b(x){=}b'(x)]\!] \Longrightarrow Lambda(A,b) = Lambda(A',b')$
by (*simp only*: *lam-def cong add*: *RepFun-cong*)

lemma *lam-theI*:

$(\bigwedge x.\ x \in A \Longrightarrow \exists !y.\ Q(x,y)) \Longrightarrow \exists f.\ \forall x{\in}A.\ Q(x, f`x)$
apply (*rule-tac x = λx∈A. THE y. Q (x,y)* **in** *exI*)
apply *simp*
apply (*blast intro*: *theI*)
done

lemma *lam-eqE*: $[\![(\lambda x{\in}A.\ f(x)) = (\lambda x{\in}A.\ g(x));\ \ a \in A]\!] \Longrightarrow f(a)=g(a)$
by (*fast intro*!: *lamI elim*: *equalityE lamE*)

lemma *Pi-empty1* [*simp*]: $Pi(0,A) = \{0\}$
by (*unfold Pi-def function-def*, *blast*)

lemma *singleton-fun* [*simp*]: $\{\langle a,b\rangle\} \in \{a\} \text{ -> } \{b\}$
by (*unfold Pi-def function-def*, *blast*)

lemma *Pi-empty2* [*simp*]: $(A\text{->}0) = (\textit{if } A{=}0 \textit{ then } \{0\} \textit{ else } 0)$
by (*unfold Pi-def function-def*, *force*)

lemma *fun-space-empty-iff* [*iff*]: $(A\text{->}X){=}0 \longleftrightarrow X{=}0 \wedge (A \neq 0)$
apply *auto*
apply (*fast intro*!: *equals0I intro*: *lam-type*)
done

8.4 Extensionality

lemma *fun-subset*:
$[\![f \in Pi(A,B);\ \ g \in Pi(C,D);\ \ A{<=}C;$
$\bigwedge x.\ x \in A \Longrightarrow f`x = g`x]\!] \Longrightarrow f{<=}g$
by (*force dest*: *Pi-memberD intro*: *apply-Pair*)

lemma *fun-extension*:
$[\![f \in Pi(A,B);\ \ g \in Pi(A,D);$
$\bigwedge x.\ x \in A \Longrightarrow f`x = g`x]\!] \Longrightarrow f{=}g$
by (*blast del*: *subsetI intro*: *subset-refl sym fun-subset*)

lemma *eta* [*simp*]: $f \in Pi(A,B) \Longrightarrow (\lambda x{\in}A.\ f`x) = f$
apply (*rule fun-extension*)
apply (*auto simp add*: *lam-type apply-type beta*)
done

lemma *fun-extension-iff*:
$[\![f \in Pi(A,B);\ g \in Pi(A,C)]\!] \Longrightarrow (\forall a{\in}A.\ f`a = g`a) \longleftrightarrow f{=}g$
by (*blast intro*: *fun-extension*)

lemma *fun-subset-eq*: $[\![f \in Pi(A,B);\ g \in Pi(A,C)]\!] \Longrightarrow f \subseteq g \longleftrightarrow (f = g)$

by (*blast dest*: *apply-Pair*
intro: *fun-extension apply-equality* [*symmetric*])

lemma *Pi-lamE*:
assumes *major*: $f \in Pi(A,B)$
and *minor*: $\bigwedge b.\ [\![\forall x{\in}A.\ b(x){:}B(x);\ \ f = (\lambda x{\in}A.\ b(x))]\!] \Longrightarrow P$
shows P
apply (*rule minor*)
apply (*rule-tac* [*2*] *eta* [*symmetric*])
apply (*blast intro*: *major apply-type*)+
done

8.5 Images of Functions

lemma *image-lam*: $C \subseteq A \Longrightarrow (\lambda x{\in}A.\ b(x))\ ``\ C = \{b(x).\ x \in C\}$
by (*unfold lam-def*, *blast*)

lemma *Repfun-function-if*:
function(f)
$\Longrightarrow \{f`x.\ x \in C\} = (if\ C \subseteq domain(f)\ then\ f``C\ else\ cons(0,f``C))$
apply *simp*
apply (*intro conjI impI*)
apply (*blast dest*: *function-apply-equality intro*: *function-apply-Pair*)
apply (*rule equalityI*)
apply (*blast intro*!: *function-apply-Pair apply-0*)
apply (*blast dest*: *function-apply-equality intro*: *apply-0* [*symmetric*])
done

lemma *image-function*:
$[\![function(f);\ \ C \subseteq domain(f)]\!] \Longrightarrow f``C = \{f`x.\ x \in C\}$
by (*simp add*: *Repfun-function-if*)

lemma *image-fun*: $[\![f \in Pi(A,B);\ \ C \subseteq A]\!] \Longrightarrow f``C = \{f`x.\ x \in C\}$
apply (*simp add*: *Pi-iff*)
apply (*blast intro*: *image-function*)
done

lemma *image-eq-UN*:
assumes *f*: $f \in Pi(A,B)$ $C \subseteq A$ **shows** $f``C = (\bigcup x{\in}C.\ \{f\ `\ x\})$
by (*auto simp add*: *image-fun* [*OF f*])

lemma *Pi-image-cons*:
$[\![f \in Pi(A,B);\ \ x \in A]\!] \Longrightarrow f\ ``\ cons(x,y) = cons(f`x,\ f``y)$
by (*blast dest*: *apply-equality apply-Pair*)

8.6 Properties of *restrict*(*f*, *A*)

lemma *restrict-subset*: $restrict(f,A) \subseteq f$
by (*unfold restrict-def*, *blast*)

lemma *function-restrictI*:
$function(f) \Longrightarrow function(restrict(f,A))$
by (*unfold restrict-def function-def*, *blast*)

lemma *restrict-type2*: $[\![f \in Pi(C,B);\ \ A<=C]\!] \Longrightarrow restrict(f,A) \in Pi(A,B)$
by (*simp add*: *Pi-iff function-def restrict-def*, *blast*)

lemma *restrict*: $restrict(f,A)$ ' $a = (if\ a \in A\ then\ f`a\ else\ 0)$
by (*simp add*: *apply-def restrict-def*, *blast*)

lemma *restrict-empty* [*simp*]: $restrict(f,0) = 0$
by (*unfold restrict-def*, *simp*)

lemma *restrict-iff*: $z \in restrict(r,A) \longleftrightarrow z \in r \wedge (\exists x\in A.\ \exists y.\ z = \langle x,\ y\rangle)$
by (*simp add*: *restrict-def*)

lemma *restrict-restrict* [*simp*]:
$restrict(restrict(r,A),B) = restrict(r,\ A \cap B)$
by (*unfold restrict-def*, *blast*)

lemma *domain-restrict* [*simp*]: $domain(restrict(f,C)) = domain(f) \cap C$
unfolding *restrict-def*
apply (*auto simp add*: *domain-def*)
done

lemma *restrict-idem*: $f \subseteq Sigma(A,B) \Longrightarrow restrict(f,A) = f$
by (*simp add*: *restrict-def*, *blast*)

lemma *domain-restrict-idem*:
$[\![domain(r) \subseteq A;\ relation(r)]\!] \Longrightarrow restrict(r,A) = r$
by (*simp add*: *restrict-def relation-def*, *blast*)

lemma *domain-restrict-lam* [*simp*]: $domain(restrict(Lambda(A,f),C)) = A \cap C$
unfolding *restrict-def lam-def*
apply (*rule equalityI*)
apply (*auto simp add*: *domain-iff*)
done

lemma *restrict-if* [*simp*]: $restrict(f,A)$ ' $a = (if\ a \in A\ then\ f`a\ else\ 0)$
by (*simp add*: *restrict apply-0*)

lemma *restrict-lam-eq*:
$A<=C \Longrightarrow restrict(\lambda x\in C.\ b(x),\ A) = (\lambda x\in A.\ b(x))$

by (*unfold restrict-def lam-def*, *auto*)

lemma *fun-cons-restrict-eq*:
  f ∈ *cons*(*a*, *b*) −> *B* ⟹ *f* = *cons*(<*a*, *f* ‘ *a*>, *restrict*(*f*, *b*))
apply (*rule equalityI*)
 prefer *2* **apply** (*blast intro*: *apply-Pair restrict-subset* [*THEN subsetD*])
apply (*auto dest*!: *Pi-memberD simp add*: *restrict-def lam-def*)
done

8.7 Unions of Functions

lemma *function-Union*:
  ⟦∀ *x*∈*S*. *function*(*x*);
    ∀ *x*∈*S*. ∀ *y*∈*S*. *x*<=*y* | *y*<=*x*⟧
   ⟹ *function*(⋃(*S*))
by (*unfold function-def*, *blast*)

lemma *fun-Union*:
  ⟦∀ *f*∈*S*. ∃ *C D*. *f* ∈ *C*−>*D*;
      ∀ *f*∈*S*. ∀ *y*∈*S*. *f*<=*y* | *y*<=*f*⟧ ⟹
    ⋃(*S*) ∈ *domain*(⋃(*S*)) −> *range*(⋃(*S*))
 unfolding *Pi-def*
apply (*blast intro*!: *rel-Union function-Union*)
done

lemma *gen-relation-Union*:
  (⋀*f*. *f*∈*F* ⟹ *relation*(*f*)) ⟹ *relation*(⋃(*F*))
by (*simp add*: *relation-def*)

lemmas *Un-rls* = *Un-subset-iff SUM-Un-distrib1 prod-Un-distrib2*
    subset-trans [*OF - Un-upper1*]
    subset-trans [*OF - Un-upper2*]

lemma *fun-disjoint-Un*:
  ⟦*f* ∈ *A*−>*B*; *g* ∈ *C*−>*D*; *A* ∩ *C* = *0*⟧
   ⟹ (*f* ∪ *g*) ∈ (*A* ∪ *C*) −> (*B* ∪ *D*)

apply (*simp add*: *Pi-iff extension Un-rls*)
apply (*unfold function-def*, *blast*)
done

lemma *fun-disjoint-apply1*: *a* ∉ *domain*(*g*) ⟹ (*f* ∪ *g*)‘*a* = *f*‘*a*
by (*simp add*: *apply-def*, *blast*)

lemma *fun-disjoint-apply2*: *c* ∉ *domain*(*f*) ⟹ (*f* ∪ *g*)‘*c* = *g*‘*c*
by (*simp add*: *apply-def*, *blast*)

8.8 Domain and Range of a Function or Relation

lemma *domain-of-fun*: $f \in Pi(A,B) \Longrightarrow domain(f)=A$
by (*unfold Pi-def*, *blast*)

lemma *apply-rangeI*: $[\![f \in Pi(A,B);\ \ a \in A]\!] \Longrightarrow f`a \in range(f)$
by (*erule apply-Pair* [*THEN rangeI*], *assumption*)

lemma *range-of-fun*: $f \in Pi(A,B) \Longrightarrow f \in A{-}{>}range(f)$
by (*blast intro*: *Pi-type apply-rangeI*)

8.9 Extensions of Functions

lemma *fun-extend*:
  $[\![f \in A{-}{>}B;\ \ c{\notin}A]\!] \Longrightarrow cons(\langle c,b\rangle,f) \in cons(c,A)\ {-}{>}\ cons(b,B)$
apply (*frule singleton-fun* [*THEN fun-disjoint-Un*], *blast*)
apply (*simp add*: *cons-eq*)
done

lemma *fun-extend3*:
  $[\![f \in A{-}{>}B;\ \ c{\notin}A;\ \ b \in B]\!] \Longrightarrow cons(\langle c,b\rangle,f) \in cons(c,A)\ {-}{>}\ B$
by (*blast intro*: *fun-extend* [*THEN fun-weaken-type*])

lemma *extend-apply*:
  $c \notin domain(f) \Longrightarrow cons(\langle c,b\rangle,f)`a = (if\ a{=}c\ then\ b\ else\ f`a)$
by (*auto simp add*: *apply-def*)

lemma *fun-extend-apply* [*simp*]:
  $[\![f \in A{-}{>}B;\ \ c{\notin}A]\!] \Longrightarrow cons(\langle c,b\rangle,f)`a = (if\ a{=}c\ then\ b\ else\ f`a)$
apply (*rule extend-apply*)
apply (*simp add*: *Pi-def*, *blast*)
done

lemmas *singleton-apply* = *apply-equality* [*OF singletonI singleton-fun*, *simp*]

lemma *cons-fun-eq*:
  $c \notin A \Longrightarrow cons(c,A)\ {-}{>}\ B = (\bigcup f \in A{-}{>}B.\ \bigcup b{\in}B.\ \{cons(\langle c,b\rangle,\ f)\})$
apply (*rule equalityI*)
apply (*safe elim*!: *fun-extend3*)

apply (*subgoal-tac restrict* $(x,\ A) \in A\ {-}{>}\ B$)
 prefer *2* **apply** (*blast intro*: *restrict-type2*)
apply (*rule UN-I*, *assumption*)
apply (*rule apply-funtype* [*THEN UN-I*])
  apply *assumption*
 apply (*rule consI1*)
apply (*simp* (*no-asm*))
apply (*rule fun-extension*)
  apply *assumption*

apply (*blast intro*: *fun-extend*)
apply (*erule consE*, *simp-all*)
done

lemma *succ-fun-eq*: *succ*(*n*) −> *B* = (⋃*f* ∈ *n*−>*B*. ⋃ *b*∈*B*. {*cons*(⟨*n*,*b*⟩, *f*)})
by (*simp add*: *succ-def mem-not-refl cons-fun-eq*)

8.10 Function Updates

definition
update :: [*i*,*i*,*i*] ⇒ *i* **where**
update(*f*,*a*,*b*) ≡ λ*x*∈*cons*(*a*, *domain*(*f*)). *if*(*x*=*a*, *b*, *f*‘*x*)

nonterminal *updbinds* **and** *updbind*

syntax
-*updbind* :: [*i*, *i*] ⇒ *updbind* (‹(‹*indent=2 notation=*‹*infix update*››- :=/ -)›)
:: *updbind* ⇒ *updbinds* (‹-›)
-*updbinds* :: [*updbind*, *updbinds*] ⇒ *updbinds* (‹-,/ -›)
-*Update* :: [*i*, *updbinds*] ⇒ *i* (‹(‹*open-block notation=*‹*mixfix function update*››-/′((-)′))› [*900*,*0*] *900*)
syntax-consts
-*Update* ⇌ *update*
translations
-*Update* (*f*, -*updbinds*(*b*,*bs*)) == -*Update* (-*Update*(*f*,*b*), *bs*)
f(*x*:=*y*) == *CONST update*(*f*,*x*,*y*)

lemma *update-apply* [*simp*]: *f*(*x*:=*y*) ‘ *z* = (*if z*=*x then y else f*‘*z*)
apply (*simp add*: *update-def*)
apply (*case-tac z* ∈ *domain*(*f*))
apply (*simp-all add*: *apply-0*)
done

lemma *update-idem*: ⟦*f*‘*x* = *y*; *f* ∈ *Pi*(*A*,*B*); *x* ∈ *A*⟧ ⟹ *f*(*x*:=*y*) = *f*
unfolding *update-def*
apply (*simp add*: *domain-of-fun cons-absorb*)
apply (*rule fun-extension*)
apply (*best intro*: *apply-type if-type lam-type*, *assumption*, *simp*)
done

declare *refl* [*THEN update-idem*, *simp*]

lemma *domain-update* [*simp*]: *domain*(*f*(*x*:=*y*)) = *cons*(*x*, *domain*(*f*))
by (*unfold update-def*, *simp*)

lemma *update-type*: ⟦*f* ∈ *Pi*(*A*,*B*); *x* ∈ *A*; *y* ∈ *B*(*x*)⟧ ⟹ *f*(*x*:=*y*) ∈ *Pi*(*A*, *B*)
unfolding *update-def*

apply (*simp add*: *domain-of-fun cons-absorb apply-funtype lam-type*)
done

8.11 Monotonicity Theorems

8.11.1 Replacement in its Various Forms

lemma *Replace-mono*: $A<=B \Longrightarrow Replace(A,P) \subseteq Replace(B,P)$
by (*blast elim*!: *ReplaceE*)

lemma *RepFun-mono*: $A<=B \Longrightarrow \{f(x).\ x \in A\} \subseteq \{f(x).\ x \in B\}$
by *blast*

lemma *Pow-mono*: $A<=B \Longrightarrow Pow(A) \subseteq Pow(B)$
by *blast*

lemma *Union-mono*: $A<=B \Longrightarrow \bigcup(A) \subseteq \bigcup(B)$
by *blast*

lemma *UN-mono*:
$[\![A<=C;\ \bigwedge x.\ x \in A \Longrightarrow B(x)<=D(x)]\!] \Longrightarrow (\bigcup x\in A.\ B(x)) \subseteq (\bigcup x\in C.\ D(x))$
by *blast*

lemma *Inter-anti-mono*: $[\![A<=B;\ A\neq 0]\!] \Longrightarrow \bigcap(B) \subseteq \bigcap(A)$
by *blast*

lemma *cons-mono*: $C<=D \Longrightarrow cons(a,C) \subseteq cons(a,D)$
by *blast*

lemma *Un-mono*: $[\![A<=C;\ B<=D]\!] \Longrightarrow A \cup B \subseteq C \cup D$
by *blast*

lemma *Int-mono*: $[\![A<=C;\ B<=D]\!] \Longrightarrow A \cap B \subseteq C \cap D$
by *blast*

lemma *Diff-mono*: $[\![A<=C;\ D<=B]\!] \Longrightarrow A-B \subseteq C-D$
by *blast*

8.11.2 Standard Products, Sums and Function Spaces

lemma *Sigma-mono* [*rule-format*]:
$[\![A<=C;\ \bigwedge x.\ x \in A \longrightarrow B(x) \subseteq D(x)]\!] \Longrightarrow Sigma(A,B) \subseteq Sigma(C,D)$
by *blast*

lemma *sum-mono*: $[\![A<=C;\ B<=D]\!] \Longrightarrow A+B \subseteq C+D$
by (*unfold sum-def*, *blast*)

lemma *Pi-mono*: $B<=C \Longrightarrow A->B \subseteq A->C$

by (*blast intro*: *lam-type elim*: *Pi-lamE*)

lemma *lam-mono*: $A<=B \Longrightarrow Lambda(A,c) \subseteq Lambda(B,c)$
  unfolding *lam-def*
apply (*erule RepFun-mono*)
done

8.11.3 Converse, Domain, Range, Field

lemma *converse-mono*: $r<=s \Longrightarrow converse(r) \subseteq converse(s)$
by *blast*

lemma *domain-mono*: $r<=s \Longrightarrow domain(r)<=domain(s)$
by *blast*

lemmas *domain-rel-subset* = *subset-trans* [*OF domain-mono domain-subset*]

lemma *range-mono*: $r<=s \Longrightarrow range(r)<=range(s)$
by *blast*

lemmas *range-rel-subset* = *subset-trans* [*OF range-mono range-subset*]

lemma *field-mono*: $r<=s \Longrightarrow field(r)<=field(s)$
by *blast*

lemma *field-rel-subset*: $r \subseteq A*A \Longrightarrow field(r) \subseteq A$
by (*erule field-mono* [*THEN subset-trans*], *blast*)

8.11.4 Images

lemma *image-pair-mono*:
  $[\![\bigwedge x\ y.\ \langle x,y\rangle{:}r \Longrightarrow \langle x,y\rangle{:}s;\ \ A<=B]\!] \Longrightarrow r``A \subseteq s``B$
by *blast*

lemma *vimage-pair-mono*:
  $[\![\bigwedge x\ y.\ \langle x,y\rangle{:}r \Longrightarrow \langle x,y\rangle{:}s;\ \ A<=B]\!] \Longrightarrow r-``A \subseteq s-``B$
by *blast*

lemma *image-mono*: $[\![r<=s;\ \ A<=B]\!] \Longrightarrow r``A \subseteq s``B$
by *blast*

lemma *vimage-mono*: $[\![r<=s;\ \ A<=B]\!] \Longrightarrow r-``A \subseteq s-``B$
by *blast*

lemma *Collect-mono*:
  $[\![A<=B;\ \ \bigwedge x.\ x \in A \Longrightarrow P(x) \longrightarrow Q(x)]\!] \Longrightarrow Collect(A,P) \subseteq Collect(B,Q)$
by *blast*

lemmas *basic-monos* = *subset-refl imp-refl disj-mono conj-mono ex-mono*

```
Collect-mono Part-mono in-mono
```

```
lemma bex-image-simp:
  ⟦f ∈ Pi(X, Y); A ⊆ X⟧ ⟹ (∃x∈f``A. P(x)) ⟷ (∃x∈A. P(f`x))
  apply safe
   apply rule
    prefer 2 apply assumption
   apply (simp add: apply-equality)
  apply (blast intro: apply-Pair)
  done

lemma ball-image-simp:
  ⟦f ∈ Pi(X, Y); A ⊆ X⟧ ⟹ (∀x∈f``A. P(x)) ⟷ (∀x∈A. P(f`x))
  apply safe
   apply (blast intro: apply-Pair)
  apply (drule bspec) apply assumption
  apply (simp add: apply-equality)
  done

end
```

9 Quine-Inspired Ordered Pairs and Disjoint Sums

```
theory QPair imports Sum func begin
```

For non-well-founded data structures in ZF. Does not precisely follow Quine's construction. Thanks to Thomas Forster for suggesting this approach!

W. V. Quine, On Ordered Pairs and Relations, in Selected Logic Papers, 1966.

```
definition
  QPair     :: [i, i] ⇒ i  (‹(‹indent=1 notation=‹mixfix Quine pair››<-;/ ->)›)
  where <a;b> ≡ a+b

definition
  qfst :: i ⇒ i  where
    qfst(p) ≡ THE a. ∃b. p=<a;b>

definition
  qsnd :: i ⇒ i  where
    qsnd(p) ≡ THE b. ∃a. p=<a;b>

definition
  qsplit    :: [[i, i] ⇒ 'a, i] ⇒ 'a::{}     where
    qsplit(c,p) ≡ c(qfst(p), qsnd(p))

definition
```

qconverse :: *i* ⇒ *i* **where**
qconverse(*r*) ≡ {*z*. *w* ∈ *r*, ∃ *x y*. *w*=<*x*;*y*> ∧ *z*=<*y*;*x*>}

definition
QSigma :: [*i*, *i* ⇒ *i*] ⇒ *i* **where**
QSigma(*A*,*B*) ≡ ⋃*x*∈*A*. ⋃*y*∈*B*(*x*). {<*x*;*y*>}

syntax
-*QSUM* :: [*idt*, *i*, *i*] ⇒ *i* (‹(‹*indent*=*3* *notation*=‹*binder QSUM*∈››*QSUM* - ∈ -./ -)› *10*)
syntax-consts
-*QSUM* ⇌ *QSigma*
translations
QSUM x ∈ *A*. *B* => *CONST QSigma*(*A*, λ*x*. *B*)

abbreviation
qprod (**infixr** ‹<∗>› *80*) **where**
A <∗> *B* ≡ *QSigma*(*A*, λ-. *B*)

definition
qsum :: [*i*,*i*]⇒*i* (**infixr** ‹<+>› *65*) **where**
A <+> *B* ≡ ({*0*} <∗> *A*) ∪ ({*1*} <∗> *B*)

definition
QInl :: *i*⇒*i* **where**
QInl(*a*) ≡ <*0*;*a*>

definition
QInr :: *i*⇒*i* **where**
QInr(*b*) ≡ <*1*;*b*>

definition
qcase :: [*i*⇒*i*, *i*⇒*i*, *i*]⇒*i* **where**
qcase(*c*,*d*) ≡ *qsplit*(λ*y z*. *cond*(*y*, *d*(*z*), *c*(*z*)))

9.1 Quine ordered pairing

lemma *QPair-empty* [*simp*]: <*0*;*0*> = *0*
by (*simp add*: *QPair-def*)

lemma *QPair-iff* [*simp*]: <*a*;*b*> = <*c*;*d*> ⟷ *a*=*c* ∧ *b*=*d*
apply (*simp add*: *QPair-def*)
apply (*rule sum-equal-iff*)
done

lemmas *QPair-inject* = *QPair-iff* [*THEN iffD1*, *THEN conjE*, *elim*!]

lemma *QPair-inject1*: <*a*;*b*> = <*c*;*d*> ⟹ *a*=*c*
by *blast*

lemma *QPair-inject2*: <*a*;*b*> = <*c*;*d*> ⟹ *b*=*d*
by *blast*

9.1.1 QSigma: Disjoint union of a family of sets Generalizes Cartesian product

lemma *QSigmaI* [*intro*!]: ⟦*a* ∈ *A*; *b* ∈ *B*(*a*)⟧ ⟹ <*a*;*b*> ∈ *QSigma*(*A*,*B*)
by (*simp add*: *QSigma-def*)

lemma *QSigmaE* [*elim*!]:
⟦*c* ∈ *QSigma*(*A*,*B*);
⋀*x y*.⟦*x* ∈ *A*; *y* ∈ *B*(*x*); *c*=<*x*;*y*>⟧ ⟹ *P*
⟧ ⟹ *P*
by (*simp add*: *QSigma-def*, *blast*)

lemma *QSigmaE2* [*elim*!]:
⟦<*a*;*b*>: *QSigma*(*A*,*B*); ⟦*a* ∈ *A*; *b* ∈ *B*(*a*)⟧ ⟹ *P*⟧ ⟹ *P*
by (*simp add*: *QSigma-def*)

lemma *QSigmaD1*: <*a*;*b*> ∈ *QSigma*(*A*,*B*) ⟹ *a* ∈ *A*
by *blast*

lemma *QSigmaD2*: <*a*;*b*> ∈ *QSigma*(*A*,*B*) ⟹ *b* ∈ *B*(*a*)
by *blast*

lemma *QSigma-cong*:
⟦*A*=*A*′; ⋀*x*. *x* ∈ *A*′ ⟹ *B*(*x*)=*B*′(*x*)⟧ ⟹
QSigma(*A*,*B*) = *QSigma*(*A*′,*B*′)
by (*simp add*: *QSigma-def*)

lemma *QSigma-empty1* [*simp*]: *QSigma*(*0*,*B*) = *0*
by *blast*

lemma *QSigma-empty2* [*simp*]: *A* <∗> *0* = *0*
by *blast*

9.1.2 Projections: qfst, qsnd

lemma *qfst-conv* [*simp*]: *qfst*(<*a*;*b*>) = *a*
by (*simp add*: *qfst-def*)

lemma *qsnd-conv* [*simp*]: *qsnd*(<*a*;*b*>) = *b*
by (*simp add*: *qsnd-def*)

lemma *qfst-type* [*TC*]: *p* ∈ *QSigma*(*A*,*B*) ⟹ *qfst*(*p*) ∈ *A*
by *auto*

lemma *qsnd-type* [*TC*]: *p* ∈ *QSigma*(*A*,*B*) ⟹ *qsnd*(*p*) ∈ *B*(*qfst*(*p*))
by *auto*

lemma *QPair-qfst-qsnd-eq*: *a* ∈ *QSigma*(*A*,*B*) ⟹ <*qfst*(*a*); *qsnd*(*a*)> = *a*
by *auto*

9.1.3 Eliminator: qsplit

lemma *qsplit* [*simp*]: *qsplit*(λ*x y*. *c*(*x*,*y*), <*a*;*b*>) ≡ *c*(*a*,*b*)
by (*simp add*: *qsplit-def*)

lemma *qsplit-type* [*elim*!]:
⟦*p* ∈ *QSigma*(*A*,*B*);
⋀*x y*.⟦*x* ∈ *A*; *y* ∈ *B*(*x*)⟧ ⟹ *c*(*x*,*y*):*C*(<*x*;*y*>)
⟧ ⟹ *qsplit*(λ*x y*. *c*(*x*,*y*), *p*) ∈ *C*(*p*)
by *auto*

lemma *expand-qsplit*:
u ∈ *A*<∗>*B* ⟹ *R*(*qsplit*(*c*,*u*)) ⟷ (∀ *x*∈*A*. ∀ *y*∈*B*. *u* = <*x*;*y*> ⟶ *R*(*c*(*x*,*y*)))
apply (*simp add*: *qsplit-def*, *auto*)
done

9.1.4 qsplit for predicates: result type o

lemma *qsplitI*: *R*(*a*,*b*) ⟹ *qsplit*(*R*, <*a*;*b*>)
by (*simp add*: *qsplit-def*)

lemma *qsplitE*:
⟦*qsplit*(*R*,*z*); *z* ∈ *QSigma*(*A*,*B*);
⋀*x y*. ⟦*z* = <*x*;*y*>; *R*(*x*,*y*)⟧ ⟹ *P*
⟧ ⟹ *P*
by (*simp add*: *qsplit-def*, *auto*)

lemma *qsplitD*: *qsplit*(*R*,<*a*;*b*>) ⟹ *R*(*a*,*b*)
by (*simp add*: *qsplit-def*)

9.1.5 qconverse

lemma *qconverseI* [*intro*!]: <*a*;*b*>:*r* ⟹ <*b*;*a*>:*qconverse*(*r*)
by (*simp add*: *qconverse-def*, *blast*)

lemma *qconverseD* [*elim*!]: <*a*;*b*> ∈ *qconverse*(*r*) ⟹ <*b*;*a*> ∈ *r*
by (*simp add*: *qconverse-def*, *blast*)

lemma *qconverseE* [*elim*!]:
⟦*yx* ∈ *qconverse*(*r*);
⋀*x y*. ⟦*yx*=<*y*;*x*>; <*x*;*y*>:*r*⟧ ⟹ *P*

$]\!] \Longrightarrow P$
by (*simp add*: *qconverse-def*, *blast*)

lemma *qconverse-qconverse*: $r<=QSigma(A,B) \Longrightarrow qconverse(qconverse(r)) = r$
by *blast*

lemma *qconverse-type*: $r \subseteq A <*> B \Longrightarrow qconverse(r) \subseteq B <*> A$
by *blast*

lemma *qconverse-prod*: $qconverse(A <*> B) = B <*> A$
by *blast*

lemma *qconverse-empty*: $qconverse(0) = 0$
by *blast*

9.2 The Quine-inspired notion of disjoint sum

lemmas *qsum-defs* = *qsum-def QInl-def QInr-def qcase-def*

lemma *QInlI* [*intro*!]: $a \in A \Longrightarrow QInl(a) \in A <+> B$
by (*simp add*: *qsum-defs*, *blast*)

lemma *QInrI* [*intro*!]: $b \in B \Longrightarrow QInr(b) \in A <+> B$
by (*simp add*: *qsum-defs*, *blast*)

lemma *qsumE* [*elim*!]:
$[\![u \in A <+> B;$
$\bigwedge x.\ [\![x \in A;\ \ u=QInl(x)]\!] \Longrightarrow P;$
$\bigwedge y.\ [\![y \in B;\ \ u=QInr(y)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
by (*simp add*: *qsum-defs*, *blast*)

lemma *QInl-iff* [*iff*]: $QInl(a)=QInl(b) \longleftrightarrow a=b$
by (*simp add*: *qsum-defs*)

lemma *QInr-iff* [*iff*]: $QInr(a)=QInr(b) \longleftrightarrow a=b$
by (*simp add*: *qsum-defs*)

lemma *QInl-QInr-iff* [*simp*]: $QInl(a)=QInr(b) \longleftrightarrow False$
by (*simp add*: *qsum-defs*)

lemma *QInr-QInl-iff* [*simp*]: $QInr(b)=QInl(a) \longleftrightarrow False$

by (*simp add*: *qsum-defs*)

lemma *qsum-empty* [*simp*]: *0<+>0 = 0*
by (*simp add*: *qsum-defs*)

lemmas *QInl-inject = QInl-iff* [*THEN iffD1*]
lemmas *QInr-inject = QInr-iff* [*THEN iffD1*]
lemmas *QInl-neq-QInr = QInl-QInr-iff* [*THEN iffD1*, *THEN FalseE*, *elim*!]
lemmas *QInr-neq-QInl = QInr-QInl-iff* [*THEN iffD1*, *THEN FalseE*, *elim*!]

lemma *QInlD*: *QInl*(*a*): *A<+>B* ⟹ *a* ∈ *A*
by *blast*

lemma *QInrD*: *QInr*(*b*): *A<+>B* ⟹ *b* ∈ *B*
by *blast*

lemma *qsum-iff*:
  u ∈ *A <+> B* ⟷ (∃ *x*. *x* ∈ *A* ∧ *u=QInl*(*x*)) | (∃ *y*. *y* ∈ *B* ∧ *u=QInr*(*y*))
by *blast*

lemma *qsum-subset-iff*: *A <+> B* ⊆ *C <+> D* ⟷ *A<=C* ∧ *B<=D*
by *blast*

lemma *qsum-equal-iff*: *A <+> B = C <+> D* ⟷ *A=C* ∧ *B=D*
apply (*simp* (*no-asm*) *add*: *extension qsum-subset-iff*)
apply *blast*
done

9.2.1 Eliminator – qcase

lemma *qcase-QInl* [*simp*]: *qcase*(*c*, *d*, *QInl*(*a*)) = *c*(*a*)
by (*simp add*: *qsum-defs*)

lemma *qcase-QInr* [*simp*]: *qcase*(*c*, *d*, *QInr*(*b*)) = *d*(*b*)
by (*simp add*: *qsum-defs*)

lemma *qcase-type*:
  ⟦*u* ∈ *A <+> B*;
    ⋀*x*. *x* ∈ *A* ⟹ *c*(*x*): *C*(*QInl*(*x*));
    ⋀*y*. *y* ∈ *B* ⟹ *d*(*y*): *C*(*QInr*(*y*))
⟧ ⟹ *qcase*(*c*,*d*,*u*) ∈ *C*(*u*)
by (*simp add*: *qsum-defs*, *auto*)

lemma *Part-QInl*: $Part(A <+> B,QInl) = \{QInl(x).\ x \in A\}$
by *blast*

lemma *Part-QInr*: $Part(A <+> B,QInr) = \{QInr(y).\ y \in B\}$
by *blast*

lemma *Part-QInr2*: $Part(A <+> B, \lambda x.\ QInr(h(x))) = \{QInr(y).\ y \in Part(B,h)\}$
by *blast*

lemma *Part-qsum-equality*: $C \subseteq A <+> B \Longrightarrow Part(C,QInl) \cup Part(C,QInr) = C$
by *blast*

9.2.2 Monotonicity

lemma *QPair-mono*: $[\![a<=c;\ \ b<=d]\!] \Longrightarrow <a;b> \subseteq <c;d>$
by (*simp add*: *QPair-def sum-mono*)

lemma *QSigma-mono* [*rule-format*]:
$[\![A<=C;\ \ \forall x \in A.\ B(x) \subseteq D(x)]\!] \Longrightarrow QSigma(A,B) \subseteq QSigma(C,D)$
by *blast*

lemma *QInl-mono*: $a<=b \Longrightarrow QInl(a) \subseteq QInl(b)$
by (*simp add*: *QInl-def subset-refl* [*THEN QPair-mono*])

lemma *QInr-mono*: $a<=b \Longrightarrow QInr(a) \subseteq QInr(b)$
by (*simp add*: *QInr-def subset-refl* [*THEN QPair-mono*])

lemma *qsum-mono*: $[\![A<=C;\ \ B<=D]\!] \Longrightarrow A <+> B \subseteq C <+> D$
by *blast*

end

10 Injections, Surjections, Bijections, Composition

theory *Perm* **imports** *func* **begin**

definition

comp :: $[i,i]\Rightarrow i$ (**infixr** ‹*O*› *60*) **where**
$r\ O\ s \equiv \{xz \in domain(s)*range(r)\ .$
$\exists x\ y\ z.\ xz=\langle x,z\rangle \wedge \langle x,y\rangle{:}s \wedge \langle y,z\rangle{:}r\}$

definition

id :: $i\Rightarrow i$ **where**
$id(A) \equiv (\lambda x\in A.\ x)$

definition

inj :: $[i,i]\Rightarrow i$ **where**
$inj(A,B) \equiv \{ f \in A{-}{>}B.\ \forall w{\in}A.\ \forall x{\in}A.\ f`w{=}f`x \longrightarrow w{=}x\}$

definition

surj :: $[i,i]\Rightarrow i$ **where**
$surj(A,B) \equiv \{ f \in A{-}{>}B\ .\ \forall y{\in}B.\ \exists x{\in}A.\ f`x{=}y\}$

definition

bij :: $[i,i]\Rightarrow i$ **where**
$bij(A,B) \equiv inj(A,B) \cap surj(A,B)$

10.1 Surjective Function Space

lemma *surj-is-fun*: $f \in surj(A,B) \Longrightarrow f \in A{-}{>}B$
unfolding *surj-def*
apply (*erule CollectD1*)
done

lemma *fun-is-surj*: $f \in Pi(A,B) \Longrightarrow f \in surj(A,range(f))$
unfolding *surj-def*
apply (*blast intro*: *apply-equality range-of-fun domain-type*)
done

lemma *surj-range*: $f \in surj(A,B) \Longrightarrow range(f){=}B$
unfolding *surj-def*
apply (*best intro*: *apply-Pair elim*: *range-type*)
done

A function with a right inverse is a surjection

lemma *f-imp-surjective*:
$[\![f \in A{-}{>}B;\ \bigwedge y.\ y \in B \Longrightarrow d(y){:}\ A;\ \bigwedge y.\ y \in B \Longrightarrow f`d(y) = y]\!]$
$\Longrightarrow f \in surj(A,B)$
by (*simp add*: *surj-def*, *blast*)

lemma *lam-surjective*:
$[\![\bigwedge x.\ x \in A \Longrightarrow c(x){:}\ B;$
$\bigwedge y.\ y \in B \Longrightarrow d(y){:}\ A;$
$\bigwedge y.\ y \in B \Longrightarrow c(d(y)) = y$
$]\!] \Longrightarrow (\lambda x{\in}A.\ c(x)) \in surj(A,B)$
apply (*rule-tac d = d* **in** *f-imp-surjective*)
apply (*simp-all add*: *lam-type*)
done

Cantor's theorem revisited

lemma *cantor-surj*: $f \notin surj(A,Pow(A))$

apply (*unfold surj-def*, *safe*)
apply (*cut-tac cantor*)
apply (*best del*: *subsetI*)
done

10.2 Injective Function Space

lemma *inj-is-fun*: $f \in inj(A,B) \Longrightarrow f \in A{-}{>}B$
 unfolding *inj-def*
apply (*erule CollectD1*)
done

Good for dealing with sets of pairs, but a bit ugly in use [used in AC]

lemma *inj-equality*:
 $[\![\langle a,b\rangle{:}f;\ \ \langle c,b\rangle{:}f;\ \ f \in inj(A,B)]\!] \Longrightarrow a{=}c$
 unfolding *inj-def*
apply (*blast dest*: *Pair-mem-PiD*)
done

lemma *inj-apply-equality*: $[\![f \in inj(A,B);\ \ f`a{=}f`b;\ \ a \in A;\ \ b \in A]\!] \Longrightarrow a{=}b$
by (*unfold inj-def*, *blast*)

A function with a left inverse is an injection

lemma *f-imp-injective*: $[\![f \in A{-}{>}B;\ \ \forall x{\in}A.\ d(f`x){=}x]\!] \Longrightarrow f \in inj(A,B)$
apply (*simp* (*no-asm-simp*) *add*: *inj-def*)
apply (*blast intro*: *subst-context* [*THEN box-equals*])
done

lemma *lam-injective*:
 $[\![\bigwedge x.\ x \in A \Longrightarrow c(x){:}\ B;$
  $\bigwedge x.\ x \in A \Longrightarrow d(c(x)) = x]\!]$
 $\Longrightarrow (\lambda x{\in}A.\ c(x)) \in inj(A,B)$
apply (*rule-tac d* = *d* **in** *f-imp-injective*)
apply (*simp-all add*: *lam-type*)
done

10.3 Bijections

lemma *bij-is-inj*: $f \in bij(A,B) \Longrightarrow f \in inj(A,B)$
 unfolding *bij-def*
apply (*erule IntD1*)
done

lemma *bij-is-surj*: $f \in bij(A,B) \Longrightarrow f \in surj(A,B)$
 unfolding *bij-def*
apply (*erule IntD2*)
done

lemma *bij-is-fun*: $f \in bij(A,B) \Longrightarrow f \in A{-}{>}B$

by (*rule bij-is-inj* [*THEN inj-is-fun*])

lemma *lam-bijective*:
$[\![\bigwedge x.\ x \in A \Longrightarrow c(x): B;$
$\bigwedge y.\ y \in B \Longrightarrow d(y): A;$
$\bigwedge x.\ x \in A \Longrightarrow d(c(x)) = x;$
$\bigwedge y.\ y \in B \Longrightarrow c(d(y)) = y$
$]\!] \Longrightarrow (\lambda x{\in}A.\ c(x)) \in bij(A,B)$
unfolding *bij-def*
apply (*blast intro*!: *lam-injective lam-surjective*)
done

lemma *RepFun-bijective*: $(\forall y{\in}x.\ \exists!y'.\ f(y') = f(y))$
$\Longrightarrow (\lambda z{\in}\{f(y).\ y \in x\}.\ THE\ y.\ f(y) = z) \in bij(\{f(y).\ y \in x\},\ x)$
apply (*rule-tac d* = *f* **in** *lam-bijective*)
apply (*auto simp add*: *the-equality2*)
done

10.4 Identity Function

lemma *idI* [*intro*!]: $a \in A \Longrightarrow \langle a,a\rangle \in id(A)$
unfolding *id-def*
apply (*erule lamI*)
done

lemma *idE* [*elim*!]: $[\![p \in id(A);\ \ \bigwedge x.[\![x \in A;\ p{=}\langle x,x\rangle]\!] \Longrightarrow P]\!] \Longrightarrow\ \ P$
by (*simp add*: *id-def lam-def*, *blast*)

lemma *id-type*: $id(A) \in A{-}{>}A$
unfolding *id-def*
apply (*rule lam-type*, *assumption*)
done

lemma *id-conv* [*simp*]: $x \in A \Longrightarrow id(A)`x = x$
unfolding *id-def*
apply (*simp* (*no-asm-simp*))
done

lemma *id-mono*: $A{<}{=}B \Longrightarrow id(A) \subseteq id(B)$
unfolding *id-def*
apply (*erule lam-mono*)
done

lemma *id-subset-inj*: $A{<}{=}B \Longrightarrow id(A): inj(A,B)$
apply (*simp add*: *inj-def id-def*)
apply (*blast intro*: *lam-type*)
done

lemmas *id-inj* = *subset-refl* [*THEN id-subset-inj*]

lemma *id-surj*: *id*(*A*): *surj*(*A*,*A*)
  unfolding *id-def surj-def*
apply (*simp* (*no-asm-simp*))
done

lemma *id-bij*: *id*(*A*): *bij*(*A*,*A*)
  unfolding *bij-def*
apply (*blast intro*: *id-inj id-surj*)
done

lemma *subset-iff-id*: $A \subseteq B \longleftrightarrow id(A) \in A{-}{>}B$
  unfolding *id-def*
apply (*force intro*!: *lam-type dest*: *apply-type*)
done

id as the identity relation

lemma *id-iff* [*simp*]: $\langle x,y\rangle \in id(A) \longleftrightarrow x=y \land y \in A$
by *auto*

10.5 Converse of a Function

lemma *inj-converse-fun*: $f \in inj(A,B) \Longrightarrow converse(f) \in range(f){-}{>}A$
  unfolding *inj-def*
apply (*simp* (*no-asm-simp*) *add*: *Pi-iff function-def*)
apply (*erule CollectE*)
apply (*simp* (*no-asm-simp*) *add*: *apply-iff*)
apply (*blast dest*: *fun-is-rel*)
done

Equations for converse(f)

The premises are equivalent to saying that f is injective...

lemma *left-inverse-lemma*:
    $[\![f \in A{-}{>}B;\ \ converse(f): C{-}{>}A;\ \ a \in A]\!] \Longrightarrow converse(f)`(f`a) = a$
by (*blast intro*: *apply-Pair apply-equality converseI*)

lemma *left-inverse* [*simp*]: $[\![f \in inj(A,B);\ \ a \in A]\!] \Longrightarrow converse(f)`(f`a) = a$
by (*blast intro*: *left-inverse-lemma inj-converse-fun inj-is-fun*)

lemma *left-inverse-eq*:
    $[\![f \in inj(A,B);\ f\ `\ x = y;\ x \in A]\!] \Longrightarrow converse(f)\ `\ y = x$
by *auto*

lemmas *left-inverse-bij* = *bij-is-inj* [*THEN left-inverse*]

lemma *right-inverse-lemma*:
    $[\![f \in A{-}{>}B;\ \ converse(f): C{-}{>}A;\ \ b \in C]\!] \Longrightarrow f`(converse(f)`b) = b$
by (*rule apply-Pair* [*THEN converseD* [*THEN apply-equality*]], *auto*)

lemma *right-inverse* [*simp*]:
$[\![f \in inj(A,B);\ \ b \in range(f)]\!] \Longrightarrow f`(converse(f)`b) = b$
by (*blast intro*: *right-inverse-lemma inj-converse-fun inj-is-fun*)

lemma *right-inverse-bij*: $[\![f \in bij(A,B);\ \ b \in B]\!] \Longrightarrow f`(converse(f)`b) = b$
by (*force simp add*: *bij-def surj-range*)

10.6 Converses of Injections, Surjections, Bijections

lemma *inj-converse-inj*: $f \in inj(A,B) \Longrightarrow converse(f): inj(range(f), A)$
apply (*rule f-imp-injective*)
apply (*erule inj-converse-fun*, *clarify*)
apply (*rule right-inverse*)
apply *assumption*
apply *blast*
done

lemma *inj-converse-surj*: $f \in inj(A,B) \Longrightarrow converse(f): surj(range(f), A)$
by (*blast intro*: *f-imp-surjective inj-converse-fun left-inverse inj-is-fun range-of-fun* [*THEN apply-type*])

Adding this as an intro! rule seems to cause looping

lemma *bij-converse-bij* [*TC*]: $f \in bij(A,B) \Longrightarrow converse(f): bij(B,A)$
unfolding *bij-def*
apply (*fast elim*: *surj-range* [*THEN subst*] *inj-converse-inj inj-converse-surj*)
done

10.7 Composition of Two Relations

The inductive definition package could derive these theorems for $r\ O\ s$

lemma *compI* [*intro*]: $[\![\langle a,b\rangle{:}s;\ \langle b,c\rangle{:}r]\!] \Longrightarrow \langle a,c\rangle \in r\ O\ s$
by (*unfold comp-def*, *blast*)

lemma *compE* [*elim*!]:
$[\![xz \in r\ O\ s;$
$\bigwedge x\ y\ z.\ [\![xz=\langle x,z\rangle;\ \ \langle x,y\rangle{:}s;\ \ \langle y,z\rangle{:}r]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
by (*unfold comp-def*, *blast*)

lemma *compEpair*:
$[\![\langle a,c\rangle \in r\ O\ s;$
$\bigwedge y.\ [\![\langle a,y\rangle{:}s;\ \ \langle y,c\rangle{:}r]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
by (*erule compE*, *simp*)

lemma *converse-comp*: $converse(R\ O\ S) = converse(S)\ O\ converse(R)$
by *blast*

10.8 Domain and Range – see Suppes, Section 3.1

Boyer et al., Set Theory in First-Order Logic, JAR 2 (1986), 287-327

lemma *range-comp*: $range(r\ O\ s) \subseteq range(r)$
by *blast*

lemma *range-comp-eq*: $domain(r) \subseteq range(s) \Longrightarrow range(r\ O\ s) = range(r)$
by (*rule range-comp* [*THEN equalityI*], *blast*)

lemma *domain-comp*: $domain(r\ O\ s) \subseteq domain(s)$
by *blast*

lemma *domain-comp-eq*: $range(s) \subseteq domain(r) \Longrightarrow domain(r\ O\ s) = domain(s)$
by (*rule domain-comp* [*THEN equalityI*], *blast*)

lemma *image-comp*: $(r\ O\ s)``A = r``(s``A)$
by *blast*

lemma *inj-inj-range*: $f \in inj(A,B) \Longrightarrow f \in inj(A,range(f))$
by (*auto simp add*: *inj-def Pi-iff function-def*)

lemma *inj-bij-range*: $f \in inj(A,B) \Longrightarrow f \in bij(A,range(f))$
by (*auto simp add*: *bij-def intro*: *inj-inj-range inj-is-fun fun-is-surj*)

10.9 Other Results

lemma *comp-mono*: $[\![r'<=r;\ s'<=s]\!] \Longrightarrow (r'\ O\ s') \subseteq (r\ O\ s)$
by *blast*

composition preserves relations

lemma *comp-rel*: $[\![s<=A*B;\ \ r<=B*C]\!] \Longrightarrow (r\ O\ s) \subseteq A*C$
by *blast*

associative law for composition

lemma *comp-assoc*: $(r\ O\ s)\ O\ t = r\ O\ (s\ O\ t)$
by *blast*

lemma *left-comp-id*: $r<=A*B \Longrightarrow id(B)\ O\ r = r$
by *blast*

lemma *right-comp-id*: $r<=A*B \Longrightarrow r\ O\ id(A) = r$
by *blast*

10.10 Composition Preserves Functions, Injections, and Surjections

lemma *comp-function*: $[\![function(g);\ \ function(f)]\!] \Longrightarrow function(f\ O\ g)$

by (*unfold function-def*, *blast*)

Don't think the premises can be weakened much

lemma *comp-fun*: $[\![g \in A{-}{>}B;\ f \in B{-}{>}C]\!] \Longrightarrow (f\ O\ g) \in A{-}{>}C$
apply (*auto simp add*: *Pi-def comp-function Pow-iff comp-rel*)
apply (*subst range-rel-subset* [*THEN domain-comp-eq*], *auto*)
done

lemma *comp-fun-apply* [*simp*]:
$[\![g \in A{-}{>}B;\ a \in A]\!] \Longrightarrow (f\ O\ g)`a = f`(g`a)$
apply (*frule apply-Pair*, *assumption*)
apply (*simp add*: *apply-def image-comp*)
apply (*blast dest*: *apply-equality*)
done

Simplifies compositions of lambda-abstractions

lemma *comp-lam*:
$[\![\bigwedge x.\ x \in A \Longrightarrow b(x) \colon B]\!]$
$\Longrightarrow (\lambda y{\in}B.\ c(y))\ O\ (\lambda x{\in}A.\ b(x)) = (\lambda x{\in}A.\ c(b(x)))$
apply (*subgoal-tac* $(\lambda x{\in}A.\ b(x)) \in A\ {-}{>}\ B$)
apply (*rule fun-extension*)
apply (*blast intro*: *comp-fun lam-funtype*)
apply (*rule lam-funtype*)
apply *simp*
apply (*simp add*: *lam-type*)
done

lemma *comp-inj*:
$[\![g \in inj(A,B);\ f \in inj(B,C)]\!] \Longrightarrow (f\ O\ g) \in inj(A,C)$
apply (*frule inj-is-fun* [*of g*])
apply (*frule inj-is-fun* [*of f*])
apply (*rule-tac* $d = \lambda y.\ converse\ (g)\ `\ (converse\ (f)\ `\ y)$ **in** *f-imp-injective*)
apply (*blast intro*: *comp-fun*, *simp*)
done

lemma *comp-surj*:
$[\![g \in surj(A,B);\ f \in surj(B,C)]\!] \Longrightarrow (f\ O\ g) \in surj(A,C)$
unfolding *surj-def*
apply (*blast intro*!: *comp-fun comp-fun-apply*)
done

lemma *comp-bij*:
$[\![g \in bij(A,B);\ f \in bij(B,C)]\!] \Longrightarrow (f\ O\ g) \in bij(A,C)$
unfolding *bij-def*
apply (*blast intro*: *comp-inj comp-surj*)
done

10.11 Dual Properties of *inj* and *surj*

Useful for proofs from D Pastre. Automatic theorem proving in set theory. Artificial Intelligence, 10:1–27, 1978.

lemma *comp-mem-injD1*:
  ⟦(*f O g*): *inj*(*A*,*C*); *g* ∈ *A*−>*B*; *f* ∈ *B*−>*C*⟧ ⟹ *g* ∈ *inj*(*A*,*B*)
by (*unfold inj-def*, *force*)

lemma *comp-mem-injD2*:
  ⟦(*f O g*): *inj*(*A*,*C*); *g* ∈ *surj*(*A*,*B*); *f* ∈ *B*−>*C*⟧ ⟹ *f* ∈ *inj*(*B*,*C*)
apply (*unfold inj-def surj-def*, *safe*)
apply (*rule-tac x1* = *x* **in** *bspec* [*THEN bexE*])
apply (*erule-tac* [*3*] *x1* = *w* **in** *bspec* [*THEN bexE*], *assumption*+, *safe*)
apply (*rule-tac t* = (‘) (*g*) **in** *subst-context*)
apply (*erule asm-rl bspec* [*THEN bspec*, *THEN mp*])+
apply (*simp* (*no-asm-simp*))
done

lemma *comp-mem-surjD1*:
  ⟦(*f O g*): *surj*(*A*,*C*); *g* ∈ *A*−>*B*; *f* ∈ *B*−>*C*⟧ ⟹ *f* ∈ *surj*(*B*,*C*)
 unfolding *surj-def*
apply (*blast intro*!: *comp-fun-apply* [*symmetric*] *apply-funtype*)
done

lemma *comp-mem-surjD2*:
  ⟦(*f O g*): *surj*(*A*,*C*); *g* ∈ *A*−>*B*; *f* ∈ *inj*(*B*,*C*)⟧ ⟹ *g* ∈ *surj*(*A*,*B*)
apply (*unfold inj-def surj-def*, *safe*)
apply (*drule-tac x* = *f*‘*y* **in** *bspec*, *auto*)
apply (*blast intro*: *apply-funtype*)
done

10.11.1 Inverses of Composition

left inverse of composition; one inclusion is $f \in A \to B \Longrightarrow id(A) \subseteq converse(f)\ O\ f$

lemma *left-comp-inverse*: *f* ∈ *inj*(*A*,*B*) ⟹ *converse*(*f*) *O f* = *id*(*A*)
apply (*unfold inj-def*, *clarify*)
apply (*rule equalityI*)
 apply (*auto simp add*: *apply-iff*, *blast*)
done

right inverse of composition; one inclusion is $f \in A \to B \Longrightarrow f\ O\ converse(f) \subseteq id(B)$

lemma *right-comp-inverse*:
  f ∈ *surj*(*A*,*B*) ⟹ *f O converse*(*f*) = *id*(*B*)
apply (*simp add*: *surj-def*, *clarify*)
apply (*rule equalityI*)

apply (*best elim*: *domain-type range-type dest*: *apply-equality2*)
apply (*blast intro*: *apply-Pair*)
done

10.11.2 Proving that a Function is a Bijection

lemma *comp-eq-id-iff*:
$[\![f \in A{-}{>}B;\ \ g \in B{-}{>}A]\!] \Longrightarrow f\ O\ g = id(B) \longleftrightarrow (\forall\, y{\in}B.\ f`(g`y){=}y)$
apply (*unfold id-def*, *safe*)
apply (*drule-tac* $t = \lambda h.\ h`y$ **in** *subst-context*)
apply *simp*
apply (*rule fun-extension*)
apply (*blast intro*: *comp-fun lam-type*)
apply *auto*
done

lemma *fg-imp-bijective*:
$[\![f \in A{-}{>}B;\ \ g \in B{-}{>}A;\ \ f\ O\ g = id(B);\ \ g\ O\ f = id(A)]\!] \Longrightarrow f \in bij(A,B)$
unfolding *bij-def*
apply (*simp add*: *comp-eq-id-iff*)
apply (*blast intro*: *f-imp-injective f-imp-surjective apply-funtype*)
done

lemma *nilpotent-imp-bijective*: $[\![f \in A{-}{>}A;\ \ f\ O\ f = id(A)]\!] \Longrightarrow f \in bij(A,A)$
by (*blast intro*: *fg-imp-bijective*)

lemma *invertible-imp-bijective*:
$[\![converse(f){:}\ B{-}{>}A;\ \ f \in A{-}{>}B]\!] \Longrightarrow f \in bij(A,B)$
by (*simp add*: *fg-imp-bijective comp-eq-id-iff*
left-inverse-lemma right-inverse-lemma)

10.11.3 Unions of Functions

See similar theorems in func.thy

Theorem by KG, proof by LCP

lemma *inj-disjoint-Un*:
$[\![f \in inj(A,B);\ \ g \in inj(C,D);\ \ B \cap D = 0]\!]$
$\Longrightarrow (\lambda a{\in}A \cup C.\ if\ a \in A\ then\ f`a\ else\ g`a) \in inj(A \cup C,\ B \cup D)$
apply (*rule-tac* $d = \lambda z.\ if\ z \in B\ then\ converse\ (f)\ `z\ else\ converse\ (g)\ `z$
in *lam-injective*)
apply (*auto simp add*: *inj-is-fun* [*THEN apply-type*])
done

lemma *surj-disjoint-Un*:
$[\![f \in surj(A,B);\ \ g \in surj(C,D);\ \ A \cap C = 0]\!]$
$\Longrightarrow (f \cup g) \in surj(A \cup C,\ B \cup D)$
apply (*simp add*: *surj-def fun-disjoint-Un*)
apply (*blast dest*!: *domain-of-fun*

intro!: *fun-disjoint-apply1 fun-disjoint-apply2*)
done

A simple, high-level proof; the version for injections follows from it, using $f \in inj(A, B) \longleftrightarrow f \in bij(A, range(f))$

lemma *bij-disjoint-Un*:
  $[\![f \in bij(A,B); \; g \in bij(C,D); \; A \cap C = 0; \; B \cap D = 0]\!]$
  $\Longrightarrow (f \cup g) \in bij(A \cup C, B \cup D)$
apply (*rule invertible-imp-bijective*)
apply (*subst converse-Un*)
apply (*auto intro*: *fun-disjoint-Un bij-is-fun bij-converse-bij*)
done

10.11.4 Restrictions as Surjections and Bijections

lemma *surj-image*:
  $f \in Pi(A,B) \Longrightarrow f \in surj(A, f``A)$
apply (*simp add*: *surj-def*)
apply (*blast intro*: *apply-equality apply-Pair Pi-type*)
done

lemma *surj-image-eq*: $f \in surj(A, B) \Longrightarrow f``A = B$
 by (*auto simp add*: *surj-def image-fun*) (*blast dest*: *apply-type*)

lemma *restrict-image* [*simp*]: $restrict(f,A)$ `` $B = f$ `` $(A \cap B)$
by (*auto simp add*: *restrict-def*)

lemma *restrict-inj*:
  $[\![f \in inj(A,B); \; C<=A]\!] \Longrightarrow restrict(f,C): inj(C,B)$
 unfolding *inj-def*
apply (*safe elim*!: *restrict-type2*, *auto*)
done

lemma *restrict-surj*: $[\![f \in Pi(A,B); \; C<=A]\!] \Longrightarrow restrict(f,C): surj(C, f``C)$
apply (*insert restrict-type2* [*THEN surj-image*])
apply (*simp add*: *restrict-image*)
done

lemma *restrict-bij*:
  $[\![f \in inj(A,B); \; C<=A]\!] \Longrightarrow restrict(f,C): bij(C, f``C)$
apply (*simp add*: *inj-def bij-def*)
apply (*blast intro*: *restrict-surj surj-is-fun*)
done

10.11.5 Lemmas for Ramsey's Theorem

lemma *inj-weaken-type*: $[\![f \in inj(A,B); \; B<=D]\!] \Longrightarrow f \in inj(A,D)$
 unfolding *inj-def*
apply (*blast intro*: *fun-weaken-type*)

done

lemma *inj-succ-restrict*:
  $[\![f \in inj(succ(m),\ A)]\!] \Longrightarrow restrict(f,m) \in inj(m,\ A-\{f`m\})$
apply (*rule restrict-bij* [*THEN bij-is-inj*, *THEN inj-weaken-type*], *assumption*, *blast*)
 unfolding *inj-def*
apply (*fast elim*: *range-type mem-irrefl dest*: *apply-equality*)
done

lemma *inj-extend*:
  $[\![f \in inj(A,B);\ \ a{\notin}A;\ \ b{\notin}B]\!]$
  $\Longrightarrow cons(\langle a,b\rangle,f) \in inj(cons(a,A),\ cons(b,B))$
 unfolding *inj-def*
apply (*force intro*: *apply-type* *simp add*: *fun-extend*)
done

end

11 Relations: Their General Properties and Transitive Closure

theory *Trancl* **imports** *Fixedpt Perm* **begin**

definition
 refl :: $[i,i]\Rightarrow o$ **where**
  $refl(A,r) \equiv (\forall x{\in}A.\ \langle x,x\rangle \in r)$

definition
 irrefl :: $[i,i]\Rightarrow o$ **where**
  $irrefl(A,r) \equiv \forall x{\in}A.\ \langle x,x\rangle \notin r$

definition
 sym :: $i\Rightarrow o$ **where**
  $sym(r) \equiv \forall x\ y.\ \langle x,y\rangle{:}\ r \longrightarrow \langle y,x\rangle{:}\ r$

definition
 asym :: $i\Rightarrow o$ **where**
  $asym(r) \equiv \forall x\ y.\ \langle x,y\rangle{:}r \longrightarrow \neg\ \langle y,x\rangle{:}r$

definition
 antisym :: $i\Rightarrow o$ **where**
  $antisym(r) \equiv \forall x\ y.\langle x,y\rangle{:}r \longrightarrow \langle y,x\rangle{:}r \longrightarrow x=y$

definition
 trans :: $i\Rightarrow o$ **where**
  $trans(r) \equiv \forall x\ y\ z.\ \langle x,y\rangle{:}\ r \longrightarrow \langle y,z\rangle{:}\ r \longrightarrow \langle x,z\rangle{:}\ r$

definition
trans-on :: $[i,i]\Rightarrow o$ (‹(‹open-block notation=‹mixfix trans-on›› trans[-]′(-′))›) **where**
$trans[A](r) \equiv \forall x{\in}A.\ \forall y{\in}A.\ \forall z{\in}A.$
$\langle x,y\rangle : r \longrightarrow \langle y,z\rangle : r \longrightarrow \langle x,z\rangle : r$

definition
rtrancl :: $i\Rightarrow i$ (‹(‹notation=‹postfix ^*››-^*)› [100] 100) **where**
$r\hat{}* \equiv lfp(field(r)*field(r),\ \lambda s.\ id(field(r)) \cup (r\ O\ s))$

definition
trancl :: $i\Rightarrow i$ (‹(‹notation=‹postfix ^+››-^+)› [100] 100) **where**
$r\hat{}+ \equiv r\ O\ r\hat{}*$

definition
equiv :: $[i,i]\Rightarrow o$ **where**
$equiv(A,r) \equiv r \subseteq A*A \wedge refl(A,r) \wedge sym(r) \wedge trans(r)$

11.1 General properties of relations

11.1.1 irreflexivity

lemma *irreflI*:
$[\![\bigwedge x.\ x \in A \Longrightarrow \langle x,x\rangle \notin r]\!] \Longrightarrow irrefl(A,r)$
by (*simp add*: *irrefl-def*)

lemma *irreflE*: $[\![irrefl(A,r);\ \ x \in A]\!] \Longrightarrow \langle x,x\rangle \notin r$
by (*simp add*: *irrefl-def*)

11.1.2 symmetry

lemma *symI*:
$[\![\bigwedge x\ y.\langle x,y\rangle : r \Longrightarrow \langle y,x\rangle : r]\!] \Longrightarrow sym(r)$
by (*unfold sym-def*, *blast*)

lemma *symE*: $[\![sym(r);\ \langle x,y\rangle : r]\!] \Longrightarrow \langle y,x\rangle : r$
by (*unfold sym-def*, *blast*)

11.1.3 antisymmetry

lemma *antisymI*:
$[\![\bigwedge x\ y.[\![\langle x,y\rangle : r;\ \ \langle y,x\rangle : r]\!] \Longrightarrow x{=}y]\!] \Longrightarrow antisym(r)$
by (*simp add*: *antisym-def*, *blast*)

lemma *antisymE*: $[\![antisym(r);\ \langle x,y\rangle : r;\ \ \langle y,x\rangle : r]\!] \Longrightarrow x{=}y$
by (*simp add*: *antisym-def*, *blast*)

11.1.4 transitivity

lemma *transD*: $[\![trans(r);\ \ \langle a,b\rangle{:}r;\ \ \langle b,c\rangle{:}r]\!] \Longrightarrow \langle a,c\rangle{:}r$
by (*unfold trans-def*, *blast*)

lemma *trans-onD*:
$[\![trans[A](r);\ \langle a,b\rangle{:}r;\ \langle b,c\rangle{:}r;\ a \in A;\ b \in A;\ c \in A]\!] \Longrightarrow \langle a,c\rangle{:}r$
by (*unfold trans-on-def*, *blast*)

lemma *trans-imp-trans-on*: $trans(r) \Longrightarrow trans[A](r)$
by (*unfold trans-def trans-on-def*, *blast*)

lemma *trans-on-imp-trans*: $[\![trans[A](r);\ r \subseteq A{*}A]\!] \Longrightarrow trans(r)$
by (*simp add*: *trans-on-def trans-def*, *blast*)

11.2 Transitive closure of a relation

lemma *rtrancl-bnd-mono*:
$bnd\text{-}mono(field(r){*}field(r),\ \lambda s.\ id(field(r)) \cup (r\ O\ s))$
by (*rule bnd-monoI*, *blast*+)

lemma *rtrancl-mono*: $r<=s \Longrightarrow r\hat{}* \subseteq s\hat{}*$
unfolding *rtrancl-def*
apply (*rule lfp-mono*)
apply (*rule rtrancl-bnd-mono*)+
apply *blast*
done

lemmas *rtrancl-unfold* =
rtrancl-bnd-mono [*THEN rtrancl-def* [*THEN def-lfp-unfold*]]

lemmas *rtrancl-type* = *rtrancl-def* [*THEN def-lfp-subset*]

lemma *relation-rtrancl*: $relation(r\hat{}*)$
apply (*simp add*: *relation-def*)
apply (*blast dest*: *rtrancl-type* [*THEN subsetD*])
done

lemma *rtrancl-refl*: $[\![a \in field(r)]\!] \Longrightarrow \langle a,a\rangle \in r\hat{}*$
apply (*rule rtrancl-unfold* [*THEN ssubst*])
apply (*erule idI* [*THEN UnI1*])
done

lemma *rtrancl-into-rtrancl*: $[\![\langle a,b\rangle \in r\hat{}*;\ \langle b,c\rangle \in r]\!] \Longrightarrow \langle a,c\rangle \in r\hat{}*$
apply (*rule rtrancl-unfold* [*THEN ssubst*])
apply (*rule compI* [*THEN UnI2*], *assumption*, *assumption*)
done

lemma *r-into-rtrancl*: $\langle a,b\rangle \in r \Longrightarrow \langle a,b\rangle \in r\hat{}*$
by (*rule rtrancl-refl* [*THEN rtrancl-into-rtrancl*], *blast*+)

lemma *r-subset-rtrancl*: *relation*(r) $\Longrightarrow r \subseteq r\hat{}*$
by (*simp add*: *relation-def*, *blast intro*: *r-into-rtrancl*)

lemma *rtrancl-field*: *field*($r\hat{}*$) = *field*(r)
by (*blast intro*: *r-into-rtrancl dest*!: *rtrancl-type* [*THEN subsetD*])

lemma *rtrancl-full-induct* [*case-names initial step*, *consumes 1*]:
$[\![\langle a,b\rangle \in r\hat{}*;$
$\bigwedge x.\ x \in field(r) \Longrightarrow P(\langle x,x\rangle);$
$\bigwedge x\ y\ z.[\![P(\langle x,y\rangle);\ \langle x,y\rangle: r\hat{}*;\ \langle y,z\rangle: r]\!] \Longrightarrow P(\langle x,z\rangle)]\!]$
$\Longrightarrow P(\langle a,b\rangle)$
by (*erule def-induct* [*OF rtrancl-def rtrancl-bnd-mono*], *blast*)

lemma *rtrancl-induct* [*case-names initial step*, *induct set*: *rtrancl*]:
$[\![\langle a,b\rangle \in r\hat{}*;$
$P(a);$
$\bigwedge y\ z.[\![\langle a,y\rangle \in r\hat{}*;\ \ \langle y,z\rangle \in r;\ \ P(y)]\!] \Longrightarrow P(z)$
$]\!] \Longrightarrow P(b)$

apply (*subgoal-tac* $\forall\, y.\ \langle a,b\rangle = \langle a,y\rangle \longrightarrow P\ (y)$)

apply (*erule spec* [*THEN mp*], *rule refl*)

apply (*erule rtrancl-full-induct*, *blast*+)
done

lemma *trans-rtrancl*: *trans*($r\hat{}*$)
unfolding *trans-def*
apply (*intro allI impI*)
apply (*erule-tac* $b = z$ **in** *rtrancl-induct*, *assumption*)
apply (*blast intro*: *rtrancl-into-rtrancl*)
done

lemmas *rtrancl-trans* = *trans-rtrancl* [*THEN transD*]

lemma *rtranclE*:
$[\![\langle a,b\rangle \in r\hat{}*;\ \ (a{=}b) \Longrightarrow P;$

$\bigwedge y. [\![\langle a,y\rangle \in r\hat{\ }*; \quad \langle y,b\rangle \in r]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
apply (*subgoal-tac* $a = b \mid (\exists\, y.\ \langle a,y\rangle \in r\hat{\ }* \wedge \langle y,b\rangle \in r)$)

apply *blast*
apply (*erule rtrancl-induct*, *blast*+)
done

lemma *trans-trancl*: *trans*($r\hat{\ }+$)
unfolding *trans-def trancl-def*
apply (*blast intro*: *rtrancl-into-rtrancl*
trans-rtrancl [*THEN transD*, *THEN compI*])
done

lemmas *trans-on-trancl* = *trans-trancl* [*THEN trans-imp-trans-on*]

lemmas *trancl-trans* = *trans-trancl* [*THEN transD*]

lemma *trancl-into-rtrancl*: $\langle a,b\rangle \in r\hat{\ }+ \Longrightarrow \langle a,b\rangle \in r\hat{\ }*$
unfolding *trancl-def*
apply (*blast intro*: *rtrancl-into-rtrancl*)
done

lemma *r-into-trancl*: $\langle a,b\rangle \in r \Longrightarrow \langle a,b\rangle \in r\hat{\ }+$
unfolding *trancl-def*
apply (*blast intro*!: *rtrancl-refl*)
done

lemma *r-subset-trancl*: *relation*(r) $\Longrightarrow r \subseteq r\hat{\ }+$
by (*simp add*: *relation-def*, *blast intro*: *r-into-trancl*)

lemma *rtrancl-into-trancl1*: $[\![\langle a,b\rangle \in r\hat{\ }*;\ \langle b,c\rangle \in r]\!] \Longrightarrow \langle a,c\rangle \in r\hat{\ }+$
by (*unfold trancl-def*, *blast*)

lemma *rtrancl-into-trancl2*:
$[\![\langle a,b\rangle \in r;\ \langle b,c\rangle \in r\hat{\ }*]\!] \Longrightarrow \langle a,c\rangle \in r\hat{\ }+$
apply (*erule rtrancl-induct*)
apply (*erule r-into-trancl*)

apply (*blast intro*: *r-into-trancl trancl-trans*)
done

lemma *trancl-induct* [*case-names initial step*, *induct set*: *trancl*]:
$[\![\langle a,b\rangle \in r\hat{~}+;$
$\bigwedge y.\ [\![\langle a,y\rangle \in r]\!] \Longrightarrow P(y);$
$\bigwedge y\ z.[\![\langle a,y\rangle \in r\hat{~}+;\ \langle y,z\rangle \in r;\ P(y)]\!] \Longrightarrow P(z)$
$]\!] \Longrightarrow P(b)$
apply (*rule compEpair*)
apply (*unfold trancl-def*, *assumption*)

apply (*subgoal-tac* $\forall z.\ \langle y,z\rangle \in r \longrightarrow P\ (z)$)

apply *blast*
apply (*erule rtrancl-induct*)
apply (*blast intro*: *rtrancl-into-trancl1*)+
done

lemma *tranclE*:
$[\![\langle a,b\rangle \in r\hat{~}+;$
$\langle a,b\rangle \in r \Longrightarrow P;$
$\bigwedge y.[\![\langle a,y\rangle \in r\hat{~}+;\ \langle y,b\rangle \in r]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
apply (*subgoal-tac* $\langle a,b\rangle \in r\ |\ (\exists y.\ \langle a,y\rangle \in r\hat{~}+ \wedge \langle y,b\rangle \in r)$)
apply *blast*
apply (*rule compEpair*)
apply (*unfold trancl-def*, *assumption*)
apply (*erule rtranclE*)
apply (*blast intro*: *rtrancl-into-trancl1*)+
done

lemma *trancl-type*: $r\hat{~}+ \subseteq field(r)*field(r)$
unfolding *trancl-def*
apply (*blast elim*: *rtrancl-type* [*THEN subsetD*, *THEN SigmaE2*])
done

lemma *relation-trancl*: $relation(r\hat{~}+)$
apply (*simp add*: *relation-def*)
apply (*blast dest*: *trancl-type* [*THEN subsetD*])
done

lemma *trancl-subset-times*: $r \subseteq A * A \Longrightarrow r\hat{~}+ \subseteq A * A$
by (*insert trancl-type* [*of r*], *blast*)

lemma *trancl-mono*: $r<=s \Longrightarrow r\hat{~}+ \subseteq s\hat{~}+$
by (*unfold trancl-def*, *intro comp-mono rtrancl-mono*)

lemma *trancl-eq-r*: $[\![relation(r); trans(r)]\!] \Longrightarrow r\hat{\ }+ = r$
apply (*rule equalityI*)
prefer *2* **apply** (*erule r-subset-trancl*, *clarify*)
apply (*frule trancl-type* [*THEN subsetD*], *clarify*)
apply (*erule trancl-induct*, *assumption*)
apply (*blast dest*: *transD*)
done

lemma *rtrancl-idemp* [*simp*]: $(r\hat{\ }*)\hat{\ }* = r\hat{\ }*$
apply (*rule equalityI*, *auto*)
prefer *2*
apply (*frule rtrancl-type* [*THEN subsetD*])
apply (*blast intro*: *r-into-rtrancl*)

converse direction

apply (*frule rtrancl-type* [*THEN subsetD*], *clarify*)
apply (*erule rtrancl-induct*)
apply (*simp add*: *rtrancl-refl rtrancl-field*)
apply (*blast intro*: *rtrancl-trans*)
done

lemma *rtrancl-subset*: $[\![R \subseteq S; S \subseteq R\hat{\ }*]\!] \Longrightarrow S\hat{\ }* = R\hat{\ }*$
apply (*drule rtrancl-mono*)
apply (*drule rtrancl-mono*, *simp-all*, *blast*)
done

lemma *rtrancl-Un-rtrancl*:
$[\![relation(r); relation(s)]\!] \Longrightarrow (r\hat{\ }* \cup s\hat{\ }*)\hat{\ }* = (r \cup s)\hat{\ }*$
apply (*rule rtrancl-subset*)
apply (*blast dest*: *r-subset-rtrancl*)
apply (*blast intro*: *rtrancl-mono* [*THEN subsetD*])
done

lemma *rtrancl-converseD*: $\langle x,y\rangle:converse(r)\hat{\ }* \Longrightarrow \langle x,y\rangle:converse(r\hat{\ }*)$
apply (*rule converseI*)
apply (*frule rtrancl-type* [*THEN subsetD*])
apply (*erule rtrancl-induct*)
apply (*blast intro*: *rtrancl-refl*)
apply (*blast intro*: *r-into-rtrancl rtrancl-trans*)
done

lemma *rtrancl-converseI*: $\langle x,y\rangle:converse(r\hat{\ }*) \Longrightarrow \langle x,y\rangle:converse(r)\hat{\ }*$

apply (*drule converseD*)
apply (*frule rtrancl-type* [*THEN subsetD*])
apply (*erule rtrancl-induct*)
apply (*blast intro*: *rtrancl-refl*)
apply (*blast intro*: *r-into-rtrancl rtrancl-trans*)
done

lemma *rtrancl-converse*: $converse(r)\hat{}* = converse(r\hat{}*)$
apply (*safe intro*!: *equalityI*)
apply (*frule rtrancl-type* [*THEN subsetD*])
apply (*safe dest*!: *rtrancl-converseD intro*!: *rtrancl-converseI*)
done

lemma *trancl-converseD*: $\langle a, b\rangle:converse(r)\hat{}+ \Longrightarrow \langle a, b\rangle:converse(r\hat{}+)$
apply (*erule trancl-induct*)
apply (*auto intro*: *r-into-trancl trancl-trans*)
done

lemma *trancl-converseI*: $\langle x,y\rangle:converse(r\hat{}+) \Longrightarrow \langle x,y\rangle:converse(r)\hat{}+$
apply (*drule converseD*)
apply (*erule trancl-induct*)
apply (*auto intro*: *r-into-trancl trancl-trans*)
done

lemma *trancl-converse*: $converse(r)\hat{}+ = converse(r\hat{}+)$
apply (*safe intro*!: *equalityI*)
apply (*frule trancl-type* [*THEN subsetD*])
apply (*safe dest*!: *trancl-converseD intro*!: *trancl-converseI*)
done

lemma *converse-trancl-induct* [*case-names initial step*, *consumes 1*]:
$[\![\langle a, b\rangle:r\hat{}+; \bigwedge y. \langle y, b\rangle :r \Longrightarrow P(y);$
$\bigwedge y\ z. [\![\langle y, z\rangle \in r; \langle z, b\rangle \in r\hat{}+; P(z)]\!] \Longrightarrow P(y)]\!]$
$\Longrightarrow P(a)$
apply (*drule converseI*)
apply (*simp* (*no-asm-use*) *add*: *trancl-converse* [*symmetric*])
apply (*erule trancl-induct*)
apply (*auto simp add*: *trancl-converse*)
done

end

12 Well-Founded Recursion

theory *WF* **imports** *Trancl* **begin**

definition

wf :: *i*⇒*o* **where**

$wf(r) \equiv \forall Z.\ Z{=}0 \mid (\exists x{\in}Z.\ \forall y.\ \langle y{,}x\rangle{:}r \longrightarrow \neg\ y \in Z)$

definition
wf-on :: [*i*,*i*]⇒*o* (‹(‹open-block notation=‹mixfix wf-on››wf[-]′(-′))›) **where**

$wf\text{-}on(A{,}r) \equiv wf(r \cap A{*}A)$

definition
is-recfun :: [*i*, *i*, [*i*,*i*]⇒*i*, *i*] ⇒*o* **where**
$is\text{-}recfun(r{,}a{,}H{,}f) \equiv (f = (\lambda x{\in}r{-}``\{a\}.\ H(x,\ restrict(f,\ r{-}``\{x\}))))$

definition
the-recfun :: [*i*, *i*, [*i*,*i*]⇒*i*] ⇒*i* **where**
$the\text{-}recfun(r{,}a{,}H) \equiv (THE\ f.\ is\text{-}recfun(r{,}a{,}H{,}f))$

definition
wftrec :: [*i*, *i*, [*i*,*i*]⇒*i*] ⇒*i* **where**
$wftrec(r{,}a{,}H) \equiv H(a,\ the\text{-}recfun(r{,}a{,}H))$

definition
wfrec :: [*i*, *i*, [*i*,*i*]⇒*i*] ⇒*i* **where**

$wfrec(r{,}a{,}H) \equiv wftrec(r\hat{}{+},\ a,\ \lambda x\ f.\ H(x,\ restrict(f{,}r{-}``\{x\})))$

definition
wfrec-on :: [*i*, *i*, *i*, [*i*,*i*]⇒*i*] ⇒*i* (‹(‹open-block notation=‹mixfix wfrec-on››wfrec[-]′(-,-,-′))›)
where $wfrec[A](r{,}a{,}H) \equiv wfrec(r \cap A{*}A,\ a,\ H)$

12.1 Well-Founded Relations

12.1.1 Equivalences between *wf* and *wf-on*

lemma *wf-imp-wf-on*: $wf(r) \Longrightarrow wf[A](r)$
by (*unfold wf-def wf-on-def*, *force*)

lemma *wf-on-imp-wf*: $[\![wf[A](r);\ r \subseteq A{*}A]\!] \Longrightarrow wf(r)$
by (*simp add*: *wf-on-def subset-Int-iff*)

lemma *wf-on-field-imp-wf*: $wf[field(r)](r) \Longrightarrow wf(r)$
by (*unfold wf-def wf-on-def*, *fast*)

lemma *wf-iff-wf-on-field*: $wf(r) \longleftrightarrow wf[field(r)](r)$
by (*blast intro*: *wf-imp-wf-on wf-on-field-imp-wf*)

lemma *wf-on-subset-A*: $[\![wf[A](r);\ B{<=}A]\!] \Longrightarrow wf[B](r)$
by (*unfold wf-on-def wf-def*, *fast*)

lemma *wf-on-subset-r*: $[\![wf[A](r);\ s{<=}r]\!] \Longrightarrow wf[A](s)$

by (*unfold wf-on-def wf-def*, *fast*)

lemma *wf-subset*: $[\![wf(s);\ r<=s]\!] \Longrightarrow wf(r)$
by (*simp add*: *wf-def*, *fast*)

12.1.2 Introduction Rules for *wf-on*

If every non-empty subset of A has an r-minimal element then we have $wf[A](r)$.

lemma *wf-onI*:
assumes *prem*: $\bigwedge Z\ u.\ [\![Z<=A;\ \ u \in Z;\ \ \forall x{\in}Z.\ \exists y{\in}Z.\ \langle y,x\rangle{:}r]\!] \Longrightarrow False$
shows $wf[A](r)$
unfolding *wf-on-def wf-def*
apply (*rule equals0I* [*THEN disjCI*, *THEN allI*])
apply (*rule-tac* $Z = Z$ **in** *prem*, *blast*+)
done

If r allows well-founded induction over A then we have $wf[A](r)$. Premise is equivalent to $\bigwedge B.\ \forall x{\in}A.\ (\forall y.\ \langle y,\ x\rangle \in r \longrightarrow y \in B) \longrightarrow x \in B \Longrightarrow A \subseteq B$

lemma *wf-onI2*:
assumes *prem*: $\bigwedge y\ B.\ [\![\forall x{\in}A.\ (\forall y{\in}A.\ \langle y,x\rangle{:}r \longrightarrow y \in B) \longrightarrow x \in B;\ \ \ y \in A]\!]$
$\Longrightarrow y \in B$
shows $wf[A](r)$
apply (*rule wf-onI*)
apply (*rule-tac* $c{=}u$ **in** *prem* [*THEN DiffE*])
prefer *3* **apply** *blast*
apply *fast*+
done

12.1.3 Well-founded Induction

Consider the least z in $domain(r)$ such that $P(z)$ does not hold...

lemma *wf-induct-raw*:
$[\![wf(r);$
$\bigwedge x.[\![\forall y.\ \langle y,x\rangle{:}\ r \longrightarrow P(y)]\!] \Longrightarrow P(x)]\!]$
$\Longrightarrow P(a)$
unfolding *wf-def*
apply (*erule-tac* $x = \{z \in domain(r).\ \neg\ P(z)\}$ **in** *allE*)
apply *blast*
done

lemmas *wf-induct* = *wf-induct-raw* [*rule-format*, *consumes 1*, *case-names step*, *induct set*: *wf*]

The form of this rule is designed to match *wfI*

lemma *wf-induct2*:

$[\![wf(r);\ \ a \in A;\ \ field(r) <= A;$
$\quad \bigwedge x. [\![x \in A;\ \ \forall y.\ \langle y,x\rangle : r \longrightarrow P(y)]\!] \Longrightarrow P(x)]\!]$
$\Longrightarrow\ P(a)$
apply (*erule-tac P=a ∈ A* **in** *rev-mp*)
apply (*erule-tac a=a* **in** *wf-induct*, *blast*)
done

lemma *field-Int-square*: $field(r \cap A*A) \subseteq A$
by *blast*

lemma *wf-on-induct-raw* [*consumes 2*, *induct set*: *wf-on*]:
$[\![wf[A](r);\ \ a \in A;$
$\quad \bigwedge x. [\![x \in A;\ \ \forall y{\in}A.\ \langle y,x\rangle : r \longrightarrow P(y)]\!] \Longrightarrow P(x)$
$]\!]\ \Longrightarrow\ P(a)$
unfolding *wf-on-def*
apply (*erule wf-induct2*, *assumption*)
apply (*rule field-Int-square*, *blast*)
done

lemma *wf-on-induct* [*consumes 2*, *case-names step*, *induct set*: *wf-on*]:
$wf[A](r) \Longrightarrow a \in A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow (\bigwedge y.\ y \in A \Longrightarrow \langle y, x\rangle \in r \Longrightarrow P(y))$
$\Longrightarrow P(x)) \Longrightarrow P(a)$
using *wf-on-induct-raw* [*of A r a P*] **by** *simp*

If r allows well-founded induction then we have $wf(r)$.

lemma *wfI*:
$[\![field(r) <= A;$
$\quad \bigwedge y\ B.\ [\![\forall x{\in}A.\ (\forall y{\in}A.\ \langle y,x\rangle : r \longrightarrow y \in B) \longrightarrow x \in B;\ \ y \in A]\!]$
$\qquad \Longrightarrow y \in B]\!]$
$\Longrightarrow\ wf(r)$
apply (*rule wf-on-subset-A* [*THEN wf-on-field-imp-wf*])
apply (*rule wf-onI2*)
prefer *2* **apply** *blast*
apply *blast*
done

12.2 Basic Properties of Well-Founded Relations

lemma *wf-not-refl*: $wf(r) \Longrightarrow \langle a,a\rangle \notin r$
by (*erule-tac a=a* **in** *wf-induct*, *blast*)

lemma *wf-not-sym* [*rule-format*]: $wf(r) \Longrightarrow \forall x.\ \langle a,x\rangle : r \longrightarrow \langle x,a\rangle \notin r$
by (*erule-tac a=a* **in** *wf-induct*, *blast*)

lemmas *wf-asym* = *wf-not-sym* [*THEN swap*]

lemma *wf-on-not-refl*: $[\![wf[A](r);\ a \in A]\!] \Longrightarrow \langle a,a\rangle \notin r$
by (*erule-tac a=a* **in** *wf-on-induct*, *assumption*, *blast*)

lemma *wf-on-not-sym*:
$[\![wf[A](r);\ \ a \in A]\!] \Longrightarrow (\bigwedge b.\ b{\in}A \Longrightarrow \langle a,b\rangle{:}r \Longrightarrow \langle b,a\rangle{\notin}r)$
apply (*atomize* (*full*), *intro impI*)
apply (*erule-tac a=a* **in** *wf-on-induct*, *assumption*, *blast*)
done

lemma *wf-on-asym*:
$[\![wf[A](r);\ \ \neg Z \Longrightarrow \langle a,b\rangle \in r;$
$\langle b,a\rangle \notin r \Longrightarrow Z;\ \neg Z \Longrightarrow a \in A;\ \neg Z \Longrightarrow b \in A]\!] \Longrightarrow Z$
by (*blast dest*: *wf-on-not-sym*)

lemma *wf-on-chain3*:
$[\![wf[A](r);\ \langle a,b\rangle{:}r;\ \langle b,c\rangle{:}r;\ \langle c,a\rangle{:}r;\ a \in A;\ b \in A;\ c \in A]\!] \Longrightarrow P$
apply (*subgoal-tac* $\forall\, y{\in}A.\ \forall\, z{\in}A.\ \langle a,y\rangle{:}r \longrightarrow \langle y,z\rangle{:}r \longrightarrow \langle z,a\rangle{:}r \longrightarrow P$,
blast)
apply (*erule-tac a=a* **in** *wf-on-induct*, *assumption*, *blast*)
done

transitive closure of a WF relation is WF provided A is downward closed

lemma *wf-on-trancl*:
$[\![wf[A](r);\ \ r-``A \subseteq A]\!] \Longrightarrow wf[A](r\hat{\ }+)$
apply (*rule wf-onI2*)
apply (*frule bspec* [*THEN mp*], *assumption+*)
apply (*erule-tac a = y* **in** *wf-on-induct*, *assumption*)
apply (*blast elim*: *tranclE*, *blast*)
done

lemma *wf-trancl*: $wf(r) \Longrightarrow wf(r\hat{\ }+)$
apply (*simp add*: *wf-iff-wf-on-field*)
apply (*rule wf-on-subset-A*)
apply (*erule wf-on-trancl*)
apply *blast*
apply (*rule trancl-type* [*THEN field-rel-subset*])
done

$r\ -``\ \{a\}$ is the set of everything under a in r

lemmas *underI* = *vimage-singleton-iff* [*THEN iffD2*]
lemmas *underD* = *vimage-singleton-iff* [*THEN iffD1*]

12.3 The Predicate *is-recfun*

lemma *is-recfun-type*: *is-recfun*$(r,a,H,f) \Longrightarrow f \in r-``\{a\}$ -> *range*(f)
unfolding *is-recfun-def*
apply (*erule ssubst*)
apply (*rule lamI* [*THEN rangeI*, *THEN lam-type*], *assumption*)
done

lemmas *is-recfun-imp-function* = *is-recfun-type* [*THEN fun-is-function*]

lemma *apply-recfun*:
  ⟦*is-recfun*(*r*,*a*,*H*,*f*); ⟨*x*,*a*⟩:*r*⟧ ⟹ *f*‘*x* = *H*(*x*, *restrict*(*f*,*r*−‘‘{*x*}))
 unfolding *is-recfun-def*

replace f only on the left-hand side

apply (*erule-tac P* = *λx*. *t*(*x*) = *u* **for** *t u* **in** *ssubst*)
apply (*simp add*: *underI*)
done

lemma *is-recfun-equal* [*rule-format*]:
   ⟦*wf*(*r*); *trans*(*r*); *is-recfun*(*r*,*a*,*H*,*f*); *is-recfun*(*r*,*b*,*H*,*g*)⟧
   ⟹ ⟨*x*,*a*⟩:*r* ⟶ ⟨*x*,*b*⟩:*r* ⟶ *f*‘*x*=*g*‘*x*
apply (*frule-tac f* = *f* **in** *is-recfun-type*)
apply (*frule-tac f* = *g* **in** *is-recfun-type*)
apply (*simp add*: *is-recfun-def*)
apply (*erule-tac a*=*x* **in** *wf-induct*)
apply (*intro impI*)
apply (*elim ssubst*)
apply (*simp* (*no-asm-simp*) *add*: *vimage-singleton-iff restrict-def*)
apply (*rule-tac t* = *λz*. *H* (*x*, *z*) **for** *x* **in** *subst-context*)
apply (*subgoal-tac* ∀ *y*∈*r*−‘‘{*x*}. ∀ *z*. ⟨*y*,*z*⟩:*f* ⟷ ⟨*y*,*z*⟩:*g*)
 apply (*blast dest*: *transD*)
apply (*simp add*: *apply-iff*)
apply (*blast dest*: *transD intro*: *sym*)
done

lemma *is-recfun-cut*:
   ⟦*wf*(*r*); *trans*(*r*);
     is-recfun(*r*,*a*,*H*,*f*); *is-recfun*(*r*,*b*,*H*,*g*); ⟨*b*,*a*⟩:*r*⟧
   ⟹ *restrict*(*f*, *r*−‘‘{*b*}) = *g*
apply (*frule-tac f* = *f* **in** *is-recfun-type*)
apply (*rule fun-extension*)
  apply (*blast dest*: *transD intro*: *restrict-type2*)
 apply (*erule is-recfun-type*, *simp*)
apply (*blast dest*: *transD intro*: *is-recfun-equal*)
done

12.4 Recursion: Main Existence Lemma

lemma *is-recfun-functional*:
   ⟦*wf*(*r*); *trans*(*r*); *is-recfun*(*r*,*a*,*H*,*f*); *is-recfun*(*r*,*a*,*H*,*g*)⟧ ⟹ *f*=*g*
by (*blast intro*: *fun-extension is-recfun-type is-recfun-equal*)

lemma *the-recfun-eq*:
  ⟦*is-recfun*(*r*,*a*,*H*,*f*); *wf*(*r*); *trans*(*r*)⟧ ⟹ *the-recfun*(*r*,*a*,*H*) = *f*
 unfolding *the-recfun-def*
apply (*blast intro*: *is-recfun-functional*)

done

lemma *is-the-recfun*:
  $[\![$*is-recfun*(r,a,H,f); *wf*(r); *trans*$(r)]\!]$
  $\Longrightarrow$ *is-recfun*$(r, a, H,$ *the-recfun*$(r,a,H))$
by (*simp add*: *the-recfun-eq*)

lemma *unfold-the-recfun*:
  $[\![$*wf*(r); *trans*$(r)]\!] \Longrightarrow$ *is-recfun*$(r, a, H,$ *the-recfun*$(r,a,H))$
apply (*rule-tac a=a* **in** *wf-induct*, *assumption*)
apply (*rename-tac a1*)
apply (*rule-tac f* = $\lambda y \in r-``\{a1\}$. *wftrec* (r,y,H) **in** *is-the-recfun*)
 apply *typecheck*
 unfolding *is-recfun-def wftrec-def*
 — Applying the substitution: must keep the quantified assumption!
apply (*rule lam-cong* [*OF refl*])
apply (*drule underD*)
apply (*fold is-recfun-def*)
apply (*rule-tac t* = $\lambda z. H(x, z)$ **for** x **in** *subst-context*)
apply (*rule fun-extension*)
 apply (*blast intro*: *is-recfun-type*)
 apply (*rule lam-type* [*THEN restrict-type2*])
 apply *blast*
 apply (*blast dest*: *transD*)
apply *atomize*
apply (*frule spec* [*THEN mp*], *assumption*)
apply (*subgoal-tac* $\langle xa,a1\rangle \in r$)
 apply (*drule-tac x1* = *xa* **in** *spec* [*THEN mp*], *assumption*)
apply (*simp add*: *vimage-singleton-iff*
    apply-recfun is-recfun-cut)
apply (*blast dest*: *transD*)
done

12.5 Unfolding *wftrec*(r, a, H)

lemma *the-recfun-cut*:
  $[\![$*wf*(r); *trans*(r); $\langle b,a\rangle:r]\!]$
  $\Longrightarrow$ *restrict*(*the-recfun*(r,a,H), $r-``\{b\})$ = *the-recfun*(r,b,H)
by (*blast intro*: *is-recfun-cut unfold-the-recfun*)

lemma *wftrec*:
  $[\![$*wf*(r); *trans*$(r)]\!] \Longrightarrow$
    wftrec$(r,a,H) = H(a, \lambda x \in r-``\{a\}$. *wftrec*$(r,x,H))$
 unfolding *wftrec-def*
apply (*subst unfold-the-recfun* [*unfolded is-recfun-def*])
apply (*simp-all add*: *vimage-singleton-iff* [*THEN iff-sym*] *the-recfun-cut*)
done

12.5.1 Removal of the Premise *trans*(r)

lemma *wfrec*:
  $wf(r) \Longrightarrow wfrec(r,a,H) = H(a, \lambda x{\in}r-``\{a\}.\ wfrec(r,x,H))$
 unfolding *wfrec-def*
apply (*erule wf-trancl* [*THEN wftrec*, *THEN ssubst*])
 apply (*rule trans-trancl*)
apply (*rule vimage-pair-mono* [*THEN restrict-lam-eq*, *THEN subst-context*])
 apply (*erule r-into-trancl*)
apply (*rule subset-refl*)
done

lemma *def-wfrec*:
  $[\![\bigwedge x.\ h(x){\equiv}wfrec(r,x,H);\ \ wf(r)]\!] \Longrightarrow$
  $h(a) = H(a, \lambda x{\in}r-``\{a\}.\ h(x))$
apply *simp*
apply (*elim wfrec*)
done

lemma *wfrec-type*:
  $[\![wf(r);\ \ a \in A;\ \ field(r){<=}A;$
   $\bigwedge x\ u.\ [\![x \in A;\ \ u \in Pi(r-``\{x\}, B)]\!] \Longrightarrow H(x,u) \in B(x)$
$]\!] \Longrightarrow wfrec(r,a,H) \in B(a)$
apply (*rule-tac a = a* **in** *wf-induct2*, *assumption*+)
apply (*subst wfrec*, *assumption*)
apply (*simp add*: *lam-type underD*)
done

lemma *wfrec-on*:
 $[\![wf[A](r);\ \ a \in A]\!] \Longrightarrow$
   $wfrec[A](r,a,H) = H(a, \lambda x{\in}(r-``\{a\}) \cap A.\ wfrec[A](r,x,H))$
 unfolding *wf-on-def wfrec-on-def*
apply (*erule wfrec* [*THEN trans*])
apply (*simp add*: *vimage-Int-square*)
done

Minimal-element characterization of well-foundedness

lemma *wf-eq-minimal*: $wf(r) \longleftrightarrow (\forall\, Q\ x.\ x \in Q \longrightarrow (\exists\, z{\in}Q.\ \forall\, y.\ \langle y,z\rangle{:}r \longrightarrow y{\notin}Q))$
 unfolding *wf-def* **by** *blast*

end

13 Transitive Sets and Ordinals

theory *Ordinal* **imports** *WF Bool equalities* **begin**

definition
Memrel :: $i \Rightarrow i$ **where**
Memrel(A) $\equiv \{z \in A * A .\ \exists x\ y.\ z = \langle x,y \rangle \wedge x \in y\ \}$

definition
Transset :: $i \Rightarrow o$ **where**
Transset$(i) \equiv \forall x \in i.\ x <= i$

definition
Ord :: $i \Rightarrow o$ **where**
Ord(i) $\equiv$ *Transset*$(i) \wedge (\forall x \in i.$ *Transset*$(x))$

definition
lt :: $[i,i] \Rightarrow o$ (**infixl** ‹<› *50*) **where**
$i<j$ $\equiv i \in j \wedge$ *Ord*(j)

definition
Limit :: $i \Rightarrow o$ **where**
Limit(i) $\equiv$ *Ord*$(i) \wedge 0<i \wedge (\forall y.\ y<i \longrightarrow succ(y)<i)$

abbreviation
le (**infixl** ‹≤› *50*) **where**
$x \le y \equiv x < succ(y)$

13.1 Rules for Transset

13.1.1 Three Neat Characterisations of Transset

lemma *Transset-iff-Pow*: *Transset*(A) $<->$ $A<=Pow(A)$
by (*unfold Transset-def*, *blast*)

lemma *Transset-iff-Union-succ*: *Transset*(A) $<->$ $\bigcup(succ(A)) = A$
unfolding *Transset-def*
apply (*blast elim*!: *equalityE*)
done

lemma *Transset-iff-Union-subset*: *Transset*(A) $<->$ $\bigcup(A) \subseteq A$
by (*unfold Transset-def*, *blast*)

13.1.2 Consequences of Downwards Closure

lemma *Transset-doubleton-D*:
$[\![$*Transset*(C); $\{a,b\}$: $C]\!] \Longrightarrow a \in C \wedge b \in C$
by (*unfold Transset-def*, *blast*)

lemma *Transset-Pair-D*:
$[\![$*Transset*(C); $\langle a,b \rangle \in C]\!] \Longrightarrow a \in C \wedge b \in C$
apply (*simp add*: *Pair-def*)
apply (*blast dest*: *Transset-doubleton-D*)
done

lemma *Transset-includes-domain*:
  $[\![Transset(C); A*B \subseteq C; b \in B]\!] \Longrightarrow A \subseteq C$
by (*blast dest*: *Transset-Pair-D*)

lemma *Transset-includes-range*:
  $[\![Transset(C); A*B \subseteq C; a \in A]\!] \Longrightarrow B \subseteq C$
by (*blast dest*: *Transset-Pair-D*)

13.1.3 Closure Properties

lemma *Transset-0*: $Transset(0)$
by (*unfold Transset-def*, *blast*)

lemma *Transset-Un*:
  $[\![Transset(i); \; Transset(j)]\!] \Longrightarrow Transset(i \cup j)$
by (*unfold Transset-def*, *blast*)

lemma *Transset-Int*:
  $[\![Transset(i); \; Transset(j)]\!] \Longrightarrow Transset(i \cap j)$
by (*unfold Transset-def*, *blast*)

lemma *Transset-succ*: $Transset(i) \Longrightarrow Transset(succ(i))$
by (*unfold Transset-def*, *blast*)

lemma *Transset-Pow*: $Transset(i) \Longrightarrow Transset(Pow(i))$
by (*unfold Transset-def*, *blast*)

lemma *Transset-Union*: $Transset(A) \Longrightarrow Transset(\bigcup(A))$
by (*unfold Transset-def*, *blast*)

lemma *Transset-Union-family*:
  $[\![\bigwedge i. \; i \in A \Longrightarrow Transset(i)]\!] \Longrightarrow Transset(\bigcup(A))$
by (*unfold Transset-def*, *blast*)

lemma *Transset-Inter-family*:
  $[\![\bigwedge i. \; i \in A \Longrightarrow Transset(i)]\!] \Longrightarrow Transset(\bigcap(A))$
by (*unfold Inter-def Transset-def*, *blast*)

lemma *Transset-UN*:
  $(\bigwedge x. \; x \in A \Longrightarrow Transset(B(x))) \Longrightarrow Transset \; (\bigcup x \in A. \; B(x))$
by (*rule Transset-Union-family*, *auto*)

lemma *Transset-INT*:
  $(\bigwedge x. \; x \in A \Longrightarrow Transset(B(x))) \Longrightarrow Transset \; (\bigcap x \in A. \; B(x))$
by (*rule Transset-Inter-family*, *auto*)

13.2 Lemmas for Ordinals

lemma *OrdI*:

⟦*Transset*(*i*); ⋀*x*. *x*∈*i* ⟹ *Transset*(*x*)⟧ ⟹ *Ord*(*i*)
by (*simp add*: *Ord-def*)

lemma *Ord-is-Transset*: *Ord*(*i*) ⟹ *Transset*(*i*)
by (*simp add*: *Ord-def*)

lemma *Ord-contains-Transset*:
⟦*Ord*(*i*); *j*∈*i*⟧ ⟹ *Transset*(*j*)
by (*unfold Ord-def*, *blast*)

lemma *Ord-in-Ord*: ⟦*Ord*(*i*); *j*∈*i*⟧ ⟹ *Ord*(*j*)
by (*unfold Ord-def Transset-def*, *blast*)

lemma *Ord-in-Ord′*: ⟦*j*∈*i*; *Ord*(*i*)⟧ ⟹ *Ord*(*j*)
by (*blast intro*: *Ord-in-Ord*)

lemmas *Ord-succD* = *Ord-in-Ord* [*OF - succI1*]

lemma *Ord-subset-Ord*: ⟦*Ord*(*i*); *Transset*(*j*); *j*<=*i*⟧ ⟹ *Ord*(*j*)
by (*simp add*: *Ord-def Transset-def*, *blast*)

lemma *OrdmemD*: ⟦*j*∈*i*; *Ord*(*i*)⟧ ⟹ *j*<=*i*
by (*unfold Ord-def Transset-def*, *blast*)

lemma *Ord-trans*: ⟦*i*∈*j*; *j*∈*k*; *Ord*(*k*)⟧ ⟹ *i*∈*k*
by (*blast dest*: *OrdmemD*)

lemma *Ord-succ-subsetI*: ⟦*i*∈*j*; *Ord*(*j*)⟧ ⟹ *succ*(*i*) ⊆ *j*
by (*blast dest*: *OrdmemD*)

13.3 The Construction of Ordinals: 0, succ, Union

lemma *Ord-0* [*iff*,*TC*]: *Ord*(*0*)
by (*blast intro*: *OrdI Transset-0*)

lemma *Ord-succ* [*TC*]: *Ord*(*i*) ⟹ *Ord*(*succ*(*i*))
by (*blast intro*: *OrdI Transset-succ Ord-is-Transset Ord-contains-Transset*)

lemmas *Ord-1* = *Ord-0* [*THEN Ord-succ*]

lemma *Ord-succ-iff* [*iff*]: *Ord*(*succ*(*i*)) <−> *Ord*(*i*)
by (*blast intro*: *Ord-succ dest*!: *Ord-succD*)

lemma *Ord-Un* [*intro*,*simp*,*TC*]: ⟦*Ord*(*i*); *Ord*(*j*)⟧ ⟹ *Ord*(*i* ∪ *j*)
unfolding *Ord-def*
apply (*blast intro*!: *Transset-Un*)

done

lemma *Ord-Int* [*TC*]: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow Ord(i \cap j)$
  unfolding *Ord-def*
apply (*blast intro*!: *Transset-Int*)
done

There is no set of all ordinals, for then it would contain itself

lemma *ON-class*: $\neg\ (\forall\, i.\ i{\in}X <-> Ord(i))$
proof (*rule notI*)
  assume X: $\forall\, i.\ i \in X \longleftrightarrow Ord(i)$
  have $\forall\, x\ y.\ x{\in}X \longrightarrow y{\in}x \longrightarrow y{\in}X$
    by (*simp add*: *X*, *blast intro*: *Ord-in-Ord*)
  hence $Transset(X)$
     by (*auto simp add*: *Transset-def*)
  moreover have $\bigwedge x.\ x \in X \Longrightarrow Transset(x)$
     by (*simp add*: *X Ord-def*)
  ultimately have $Ord(X)$ **by** (*rule OrdI*)
  hence $X \in X$ **by** (*simp add*: *X*)
  thus *False* **by** (*rule mem-irrefl*)
qed

13.4 < is 'less Than' for Ordinals

lemma *ltI*: $[\![i{\in}j;\ \ Ord(j)]\!] \Longrightarrow i{<}j$
by (*unfold lt-def*, *blast*)

lemma *ltE*:
    $[\![i{<}j;\ \ [\![i{\in}j;\ \ Ord(i);\ \ Ord(j)]\!] \Longrightarrow P]\!] \Longrightarrow P$
  unfolding *lt-def*
apply (*blast intro*: *Ord-in-Ord*)
done

lemma *ltD*: $i{<}j \Longrightarrow i{\in}j$
by (*erule ltE*, *assumption*)

lemma *not-lt0* [*simp*]: $\neg\ i{<}0$
by (*unfold lt-def*, *blast*)

lemma *lt-Ord*: $j{<}i \Longrightarrow Ord(j)$
by (*erule ltE*, *assumption*)

lemma *lt-Ord2*: $j{<}i \Longrightarrow Ord(i)$
by (*erule ltE*, *assumption*)

lemmas *le-Ord2* = *lt-Ord2* [*THEN Ord-succD*]

lemmas *lt0E* = *not-lt0* [*THEN notE*, *elim*!]

lemma *lt-trans* [*trans*]: $[\![i<j;\ \ j<k]\!] \Longrightarrow i<k$
by (*blast intro*!: *ltI elim*!: *ltE intro*: *Ord-trans*)

lemma *lt-not-sym*: $i<j \Longrightarrow \neg\ (j<i)$
 unfolding *lt-def*
apply (*blast elim*: *mem-asym*)
done

lemmas *lt-asym* = *lt-not-sym* [*THEN swap*]

lemma *lt-irrefl* [*elim*!]: $i<i \Longrightarrow P$
by (*blast intro*: *lt-asym*)

lemma *lt-not-refl*: $\neg\ i<i$
apply (*rule notI*)
apply (*erule lt-irrefl*)
done

Recall that $i \le j$ abbreviates $i \le j$!

lemma *le-iff*: $i \le j <\!-\!> i<j \mid (i=j \wedge Ord(j))$
by (*unfold lt-def*, *blast*)

lemma *leI*: $i<j \Longrightarrow i \le j$
by (*simp add*: *le-iff*)

lemma *le-eqI*: $[\![i=j;\ \ Ord(j)]\!] \Longrightarrow i \le j$
by (*simp add*: *le-iff*)

lemmas *le-refl* = *refl* [*THEN le-eqI*]

lemma *le-refl-iff* [*iff*]: $i \le i <\!-\!> Ord(i)$
by (*simp* (*no-asm-simp*) *add*: *lt-not-refl le-iff*)

lemma *leCI*: $(\neg\ (i=j \wedge Ord(j)) \Longrightarrow i<j) \Longrightarrow i \le j$
by (*simp add*: *le-iff*, *blast*)

lemma *leE*:
  $[\![i \le j;\ \ i<j \Longrightarrow P;\ \ [\![i=j;\ \ Ord(j)]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*simp add*: *le-iff*, *blast*)

lemma *le-anti-sym*: $[\![i \le j;\ \ j \le i]\!] \Longrightarrow i=j$
apply (*simp add*: *le-iff*)
apply (*blast elim*: *lt-asym*)
done

lemma *le0-iff* [*simp*]: $i \leq 0$ <−> $i{=}0$
by (*blast elim*!: *leE*)

lemmas *le0D* = *le0-iff* [*THEN iffD1*, *dest*!]

13.5 Natural Deduction Rules for Memrel

lemma *Memrel-iff* [*simp*]: $\langle a,b\rangle \in Memrel(A)$ <−> $a{\in}b \wedge a{\in}A \wedge b{\in}A$
by (*unfold Memrel-def*, *blast*)

lemma *MemrelI* [*intro*!]: $[\![a \in b;\ \ a \in A;\ \ b \in A]\!] \Longrightarrow \langle a,b\rangle \in Memrel(A)$
by *auto*

lemma *MemrelE* [*elim*!]:
$[\![\langle a,b\rangle \in Memrel(A);$
$[\![a \in A;\ \ b \in A;\ \ a{\in}b]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
by *auto*

lemma *Memrel-type*: $Memrel(A) \subseteq A{*}A$
by (*unfold Memrel-def*, *blast*)

lemma *Memrel-mono*: A<=$B \Longrightarrow Memrel(A) \subseteq Memrel(B)$
by (*unfold Memrel-def*, *blast*)

lemma *Memrel-0* [*simp*]: $Memrel(0) = 0$
by (*unfold Memrel-def*, *blast*)

lemma *Memrel-1* [*simp*]: $Memrel(1) = 0$
by (*unfold Memrel-def*, *blast*)

lemma *relation-Memrel*: *relation*(*Memrel*(*A*))
by (*simp add*: *relation-def Memrel-def*)

lemma *wf-Memrel*: *wf*(*Memrel*(*A*))
unfolding *wf-def*
apply (*rule foundation* [*THEN disjE*, *THEN allI*], *erule disjI1*, *blast*)
done

The premise *Ord*(*i*) does not suffice.

lemma *trans-Memrel*:
$Ord(i) \Longrightarrow trans(Memrel(i))$
by (*unfold Ord-def Transset-def trans-def*, *blast*)

However, the following premise is strong enough.

lemma *Transset-trans-Memrel*:
$\forall j{\in}i.\ Transset(j) \Longrightarrow trans(Memrel(i))$
by (*unfold Transset-def trans-def*, *blast*)

lemma *Transset-Memrel-iff*:
  Transset(*A*) $\Longrightarrow$ ⟨*a*,*b*⟩ ∈ *Memrel*(*A*) <−> *a*∈*b* ∧ *b*∈*A*
by (*unfold Transset-def*, *blast*)

13.6 Transfinite Induction

lemma *Transset-induct*:
  ⟦*i* ∈ *k*; *Transset*(*k*);
    ⋀*x*.⟦*x* ∈ *k*; ∀ *y*∈*x*. *P*(*y*)⟧ $\Longrightarrow$ *P*(*x*)⟧
  $\Longrightarrow$ *P*(*i*)
apply (*simp add*: *Transset-def*)
apply (*erule wf-Memrel* [*THEN wf-induct2*], *blast*+)
done

lemma *Ord-induct* [*consumes 2*]:
 i ∈ *k* $\Longrightarrow$ *Ord*(*k*) $\Longrightarrow$ (⋀*x*. *x* ∈ *k* $\Longrightarrow$ (⋀*y*. *y* ∈ *x* $\Longrightarrow$ *P*(*y*)) $\Longrightarrow$ *P*(*x*)) $\Longrightarrow$ *P*(*i*)
 using *Transset-induct* [*OF - Ord-is-Transset*, *of i k P*] **by** *simp*

lemma *trans-induct* [*consumes 1*, *case-names step*]:
 Ord(*i*) $\Longrightarrow$ (⋀*x*. *Ord*(*x*) $\Longrightarrow$ (⋀*y*. *y* ∈ *x* $\Longrightarrow$ *P*(*y*)) $\Longrightarrow$ *P*(*x*)) $\Longrightarrow$ *P*(*i*)
 apply (*rule Ord-succ* [*THEN succI1* [*THEN Ord-induct*]], *assumption*)
 apply (*blast intro*: *Ord-succ* [*THEN Ord-in-Ord*])
 done

14 Fundamental properties of the epsilon ordering (< on ordinals)

14.0.1 Proving That < is a Linear Ordering on the Ordinals

lemma *Ord-linear*:
  Ord(*i*) $\Longrightarrow$ *Ord*(*j*) $\Longrightarrow$ *i*∈*j* | *i*=*j* | *j*∈*i*
proof (*induct i arbitrary*: *j rule*: *trans-induct*)
 case (*step i*)
 note *step-i* = *step*
 show *?case* **using** ‹*Ord*(*j*)›
  proof (*induct j rule*: *trans-induct*)
   case (*step j*)
   thus *?case* **using** *step-i*
    by (*blast dest*: *Ord-trans*)
  qed
qed

The trichotomy law for ordinals

lemma *Ord-linear-lt*:
 assumes *o*: *Ord*(*i*) *Ord*(*j*)

obtains (*lt*) $i<j$ | (*eq*) $i=j$ | (*gt*) $j<i$
apply (*simp add*: *lt-def*)
apply (*rule-tac* $i1=i$ **and** $j1=j$ **in** *Ord-linear* [*THEN disjE*])
apply (*blast intro*: *o*)+
done

lemma *Ord-linear2*:
assumes *o*: $Ord(i)$ $Ord(j)$
obtains (*lt*) $i<j$ | (*ge*) $j \leq i$
apply (*rule-tac* $i = i$ **and** $j = j$ **in** *Ord-linear-lt*)
apply (*blast intro*: *leI le-eqI sym o*) +
done

lemma *Ord-linear-le*:
assumes *o*: $Ord(i)$ $Ord(j)$
obtains (*le*) $i \leq j$ | (*ge*) $j \leq i$
apply (*rule-tac* $i = i$ **and** $j = j$ **in** *Ord-linear-lt*)
apply (*blast intro*: *leI le-eqI o*) +
done

lemma *le-imp-not-lt*: $j \leq i \Longrightarrow \neg\ i<j$
by (*blast elim*!: *leE elim*: *lt-asym*)

lemma *not-lt-imp-le*: $[\![\neg\ i<j;\ \ Ord(i);\ \ Ord(j)]\!] \Longrightarrow j \leq i$
by (*rule-tac* $i = i$ **and** $j = j$ **in** *Ord-linear2*, *auto*)

14.0.2 Some Rewrite Rules for $<$, $\leq$

lemma *Ord-mem-iff-lt*: $Ord(j) \Longrightarrow i{\in}j <-> i<j$
by (*unfold lt-def*, *blast*)

lemma *not-lt-iff-le*: $[\![Ord(i);\ \ Ord(j)]\!] \Longrightarrow \neg\ i<j <-> j \leq i$
by (*blast dest*: *le-imp-not-lt not-lt-imp-le*)

lemma *not-le-iff-lt*: $[\![Ord(i);\ \ Ord(j)]\!] \Longrightarrow \neg\ i \leq j <-> j<i$
by (*simp* (*no-asm-simp*) *add*: *not-lt-iff-le* [*THEN iff-sym*])

lemma *Ord-0-le*: $Ord(i) \Longrightarrow 0 \leq i$
by (*erule not-lt-iff-le* [*THEN iffD1*], *auto*)

lemma *Ord-0-lt*: $[\![Ord(i);\ \ i{\neq}0]\!] \Longrightarrow 0<i$
apply (*erule not-le-iff-lt* [*THEN iffD1*])
apply (*rule Ord-0*, *blast*)
done

lemma *Ord-0-lt-iff*: $Ord(i) \Longrightarrow i{\neq}0 <-> 0<i$
by (*blast intro*: *Ord-0-lt*)

14.1 Results about Less-Than or Equals

lemma *zero-le-succ-iff* [*iff*]: $0 \le succ(x) <-> Ord(x)$
by (*blast intro*: *Ord-0-le elim*: *ltE*)

lemma *subset-imp-le*: $[\![j<=i; \;\; Ord(i); \;\; Ord(j)]\!] \Longrightarrow j \le i$
apply (*rule not-lt-iff-le* [*THEN iffD1*], *assumption*+)
apply (*blast elim*: *ltE mem-irrefl*)
done

lemma *le-imp-subset*: $i \le j \Longrightarrow i<=j$
by (*blast dest*: *OrdmemD elim*: *ltE leE*)

lemma *le-subset-iff*: $j \le i <-> j<=i \wedge Ord(i) \wedge Ord(j)$
by (*blast dest*: *subset-imp-le le-imp-subset elim*: *ltE*)

lemma *le-succ-iff*: $i \le succ(j) <-> i \le j \mid i=succ(j) \wedge Ord(i)$
apply (*simp* (*no-asm*) *add*: *le-iff*)
apply *blast*
done

lemma *all-lt-imp-le*: $[\![Ord(i); \;\; Ord(j); \;\; \bigwedge x. \; x<j \Longrightarrow x<i]\!] \Longrightarrow j \le i$
by (*blast intro*: *not-lt-imp-le dest*: *lt-irrefl*)

14.1.1 Transitivity Laws

lemma *lt-trans1*: $[\![i \le j; \;\; j<k]\!] \Longrightarrow i<k$
by (*blast elim*!: *leE intro*: *lt-trans*)

lemma *lt-trans2*: $[\![i<j; \;\; j \le k]\!] \Longrightarrow i<k$
by (*blast elim*!: *leE intro*: *lt-trans*)

lemma *le-trans*: $[\![i \le j; \;\; j \le k]\!] \Longrightarrow i \le k$
by (*blast intro*: *lt-trans1*)

lemma *succ-leI*: $i<j \Longrightarrow succ(i) \le j$
apply (*rule not-lt-iff-le* [*THEN iffD1*])
apply (*blast elim*: *ltE leE lt-asym*)+
done

lemma *succ-leE*: $succ(i) \le j \Longrightarrow i<j$
apply (*rule not-le-iff-lt* [*THEN iffD1*])
apply (*blast elim*: *ltE leE lt-asym*)+
done

lemma *succ-le-iff* [*iff*]: $succ(i) \le j <-> i<j$
by (*blast intro*: *succ-leI succ-leE*)

lemma *succ-le-imp-le*: $succ(i) \le succ(j) \Longrightarrow i \le j$
by (*blast dest*!: *succ-leE*)

lemma *lt-subset-trans*: $[\![i \subseteq j;\ \ j<k;\ \ Ord(i)]\!] \Longrightarrow i<k$
apply (*rule subset-imp-le* [*THEN lt-trans1*])
apply (*blast intro*: *elim*: *ltE*) +
done

lemma *lt-imp-0-lt*: $j<i \Longrightarrow 0<i$
by (*blast intro*: *lt-trans1 Ord-0-le* [*OF lt-Ord*])

lemma *succ-lt-iff*: $succ(i) < j <-> i<j \wedge succ(i) \ne j$
apply *auto*
apply (*blast intro*: *lt-trans le-refl dest*: *lt-Ord*)
apply (*frule lt-Ord*)
apply (*rule not-le-iff-lt* [*THEN iffD1*])
  apply (*blast intro*: *lt-Ord2*)
 apply *blast*
apply (*simp add*: *lt-Ord lt-Ord2 le-iff*)
apply (*blast dest*: *lt-asym*)
done

lemma *Ord-succ-mem-iff*: $Ord(j) \Longrightarrow succ(i) \in succ(j) <-> i \in j$
apply (*insert succ-le-iff* [*of i j*])
apply (*simp add*: *lt-def*)
done

14.1.2 Union and Intersection

lemma *Un-upper1-le*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow i \le i \cup j$
by (*rule Un-upper1* [*THEN subset-imp-le*], *auto*)

lemma *Un-upper2-le*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow j \le i \cup j$
by (*rule Un-upper2* [*THEN subset-imp-le*], *auto*)

lemma *Un-least-lt*: $[\![i<k;\ \ j<k]\!] \Longrightarrow i \cup j < k$
apply (*rule-tac* $i = i$ **and** $j = j$ **in** *Ord-linear-le*)
apply (*auto simp add*: *Un-commute le-subset-iff subset-Un-iff lt-Ord*)
done

lemma *Un-least-lt-iff*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow i \cup j < k\ \ <->\ \ i<k \wedge j<k$
apply (*safe intro*!: *Un-least-lt*)
apply (*rule-tac* [*2*] *Un-upper2-le* [*THEN lt-trans1*])
apply (*rule Un-upper1-le* [*THEN lt-trans1*], *auto*)
done

lemma *Un-least-mem-iff*:
    $[\![Ord(i);\ Ord(j);\ Ord(k)]\!] \Longrightarrow i \cup j \in k\ \ <->\ \ i \in k \wedge j \in k$

apply (*insert Un-least-lt-iff* [*of i j k*])
apply (*simp add*: *lt-def*)
done

lemma *Int-greatest-lt*: $[\![i<k;\ \ j<k]\!] \Longrightarrow i \cap j < k$
apply (*rule-tac* $i = i$ **and** $j = j$ **in** *Ord-linear-le*)
apply (*auto simp add*: *Int-commute le-subset-iff subset-Int-iff lt-Ord*)
done

lemma *Ord-Un-if*:
$[\![Ord(i);\ Ord(j)]\!] \Longrightarrow i \cup j = (if\ j<i\ then\ i\ else\ j)$
by (*simp add*: *not-lt-iff-le le-imp-subset leI*
subset-Un-iff [*symmetric*] *subset-Un-iff2* [*symmetric*])

lemma *succ-Un-distrib*:
$[\![Ord(i);\ Ord(j)]\!] \Longrightarrow succ(i \cup j) = succ(i) \cup succ(j)$
by (*simp add*: *Ord-Un-if lt-Ord le-Ord2*)

lemma *lt-Un-iff*:
$[\![Ord(i);\ Ord(j)]\!] \Longrightarrow k < i \cup j <-> k < i \mid k < j$
apply (*simp add*: *Ord-Un-if not-lt-iff-le*)
apply (*blast intro*: *leI lt-trans2*)+
done

lemma *le-Un-iff*:
$[\![Ord(i);\ Ord(j)]\!] \Longrightarrow k \le i \cup j <-> k \le i \mid k \le j$
by (*simp add*: *succ-Un-distrib lt-Un-iff* [*symmetric*])

lemma *Un-upper1-lt*: $[\![k < i;\ Ord(j)]\!] \Longrightarrow k < i \cup j$
by (*simp add*: *lt-Un-iff lt-Ord2*)

lemma *Un-upper2-lt*: $[\![k < j;\ Ord(i)]\!] \Longrightarrow k < i \cup j$
by (*simp add*: *lt-Un-iff lt-Ord2*)

lemma *Ord-Union-succ-eq*: $Ord(i) \Longrightarrow \bigcup(succ(i)) = i$
by (*blast intro*: *Ord-trans*)

14.2 Results about Limits

lemma *Ord-Union* [*intro,simp,TC*]: $[\![\bigwedge i.\ i{\in}A \Longrightarrow Ord(i)]\!] \Longrightarrow Ord(\bigcup(A))$
apply (*rule Ord-is-Transset* [*THEN Transset-Union-family*, *THEN OrdI*])
apply (*blast intro*: *Ord-contains-Transset*)+
done

lemma *Ord-UN* [*intro,simp,TC*]:
$[\![\bigwedge x.\ x{\in}A \Longrightarrow Ord(B(x))]\!] \Longrightarrow Ord(\bigcup x{\in}A.\ B(x))$
by (*rule Ord-Union*, *blast*)

lemma *Ord-Inter* [*intro*,*simp*, *TC*]:
$[\![\bigwedge i.\ i{\in}A \Longrightarrow Ord(i)]\!] \Longrightarrow Ord(\bigcap(A))$
apply (*rule Transset-Inter-family* [*THEN OrdI*])
apply (*blast intro*: *Ord-is-Transset*)
apply (*simp add*: *Inter-def*)
apply (*blast intro*: *Ord-contains-Transset*)
done

lemma *Ord-INT* [*intro*,*simp*, *TC*]:
$[\![\bigwedge x.\ x{\in}A \Longrightarrow Ord(B(x))]\!] \Longrightarrow Ord(\bigcap x{\in}A.\ B(x))$
by (*rule Ord-Inter*, *blast*)

lemma *UN-least-le*:
$[\![Ord(i);\ \ \bigwedge x.\ x{\in}A \Longrightarrow b(x) \le i]\!] \Longrightarrow (\bigcup x{\in}A.\ b(x)) \le i$
apply (*rule le-imp-subset* [*THEN UN-least*, *THEN subset-imp-le*])
apply (*blast intro*: *Ord-UN elim*: *ltE*)+
done

lemma *UN-succ-least-lt*:
$[\![j{<}i;\ \ \bigwedge x.\ x{\in}A \Longrightarrow b(x){<}j]\!] \Longrightarrow (\bigcup x{\in}A.\ succ(b(x))) < i$
apply (*rule ltE*, *assumption*)
apply (*rule UN-least-le* [*THEN lt-trans2*])
apply (*blast intro*: *succ-leI*)+
done

lemma *UN-upper-lt*:
$[\![a{\in}A;\ \ i < b(a);\ \ Ord(\bigcup x{\in}A.\ b(x))]\!] \Longrightarrow i < (\bigcup x{\in}A.\ b(x))$
by (*unfold lt-def*, *blast*)

lemma *UN-upper-le*:
$[\![a \in A;\ \ i \le b(a);\ \ Ord(\bigcup x{\in}A.\ b(x))]\!] \Longrightarrow i \le (\bigcup x{\in}A.\ b(x))$
apply (*frule ltD*)
apply (*rule le-imp-subset* [*THEN subset-trans*, *THEN subset-imp-le*])
apply (*blast intro*: *lt-Ord UN-upper*)+
done

lemma *lt-Union-iff*: $\forall i{\in}A.\ Ord(i) \Longrightarrow (j < \bigcup(A))$ <−> $(\exists i{\in}A.\ j{<}i)$
by (*auto simp*: *lt-def Ord-Union*)

lemma *Union-upper-le*:
$[\![j \in J;\ \ i{\le}j;\ \ Ord(\bigcup(J))]\!] \Longrightarrow i \le \bigcup J$
apply (*subst Union-eq-UN*)
apply (*rule UN-upper-le*, *auto*)
done

lemma *le-implies-UN-le-UN*:

$[\![\bigwedge x.\ x \in A \Longrightarrow c(x) \le d(x)]\!] \Longrightarrow (\bigcup x \in A.\ c(x)) \le (\bigcup x \in A.\ d(x))$
apply (*rule UN-least-le*)
apply (*rule-tac* [*2*] *UN-upper-le*)
apply (*blast intro*: *Ord-UN le-Ord2*)+
done

lemma *Ord-equality*: $Ord(i) \Longrightarrow (\bigcup y \in i.\ succ(y)) = i$
by (*blast intro*: *Ord-trans*)

lemma *Ord-Union-subset*: $Ord(i) \Longrightarrow \bigcup(i) \subseteq i$
by (*blast intro*: *Ord-trans*)

14.3 Limit Ordinals – General Properties

lemma *Limit-Union-eq*: $Limit(i) \Longrightarrow \bigcup(i) = i$
 unfolding *Limit-def*
apply (*fast intro*!: *ltI elim*!: *ltE elim*: *Ord-trans*)
done

lemma *Limit-is-Ord*: $Limit(i) \Longrightarrow Ord(i)$
 unfolding *Limit-def*
apply (*erule conjunct1*)
done

lemma *Limit-has-0*: $Limit(i) \Longrightarrow 0 < i$
 unfolding *Limit-def*
apply (*erule conjunct2* [*THEN conjunct1*])
done

lemma *Limit-nonzero*: $Limit(i) \Longrightarrow i \neq 0$
by (*drule Limit-has-0*, *blast*)

lemma *Limit-has-succ*: $[\![Limit(i);\ j<i]\!] \Longrightarrow succ(j) < i$
by (*unfold Limit-def*, *blast*)

lemma *Limit-succ-lt-iff* [*simp*]: $Limit(i) \Longrightarrow succ(j) < i <-> (j<i)$
apply (*safe intro*!: *Limit-has-succ*)
apply (*frule lt-Ord*)
apply (*blast intro*: *lt-trans*)
done

lemma *zero-not-Limit* [*iff*]: $\neg\ Limit(0)$
by (*simp add*: *Limit-def*)

lemma *Limit-has-1*: $Limit(i) \Longrightarrow 1 < i$
by (*blast intro*: *Limit-has-0 Limit-has-succ*)

lemma *increasing-LimitI*: $[\![0<l;\ \forall x \in l.\ \exists y \in l.\ x<y]\!] \Longrightarrow Limit(l)$

```
apply (unfold Limit-def, simp add: lt-Ord2, clarify)
apply (drule-tac i=y in ltD)
apply (blast intro: lt-trans1 [OF - ltI] lt-Ord2)
done

lemma non-succ-LimitI:
  assumes i: 0<i and nsucc: ⋀y. succ(y) ≠ i
  shows Limit(i)
proof −
  have Oi: Ord(i) using i by (simp add: lt-def)
  { fix y
    assume yi: y<i
    hence Osy: Ord(succ(y)) by (simp add: lt-Ord Ord-succ)
    have ¬ i ≤ y using yi by (blast dest: le-imp-not-lt)
    hence succ(y) < i using nsucc [of y]
      by (blast intro: Ord-linear-lt [OF Osy Oi]) }
  thus ?thesis using i Oi by (auto simp add: Limit-def)
qed

lemma succ-LimitE [elim!]: Limit(succ(i)) ⟹ P
apply (rule lt-irrefl)
apply (rule Limit-has-succ, assumption)
apply (erule Limit-is-Ord [THEN Ord-succD, THEN le-refl])
done

lemma not-succ-Limit [simp]: ¬ Limit(succ(i))
by blast

lemma Limit-le-succD: ⟦Limit(i); i ≤ succ(j)⟧ ⟹ i ≤ j
by (blast elim!: leE)
```

14.3.1 Traditional 3-Way Case Analysis on Ordinals

```
lemma Ord-cases-disj: Ord(i) ⟹ i=0 | (∃ j. Ord(j) ∧ i=succ(j)) | Limit(i)
by (blast intro!: non-succ-LimitI Ord-0-lt)

lemma Ord-cases:
 assumes i: Ord(i)
 obtains (0) i=0 | (succ) j where Ord(j) i=succ(j) | (limit) Limit(i)
by (insert Ord-cases-disj [OF i], auto)

lemma trans-induct3-raw:
    ⟦Ord(i);
        P(0);
        ⋀x. ⟦Ord(x); P(x)⟧ ⟹ P(succ(x));
        ⋀x. ⟦Limit(x); ∀ y∈x. P(y)⟧ ⟹ P(x)
⟧ ⟹ P(i)
apply (erule trans-induct)
apply (erule Ord-cases, blast+)
```

done

lemma *trans-induct3* [*case-names 0 succ limit*, *consumes 1*]:
$Ord(i) \Longrightarrow P(0) \Longrightarrow (\bigwedge x.\ Ord(x) \Longrightarrow P(x) \Longrightarrow P(succ(x))) \Longrightarrow (\bigwedge x.\ Limit(x) \Longrightarrow (\bigwedge y.\ y \in x \Longrightarrow P(y)) \Longrightarrow P(x)) \Longrightarrow P(i)$
using *trans-induct3-raw* [*of i P*] **by** *simp*

A set of ordinals is either empty, contains its own union, or its union is a limit ordinal.

lemma *Union-le*: $[\![\bigwedge x.\ x{\in}I \Longrightarrow x{\le}j;\ Ord(j)]\!] \Longrightarrow \bigcup(I) \le j$
by (*auto simp add*: *le-subset-iff Union-least*)

lemma *Ord-set-cases*:
assumes I: $\forall i{\in}I.\ Ord(i)$
shows $I{=}0 \lor \bigcup(I) \in I \lor (\bigcup(I) \notin I \land Limit(\bigcup(I)))$
proof (*cases* $\bigcup(I)$ *rule*: *Ord-cases*)
show $Ord(\bigcup I)$ **using** I **by** (*blast intro*: *Ord-Union*)
next
assume $\bigcup I = 0$ **thus** *?thesis* **by** (*simp*, *blast intro*: *subst-elem*)
next
fix j
assume j: $Ord(j)$ **and** *UIj*:$\bigcup(I) = succ(j)$
{ assume $\forall i{\in}I.\ i{\le}j$
hence $\bigcup(I) \le j$
by (*simp add*: *Union-le j*)
hence *False*
by (*simp add*: *UIj lt-not-refl*) **}**
then obtain i **where** i: $i \in I$ $succ(j) \le i$ **using** I j
by (*atomize*, *auto simp add*: *not-le-iff-lt*)
have $\bigcup(I) \le succ(j)$ **using** *UIj j* **by** *auto*
hence $i \le succ(j)$ **using** i
by (*simp add*: *le-subset-iff Union-subset-iff*)
hence $succ(j) = i$ **using** i
by (*blast intro*: *le-anti-sym*)
hence $succ(j) \in I$ **by** (*simp add*: *i*)
thus *?thesis* **by** (*simp add*: *UIj*)
next
assume $Limit(\bigcup I)$ **thus** *?thesis* **by** *auto*
qed

If the union of a set of ordinals is a successor, then it is an element of that set.

lemma *Ord-Union-eq-succD*: $[\![\forall x{\in}X.\ Ord(x);\ \bigcup X = succ(j)]\!] \Longrightarrow succ(j) \in X$
by (*drule Ord-set-cases*, *auto*)

lemma *Limit-Union* [*rule-format*]: $[\![I \neq 0;\ (\bigwedge i.\ i{\in}I \Longrightarrow Limit(i))]\!] \Longrightarrow Limit(\bigcup I)$
apply (*simp add*: *Limit-def lt-def*)
apply (*blast intro*!: *equalityI*)
done

end

15 Special quantifiers

theory *OrdQuant* **imports** *Ordinal* **begin**

15.1 Quantifiers and union operator for ordinals

definition

oall :: [*i*, *i* ⇒ *o*] ⇒ *o* **where**
oall(*A*, *P*) ≡ ∀ *x*. *x*<*A* ⟶ *P*(*x*)

definition
oex :: [*i*, *i* ⇒ *o*] ⇒ *o* **where**
oex(*A*, *P*) ≡ ∃ *x*. *x*<*A* ∧ *P*(*x*)

definition

OUnion :: [*i*, *i* ⇒ *i*] ⇒ *i* **where**
OUnion(*i*,*B*) ≡ {*z*: ⋃*x*∈*i*. *B*(*x*). *Ord*(*i*)}

syntax
-*oall* :: [*idt*, *i*, *o*] ⇒ *o* (‹(‹*indent=3 notation=*‹*binder* ∀ <››∀ -<-./ -)› *10*)
-*oex* :: [*idt*, *i*, *o*] ⇒ *o* (‹(‹*indent=3 notation=*‹*binder* ∃ <››∃ -<-./ -)› *10*)
-*OUNION* :: [*idt*, *i*, *i*] ⇒ *i* (‹(‹*indent=3 notation=*‹*binder* ⋃ <››⋃ -<-./ -)› *10*)
syntax-consts
-*oall* ⇌ *oall* **and**
-*oex* ⇌ *oex* **and**
-*OUNION* ⇌ *OUnion*
translations
∀ *x*<*a*. *P* ⇌ *CONST oall*(*a*, λ*x*. *P*)
∃ *x*<*a*. *P* ⇌ *CONST oex*(*a*, λ*x*. *P*)
⋃ *x*<*a*. *B* ⇌ *CONST OUnion*(*a*, λ*x*. *B*)

15.1.1 simplification of the new quantifiers

lemma [*simp*]: (∀ *x*<*0*. *P*(*x*))
by (*simp add*: *oall-def*)

lemma [*simp*]: ¬(∃ *x*<*0*. *P*(*x*))
by (*simp add*: *oex-def*)

lemma [*simp*]: (∀ *x*<*succ*(*i*). *P*(*x*)) <−> (*Ord*(*i*) ⟶ *P*(*i*) ∧ (∀ *x*<*i*. *P*(*x*)))
apply (*simp add*: *oall-def le-iff*)
apply (*blast intro*: *lt-Ord2*)
done

lemma [*simp*]: $(\exists\, x<succ(i).\ P(x))$ <–> $(Ord(i) \wedge (P(i) \mid (\exists\, x<i.\ P(x))))$
apply (*simp add*: *oex-def le-iff*)
apply (*blast intro*: *lt-Ord2*)
done

15.1.2 Union over ordinals

lemma *Ord-OUN* [*intro*,*simp*]:
$[\![\bigwedge x.\ x<A \Longrightarrow Ord(B(x))]\!] \Longrightarrow Ord(\bigcup x<A.\ B(x))$
by (*simp add*: *OUnion-def ltI Ord-UN*)

lemma *OUN-upper-lt*:
$[\![a<A;\ \ i < b(a);\ \ Ord(\bigcup x<A.\ b(x))]\!] \Longrightarrow i < (\bigcup x<A.\ b(x))$
by (*unfold OUnion-def lt-def*, *blast*)

lemma *OUN-upper-le*:
$[\![a<A;\ \ i\le b(a);\ \ Ord(\bigcup x<A.\ b(x))]\!] \Longrightarrow i \le (\bigcup x<A.\ b(x))$
apply (*unfold OUnion-def*, *auto*)
apply (*rule UN-upper-le*)
apply (*auto simp add*: *lt-def*)
done

lemma *Limit-OUN-eq*: $Limit(i) \Longrightarrow (\bigcup x<i.\ x) = i$
by (*simp add*: *OUnion-def Limit-Union-eq Limit-is-Ord*)

lemma *OUN-least*:
$(\bigwedge x.\ x<A \Longrightarrow B(x) \subseteq C) \Longrightarrow (\bigcup x<A.\ B(x)) \subseteq C$
by (*simp add*: *OUnion-def UN-least ltI*)

lemma *OUN-least-le*:
$[\![Ord(i);\ \ \bigwedge x.\ x<A \Longrightarrow b(x) \le i]\!] \Longrightarrow (\bigcup x<A.\ b(x)) \le i$
by (*simp add*: *OUnion-def UN-least-le ltI Ord-0-le*)

lemma *le-implies-OUN-le-OUN*:
$[\![\bigwedge x.\ x<A \Longrightarrow c(x) \le d(x)]\!] \Longrightarrow (\bigcup x<A.\ c(x)) \le (\bigcup x<A.\ d(x))$
by (*blast intro*: *OUN-least-le OUN-upper-le le-Ord2 Ord-OUN*)

lemma *OUN-UN-eq*:
$(\bigwedge x.\ x \in A \Longrightarrow Ord(B(x)))$
$\Longrightarrow (\bigcup z < (\bigcup x\in A.\ B(x)).\ C(z)) = (\bigcup x\in A.\ \bigcup z < B(x).\ C(z))$
by (*simp add*: *OUnion-def*)

lemma *OUN-Union-eq*:
$(\bigwedge x.\ x \in X \Longrightarrow Ord(x))$
$\Longrightarrow (\bigcup z < \bigcup(X).\ C(z)) = (\bigcup x\in X.\ \bigcup z < x.\ C(z))$
by (*simp add*: *OUnion-def*)

lemma *atomize-oall* [*symmetric*, *rulify*]:
$(\bigwedge x.\ x{<}A \Longrightarrow P(x)) \equiv \mathit{Trueprop}\ (\forall\, x{<}A.\ P(x))$
by (*simp add*: *oall-def atomize-all atomize-imp*)

15.1.3 universal quantifier for ordinals

lemma *oallI* [*intro*!]:
$[\![\bigwedge x.\ x{<}A \Longrightarrow P(x)]\!] \Longrightarrow \forall\, x{<}A.\ P(x)$
by (*simp add*: *oall-def*)

lemma *ospec*: $[\![\forall\, x{<}A.\ P(x);\ \ x{<}A]\!] \Longrightarrow P(x)$
by (*simp add*: *oall-def*)

lemma *oallE*:
$[\![\forall\, x{<}A.\ P(x);\ \ P(x) \Longrightarrow Q;\ \ \neg x{<}A \Longrightarrow Q]\!] \Longrightarrow Q$
by (*simp add*: *oall-def*, *blast*)

lemma *rev-oallE* [*elim*]:
$[\![\forall\, x{<}A.\ P(x);\ \ \neg x{<}A \Longrightarrow Q;\ \ P(x) \Longrightarrow Q]\!] \Longrightarrow Q$
by (*simp add*: *oall-def*, *blast*)

lemma *oall-simp* [*simp*]: $(\forall\, x{<}a.\ \mathit{True})$ <-> *True*
by *blast*

lemma *oall-cong* [*cong*]:
$[\![a{=}a';\ \ \bigwedge x.\ x{<}a' \Longrightarrow P(x) <\!-\!> P'(x)]\!]$
$\Longrightarrow \mathit{oall}(a,\ \lambda x.\ P(x)) <\!-\!> \mathit{oall}(a',\ \lambda x.\ P'(x))$
by (*simp add*: *oall-def*)

15.1.4 existential quantifier for ordinals

lemma *oexI* [*intro*]:
$[\![P(x);\ \ x{<}A]\!] \Longrightarrow \exists\, x{<}A.\ P(x)$
apply (*simp add*: *oex-def*, *blast*)
done

lemma *oexCI*:
$[\![\forall\, x{<}A.\ \neg P(x) \Longrightarrow P(a);\ \ a{<}A]\!] \Longrightarrow \exists\, x{<}A.\ P(x)$
apply (*simp add*: *oex-def*, *blast*)
done

lemma *oexE* [*elim*!]:
$[\![\exists\, x{<}A.\ P(x);\ \ \bigwedge x.\ [\![x{<}A;\ P(x)]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
apply (*simp add*: *oex-def*, *blast*)
done

lemma *oex-cong* [*cong*]:
$[\![a=a'; \ \bigwedge x.\ x<a' \Longrightarrow P(x) <-> P'(x)]\!]$
$\Longrightarrow oex(a, \lambda x.\ P(x)) <-> oex(a', \lambda x.\ P'(x))$
apply (*simp add*: *oex-def cong add*: *conj-cong*)
done

15.1.5 Rules for Ordinal-Indexed Unions

lemma *OUN-I* [*intro*]: $[\![a<i; \ \ b \in B(a)]\!] \Longrightarrow b: (\bigcup z<i.\ B(z))$
by (*unfold OUnion-def lt-def*, *blast*)

lemma *OUN-E* [*elim!*]:
$[\![b \in (\bigcup z<i.\ B(z)); \ \bigwedge a.[\![b \in B(a); \ \ a<i]\!] \Longrightarrow R]\!] \Longrightarrow R$
apply (*unfold OUnion-def lt-def*, *blast*)
done

lemma *OUN-iff*: $b \in (\bigcup x<i.\ B(x)) <-> (\exists x<i.\ b \in B(x))$
by (*unfold OUnion-def oex-def lt-def*, *blast*)

lemma *OUN-cong* [*cong*]:
$[\![i=j; \ \bigwedge x.\ x<j \Longrightarrow C(x)=D(x)]\!] \Longrightarrow (\bigcup x<i.\ C(x)) = (\bigcup x<j.\ D(x))$
by (*simp add*: *OUnion-def lt-def OUN-iff*)

lemma *lt-induct*:
$[\![i<k; \ \bigwedge x.[\![x<k; \ \forall y<x.\ P(y)]\!] \Longrightarrow P(x)]\!] \ \Longrightarrow \ P(i)$
apply (*simp add*: *lt-def oall-def*)
apply (*erule conjE*)
apply (*erule Ord-induct*, *assumption*, *blast*)
done

15.2 Quantification over a class

definition
rall :: $[i \Rightarrow o, i \Rightarrow o] \Rightarrow o$ **where**
$rall(M, P) \equiv \forall x.\ M(x) \longrightarrow P(x)$

definition
rex :: $[i \Rightarrow o, i \Rightarrow o] \Rightarrow o$ **where**
$rex(M, P) \equiv \exists x.\ M(x) \wedge P(x)$

syntax
-rall :: [*pttrn*, $i \Rightarrow o$, *o*] $\Rightarrow$ *o* (‹(‹*indent=3 notation*=‹*binder* ∀ []››∀ -[-]./ -)›
10)
-rex :: [*pttrn*, $i \Rightarrow o$, *o*] $\Rightarrow$ *o* (‹(‹*indent=3 notation*=‹*binder* ∃ []››∃ -[-]./ -)›
10)
syntax-consts
-rall ⇌ *rall* **and**
-rex ⇌ *rex*
translations

$\forall x[M].\ P \rightleftharpoons CONST\ rall(M, \lambda x.\ P)$
$\exists x[M].\ P \rightleftharpoons CONST\ rex(M, \lambda x.\ P)$

15.2.1 Relativized universal quantifier

lemma *rallI* [*intro*!]: $[\![\bigwedge x.\ M(x) \Longrightarrow P(x)]\!] \Longrightarrow \forall x[M].\ P(x)$
by (*simp add*: *rall-def*)

lemma *rspec*: $[\![\forall x[M].\ P(x);\ M(x)]\!] \Longrightarrow P(x)$
by (*simp add*: *rall-def*)

lemma *rev-rallE* [*elim*]:
$[\![\forall x[M].\ P(x);\ \neg M(x) \Longrightarrow Q;\ P(x) \Longrightarrow Q]\!] \Longrightarrow Q$
by (*simp add*: *rall-def*, *blast*)

lemma *rallE*: $[\![\forall x[M].\ P(x);\ P(x) \Longrightarrow Q;\ \neg M(x) \Longrightarrow Q]\!] \Longrightarrow Q$
by *blast*

lemma *rall-triv* [*simp*]: $(\forall x[M].\ P) \longleftrightarrow ((\exists x.\ M(x)) \longrightarrow P)$
by (*simp add*: *rall-def*)

lemma *rall-cong* [*cong*]:
$(\bigwedge x.\ M(x) \Longrightarrow P(x) <-> P'(x)) \Longrightarrow (\forall x[M].\ P(x)) <-> (\forall x[M].\ P'(x))$
by (*simp add*: *rall-def*)

15.2.2 Relativized existential quantifier

lemma *rexI* [*intro*]: $[\![P(x);\ M(x)]\!] \Longrightarrow \exists x[M].\ P(x)$
by (*simp add*: *rex-def*, *blast*)

lemma *rev-rexI*: $[\![M(x);\ P(x)]\!] \Longrightarrow \exists x[M].\ P(x)$
by *blast*

lemma *rexCI*: $[\![\forall x[M].\ \neg P(x) \Longrightarrow P(a);\ M(a)]\!] \Longrightarrow \exists x[M].\ P(x)$
by *blast*

lemma *rexE* [*elim*!]: $[\![\exists x[M].\ P(x);\ \bigwedge x.\ [\![M(x);\ P(x)]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
by (*simp add*: *rex-def*, *blast*)

lemma *rex-triv* [*simp*]: $(\exists x[M].\ P) \longleftrightarrow ((\exists x.\ M(x)) \wedge P)$
by (*simp add*: *rex-def*)

lemma *rex-cong* [*cong*]:
$(\bigwedge x.\ M(x) \Longrightarrow P(x) <-> P'(x)) \Longrightarrow (\exists x[M].\ P(x)) <-> (\exists x[M].\ P'(x))$

by (*simp add*: *rex-def cong*: *conj-cong*)

lemma *rall-is-ball* [*simp*]: $(\forall x[\lambda z.\ z{\in}A].\ P(x))$ <-> $(\forall x{\in}A.\ P(x))$
by *blast*

lemma *rex-is-bex* [*simp*]: $(\exists x[\lambda z.\ z{\in}A].\ P(x))$ <-> $(\exists x{\in}A.\ P(x))$
by *blast*

lemma *atomize-rall*: $(\bigwedge x.\ M(x) \Longrightarrow P(x)) \equiv$ *Trueprop* $(\forall x[M].\ P(x))$
by (*simp add*: *rall-def atomize-all atomize-imp*)

declare *atomize-rall* [*symmetric*, *rulify*]

lemma *rall-simps1*:
$(\forall x[M].\ P(x) \wedge Q)$ <-> $(\forall x[M].\ P(x)) \wedge ((\forall x[M].\ False) \mid Q)$
$(\forall x[M].\ P(x) \mid Q)$ <-> $((\forall x[M].\ P(x)) \mid Q)$
$(\forall x[M].\ P(x) \longrightarrow Q)$ <-> $((\exists x[M].\ P(x)) \longrightarrow Q)$
$(\neg(\forall x[M].\ P(x)))$ <-> $(\exists x[M].\ \neg P(x))$
by *blast+*

lemma *rall-simps2*:
$(\forall x[M].\ P \wedge Q(x))$ <-> $((\forall x[M].\ False) \mid P) \wedge (\forall x[M].\ Q(x))$
$(\forall x[M].\ P \mid Q(x))$ <-> $(P \mid (\forall x[M].\ Q(x)))$
$(\forall x[M].\ P \longrightarrow Q(x))$ <-> $(P \longrightarrow (\forall x[M].\ Q(x)))$
by *blast+*

lemmas *rall-simps* [*simp*] = *rall-simps1 rall-simps2*

lemma *rall-conj-distrib*:
$(\forall x[M].\ P(x) \wedge Q(x))$ <-> $((\forall x[M].\ P(x)) \wedge (\forall x[M].\ Q(x)))$
by *blast*

lemma *rex-simps1*:
$(\exists x[M].\ P(x) \wedge Q)$ <-> $((\exists x[M].\ P(x)) \wedge Q)$
$(\exists x[M].\ P(x) \mid Q)$ <-> $(\exists x[M].\ P(x)) \mid ((\exists x[M].\ True) \wedge Q)$
$(\exists x[M].\ P(x) \longrightarrow Q)$ <-> $((\forall x[M].\ P(x)) \longrightarrow ((\exists x[M].\ True) \wedge Q))$
$(\neg(\exists x[M].\ P(x)))$ <-> $(\forall x[M].\ \neg P(x))$
by *blast+*

lemma *rex-simps2*:
$(\exists x[M].\ P \wedge Q(x))$ <-> $(P \wedge (\exists x[M].\ Q(x)))$
$(\exists x[M].\ P \mid Q(x))$ <-> $((\exists x[M].\ True) \wedge P) \mid (\exists x[M].\ Q(x))$
$(\exists x[M].\ P \longrightarrow Q(x))$ <-> $(((\forall x[M].\ False) \mid P) \longrightarrow (\exists x[M].\ Q(x)))$
by *blast+*

lemmas *rex-simps* [*simp*] = *rex-simps1 rex-simps2*

lemma *rex-disj-distrib*:
$(\exists x[M].\ P(x) \mid Q(x))$ <-> $((\exists x[M].\ P(x)) \mid (\exists x[M].\ Q(x)))$

by *blast*

15.2.3 One-point rule for bounded quantifiers

lemma *rex-triv-one-point1* [*simp*]: (∃ *x*[*M*]. *x*=*a*) <-> (*M*(*a*))
by *blast*

lemma *rex-triv-one-point2* [*simp*]: (∃ *x*[*M*]. *a*=*x*) <-> (*M*(*a*))
by *blast*

lemma *rex-one-point1* [*simp*]: (∃ *x*[*M*]. *x*=*a* ∧ *P*(*x*)) <-> (*M*(*a*) ∧ *P*(*a*))
by *blast*

lemma *rex-one-point2* [*simp*]: (∃ *x*[*M*]. *a*=*x* ∧ *P*(*x*)) <-> (*M*(*a*) ∧ *P*(*a*))
by *blast*

lemma *rall-one-point1* [*simp*]: (∀ *x*[*M*]. *x*=*a* ⟶ *P*(*x*)) <-> (*M*(*a*) ⟶ *P*(*a*))
by *blast*

lemma *rall-one-point2* [*simp*]: (∀ *x*[*M*]. *a*=*x* ⟶ *P*(*x*)) <-> (*M*(*a*) ⟶ *P*(*a*))
by *blast*

15.2.4 Sets as Classes

definition
  setclass :: [*i*,*i*] ⇒ *o* (‹(‹*open-block notation*=‹*prefix setclass*››##-)› [*40*] *40*)
where
  setclass(*A*) ≡ λ*x*. *x* ∈ *A*

lemma *setclass-iff* [*simp*]: *setclass*(*A*,*x*) <-> *x* ∈ *A*
by (*simp add*: *setclass-def*)

lemma *rall-setclass-is-ball* [*simp*]: (∀ *x*[##*A*]. *P*(*x*)) <-> (∀ *x*∈*A*. *P*(*x*))
by *auto*

lemma *rex-setclass-is-bex* [*simp*]: (∃ *x*[##*A*]. *P*(*x*)) <-> (∃ *x*∈*A*. *P*(*x*))
by *auto*

ML
‹
val Ord-atomize =
 atomize ([(***const-name*** ‹*oall*›, @{*thms ospec*}), (***const-name*** ‹*rall*›, @{*thms rspec*})]
@
  ZF-conn-pairs, *ZF-mem-pairs*);
›
declaration ‹*fn* - =>
 Simplifier.map-ss (*Simplifier.set-mksimps* (*fn ctxt* =>
  map mk-eq o Ord-atomize o Variable.gen-all ctxt))
›

Setting up the one-point-rule simproc

simproc-setup *defined-rex* ($\exists x[M]. P(x) \wedge Q(x)) = ‹
 K (*Quantifier1.rearrange-Bex* (*fn ctxt => unfold-tac ctxt* @{*thms rex-def*}))
›

simproc-setup *defined-rall* ($\forall x[M]. P(x) \longrightarrow Q(x)$) = ‹
 K (*Quantifier1.rearrange-Ball* (*fn ctxt => unfold-tac ctxt* @{*thms rall-def*}))
›

end

16 The Natural numbers As a Least Fixed Point

theory *Nat* **imports** *OrdQuant Bool* **begin**

definition
 $nat :: i$ **where**
  $nat \equiv lfp(Inf, \lambda X. \{0\} \cup \{succ(i). i \in X\})$

definition
 $quasinat :: i \Rightarrow o$ **where**
  $quasinat(n) \equiv n=0 \mid (\exists m. n = succ(m))$

definition

 $nat\text{-}case :: [i, i\Rightarrow i, i]\Rightarrow i$ **where**
  $nat\text{-}case(a,b,k) \equiv THE\ y. k=0 \wedge y=a \mid (\exists x. k=succ(x) \wedge y=b(x))$

definition
 $nat\text{-}rec :: [i, i, [i,i]\Rightarrow i]\Rightarrow i$ **where**
  $nat\text{-}rec(k,a,b) \equiv$
    $wfrec(Memrel(nat), k, \lambda n\ f. nat\text{-}case(a, \lambda m. b(m, f`m), n))$

definition
 $Le :: i$ **where**
  $Le \equiv \{\langle x,y\rangle:nat*nat. x \le y\}$

definition
 $Lt :: i$ **where**
  $Lt \equiv \{\langle x, y\rangle:nat*nat. x < y\}$

definition
 $Ge :: i$ **where**
  $Ge \equiv \{\langle x,y\rangle:nat*nat. y \le x\}$

definition

Gt :: *i* **where**
Gt ≡ {⟨*x*,*y*⟩:*nat*∗*nat*. *y* < *x*}

definition
greater-than :: *i*⇒*i* **where**
greater-than(*n*) ≡ {*i* ∈ *nat*. *n* < *i*}

No need for a less-than operator: a natural number is its list of predecessors!

lemma *nat-bnd-mono*: *bnd-mono*(*Inf*, λ*X*. {*0*} ∪ {*succ*(*i*). *i* ∈ *X*})
apply (*rule bnd-monoI*)
apply (*cut-tac infinity*, *blast*, *blast*)
done

lemmas *nat-unfold* = *nat-bnd-mono* [*THEN nat-def* [*THEN def-lfp-unfold*]]

lemma *nat-0I* [*iff*,*TC*]: *0* ∈ *nat*
apply (*subst nat-unfold*)
apply (*rule singletonI* [*THEN UnI1*])
done

lemma *nat-succI* [*intro*!,*TC*]: *n* ∈ *nat* ⟹ *succ*(*n*) ∈ *nat*
apply (*subst nat-unfold*)
apply (*erule RepFunI* [*THEN UnI2*])
done

lemma *nat-1I* [*iff*,*TC*]: *1* ∈ *nat*
by (*rule nat-0I* [*THEN nat-succI*])

lemma *nat-2I* [*iff*,*TC*]: *2* ∈ *nat*
by (*rule nat-1I* [*THEN nat-succI*])

lemma *bool-subset-nat*: *bool* ⊆ *nat*
by (*blast elim*!: *boolE*)

lemmas *bool-into-nat* = *bool-subset-nat* [*THEN subsetD*]

16.1 Injectivity Properties and Induction

lemma *nat-induct* [*case-names 0 succ*, *induct set*: *nat*]:
⟦*n* ∈ *nat*; *P*(*0*); ⋀*x*. ⟦*x* ∈ *nat*; *P*(*x*)⟧ ⟹ *P*(*succ*(*x*))⟧ ⟹ *P*(*n*)
by (*erule def-induct* [*OF nat-def nat-bnd-mono*], *blast*)

lemma *natE*:
assumes *n* ∈ *nat*
obtains (*0*) *n*=*0* | (*succ*) *x* **where** *x* ∈ *nat* *n*=*succ*(*x*)
using *assms*

by (*rule nat-unfold* [*THEN equalityD1*, *THEN subsetD*, *THEN UnE*]) *auto*

lemma *nat-into-Ord* [*simp*]: *n* ∈ *nat* ⟹ *Ord*(*n*)
by (*erule nat-induct*, *auto*)

lemmas *nat-0-le* = *nat-into-Ord* [*THEN Ord-0-le*]

lemmas *nat-le-refl* = *nat-into-Ord* [*THEN le-refl*]

lemma *Ord-nat* [*iff*]: *Ord*(*nat*)
apply (*rule OrdI*)
apply (*erule-tac* [*2*] *nat-into-Ord* [*THEN Ord-is-Transset*])
 unfolding *Transset-def*
apply (*rule ballI*)
apply (*erule nat-induct*, *auto*)
done

lemma *Limit-nat* [*iff*]: *Limit*(*nat*)
 unfolding *Limit-def*
apply (*safe intro*!: *ltI Ord-nat*)
apply (*erule ltD*)
done

lemma *naturals-not-limit*: *a* ∈ *nat* ⟹ ¬ *Limit*(*a*)
by (*induct a rule*: *nat-induct*, *auto*)

lemma *succ-natD*: *succ*(*i*): *nat* ⟹ *i* ∈ *nat*
by (*rule Ord-trans* [*OF succI1*], *auto*)

lemma *nat-succ-iff* [*iff*]: *succ*(*n*): *nat* ⟷ *n* ∈ *nat*
by (*blast dest*!: *succ-natD*)

lemma *nat-le-Limit*: *Limit*(*i*) ⟹ *nat* ≤ *i*
apply (*rule subset-imp-le*)
apply (*simp-all add*: *Limit-is-Ord*)
apply (*rule subsetI*)
apply (*erule nat-induct*)
 apply (*erule Limit-has-0* [*THEN ltD*])
apply (*blast intro*: *Limit-has-succ* [*THEN ltD*] *ltI Limit-is-Ord*)
done

lemmas *succ-in-naturalD* = *Ord-trans* [*OF succI1 - nat-into-Ord*]

lemma *lt-nat-in-nat*: ⟦*m*<*n*; *n* ∈ *nat*⟧ ⟹ *m* ∈ *nat*
apply (*erule ltE*)
apply (*erule Ord-trans*, *assumption*, *simp*)

done

lemma *le-in-nat*: $[\![m \leq n;\ n \in nat]\!] \Longrightarrow m \in nat$
by (*blast dest*!: *lt-nat-in-nat*)

16.2 Variations on Mathematical Induction

lemmas *complete-induct* = *Ord-induct* [*OF* - *Ord-nat*, *case-names less*, *consumes 1*]

lemma *complete-induct-rule* [*case-names less*, *consumes 1*]:
$i \in nat \Longrightarrow (\bigwedge x.\ x \in nat \Longrightarrow (\bigwedge y.\ y \in x \Longrightarrow P(y)) \Longrightarrow P(x)) \Longrightarrow P(i)$
using *complete-induct* [*of i P*] **by** *simp*

lemma *nat-induct-from*:
assumes $m \leq n$ $m \in nat$ $n \in nat$
and $P(m)$
and $\bigwedge x.\ [\![x \in nat;\ \ m \leq x;\ \ P(x)]\!] \Longrightarrow P(succ(x))$
shows $P(n)$
proof −
from *assms*(*3*) **have** $m \leq n \longrightarrow P(m) \longrightarrow P(n)$
by (*rule nat-induct*) (*use assms*(*5*) **in** ‹*simp-all add*: *distrib-simps le-succ-iff*›)
with *assms*(*1*,*2*,*4*) **show** *?thesis* **by** *blast*
qed

lemma *diff-induct* [*case-names 0 0-succ succ-succ*, *consumes 2*]:
$[\![m \in nat;\ \ n \in nat;$
$\bigwedge x.\ x \in nat \Longrightarrow P(x,0);$
$\bigwedge y.\ y \in nat \Longrightarrow P(0,succ(y));$
$\bigwedge x\ y.\ [\![x \in nat;\ \ y \in nat;\ \ P(x,y)]\!] \Longrightarrow P(succ(x),succ(y))]\!]$
$\Longrightarrow P(m,n)$
apply (*erule-tac* $x = m$ **in** *rev-bspec*)
apply (*erule nat-induct*, *simp*)
apply (*rule ballI*)
apply (*rename-tac i j*)
apply (*erule-tac n=j* **in** *nat-induct*, *auto*)
done

lemma *succ-lt-induct-lemma* [*rule-format*]:
$m \in nat \Longrightarrow P(m,succ(m)) \longrightarrow (\forall x{\in}nat.\ P(m,x) \longrightarrow P(m,succ(x))) \longrightarrow$
$(\forall n{\in}nat.\ m{<}n \longrightarrow P(m,n))$
apply (*erule nat-induct*)
apply (*intro impI*, *rule nat-induct* [*THEN ballI*])
prefer *4* **apply** (*intro impI*, *rule nat-induct* [*THEN ballI*])

apply (*auto simp add*: *le-iff*)
done

lemma *succ-lt-induct*:
$\llbracket m<n;\ \ n \in nat;$
$P(m,succ(m));$
$\bigwedge x.\ \llbracket x \in nat;\ \ P(m,x)\rrbracket \Longrightarrow P(m,succ(x))\rrbracket$
$\Longrightarrow P(m,n)$
by (*blast intro*: *succ-lt-induct-lemma lt-nat-in-nat*)

16.3 quasinat: to allow a case-split rule for *nat-case*

True if the argument is zero or any successor

lemma [*iff*]: *quasinat*(*0*)
by (*simp add*: *quasinat-def*)

lemma [*iff*]: *quasinat*(*succ*(*x*))
by (*simp add*: *quasinat-def*)

lemma *nat-imp-quasinat*: $n \in nat \Longrightarrow quasinat(n)$
by (*erule natE*, *simp-all*)

lemma *non-nat-case*: $\neg\ quasinat(x) \Longrightarrow nat\text{-}case(a,b,x) = 0$
by (*simp add*: *quasinat-def nat-case-def*)

lemma *nat-cases-disj*: $k=0 \mid (\exists\, y.\ k = succ(y)) \mid \neg\ quasinat(k)$
apply (*case-tac* $k=0$, *simp*)
apply (*case-tac* $\exists\, m.\ k = succ(m)$)
apply (*simp-all add*: *quasinat-def*)
done

lemma *nat-cases*:
$\llbracket k=0 \Longrightarrow P;\ \ \bigwedge y.\ k = succ(y) \Longrightarrow P;\ \neg\ quasinat(k) \Longrightarrow P\rrbracket \Longrightarrow P$
by (*insert nat-cases-disj* [*of k*], *blast*)

lemma *nat-case-0* [*simp*]: $nat\text{-}case(a,b,0) = a$
by (*simp add*: *nat-case-def*)

lemma *nat-case-succ* [*simp*]: $nat\text{-}case(a,b,succ(n)) = b(n)$
by (*simp add*: *nat-case-def*)

lemma *nat-case-type* [*TC*]:
$\llbracket n \in nat;\ \ a \in C(0);\ \ \bigwedge m.\ m \in nat \Longrightarrow b(m){:}\ C(succ(m))\rrbracket$
$\Longrightarrow nat\text{-}case(a,b,n) \in C(n)$
by (*erule nat-induct*, *auto*)

lemma *split-nat-case*:

$P(nat\text{-}case(a,b,k)) \longleftrightarrow$
$((k=0 \longrightarrow P(a)) \wedge (\forall x.\ k=succ(x) \longrightarrow P(b(x))) \wedge (\neg\ quasinat(k) \longrightarrow P(0)))$
apply (*rule nat-cases* [*of k*])
apply (*auto simp add*: *non-nat-case*)
done

16.4 Recursion on the Natural Numbers

lemma *nat-rec-0*: $nat\text{-}rec(0,a,b) = a$
apply (*rule nat-rec-def* [*THEN def-wfrec*, *THEN trans*])
apply (*rule wf-Memrel*)
apply (*rule nat-case-0*)
done

lemma *nat-rec-succ*: $m \in nat \Longrightarrow nat\text{-}rec(succ(m),a,b) = b(m,\ nat\text{-}rec(m,a,b))$
apply (*rule nat-rec-def* [*THEN def-wfrec*, *THEN trans*])
apply (*rule wf-Memrel*)
apply (*simp add*: *vimage-singleton-iff*)
done

lemma *Un-nat-type* [*TC*]: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow i \cup j \in nat$
apply (*rule Un-least-lt* [*THEN ltD*])
apply (*simp-all add*: *lt-def*)
done

lemma *Int-nat-type* [*TC*]: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow i \cap j \in nat$
apply (*rule Int-greatest-lt* [*THEN ltD*])
apply (*simp-all add*: *lt-def*)
done

lemma *nat-nonempty* [*simp*]: $nat \neq 0$
by *blast*

A natural number is the set of its predecessors

lemma *nat-eq-Collect-lt*: $i \in nat \Longrightarrow \{j{\in}nat.\ j<i\} = i$
apply (*rule equalityI*)
apply (*blast dest*: *ltD*)
apply (*auto simp add*: *Ord-mem-iff-lt*)
apply (*blast intro*: *lt-trans*)
done

lemma *Le-iff* [*iff*]: $\langle x,y\rangle \in Le \longleftrightarrow x \le y \wedge x \in nat \wedge y \in nat$
by (*force simp add*: *Le-def*)

end

17 Inductive and Coinductive Definitions

theory *Inductive*
imports *Fixedpt QPair Nat*
keywords
inductive coinductive inductive-cases rep-datatype primrec :: *thy-decl* **and**
domains intros monos con-defs type-intros type-elims
elimination induction case-eqns recursor-eqns :: *quasi-command*
begin

lemma *def-swap-iff*: $a \equiv b \Longrightarrow a = c \longleftrightarrow c = b$
by *blast*

lemma *def-trans*: $f \equiv g \Longrightarrow g(a) = b \Longrightarrow f(a) = b$
by *simp*

lemma *refl-thin*: $\bigwedge P.\ a = a \Longrightarrow P \Longrightarrow P$ **.**

ML-file ‹*ind-syntax.ML*›
ML-file ‹*Tools/ind-cases.ML*›
ML-file ‹*Tools/cartprod.ML*›
ML-file ‹*Tools/inductive-package.ML*›
ML-file ‹*Tools/induct-tacs.ML*›
ML-file ‹*Tools/primrec-package.ML*›

ML ‹
```
structure Lfp =
  struct
  val oper       = Const ‹lfp›
  val bnd-mono   = Const ‹bnd-mono›
  val bnd-monoI  = @{thm bnd-monoI}
  val subs       = @{thm def-lfp-subset}
  val Tarski     = @{thm def-lfp-unfold}
  val induct     = @{thm def-induct}
  end;

structure Standard-Prod =
  struct
  val sigma      = Const ‹Sigma›
  val pair       = Const ‹Pair›
  val split-name = const-name ‹split›
  val pair-iff   = @{thm Pair-iff}
  val split-eq   = @{thm split}
  val fsplitI    = @{thm splitI}
  val fsplitD    = @{thm splitD}
  val fsplitE    = @{thm splitE}
  end;

structure Standard-CP = CartProd-Fun (Standard-Prod);
```

```
structure Standard-Sum =
  struct
  val sum       = Const ‹sum›
  val inl       = Const ‹Inl›
  val inr       = Const ‹Inr›
  val elim      = Const ‹case›
  val case-inl  = @{thm case-Inl}
  val case-inr  = @{thm case-Inr}
  val inl-iff   = @{thm Inl-iff}
  val inr-iff   = @{thm Inr-iff}
  val distinct  = @{thm Inl-Inr-iff}
  val distinct' = @{thm Inr-Inl-iff}
  val free-SEs  = Ind-Syntax.mk-free-SEs
            [distinct, distinct', inl-iff, inr-iff, Standard-Prod.pair-iff]
  end;


structure Ind-Package =
    Add-inductive-def-Fun
      (structure Fp=Lfp and Pr=Standard-Prod and CP=Standard-CP
       and Su=Standard-Sum val coind = false);


structure Gfp =
  struct
  val oper      = Const ‹gfp›
  val bnd-mono  = Const ‹bnd-mono›
  val bnd-monoI = @{thm bnd-monoI}
  val subs      = @{thm def-gfp-subset}
  val Tarski    = @{thm def-gfp-unfold}
  val induct    = @{thm def-Collect-coinduct}
  end;

structure Quine-Prod =
  struct
  val sigma      = Const ‹QSigma›
  val pair       = Const ‹QPair›
  val split-name = const-name ‹qsplit›
  val pair-iff   = @{thm QPair-iff}
  val split-eq   = @{thm qsplit}
  val fsplitI    = @{thm qsplitI}
  val fsplitD    = @{thm qsplitD}
  val fsplitE    = @{thm qsplitE}
  end;

structure Quine-CP = CartProd-Fun (Quine-Prod);

structure Quine-Sum =
```

```
  struct
  val sum        = Const ‹qsum›
  val inl        = Const ‹QInl›
  val inr        = Const ‹QInr›
  val elim       = Const ‹qcase›
  val case-inl  = @{thm qcase-QInl}
  val case-inr  = @{thm qcase-QInr}
  val inl-iff   = @{thm QInl-iff}
  val inr-iff   = @{thm QInr-iff}
  val distinct  = @{thm QInl-QInr-iff}
  val distinct' = @{thm QInr-QInl-iff}
  val free-SEs  = Ind-Syntax.mk-free-SEs
           [distinct, distinct', inl-iff, inr-iff, Quine-Prod.pair-iff]
  end;


structure CoInd-Package =
 Add-inductive-def-Fun(structure Fp=Gfp and Pr=Quine-Prod and CP=Quine-CP
   and Su=Quine-Sum val coind = true);

›
```

end

18 Epsilon Induction and Recursion

theory *Epsilon* **imports** *Nat* **begin**

definition
  eclose :: $i \Rightarrow i$ **where**
    $eclose(A) \equiv \bigcup n\in nat.\ nat\text{-}rec(n,\ A,\ \lambda m\ r.\ \bigcup(r))$

definition
  transrec :: $[i,\ [i,i]\Rightarrow i] \Rightarrow i$ **where**
    $transrec(a,H) \equiv wfrec(Memrel(eclose(\{a\})),\ a,\ H)$

definition
  rank :: $i \Rightarrow i$ **where**
    $rank(a) \equiv transrec(a,\ \lambda x\ f.\ \bigcup y\in x.\ succ(f`y))$

definition
  transrec2 :: $[i,\ i,\ [i,i]\Rightarrow i] \Rightarrow i$ **where**
    $transrec2(k,\ a,\ b) \equiv$
      $transrec(k,$
            $\lambda i\ r.\ if(i=0,\ a,$
                  $if(\exists j.\ i=succ(j),$
                    $b(THE\ j.\ i=succ(j),\ r`(THE\ j.\ i=succ(j))),$
                    $\bigcup j<i.\ r`j)))$

definition
recursor :: $[i, [i,i]\Rightarrow i, i]\Rightarrow i$ **where**
recursor$(a,b,k) \equiv$ *transrec*$(k, \lambda n\ f.\ nat\text{-}case(a, \lambda m.\ b(m, f`m), n))$

definition
rec :: $[i, i, [i,i]\Rightarrow i]\Rightarrow i$ **where**
rec$(k,a,b) \equiv$ *recursor*(a,b,k)

18.1 Basic Closure Properties

lemma *arg-subset-eclose*: $A \subseteq eclose(A)$
unfolding *eclose-def*
apply (*rule nat-rec-0* [*THEN equalityD2*, *THEN subset-trans*])
apply (*rule nat-0I* [*THEN UN-upper*])
done

lemmas *arg-into-eclose* = *arg-subset-eclose* [*THEN subsetD*]

lemma *Transset-eclose*: *Transset*(*eclose*(*A*))
unfolding *eclose-def Transset-def*
apply (*rule subsetI* [*THEN ballI*])
apply (*erule UN-E*)
apply (*rule nat-succI* [*THEN UN-I*], *assumption*)
apply (*erule nat-rec-succ* [*THEN ssubst*])
apply (*erule UnionI*, *assumption*)
done

lemmas *eclose-subset* =
Transset-eclose [*unfolded Transset-def*, *THEN bspec*]

lemmas *ecloseD* = *eclose-subset* [*THEN subsetD*]

lemmas *arg-in-eclose-sing* = *arg-subset-eclose* [*THEN singleton-subsetD*]
lemmas *arg-into-eclose-sing* = *arg-in-eclose-sing* [*THEN ecloseD*]

lemmas *eclose-induct* =
Transset-induct [*OF - Transset-eclose*, *induct set*: *eclose*]

lemma *eps-induct*:
$[\![\bigwedge x.\ \forall y{\in}x.\ P(y) \Longrightarrow P(x)]\!] \Longrightarrow P(a)$
by (*rule arg-in-eclose-sing* [*THEN eclose-induct*], *blast*)

18.2 Leastness of *eclose*

lemma *eclose-least-lemma*:

⟦*Transset*(*X*); *A*<=*X*; *n* ∈ *nat*⟧ ⟹ *nat-rec*(*n*, *A*, λ*m r*. ⋃(*r*)) ⊆ *X*
unfolding *Transset-def*
apply (*erule nat-induct*)
apply (*simp add*: *nat-rec-0*)
apply (*simp add*: *nat-rec-succ*, *blast*)
done

lemma *eclose-least*:
⟦*Transset*(*X*); *A*<=*X*⟧ ⟹ *eclose*(*A*) ⊆ *X*
unfolding *eclose-def*
apply (*rule eclose-least-lemma* [*THEN UN-least*], *assumption*+)
done

lemma *eclose-induct-down* [*consumes 1*]:
⟦*a* ∈ *eclose*(*b*);
⋀*y*. ⟦*y* ∈ *b*⟧ ⟹ *P*(*y*);
⋀*y z*. ⟦*y* ∈ *eclose*(*b*); *P*(*y*); *z* ∈ *y*⟧ ⟹ *P*(*z*)
⟧ ⟹ *P*(*a*)
apply (*rule eclose-least* [*THEN subsetD*, *THEN CollectD2*, *of eclose*(*b*)])
prefer *3* **apply** *assumption*
unfolding *Transset-def*
apply (*blast intro*: *ecloseD*)
apply (*blast intro*: *arg-subset-eclose* [*THEN subsetD*])
done

lemma *Transset-eclose-eq-arg*: *Transset*(*X*) ⟹ *eclose*(*X*) = *X*
apply (*erule equalityI* [*OF eclose-least arg-subset-eclose*])
apply (*rule subset-refl*)
done

A transitive set either is empty or contains the empty set.

lemma *Transset-0-lemma* [*rule-format*]: *Transset*(*A*) ⟹ *x*∈*A* ⟶ *0*∈*A*
apply (*simp add*: *Transset-def*)
apply (*rule-tac a*=*x* **in** *eps-induct*, *clarify*)
apply (*drule bspec*, *assumption*)
apply (*case-tac x*=*0*, *auto*)
done

lemma *Transset-0-disj*: *Transset*(*A*) ⟹ *A*=*0* | *0*∈*A*
by (*blast dest*: *Transset-0-lemma*)

18.3 Epsilon Recursion

lemma *mem-eclose-trans*: ⟦*A* ∈ *eclose*(*B*); *B* ∈ *eclose*(*C*)⟧ ⟹ *A* ∈ *eclose*(*C*)
by (*rule eclose-least* [*OF Transset-eclose eclose-subset*, *THEN subsetD*], *assumption*+)

lemma *mem-eclose-sing-trans*:
  $[\![A \in eclose(\{B\});\ \ B \in eclose(\{C\})]\!] \Longrightarrow A \in eclose(\{C\})$
by (*rule eclose-least* [*OF Transset-eclose singleton-subsetI*, *THEN subsetD*],
  assumption+)

lemma *under-Memrel*: $[\![Transset(i);\ \ j \in i]\!] \Longrightarrow Memrel(i)-``\{j\} = j$
by (*unfold Transset-def*, *blast*)

lemma *lt-Memrel*: $j < i \Longrightarrow Memrel(i)\ -``\ \{j\} = j$
by (*simp add*: *lt-def Ord-def under-Memrel*)

lemmas *under-Memrel-eclose* = *Transset-eclose* [*THEN under-Memrel*]

lemmas *wfrec-ssubst* = *wf-Memrel* [*THEN wfrec*, *THEN ssubst*]

lemma *wfrec-eclose-eq*:
  $[\![k \in eclose(\{j\});\ \ j \in eclose(\{i\})]\!] \Longrightarrow$
   $wfrec(Memrel(eclose(\{i\})),\ k,\ H) = wfrec(Memrel(eclose(\{j\})),\ k,\ H)$
apply (*erule eclose-induct*)
apply (*rule wfrec-ssubst*)
apply (*rule wfrec-ssubst*)
apply (*simp add*: *under-Memrel-eclose mem-eclose-sing-trans* [*of - j i*])
done

lemma *wfrec-eclose-eq2*:
  $k \in i \Longrightarrow wfrec(Memrel(eclose(\{i\})),k,H) = wfrec(Memrel(eclose(\{k\})),k,H)$
apply (*rule arg-in-eclose-sing* [*THEN wfrec-eclose-eq*])
apply (*erule arg-into-eclose-sing*)
done

lemma *transrec*: $transrec(a,H) = H(a,\ \lambda x{\in}a.\ transrec(x,H))$
 unfolding *transrec-def*
apply (*rule wfrec-ssubst*)
apply (*simp add*: *wfrec-eclose-eq2 arg-in-eclose-sing under-Memrel-eclose*)
done

lemma *def-transrec*:
  $[\![\bigwedge x.\ f(x)\equiv transrec(x,H)]\!] \Longrightarrow f(a) = H(a,\ \lambda x{\in}a.\ f(x))$
apply *simp*
apply (*rule transrec*)
done

lemma *transrec-type*:
  $[\![\bigwedge x\ u.\ [\![x \in eclose(\{a\});\ \ u \in Pi(x,B)]\!] \Longrightarrow H(x,u) \in B(x)]\!]$
   $\Longrightarrow transrec(a,H) \in B(a)$
apply (*rule-tac* $i = a$ **in** *arg-in-eclose-sing* [*THEN eclose-induct*])
apply (*subst transrec*)

apply (*simp add*: *lam-type*)
done

lemma *eclose-sing-Ord*: $Ord(i) \Longrightarrow eclose(\{i\}) \subseteq succ(i)$
apply (*erule Ord-is-Transset* [*THEN Transset-succ*, *THEN eclose-least*])
apply (*rule succI1* [*THEN singleton-subsetI*])
done

lemma *succ-subset-eclose-sing*: $succ(i) \subseteq eclose(\{i\})$
apply (*insert arg-subset-eclose* [*of* {*i*}], *simp*)
apply (*frule eclose-subset*, *blast*)
done

lemma *eclose-sing-Ord-eq*: $Ord(i) \Longrightarrow eclose(\{i\}) = succ(i)$
apply (*rule equalityI*)
apply (*erule eclose-sing-Ord*)
apply (*rule succ-subset-eclose-sing*)
done

lemma *Ord-transrec-type*:
  assumes *jini*: $j \in i$
    and *ordi*: $Ord(i)$
    and *minor*: $\bigwedge x\ u.\ [\![x \in i;\ \ u \in Pi(x,B)]\!] \Longrightarrow H(x,u) \in B(x)$
  shows $transrec(j,H) \in B(j)$
apply (*rule transrec-type*)
apply (*insert jini ordi*)
apply (*blast intro*!: *minor*
      intro: *Ord-trans*
      dest: *Ord-in-Ord* [*THEN eclose-sing-Ord*, *THEN subsetD*])
done

18.4 Rank

lemma *rank*: $rank(a) = (\bigcup y{\in}a.\ succ(rank(y)))$
by (*subst rank-def* [*THEN def-transrec*], *simp*)

lemma *Ord-rank* [*simp*]: $Ord(rank(a))$
apply (*rule-tac a*=*a* **in** *eps-induct*)
apply (*subst rank*)
apply (*rule Ord-succ* [*THEN Ord-UN*])
apply (*erule bspec*, *assumption*)
done

lemma *rank-of-Ord*: $Ord(i) \Longrightarrow rank(i) = i$
apply (*erule trans-induct*)
apply (*subst rank*)
apply (*simp add*: *Ord-equality*)
done

lemma *rank-lt*: $a \in b \Longrightarrow rank(a) < rank(b)$
apply (*rule-tac a1 = b* **in** *rank* [*THEN ssubst*])
apply (*erule UN-I* [*THEN ltI*])
apply (*rule-tac* [*2*] *Ord-UN*, *auto*)
done

lemma *eclose-rank-lt*: $a \in eclose(b) \Longrightarrow rank(a) < rank(b)$
apply (*erule eclose-induct-down*)
apply (*erule rank-lt*)
apply (*erule rank-lt* [*THEN lt-trans*], *assumption*)
done

lemma *rank-mono*: $a<=b \Longrightarrow rank(a) \leq rank(b)$
apply (*rule subset-imp-le*)
apply (*auto simp add*: *rank* [*of a*] *rank* [*of b*])
done

lemma *rank-Pow*: $rank(Pow(a)) = succ(rank(a))$
apply (*rule rank* [*THEN trans*])
apply (*rule le-anti-sym*)
apply (*rule-tac* [*2*] *UN-upper-le*)
apply (*rule UN-least-le*)
apply (*auto intro*: *rank-mono simp add*: *Ord-UN*)
done

lemma *rank-0* [*simp*]: $rank(0) = 0$
by (*rule rank* [*THEN trans*], *blast*)

lemma *rank-succ* [*simp*]: $rank(succ(x)) = succ(rank(x))$
apply (*rule rank* [*THEN trans*])
apply (*rule equalityI* [*OF UN-least succI1* [*THEN UN-upper*]])
apply (*erule succE*, *blast*)
apply (*erule rank-lt* [*THEN leI*, *THEN succ-leI*, *THEN le-imp-subset*])
done

lemma *rank-Union*: $rank(\bigcup(A)) = (\bigcup x{\in}A.\ rank(x))$
apply (*rule equalityI*)
apply (*rule-tac* [*2*] *rank-mono* [*THEN le-imp-subset*, *THEN UN-least*])
apply (*erule-tac* [*2*] *Union-upper*)
apply (*subst rank*)
apply (*rule UN-least*)
apply (*erule UnionE*)
apply (*rule subset-trans*)
apply (*erule-tac* [*2*] *RepFunI* [*THEN Union-upper*])
apply (*erule rank-lt* [*THEN succ-leI*, *THEN le-imp-subset*])
done

lemma *rank-eclose*: $rank(eclose(a)) = rank(a)$
apply (*rule le-anti-sym*)

apply (*rule-tac* [*2*] *arg-subset-eclose* [*THEN rank-mono*])
apply (*rule-tac a1* = *eclose* (*a*) **in** *rank* [*THEN ssubst*])
apply (*rule Ord-rank* [*THEN UN-least-le*])
apply (*erule eclose-rank-lt* [*THEN succ-leI*])
done

lemma *rank-pair1*: *rank*(*a*) < *rank*(⟨*a*,*b*⟩)
 unfolding *Pair-def*
apply (*rule consI1* [*THEN rank-lt*, *THEN lt-trans*])
apply (*rule consI1* [*THEN consI2*, *THEN rank-lt*])
done

lemma *rank-pair2*: *rank*(*b*) < *rank*(⟨*a*,*b*⟩)
 unfolding *Pair-def*
apply (*rule consI1* [*THEN consI2*, *THEN rank-lt*, *THEN lt-trans*])
apply (*rule consI1* [*THEN consI2*, *THEN rank-lt*])
done

lemma *the-equality-if*:
  P(*a*) ⟹ (*THE x*. *P*(*x*)) = (*if* (∃!*x*. *P*(*x*)) *then a else 0*)
by (*simp add*: *the-0 the-equality2*)

lemma *rank-apply*: ⟦*i* ∈ *domain*(*f*); *function*(*f*)⟧ ⟹ *rank*(*f*‘*i*) < *rank*(*f*)
apply *clarify*
apply (*simp add*: *function-apply-equality*)
apply (*blast intro*: *lt-trans rank-lt rank-pair2*)
done

18.5 Corollaries of Leastness

lemma *mem-eclose-subset*: *A* ∈ *B* ⟹ *eclose*(*A*)<=*eclose*(*B*)
apply (*rule Transset-eclose* [*THEN eclose-least*])
apply (*erule arg-into-eclose* [*THEN eclose-subset*])
done

lemma *eclose-mono*: *A*<=*B* ⟹ *eclose*(*A*) ⊆ *eclose*(*B*)
apply (*rule Transset-eclose* [*THEN eclose-least*])
apply (*erule subset-trans*)
apply (*rule arg-subset-eclose*)
done

lemma *eclose-idem*: *eclose*(*eclose*(*A*)) = *eclose*(*A*)
apply (*rule equalityI*)
apply (*rule eclose-least* [*OF Transset-eclose subset-refl*])
apply (*rule arg-subset-eclose*)

done

lemma *transrec2-0* [*simp*]: *transrec2*(*0*,*a*,*b*) = *a*
by (*rule transrec2-def* [*THEN def-transrec*, *THEN trans*], *simp*)

lemma *transrec2-succ* [*simp*]: *transrec2*(*succ*(*i*),*a*,*b*) = *b*(*i*, *transrec2*(*i*,*a*,*b*))
apply (*rule transrec2-def* [*THEN def-transrec*, *THEN trans*])
apply (*simp add*: *the-equality if-P*)
done

lemma *transrec2-Limit*:
  Limit(*i*) ⟹ *transrec2*(*i*,*a*,*b*) = (⋃*j*<*i*. *transrec2*(*j*,*a*,*b*))
apply (*rule transrec2-def* [*THEN def-transrec*, *THEN trans*])
apply (*auto simp add*: *OUnion-def*)
done

lemma *def-transrec2*:
  (⋀*x*. *f*(*x*)≡*transrec2*(*x*,*a*,*b*))
   ⟹ *f*(*0*) = *a* ∧
    f(*succ*(*i*)) = *b*(*i*, *f*(*i*)) ∧
    (*Limit*(*K*) ⟶ *f*(*K*) = (⋃*j*<*K*. *f*(*j*)))
by (*simp add*: *transrec2-Limit*)

lemmas *recursor-lemma* = *recursor-def* [*THEN def-transrec*, *THEN trans*]

lemma *recursor-0*: *recursor*(*a*,*b*,*0*) = *a*
by (*rule nat-case-0* [*THEN recursor-lemma*])

lemma *recursor-succ*: *recursor*(*a*,*b*,*succ*(*m*)) = *b*(*m*, *recursor*(*a*,*b*,*m*))
by (*rule recursor-lemma*, *simp*)

lemma *rec-0* [*simp*]: *rec*(*0*,*a*,*b*) = *a*
 unfolding *rec-def*
apply (*rule recursor-0*)
done

lemma *rec-succ* [*simp*]: *rec*(*succ*(*m*),*a*,*b*) = *b*(*m*, *rec*(*m*,*a*,*b*))
 unfolding *rec-def*
apply (*rule recursor-succ*)
done

lemma *rec-type*:
  $[\![n \in nat;$
    $a \in C(0);$
    $\bigwedge m\ z.\ [\![m \in nat;\ \ z \in C(m)]\!] \Longrightarrow b(m,z){:}\ C(succ(m))]\!]$
  $\Longrightarrow rec(n,a,b) \in C(n)$
by (*erule nat-induct*, *auto*)

end

19 Partial and Total Orderings: Basic Definitions and Properties

theory *Order* **imports** *WF Perm* **begin**

We adopt the following convention: *ord* is used for strict orders and *order* is used for their reflexive counterparts.

definition
  part-ord :: $[i,i]\Rightarrow o$ **where**
  $part\text{-}ord(A,r) \equiv irrefl(A,r) \wedge trans[A](r)$

definition
  linear :: $[i,i]\Rightarrow o$ **where**
  $linear(A,r) \equiv (\forall x{\in}A.\ \forall y{\in}A.\ \langle x,y\rangle{:}r \mid x{=}y \mid \langle y,x\rangle{:}r)$

definition
  tot-ord :: $[i,i]\Rightarrow o$ **where**
  $tot\text{-}ord(A,r) \equiv part\text{-}ord(A,r) \wedge linear(A,r)$

definition
  $preorder\text{-}on(A,\ r) \equiv refl(A,\ r) \wedge trans[A](r)$

definition
  $partial\text{-}order\text{-}on(A,\ r) \equiv preorder\text{-}on(A,\ r) \wedge antisym(r)$

abbreviation
  $Preorder(r) \equiv preorder\text{-}on(field(r),\ r)$

abbreviation
  $Partial\text{-}order(r) \equiv partial\text{-}order\text{-}on(field(r),\ r)$

definition
  well-ord :: $[i,i]\Rightarrow o$ **where**
  $well\text{-}ord(A,r) \equiv tot\text{-}ord(A,r) \wedge wf[A](r)$

definition
  mono-map :: $[i,i,i,i]\Rightarrow i$ **where**
  $mono\text{-}map(A,r,B,s) \equiv$

{f ∈ A−>B. ∀x∈A. ∀y∈A. ⟨x,y⟩:r ⟶ <f‘x,f‘y>:s}

definition
ord-iso :: [i,i,i,i]⇒i (‹(‹notation=‹infix ord-iso››⟨-, -⟩ ≅/ ⟨-, -⟩)› 51) **where**
⟨A,r⟩ ≅ ⟨B,s⟩ ≡
{f ∈ bij(A,B). ∀x∈A. ∀y∈A. ⟨x,y⟩:r ⟷ <f‘x,f‘y>:s}

definition
pred :: [i,i,i]⇒i **where**
pred(A,x,r) ≡ {y ∈ A. ⟨y,x⟩:r}

definition
ord-iso-map :: [i,i,i,i]⇒i **where**
ord-iso-map(A,r,B,s) ≡
⋃x∈A. ⋃y∈B. ⋃f ∈ ord-iso(pred(A,x,r), r, pred(B,y,s), s). {⟨x,y⟩}

definition
first :: [i, i, i] ⇒ o **where**
first(u, X, R) ≡ u ∈ X ∧ (∀v∈X. v≠u ⟶ ⟨u,v⟩ ∈ R)

19.1 Immediate Consequences of the Definitions

lemma *part-ord-Imp-asym*:
part-ord(A,r) ⟹ asym(r ∩ A*A)
by (unfold part-ord-def irrefl-def trans-on-def asym-def, blast)

lemma *linearE*:
⟦linear(A,r); x ∈ A; y ∈ A;
⟨x,y⟩:r ⟹ P; x=y ⟹ P; ⟨y,x⟩:r ⟹ P⟧
⟹ P
by (simp add: linear-def, blast)

lemma *well-ordI*:
⟦wf[A](r); linear(A,r)⟧ ⟹ well-ord(A,r)
apply (simp add: irrefl-def part-ord-def tot-ord-def
trans-on-def well-ord-def wf-on-not-refl)
apply (fast elim: linearE wf-on-asym wf-on-chain3)
done

lemma *well-ord-is-wf*:
well-ord(A,r) ⟹ wf[A](r)
by (unfold well-ord-def, safe)

lemma *well-ord-is-trans-on*:
well-ord(A,r) ⟹ trans[A](r)
by (unfold well-ord-def tot-ord-def part-ord-def, safe)

lemma *well-ord-is-linear*: *well-ord*(*A*,*r*) $\Longrightarrow$ *linear*(*A*,*r*)
by (*unfold well-ord-def tot-ord-def*, *blast*)

lemma *pred-iff*: $y \in pred(A,x,r) \longleftrightarrow \langle y,x\rangle{:}r \wedge y \in A$
by (*unfold pred-def*, *blast*)

lemmas *predI* = *conjI* [*THEN pred-iff* [*THEN iffD2*]]

lemma *predE*: $[\![y \in pred(A,x,r);\ [\![y \in A;\ \langle y,x\rangle{:}r]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*simp add*: *pred-def*)

lemma *pred-subset-under*: $pred(A,x,r) \subseteq r -``\ \{x\}$
by (*simp add*: *pred-def*, *blast*)

lemma *pred-subset*: $pred(A,x,r) \subseteq A$
by (*simp add*: *pred-def*, *blast*)

lemma *pred-pred-eq*:
$pred(pred(A,x,r),\ y,\ r) = pred(A,x,r) \cap pred(A,y,r)$
by (*simp add*: *pred-def*, *blast*)

lemma *trans-pred-pred-eq*:
$[\![trans[A](r);\ \langle y,x\rangle{:}r;\ x \in A;\ y \in A]\!]$
$\Longrightarrow pred(pred(A,x,r),\ y,\ r) = pred(A,y,r)$
by (*unfold trans-on-def pred-def*, *blast*)

19.2 Restricting an Ordering's Domain

lemma *part-ord-subset*:
$[\![part\text{-}ord(A,r);\ B{<=}A]\!] \Longrightarrow part\text{-}ord(B,r)$
by (*unfold part-ord-def irrefl-def trans-on-def*, *blast*)

lemma *linear-subset*:
$[\![linear(A,r);\ B{<=}A]\!] \Longrightarrow linear(B,r)$
by (*unfold linear-def*, *blast*)

lemma *tot-ord-subset*:
$[\![tot\text{-}ord(A,r);\ B{<=}A]\!] \Longrightarrow tot\text{-}ord(B,r)$
unfolding *tot-ord-def*
apply (*fast elim*!: *part-ord-subset linear-subset*)
done

lemma *well-ord-subset*:
$[\![well\text{-}ord(A,r);\ B{<=}A]\!] \Longrightarrow well\text{-}ord(B,r)$
unfolding *well-ord-def*

apply (*fast elim*!: *tot-ord-subset wf-on-subset-A*)
done

lemma *irrefl-Int-iff*: *irrefl*(A,r ∩ A∗A) ⟷ *irrefl*(A,r)
by (*unfold irrefl-def*, *blast*)

lemma *trans-on-Int-iff*: *trans*[A](r ∩ A∗A) ⟷ *trans*[A](r)
by (*unfold trans-on-def*, *blast*)

lemma *part-ord-Int-iff*: *part-ord*(A,r ∩ A∗A) ⟷ *part-ord*(A,r)
  unfolding *part-ord-def*
apply (*simp add*: *irrefl-Int-iff trans-on-Int-iff*)
done

lemma *linear-Int-iff*: *linear*(A,r ∩ A∗A) ⟷ *linear*(A,r)
by (*unfold linear-def*, *blast*)

lemma *tot-ord-Int-iff*: *tot-ord*(A,r ∩ A∗A) ⟷ *tot-ord*(A,r)
  unfolding *tot-ord-def*
apply (*simp add*: *part-ord-Int-iff linear-Int-iff*)
done

lemma *wf-on-Int-iff*: *wf*[A](r ∩ A∗A) ⟷ *wf*[A](r)
apply (*unfold wf-on-def wf-def*, *fast*)
done

lemma *well-ord-Int-iff*: *well-ord*(A,r ∩ A∗A) ⟷ *well-ord*(A,r)
  unfolding *well-ord-def*
apply (*simp add*: *tot-ord-Int-iff wf-on-Int-iff*)
done

19.3 Empty and Unit Domains

lemma *wf-on-any-0*: *wf*[A](*0*)
by (*simp add*: *wf-on-def wf-def*, *fast*)

19.3.1 Relations over the Empty Set

lemma *irrefl-0*: *irrefl*(*0*,r)
by (*unfold irrefl-def*, *blast*)

lemma *trans-on-0*: *trans*[*0*](r)
by (*unfold trans-on-def*, *blast*)

lemma *part-ord-0*: *part-ord*(*0*,r)
  unfolding *part-ord-def*
apply (*simp add*: *irrefl-0 trans-on-0*)

done

lemma *linear-0*: *linear*(*0*,*r*)
by (*unfold linear-def*, *blast*)

lemma *tot-ord-0*: *tot-ord*(*0*,*r*)
  unfolding *tot-ord-def*
apply (*simp add*: *part-ord-0 linear-0*)
done

lemma *wf-on-0*: *wf*[*0*](*r*)
by (*unfold wf-on-def wf-def*, *blast*)

lemma *well-ord-0*: *well-ord*(*0*,*r*)
  unfolding *well-ord-def*
apply (*simp add*: *tot-ord-0 wf-on-0*)
done

19.3.2 The Empty Relation Well-Orders the Unit Set

by Grabczewski

lemma *tot-ord-unit*: *tot-ord*({*a*},*0*)
by (*simp add*: *irrefl-def trans-on-def part-ord-def linear-def tot-ord-def*)

lemma *well-ord-unit*: *well-ord*({*a*},*0*)
  unfolding *well-ord-def*
apply (*simp add*: *tot-ord-unit wf-on-any-0*)
done

19.4 Order-Isomorphisms

Suppes calls them "similarities"

lemma *mono-map-is-fun*: $f \in$ *mono-map*(A,r,B,s) $\Longrightarrow f \in A{-}{>}B$
by (*simp add*: *mono-map-def*)

lemma *mono-map-is-inj*:
    ⟦*linear*(A,r); *wf*[B](s); $f \in$ *mono-map*(A,r,B,s)⟧ $\Longrightarrow f \in$ *inj*(A,B)
apply (*unfold mono-map-def inj-def*, *clarify*)
apply (*erule-tac* x=w **and** y=x **in** *linearE*, *assumption*+)
apply (*force intro*: *apply-type dest*: *wf-on-not-refl*)+
done

lemma *ord-isoI*:
    ⟦$f \in$ *bij*(A, B);
        $\bigwedge x\ y.$ ⟦$x \in A$; $y \in A$⟧ $\Longrightarrow \langle x,\ y\rangle \in r \longleftrightarrow$ <f‘x, f‘y> $\in s$⟧
     $\Longrightarrow f \in$ *ord-iso*(A,r,B,s)
by (*simp add*: *ord-iso-def*)

lemma *ord-iso-is-mono-map*:
  f ∈ *ord-iso*(*A*,*r*,*B*,*s*) ⟹ *f* ∈ *mono-map*(*A*,*r*,*B*,*s*)
apply (*simp add*: *ord-iso-def mono-map-def*)
apply (*blast dest*!: *bij-is-fun*)
done

lemma *ord-iso-is-bij*:
  f ∈ *ord-iso*(*A*,*r*,*B*,*s*) ⟹ *f* ∈ *bij*(*A*,*B*)
by (*simp add*: *ord-iso-def*)

lemma *ord-iso-apply*:
  ⟦*f* ∈ *ord-iso*(*A*,*r*,*B*,*s*); ⟨*x*,*y*⟩: *r*; *x* ∈ *A*; *y* ∈ *A*⟧ ⟹ <*f*‘*x*, *f*‘*y*> ∈ *s*
by (*simp add*: *ord-iso-def*)

lemma *ord-iso-converse*:
  ⟦*f* ∈ *ord-iso*(*A*,*r*,*B*,*s*); ⟨*x*,*y*⟩: *s*; *x* ∈ *B*; *y* ∈ *B*⟧
   ⟹ <*converse*(*f*) ‘ *x*, *converse*(*f*) ‘ *y*> ∈ *r*
apply (*simp add*: *ord-iso-def*, *clarify*)
apply (*erule bspec* [*THEN bspec*, *THEN iffD2*])
apply (*erule asm-rl bij-converse-bij* [*THEN bij-is-fun*, *THEN apply-type*])+
apply (*auto simp add*: *right-inverse-bij*)
done

lemma *ord-iso-refl*: *id*(*A*): *ord-iso*(*A*,*r*,*A*,*r*)
by (*rule id-bij* [*THEN ord-isoI*], *simp*)

lemma *ord-iso-sym*: *f* ∈ *ord-iso*(*A*,*r*,*B*,*s*) ⟹ *converse*(*f*): *ord-iso*(*B*,*s*,*A*,*r*)
apply (*simp add*: *ord-iso-def*)
apply (*auto simp add*: *right-inverse-bij bij-converse-bij*
          bij-is-fun [*THEN apply-funtype*])
done

lemma *mono-map-trans*:
  ⟦*g* ∈ *mono-map*(*A*,*r*,*B*,*s*); *f* ∈ *mono-map*(*B*,*s*,*C*,*t*)⟧
   ⟹ (*f O g*): *mono-map*(*A*,*r*,*C*,*t*)
 unfolding *mono-map-def*
apply (*auto simp add*: *comp-fun*)
done

lemma *ord-iso-trans*:
  ⟦*g* ∈ *ord-iso*(*A*,*r*,*B*,*s*); *f* ∈ *ord-iso*(*B*,*s*,*C*,*t*)⟧

⟹ (*f O g*): *ord-iso*(*A*,*r*,*C*,*t*)
apply (*unfold ord-iso-def*, *clarify*)
apply (*frule bij-is-fun* [*of f*])
apply (*frule bij-is-fun* [*of g*])
apply (*auto simp add*: *comp-bij*)
done

lemma *mono-ord-isoI*:
⟦*f* ∈ *mono-map*(*A*,*r*,*B*,*s*); *g* ∈ *mono-map*(*B*,*s*,*A*,*r*);
f O g = *id*(*B*); *g O f* = *id*(*A*)⟧ ⟹ *f* ∈ *ord-iso*(*A*,*r*,*B*,*s*)
apply (*simp add*: *ord-iso-def mono-map-def*, *safe*)
apply (*intro fg-imp-bijective*, *auto*)
apply (*subgoal-tac* <*g*' (*f*'*x*), *g*' (*f*'*y*) > ∈ *r*)
apply (*simp add*: *comp-eq-id-iff* [*THEN iffD1*])
apply (*blast intro*: *apply-funtype*)
done

lemma *well-ord-mono-ord-isoI*:
⟦*well-ord*(*A*,*r*); *well-ord*(*B*,*s*);
f ∈ *mono-map*(*A*,*r*,*B*,*s*); *converse*(*f*): *mono-map*(*B*,*s*,*A*,*r*)⟧
⟹ *f* ∈ *ord-iso*(*A*,*r*,*B*,*s*)
apply (*intro mono-ord-isoI*, *auto*)
apply (*frule mono-map-is-fun* [*THEN fun-is-rel*])
apply (*erule converse-converse* [*THEN subst*], *rule left-comp-inverse*)
apply (*blast intro*: *left-comp-inverse mono-map-is-inj well-ord-is-linear*
well-ord-is-wf)+
done

lemma *part-ord-ord-iso*:
⟦*part-ord*(*B*,*s*); *f* ∈ *ord-iso*(*A*,*r*,*B*,*s*)⟧ ⟹ *part-ord*(*A*,*r*)
apply (*simp add*: *part-ord-def irrefl-def trans-on-def ord-iso-def*)
apply (*fast intro*: *bij-is-fun* [*THEN apply-type*])
done

lemma *linear-ord-iso*:
⟦*linear*(*B*,*s*); *f* ∈ *ord-iso*(*A*,*r*,*B*,*s*)⟧ ⟹ *linear*(*A*,*r*)
apply (*simp add*: *linear-def ord-iso-def*, *safe*)
apply (*drule-tac x1* = *f*'*x* **and** *x* = *f*'*y* **in** *bspec* [*THEN bspec*])
apply (*safe elim*!: *bij-is-fun* [*THEN apply-type*])
apply (*drule-tac t* = (') (*converse* (*f*)) **in** *subst-context*)
apply (*simp add*: *left-inverse-bij*)
done

lemma *wf-on-ord-iso*:

⟦wf[B](s); f ∈ ord-iso(A,r,B,s)⟧ ⟹ wf[A](r)
apply (*simp add*: *wf-on-def wf-def ord-iso-def*, *safe*)
apply (*drule-tac x* = {*f*‘*z*. *z* ∈ *Z* ∩ *A*} **in** *spec*)
apply (*safe intro*!: *equalityI*)
apply (*blast dest*!: *equalityD1 intro*: *bij-is-fun* [*THEN apply-type*])+
done

lemma *well-ord-ord-iso*:
⟦well-ord(B,s); f ∈ ord-iso(A,r,B,s)⟧ ⟹ well-ord(A,r)
unfolding *well-ord-def tot-ord-def*
apply (*fast elim*!: *part-ord-ord-iso linear-ord-iso wf-on-ord-iso*)
done

19.5 Main results of Kunen, Chapter 1 section 6

lemma *well-ord-iso-subset-lemma*:
⟦well-ord(A,r); f ∈ ord-iso(A,r, A′,r); A′<= A; y ∈ A⟧
⟹ ¬ <f‘y, y>: r
apply (*simp add*: *well-ord-def ord-iso-def*)
apply (*elim conjE CollectE*)
apply (*rule-tac a*=*y* **in** *wf-on-induct*, *assumption*+)
apply (*blast dest*: *bij-is-fun* [*THEN apply-type*])
done

lemma *well-ord-iso-predE*:
⟦well-ord(A,r); f ∈ ord-iso(A, r, pred(A,x,r), r); x ∈ A⟧ ⟹ P
apply (*insert well-ord-iso-subset-lemma* [*of A r f pred*(*A*,*x*,*r*) *x*])
apply (*simp add*: *pred-subset*)

apply (*drule ord-iso-is-bij* [*THEN bij-is-fun*, *THEN apply-type*], *assumption*)

apply (*simp add*: *well-ord-def pred-def*)
done

lemma *well-ord-iso-pred-eq*:
⟦well-ord(A,r); f ∈ ord-iso(pred(A,a,r), r, pred(A,c,r), r);
a ∈ A; c ∈ A⟧ ⟹ a=c
apply (*frule well-ord-is-trans-on*)
apply (*frule well-ord-is-linear*)
apply (*erule-tac x*=*a* **and** *y*=*c* **in** *linearE*, *assumption*+)
apply (*drule ord-iso-sym*)

apply (*auto elim*!: *well-ord-subset* [*OF - pred-subset*, *THEN well-ord-iso-predE*]
intro!: *predI*
simp add: *trans-pred-pred-eq*)
done

lemma *ord-iso-image-pred*:
⟦f ∈ ord-iso(A,r,B,s); a ∈ A⟧ ⟹ f `` pred(A,a,r) = pred(B, f`a, s)
unfolding *ord-iso-def pred-def*
apply (*erule CollectE*)
apply (*simp* (*no-asm-simp*) *add*: *image-fun* [*OF bij-is-fun Collect-subset*])
apply (*rule equalityI*)
apply (*safe elim*!: *bij-is-fun* [*THEN apply-type*])
apply (*rule RepFun-eqI*)
apply (*blast intro*!: *right-inverse-bij* [*symmetric*])
apply (*auto simp add*: *right-inverse-bij bij-is-fun* [*THEN apply-funtype*])
done

lemma *ord-iso-restrict-image*:
⟦f ∈ ord-iso(A,r,B,s); C<=A⟧
⟹ restrict(f,C) ∈ ord-iso(C, r, f``C, s)
apply (*simp add*: *ord-iso-def*)
apply (*blast intro*: *bij-is-inj restrict-bij*)
done

lemma *ord-iso-restrict-pred*:
⟦f ∈ ord-iso(A,r,B,s); a ∈ A⟧
⟹ restrict(f, pred(A,a,r)) ∈ ord-iso(pred(A,a,r), r, pred(B, f`a, s), s)
apply (*simp add*: *ord-iso-image-pred* [*symmetric*])
apply (*blast intro*: *ord-iso-restrict-image elim*: *predE*)
done

lemma *well-ord-iso-preserving*:
⟦well-ord(A,r); well-ord(B,s); ⟨a,c⟩: r;
f ∈ ord-iso(pred(A,a,r), r, pred(B,b,s), s);
g ∈ ord-iso(pred(A,c,r), r, pred(B,d,s), s);
a ∈ A; c ∈ A; b ∈ B; d ∈ B⟧ ⟹ ⟨b,d⟩: s
apply (*frule ord-iso-is-bij* [*THEN bij-is-fun*, *THEN apply-type*], (*erule asm-rl predI predE*)+)
apply (*subgoal-tac b = g`a*)
apply (*simp* (*no-asm-simp*))
apply (*rule well-ord-iso-pred-eq*, *auto*)
apply (*frule ord-iso-restrict-pred*, (*erule asm-rl predI*)+)
apply (*simp add*: *well-ord-is-trans-on trans-pred-pred-eq*)
apply (*erule ord-iso-sym* [*THEN ord-iso-trans*], *assumption*)
done

lemma *well-ord-iso-unique-lemma*:
⟦well-ord(A,r);
f ∈ ord-iso(A,r, B,s); g ∈ ord-iso(A,r, B,s); y ∈ A⟧
⟹ ¬ <g`y, f`y> ∈ s

```
apply (frule well-ord-iso-subset-lemma)
apply (rule-tac f = converse (f)  and g = g in ord-iso-trans)
apply auto
apply (blast intro: ord-iso-sym)
apply (frule ord-iso-is-bij [of f])
apply (frule ord-iso-is-bij [of g])
apply (frule ord-iso-converse)
apply (blast intro!: bij-converse-bij
          intro: bij-is-fun apply-funtype)+
apply (erule notE)
apply (simp add: left-inverse-bij bij-is-fun comp-fun-apply [of - A B])
done
```

```
lemma well-ord-iso-unique: ⟦well-ord(A,r);
        f ∈ ord-iso(A,r, B,s);  g ∈ ord-iso(A,r, B,s)⟧ ⟹ f = g
apply (rule fun-extension)
apply (erule ord-iso-is-bij [THEN bij-is-fun])+
apply (subgoal-tac f‘x ∈ B ∧ g‘x ∈ B ∧ linear(B,s))
 apply (simp add: linear-def)
 apply (blast dest: well-ord-iso-unique-lemma)
apply (blast intro: ord-iso-is-bij bij-is-fun apply-funtype
              well-ord-is-linear well-ord-ord-iso ord-iso-sym)
done
```

19.6 Towards Kunen's Theorem 6.3: Linearity of the Similarity Relation

```
lemma ord-iso-map-subset: ord-iso-map(A,r,B,s) ⊆ A*B
by (unfold ord-iso-map-def, blast)

lemma domain-ord-iso-map: domain(ord-iso-map(A,r,B,s)) ⊆ A
by (unfold ord-iso-map-def, blast)

lemma range-ord-iso-map: range(ord-iso-map(A,r,B,s)) ⊆ B
by (unfold ord-iso-map-def, blast)

lemma converse-ord-iso-map:
   converse(ord-iso-map(A,r,B,s)) = ord-iso-map(B,s,A,r)
  unfolding ord-iso-map-def
apply (blast intro: ord-iso-sym)
done

lemma function-ord-iso-map:
   well-ord(B,s) ⟹ function(ord-iso-map(A,r,B,s))
  unfolding ord-iso-map-def function-def
apply (blast intro: well-ord-iso-pred-eq ord-iso-sym ord-iso-trans)
done
```

lemma *ord-iso-map-fun*: *well-ord*(*B*,*s*) $\Longrightarrow$ *ord-iso-map*(*A*,*r*,*B*,*s*)
  $\in$ *domain*(*ord-iso-map*(*A*,*r*,*B*,*s*)) −> *range*(*ord-iso-map*(*A*,*r*,*B*,*s*))
by (*simp add*: *Pi-iff function-ord-iso-map*
  ord-iso-map-subset [*THEN domain-times-range*])

lemma *ord-iso-map-mono-map*:
  ⟦*well-ord*(*A*,*r*); *well-ord*(*B*,*s*)⟧
  $\Longrightarrow$ *ord-iso-map*(*A*,*r*,*B*,*s*)
  $\in$ *mono-map*(*domain*(*ord-iso-map*(*A*,*r*,*B*,*s*)), *r*,
  range(*ord-iso-map*(*A*,*r*,*B*,*s*)), *s*)
 unfolding *mono-map-def*
apply (*simp* (*no-asm-simp*) *add*: *ord-iso-map-fun*)
apply *safe*
apply (*subgoal-tac* *x* $\in$ *A* $\wedge$ *ya*:*A* $\wedge$ *y* $\in$ *B* $\wedge$ *yb*:*B*)
 apply (*simp add*: *apply-equality* [*OF* - *ord-iso-map-fun*])
  unfolding *ord-iso-map-def*
 apply (*blast intro*: *well-ord-iso-preserving*, *blast*)
done

lemma *ord-iso-map-ord-iso*:
  ⟦*well-ord*(*A*,*r*); *well-ord*(*B*,*s*)⟧ $\Longrightarrow$ *ord-iso-map*(*A*,*r*,*B*,*s*)
  $\in$ *ord-iso*(*domain*(*ord-iso-map*(*A*,*r*,*B*,*s*)), *r*,
  range(*ord-iso-map*(*A*,*r*,*B*,*s*)), *s*)
apply (*rule well-ord-mono-ord-isoI*)
  prefer *4*
  apply (*rule converse-ord-iso-map* [*THEN subst*])
  apply (*simp add*: *ord-iso-map-mono-map*
  ord-iso-map-subset [*THEN converse-converse*])
apply (*blast intro*!: *domain-ord-iso-map range-ord-iso-map*
  intro: *well-ord-subset ord-iso-map-mono-map*)+
done

lemma *domain-ord-iso-map-subset*:
  ⟦*well-ord*(*A*,*r*); *well-ord*(*B*,*s*);
  a $\in$ *A*; *a* $\notin$ *domain*(*ord-iso-map*(*A*,*r*,*B*,*s*))⟧
  $\Longrightarrow$ *domain*(*ord-iso-map*(*A*,*r*,*B*,*s*)) $\subseteq$ *pred*(*A*, *a*, *r*)
 unfolding *ord-iso-map-def*
apply (*safe intro*!: *predI*)

apply (*simp* (*no-asm-simp*))
apply (*frule-tac A* = *A* **in** *well-ord-is-linear*)
apply (*rename-tac b y f*)
apply (*erule-tac x*=*b* **and** *y*=*a* **in** *linearE*, *assumption*+)

apply *clarify*
apply *blast*

```
apply (frule ord-iso-is-bij [THEN bij-is-fun, THEN apply-type],
       (erule asm-rl predI predE)+)
apply (frule ord-iso-restrict-pred)
 apply (simp add: pred-iff)
apply (simp split: split-if-asm
          add: well-ord-is-trans-on trans-pred-pred-eq domain-UN domain-Union,
blast)
done
```

```
lemma domain-ord-iso-map-cases:
     ⟦well-ord(A,r);  well-ord(B,s)⟧
      ⟹ domain(ord-iso-map(A,r,B,s)) = A |
          (∃x∈A. domain(ord-iso-map(A,r,B,s)) = pred(A,x,r))
apply (frule well-ord-is-wf)
  unfolding wf-on-def wf-def
apply (drule-tac x = A−domain (ord-iso-map (A,r,B,s)) in spec)
apply safe

apply (rule domain-ord-iso-map [THEN equalityI])
apply (erule Diff-eq-0-iff [THEN iffD1])

apply (blast del: domainI subsetI
             elim!: predE
             intro!: domain-ord-iso-map-subset
             intro: subsetI)+
done
```

```
lemma range-ord-iso-map-cases:
    ⟦well-ord(A,r);  well-ord(B,s)⟧
     ⟹ range(ord-iso-map(A,r,B,s)) = B |
         (∃y∈B. range(ord-iso-map(A,r,B,s)) = pred(B,y,s))
apply (rule converse-ord-iso-map [THEN subst])
apply (simp add: domain-ord-iso-map-cases)
done
```

Kunen's Theorem 6.3: Fundamental Theorem for Well-Ordered Sets

```
theorem well-ord-trichotomy:
   ⟦well-ord(A,r);  well-ord(B,s)⟧
    ⟹ ord-iso-map(A,r,B,s) ∈ ord-iso(A, r, B, s) |
        (∃x∈A. ord-iso-map(A,r,B,s) ∈ ord-iso(pred(A,x,r), r, B, s)) |
        (∃y∈B. ord-iso-map(A,r,B,s) ∈ ord-iso(A, r, pred(B,y,s), s))
apply (frule-tac B = B in domain-ord-iso-map-cases, assumption)
apply (frule-tac B = B in range-ord-iso-map-cases, assumption)
apply (drule ord-iso-map-ord-iso, assumption)
apply (elim disjE bexE)
   apply (simp-all add: bexI)
```

apply (*rule wf-on-not-refl* [*THEN notE*])
 apply (*erule well-ord-is-wf*)
 apply *assumption*
apply (*subgoal-tac* ⟨*x*,*y*⟩: *ord-iso-map* (*A*,*r*,*B*,*s*))
 apply (*drule rangeI*)
 apply (*simp add*: *pred-def*)
apply (*unfold ord-iso-map-def*, *blast*)
done

19.7 Miscellaneous Results by Krzysztof Grabczewski

lemma *irrefl-converse*: *irrefl*(*A*,*r*) ⟹ *irrefl*(*A*,*converse*(*r*))
by (*unfold irrefl-def*, *blast*)

lemma *trans-on-converse*: *trans*[*A*](*r*) ⟹ *trans*[*A*](*converse*(*r*))
by (*unfold trans-on-def*, *blast*)

lemma *part-ord-converse*: *part-ord*(*A*,*r*) ⟹ *part-ord*(*A*,*converse*(*r*))
 unfolding *part-ord-def*
apply (*blast intro*!: *irrefl-converse trans-on-converse*)
done

lemma *linear-converse*: *linear*(*A*,*r*) ⟹ *linear*(*A*,*converse*(*r*))
by (*unfold linear-def*, *blast*)

lemma *tot-ord-converse*: *tot-ord*(*A*,*r*) ⟹ *tot-ord*(*A*,*converse*(*r*))
 unfolding *tot-ord-def*
apply (*blast intro*!: *part-ord-converse linear-converse*)
done

lemma *first-is-elem*: *first*(*b*,*B*,*r*) ⟹ *b* ∈ *B*
by (*unfold first-def*, *blast*)

lemma *well-ord-imp-ex1-first*:
  ⟦*well-ord*(*A*,*r*); *B*<=*A*; *B*≠*0*⟧ ⟹ (∃!*b*. *first*(*b*,*B*,*r*))
 unfolding *well-ord-def wf-on-def wf-def first-def*
apply (*elim conjE allE disjE*, *blast*)
apply (*erule bexE*)
apply (*rule-tac a* = *x* **in** *ex1I*, *auto*)
apply (*unfold tot-ord-def linear-def*, *blast*)
done

lemma *the-first-in*:
 ⟦*well-ord*(*A*,*r*); *B*<=*A*; *B*≠*0*⟧ ⟹ (*THE b*. *first*(*b*,*B*,*r*)) ∈ *B*
apply (*drule well-ord-imp-ex1-first*, *assumption*+)
apply (*rule first-is-elem*)

apply (*erule theI*)
done

19.8 Lemmas for the Reflexive Orders

lemma *subset-vimage-vimage-iff*:
  ⟦*Preorder*(*r*); *A* ⊆ *field*(*r*); *B* ⊆ *field*(*r*)⟧ ⟹
  r −‘‘ *A* ⊆ *r* −‘‘ *B* ⟷ (∀ *a*∈*A*. ∃ *b*∈*B*. ⟨*a*, *b*⟩ ∈ *r*)
  apply (*auto simp*: *subset-def preorder-on-def refl-def vimage-def image-def*)
   apply *blast*
  unfolding *trans-on-def*
  apply (*erule-tac P* = (λ*x*. ∀ *y*∈*field*(*r*).
        ∀ *z*∈*field*(*r*). ⟨*x*, *y*⟩ ∈ *r* ⟶ ⟨*y*, *z*⟩ ∈ *r* ⟶ ⟨*x*, *z*⟩ ∈ *r*) **for** *r* **in** *rev-ballE*)

   apply *best*
  apply *blast*
  done

lemma *subset-vimage1-vimage1-iff*:
  ⟦*Preorder*(*r*); *a* ∈ *field*(*r*); *b* ∈ *field*(*r*)⟧ ⟹
  r −‘‘ {*a*} ⊆ *r* −‘‘ {*b*} ⟷ ⟨*a*, *b*⟩ ∈ *r*
  by (*simp add*: *subset-vimage-vimage-iff*)

lemma *Refl-antisym-eq-Image1-Image1-iff*:
  ⟦*refl*(*field*(*r*), *r*); *antisym*(*r*); *a* ∈ *field*(*r*); *b* ∈ *field*(*r*)⟧ ⟹
  r ‘‘ {*a*} = *r* ‘‘ {*b*} ⟷ *a* = *b*
  apply *rule*
   apply (*frule equality-iffD*)
   apply (*drule equality-iffD*)
   apply (*simp add*: *antisym-def refl-def*)
   apply *best*
  apply (*simp add*: *antisym-def refl-def*)
  done

lemma *Partial-order-eq-Image1-Image1-iff*:
  ⟦*Partial-order*(*r*); *a* ∈ *field*(*r*); *b* ∈ *field*(*r*)⟧ ⟹
  r ‘‘ {*a*} = *r* ‘‘ {*b*} ⟷ *a* = *b*
  by (*simp add*: *partial-order-on-def preorder-on-def*
    Refl-antisym-eq-Image1-Image1-iff)

lemma *Refl-antisym-eq-vimage1-vimage1-iff*:
  ⟦*refl*(*field*(*r*), *r*); *antisym*(*r*); *a* ∈ *field*(*r*); *b* ∈ *field*(*r*)⟧ ⟹
  r −‘‘ {*a*} = *r* −‘‘ {*b*} ⟷ *a* = *b*
  apply *rule*
   apply (*frule equality-iffD*)
   apply (*drule equality-iffD*)
   apply (*simp add*: *antisym-def refl-def*)
   apply *best*
  apply (*simp add*: *antisym-def refl-def*)

done

lemma *Partial-order-eq-vimage1-vimage1-iff*:
$[\![Partial\text{-}order(r);\ a \in field(r);\ b \in field(r)]\!] \Longrightarrow$
$r -`` \{a\} = r -`` \{b\} \longleftrightarrow a = b$
by (*simp add*: *partial-order-on-def preorder-on-def*
Refl-antisym-eq-vimage1-vimage1-iff)

end

20 Combining Orderings: Foundations of Ordinal Arithmetic

theory *OrderArith* **imports** *Order Sum Ordinal* **begin**

definition

radd :: $[i,i,i,i] \Rightarrow i$ **where**
$radd(A,r,B,s) \equiv$
$\{z: (A+B) * (A+B).$
$(\exists x\ y.\ z = \langle Inl(x),\ Inr(y)\rangle)\ \ |$
$(\exists x'\ x.\ z = \langle Inl(x'),\ Inl(x)\rangle \wedge \langle x',x\rangle{:}r)\ \ |$
$(\exists y'\ y.\ z = \langle Inr(y'),\ Inr(y)\rangle \wedge \langle y',y\rangle{:}s)\}$

definition

rmult :: $[i,i,i,i] \Rightarrow i$ **where**
$rmult(A,r,B,s) \equiv$
$\{z: (A*B) * (A*B).$
$\exists x'\ y'\ x\ y.\ z = \langle\langle x',y'\rangle,\ \langle x,y\rangle\rangle \wedge$
$(\langle x',x\rangle{:}\ r\ |\ (x'{=}x \wedge \langle y',y\rangle{:}\ s))\}$

definition

rvimage :: $[i,i,i] \Rightarrow i$ **where**
$rvimage(A,f,r) \equiv \{z \in A*A.\ \exists x\ y.\ z = \langle x,y\rangle \wedge \langle f`x,f`y\rangle{:}\ r\}$

definition
measure :: $[i,\ i \Rightarrow i] \Rightarrow i$ **where**
$measure(A,f) \equiv \{\langle x,y\rangle{:}\ A*A.\ f(x) < f(y)\}$

20.1 Addition of Relations – Disjoint Sum

20.1.1 Rewrite rules. Can be used to obtain introduction rules

lemma *radd-Inl-Inr-iff* [*iff*]:
$\langle Inl(a),\ Inr(b)\rangle \in radd(A,r,B,s) \ \longleftrightarrow\ a \in A \wedge b \in B$
by (*unfold radd-def*, *blast*)

lemma *radd-Inl-iff* [*iff*]:
$\langle Inl(a'),\ Inl(a)\rangle \in radd(A,r,B,s) \longleftrightarrow a'{:}A \wedge a \in A \wedge \langle a',a\rangle{:}r$
by (*unfold radd-def*, *blast*)

lemma *radd-Inr-iff* [*iff*]:
$\langle Inr(b'),\ Inr(b)\rangle \in radd(A,r,B,s) \longleftrightarrow b'{:}B \wedge b \in B \wedge \langle b',b\rangle{:}s$
by (*unfold radd-def*, *blast*)

lemma *radd-Inr-Inl-iff* [*simp*]:
$\langle Inr(b),\ Inl(a)\rangle \in radd(A,r,B,s) \longleftrightarrow False$
by (*unfold radd-def*, *blast*)

declare *radd-Inr-Inl-iff* [*THEN iffD1*, *dest*!]

20.1.2 Elimination Rule

lemma *raddE*:
$[\![\langle p',p\rangle \in radd(A,r,B,s);$
$\bigwedge x\ y.\ [\![p'=Inl(x);\ x \in A;\ p=Inr(y);\ y \in B]\!] \Longrightarrow Q;$
$\bigwedge x'\ x.\ [\![p'=Inl(x');\ p=Inl(x);\ \langle x',x\rangle{:}\ r;\ x'{:}A;\ x \in A]\!] \Longrightarrow Q;$
$\bigwedge y'\ y.\ [\![p'=Inr(y');\ p=Inr(y);\ \langle y',y\rangle{:}\ s;\ y'{:}B;\ y \in B]\!] \Longrightarrow Q$
$]\!] \Longrightarrow Q$
by (*unfold radd-def*, *blast*)

20.1.3 Type checking

lemma *radd-type*: $radd(A,r,B,s) \subseteq (A+B) * (A+B)$
unfolding *radd-def*
apply (*rule Collect-subset*)
done

lemmas *field-radd* = *radd-type* [*THEN field-rel-subset*]

20.1.4 Linearity

lemma *linear-radd*:
$[\![linear(A,r);\ \ linear(B,s)]\!] \Longrightarrow linear(A+B,radd(A,r,B,s))$
by (*unfold linear-def*, *blast*)

20.1.5 Well-foundedness

lemma *wf-on-radd*: $[\![wf[A](r);\ \ wf[B](s)]\!] \Longrightarrow wf[A+B](radd(A,r,B,s))$
apply (*rule wf-onI2*)
apply (*subgoal-tac* $\forall x{\in}A.\ Inl\ (x) \in Ba$)
— Proving the lemma, which is needed twice!
prefer *2*
apply (*erule-tac* $V = y \in A + B$ **in** *thin-rl*)
apply (*rule-tac ballI*)
apply (*erule-tac* $r = r$ **and** $a = x$ **in** *wf-on-induct*, *assumption*)
apply *blast*

Returning to main part of proof

apply *safe*
apply *blast*
apply (*erule-tac r = s* **and** *a = ya* **in** *wf-on-induct*, *assumption*, *blast*)
done

lemma *wf-radd*: $[\![wf(r);\ \ wf(s)]\!] \Longrightarrow wf(radd(field(r),r,field(s),s))$
apply (*simp add*: *wf-iff-wf-on-field*)
apply (*rule wf-on-subset-A* [*OF - field-radd*])
apply (*blast intro*: *wf-on-radd*)
done

lemma *well-ord-radd*:
$[\![well\text{-}ord(A,r);\ \ well\text{-}ord(B,s)]\!] \Longrightarrow well\text{-}ord(A+B,\ radd(A,r,B,s))$
apply (*rule well-ordI*)
apply (*simp add*: *well-ord-def wf-on-radd*)
apply (*simp add*: *well-ord-def tot-ord-def linear-radd*)
done

20.1.6 An *ord-iso* congruence law

lemma *sum-bij*:
$[\![f \in bij(A,C);\ \ g \in bij(B,D)]\!]$
$\Longrightarrow (\lambda z{\in}A{+}B.\ case(\lambda x.\ Inl(f`x),\ \lambda y.\ Inr(g`y),\ z)) \in bij(A{+}B,\ C{+}D)$
apply (*rule-tac d = case* ($\lambda x.$ *Inl* (*converse*(f)`$x$), $\lambda y.$ *Inr*(*converse*($g$)`y))
in *lam-bijective*)
apply (*typecheck add*: *bij-is-inj inj-is-fun*)
apply (*auto simp add*: *left-inverse-bij right-inverse-bij*)
done

lemma *sum-ord-iso-cong*:
$[\![f \in ord\text{-}iso(A,r,A',r');\ \ g \in ord\text{-}iso(B,s,B',s')]\!] \Longrightarrow$
$(\lambda z{\in}A{+}B.\ case(\lambda x.\ Inl(f`x),\ \lambda y.\ Inr(g`y),\ z))$
$\in ord\text{-}iso(A{+}B,\ radd(A,r,B,s),\ A'{+}B',\ radd(A',r',B',s'))$
unfolding *ord-iso-def*
apply (*safe intro*!: *sum-bij*)

apply (*auto cong add*: *conj-cong simp add*: *bij-is-fun* [*THEN apply-type*])
done

lemma *sum-disjoint-bij*: $A \cap B = 0 \Longrightarrow$
$(\lambda z{\in}A{+}B.\ case(\lambda x.\ x,\ \lambda y.\ y,\ z)) \in bij(A{+}B,\ A \cup B)$
apply (*rule-tac d* = $\lambda z.$ *if* $z \in A$ *then Inl* (z) *else Inr* (z) **in** *lam-bijective*)
apply *auto*
done

20.1.7 Associativity

lemma *sum-assoc-bij*:
$(\lambda z\in(A+B)+C.\ case(case(Inl,\ \lambda y.\ Inr(Inl(y))),\ \lambda y.\ Inr(Inr(y)),\ z))$
$\in bij((A+B)+C,\ A+(B+C))$
apply (*rule-tac d* = *case* ($\lambda x.$ *Inl* (*Inl* (*x*)), *case* ($\lambda x.$ *Inl* (*Inr* (*x*)), *Inr*))
in *lam-bijective*)
apply *auto*
done

lemma *sum-assoc-ord-iso*:
$(\lambda z\in(A+B)+C.\ case(case(Inl,\ \lambda y.\ Inr(Inl(y))),\ \lambda y.\ Inr(Inr(y)),\ z))$
$\in ord\text{-}iso((A+B)+C,\ radd(A+B,\ radd(A,r,B,s),\ C,\ t),$
$A+(B+C),\ radd(A,\ r,\ B+C,\ radd(B,s,C,t)))$
by (*rule sum-assoc-bij* [*THEN ord-isoI*], *auto*)

20.2 Multiplication of Relations – Lexicographic Product

20.2.1 Rewrite rule. Can be used to obtain introduction rules

lemma *rmult-iff* [*iff*]:
$\langle\langle a',b'\rangle,\ \langle a,b\rangle\rangle \in rmult(A,r,B,s) \longleftrightarrow$
$(\langle a',a\rangle: r\ \wedge\ a':A\ \wedge\ a \in A\ \wedge\ b': B\ \wedge\ b \in B)\ |$
$(\langle b',b\rangle: s\ \wedge\ a'=a\ \wedge\ a \in A\ \wedge\ b': B\ \wedge\ b \in B)$

by (*unfold rmult-def*, *blast*)

lemma *rmultE*:
$[\![\langle\langle a',b'\rangle,\ \langle a,b\rangle\rangle \in rmult(A,r,B,s);$
$[\![\langle a',a\rangle: r;\ \ a':A;\ \ a \in A;\ \ b':B;\ \ b \in B]\!] \Longrightarrow Q;$
$[\![\langle b',b\rangle: s;\ \ a \in A;\ \ a'=a;\ \ b':B;\ \ b \in B]\!] \Longrightarrow Q$
$]\!] \Longrightarrow Q$
by *blast*

20.2.2 Type checking

lemma *rmult-type*: $rmult(A,r,B,s) \subseteq (A*B) * (A*B)$
by (*unfold rmult-def*, *rule Collect-subset*)

lemmas *field-rmult* = *rmult-type* [*THEN field-rel-subset*]

20.2.3 Linearity

lemma *linear-rmult*:
$[\![linear(A,r);\ \ linear(B,s)]\!] \Longrightarrow linear(A*B,rmult(A,r,B,s))$
by (*simp add*: *linear-def*, *blast*)

20.2.4 Well-foundedness

lemma *wf-on-rmult*: $[\![wf[A](r);\ \ wf[B](s)]\!] \Longrightarrow wf[A*B](rmult(A,r,B,s))$
apply (*rule wf-onI2*)

apply (*erule SigmaE*)
apply (*erule ssubst*)
apply (*subgoal-tac* ∀ *b*∈*B*. ⟨*x*,*b*⟩: *Ba*, *blast*)
apply (*erule-tac a* = *x* **in** *wf-on-induct*, *assumption*)
apply (*rule ballI*)
apply (*erule-tac a* = *b* **in** *wf-on-induct*, *assumption*)
apply (*best elim*!: *rmultE bspec* [*THEN mp*])
done

lemma *wf-rmult*: ⟦*wf*(*r*); *wf*(*s*)⟧ ⟹ *wf*(*rmult*(*field*(*r*),*r*,*field*(*s*),*s*))
apply (*simp add*: *wf-iff-wf-on-field*)
apply (*rule wf-on-subset-A* [*OF* - *field-rmult*])
apply (*blast intro*: *wf-on-rmult*)
done

lemma *well-ord-rmult*:
⟦*well-ord*(*A*,*r*); *well-ord*(*B*,*s*)⟧ ⟹ *well-ord*(*A*∗*B*, *rmult*(*A*,*r*,*B*,*s*))
apply (*rule well-ordI*)
apply (*simp add*: *well-ord-def wf-on-rmult*)
apply (*simp add*: *well-ord-def tot-ord-def linear-rmult*)
done

20.2.5 An *ord-iso* congruence law

lemma *prod-bij*:
⟦*f* ∈ *bij*(*A*,*C*); *g* ∈ *bij*(*B*,*D*)⟧
⟹ (*lam* ⟨*x*,*y*⟩:*A*∗*B*. ⟨*f*‘*x*, *g*‘*y*⟩) ∈ *bij*(*A*∗*B*, *C*∗*D*)
apply (*rule-tac d* = λ⟨*x*,*y*⟩. ⟨*converse* (*f*) ‘*x*, *converse* (*g*) ‘*y*⟩
in *lam-bijective*)
apply (*typecheck add*: *bij-is-inj inj-is-fun*)
apply (*auto simp add*: *left-inverse-bij right-inverse-bij*)
done

lemma *prod-ord-iso-cong*:
⟦*f* ∈ *ord-iso*(*A*,*r*,*A*′,*r*′); *g* ∈ *ord-iso*(*B*,*s*,*B*′,*s*′)⟧
⟹ (*lam* ⟨*x*,*y*⟩:*A*∗*B*. ⟨*f*‘*x*, *g*‘*y*⟩)
∈ *ord-iso*(*A*∗*B*, *rmult*(*A*,*r*,*B*,*s*), *A*′∗*B*′, *rmult*(*A*′,*r*′,*B*′,*s*′))
unfolding *ord-iso-def*
apply (*safe intro*!: *prod-bij*)
apply (*simp-all add*: *bij-is-fun* [*THEN apply-type*])
apply (*blast intro*: *bij-is-inj* [*THEN inj-apply-equality*])
done

lemma *singleton-prod-bij*: (λ*z*∈*A*. ⟨*x*,*z*⟩) ∈ *bij*(*A*, {*x*}∗*A*)
by (*rule-tac d* = *snd* **in** *lam-bijective*, *auto*)

lemma *singleton-prod-ord-iso*:

well-ord({x},*xr*) ⟹
(λz∈A. ⟨x,z⟩) ∈ *ord-iso*(A, r, {x}∗A, *rmult*({x}, *xr*, A, r))
apply (*rule singleton-prod-bij* [*THEN ord-isoI*])
apply (*simp* (*no-asm-simp*))
apply (*blast dest*: *well-ord-is-wf* [*THEN wf-on-not-refl*])
done

lemma *prod-sum-singleton-bij*:
a∉C ⟹
(λx∈C∗B + D. *case*(λx. x, λy.⟨a,y⟩, x))
∈ *bij*(C∗B + D, C∗B ∪ {a}∗D)
apply (*rule subst-elem*)
apply (*rule id-bij* [*THEN sum-bij*, *THEN comp-bij*])
apply (*rule singleton-prod-bij*)
apply (*rule sum-disjoint-bij*, *blast*)
apply (*simp* (*no-asm-simp*) *cong add*: *case-cong*)
apply (*rule comp-lam* [*THEN trans*, *symmetric*])
apply (*fast elim*!: *case-type*)
apply (*simp* (*no-asm-simp*) *add*: *case-case*)
done

lemma *prod-sum-singleton-ord-iso*:
⟦a ∈ A; *well-ord*(A,r)⟧ ⟹
(λx∈*pred*(A,a,r)∗B + *pred*(B,b,s). *case*(λx. x, λy.⟨a,y⟩, x))
∈ *ord-iso*(*pred*(A,a,r)∗B + *pred*(B,b,s),
radd(A∗B, *rmult*(A,r,B,s), B, s),
pred(A,a,r)∗B ∪ {a}∗*pred*(B,b,s), *rmult*(A,r,B,s))
apply (*rule prod-sum-singleton-bij* [*THEN ord-isoI*])
apply (*simp* (*no-asm-simp*) *add*: *pred-iff well-ord-is-wf* [*THEN wf-on-not-refl*])
apply (*auto elim*!: *well-ord-is-wf* [*THEN wf-on-asym*] *predE*)
done

20.2.6 Distributive law

lemma *sum-prod-distrib-bij*:
(*lam* ⟨x,z⟩:(A+B)∗C. *case*(λy. *Inl*(⟨y,z⟩), λy. *Inr*(⟨y,z⟩), x))
∈ *bij*((A+B)∗C, (A∗C)+(B∗C))
by (*rule-tac d* = *case* (λ⟨x,y⟩.⟨*Inl* (x),y⟩, λ⟨x,y⟩.⟨*Inr* (x),y⟩)
in *lam-bijective*, *auto*)

lemma *sum-prod-distrib-ord-iso*:
(*lam* ⟨x,z⟩:(A+B)∗C. *case*(λy. *Inl*(⟨y,z⟩), λy. *Inr*(⟨y,z⟩), x))
∈ *ord-iso*((A+B)∗C, *rmult*(A+B, *radd*(A,r,B,s), C, t),
(A∗C)+(B∗C), *radd*(A∗C, *rmult*(A,r,C,t), B∗C, *rmult*(B,s,C,t)))
by (*rule sum-prod-distrib-bij* [*THEN ord-isoI*], *auto*)

20.2.7 Associativity

lemma *prod-assoc-bij*:

$(lam\ \langle\langle x,y\rangle,\ z\rangle{:}(A*B)*C.\ \langle x,\langle y,z\rangle\rangle) \in bij((A*B)*C,\ A*(B*C))$
by $(rule\text{-}tac\ d = \lambda\langle x,\ \langle y,z\rangle\rangle.\ \langle\langle x,y\rangle,\ z\rangle$ **in** $lam\text{-}bijective,\ auto)$

lemma *prod-assoc-ord-iso*:
$(lam\ \langle\langle x,y\rangle,\ z\rangle{:}(A*B)*C.\ \langle x,\langle y,z\rangle\rangle)$
$\in ord\text{-}iso((A*B)*C,\ rmult(A*B,\ rmult(A,r,B,s),\ C,\ t),$
$A*(B*C),\ rmult(A,\ r,\ B*C,\ rmult(B,s,C,t)))$
by (*rule prod-assoc-bij* [*THEN ord-isoI*], *auto*)

20.3 Inverse Image of a Relation

20.3.1 Rewrite rule

lemma *rvimage-iff*: $\langle a,b\rangle \in rvimage(A,f,r) \longleftrightarrow \langle f`a,f`b\rangle{:}\ r \wedge a \in A \wedge b \in A$
by (*unfold rvimage-def*, *blast*)

20.3.2 Type checking

lemma *rvimage-type*: $rvimage(A,f,r) \subseteq A*A$
by (*unfold rvimage-def*, *rule Collect-subset*)

lemmas *field-rvimage* = *rvimage-type* [*THEN field-rel-subset*]

lemma *rvimage-converse*: $rvimage(A,f,\ converse(r)) = converse(rvimage(A,f,r))$
by (*unfold rvimage-def*, *blast*)

20.3.3 Partial Ordering Properties

lemma *irrefl-rvimage*:
$[\![f \in inj(A,B);\ \ irrefl(B,r)]\!] \Longrightarrow irrefl(A,\ rvimage(A,f,r))$
unfolding *irrefl-def rvimage-def*
apply (*blast intro*: *inj-is-fun* [*THEN apply-type*])
done

lemma *trans-on-rvimage*:
$[\![f \in inj(A,B);\ \ trans[B](r)]\!] \Longrightarrow trans[A](rvimage(A,f,r))$
unfolding *trans-on-def rvimage-def*
apply (*blast intro*: *inj-is-fun* [*THEN apply-type*])
done

lemma *part-ord-rvimage*:
$[\![f \in inj(A,B);\ \ part\text{-}ord(B,r)]\!] \Longrightarrow part\text{-}ord(A,\ rvimage(A,f,r))$
unfolding *part-ord-def*
apply (*blast intro*!: *irrefl-rvimage trans-on-rvimage*)
done

20.3.4 Linearity

lemma *linear-rvimage*:
$[\![f \in inj(A,B);\ \ linear(B,r)]\!] \Longrightarrow linear(A,rvimage(A,f,r))$

apply (*simp add*: *inj-def linear-def rvimage-iff*)
apply (*blast intro*: *apply-funtype*)
done

lemma *tot-ord-rvimage*:
  $[\![f \in inj(A,B);\ \ tot\text{-}ord(B,r)]\!] \Longrightarrow tot\text{-}ord(A,\ rvimage(A,f,r))$
 unfolding *tot-ord-def*
apply (*blast intro*!: *part-ord-rvimage linear-rvimage*)
done

20.3.5 Well-foundedness

lemma *wf-rvimage* [*intro*!]: $wf(r) \Longrightarrow wf(rvimage(A,f,r))$
apply (*simp* (*no-asm-use*) *add*: *rvimage-def wf-eq-minimal*)
apply *clarify*
apply (*subgoal-tac* $\exists w.\ w \in \{w: \{f`x.\ x \in Q\}.\ \exists x.\ x \in Q \wedge (f`x = w)\ \}$)
 apply (*erule allE*)
 apply (*erule impE*)
 apply *assumption*
 apply *blast*
apply *blast*
done

But note that the combination of *wf-imp-wf-on* and *wf-rvimage* gives $wf(r) \Longrightarrow wf[C](rvimage(A,\ f,\ r))$

lemma *wf-on-rvimage*: $[\![f \in A \rightarrow B;\ \ wf[B](r)]\!] \Longrightarrow wf[A](rvimage(A,f,r))$
apply (*rule wf-onI2*)
apply (*subgoal-tac* $\forall z{\in}A.\ f`z{=}f`y \longrightarrow z \in Ba$)
 apply *blast*
apply (*erule-tac* $a = f`y$ **in** *wf-on-induct*)
 apply (*blast intro*!: *apply-funtype*)
apply (*blast intro*!: *apply-funtype dest*!: *rvimage-iff* [*THEN iffD1*])
done

lemma *well-ord-rvimage*:
   $[\![f \in inj(A,B);\ \ well\text{-}ord(B,r)]\!] \Longrightarrow well\text{-}ord(A,\ rvimage(A,f,r))$
apply (*rule well-ordI*)
 unfolding *well-ord-def tot-ord-def*
apply (*blast intro*!: *wf-on-rvimage inj-is-fun*)
apply (*blast intro*!: *linear-rvimage*)
done

lemma *ord-iso-rvimage*:
  $f \in bij(A,B) \Longrightarrow f \in ord\text{-}iso(A,\ rvimage(A,f,s),\ B,\ s)$
 unfolding *ord-iso-def*
apply (*simp add*: *rvimage-iff*)
done

lemma *ord-iso-rvimage-eq*:
  $f \in ord\text{-}iso(A,r,\ B,s) \Longrightarrow rvimage(A,f,s) = r \cap A*A$
by (*unfold ord-iso-def rvimage-def*, *blast*)

20.4 Every well-founded relation is a subset of some inverse image of an ordinal

lemma *wf-rvimage-Ord*: $Ord(i) \Longrightarrow wf(rvimage(A,\ f,\ Memrel(i)))$
by (*blast intro*: *wf-rvimage wf-Memrel*)

definition
  $wfrank :: [i,i]\Rightarrow i$ **where**
  $wfrank(r,a) \equiv wfrec(r,\ a,\ \lambda x\ f.\ \bigcup y \in r-``\{x\}.\ succ(f`y))$

definition
  $wftype :: i\Rightarrow i$ **where**
  $wftype(r) \equiv \bigcup y \in range(r).\ succ(wfrank(r,y))$

lemma *wfrank*: $wf(r) \Longrightarrow wfrank(r,a) = (\bigcup y \in r-``\{a\}.\ succ(wfrank(r,y)))$
by (*subst wfrank-def* [*THEN def-wfrec*], *simp-all*)

lemma *Ord-wfrank*: $wf(r) \Longrightarrow Ord(wfrank(r,a))$
apply (*rule-tac a=a* **in** *wf-induct*, *assumption*)
apply (*subst wfrank*, *assumption*)
apply (*rule Ord-succ* [*THEN Ord-UN*], *blast*)
done

lemma *wfrank-lt*: $[\![wf(r);\ \langle a,b\rangle \in r]\!] \Longrightarrow wfrank(r,a) < wfrank(r,b)$
apply (*rule-tac a1 = b* **in** *wfrank* [*THEN ssubst*], *assumption*)
apply (*rule UN-I* [*THEN ltI*])
apply (*simp add*: *Ord-wfrank vimage-iff*)+
done

lemma *Ord-wftype*: $wf(r) \Longrightarrow Ord(wftype(r))$
by (*simp add*: *wftype-def Ord-wfrank*)

lemma *wftypeI*: $[\![wf(r);\ \ x \in field(r)]\!] \Longrightarrow wfrank(r,x) \in wftype(r)$
apply (*simp add*: *wftype-def*)
apply (*blast intro*: *wfrank-lt* [*THEN ltD*])
done

lemma *wf-imp-subset-rvimage*:
  $[\![wf(r);\ r \subseteq A*A]\!] \Longrightarrow \exists\, i\ f.\ Ord(i) \wedge r \subseteq rvimage(A,\ f,\ Memrel(i))$
apply (*rule-tac x=wftype(r)* **in** *exI*)
apply (*rule-tac x=λx∈A. wfrank(r,x)* **in** *exI*)
apply (*simp add*: *Ord-wftype*, *clarify*)
apply (*frule subsetD*, *assumption*, *clarify*)

apply (*simp add*: *rvimage-iff wfrank-lt* [*THEN ltD*])
apply (*blast intro*: *wftypeI*)
done

theorem *wf-iff-subset-rvimage*:
relation(*r*) $\Longrightarrow$ *wf*(*r*) $\longleftrightarrow$ ($\exists$ *i f A*. *Ord*(*i*) $\wedge$ *r* $\subseteq$ *rvimage*(*A*, *f*, *Memrel*(*i*)))
by (*blast dest*!: *relation-field-times-field wf-imp-subset-rvimage*
intro: *wf-rvimage-Ord* [*THEN wf-subset*])

20.5 Other Results

lemma *wf-times*: *A* $\cap$ *B* = *0* $\Longrightarrow$ *wf*(*A***B*)
by (*simp add*: *wf-def*, *blast*)

Could also be used to prove *wf-radd*

lemma *wf-Un*:
$[\![$*range*(*r*) $\cap$ *domain*(*s*) = *0*; *wf*(*r*); *wf*(*s*)$]\!]$ $\Longrightarrow$ *wf*(*r* $\cup$ *s*)
apply (*simp add*: *wf-def*, *clarify*)
apply (*rule equalityI*)
prefer *2* **apply** *blast*
apply *clarify*
apply (*drule-tac x*=*Z* **in** *spec*)
apply (*drule-tac x*=*Z* $\cap$ *domain*(*s*) **in** *spec*)
apply *simp*
apply (*blast intro*: *elim*: *equalityE*)
done

20.5.1 The Empty Relation

lemma *wf0*: *wf*(*0*)
by (*simp add*: *wf-def*, *blast*)

lemma *linear0*: *linear*(*0*,*0*)
by (*simp add*: *linear-def*)

lemma *well-ord0*: *well-ord*(*0*,*0*)
by (*blast intro*: *wf-imp-wf-on well-ordI wf0 linear0*)

20.5.2 The "measure" relation is useful with wfrec

lemma *measure-eq-rvimage-Memrel*:
measure(*A*,*f*) = *rvimage*(*A*,*Lambda*(*A*,*f*),*Memrel*(*Collect*(*RepFun*(*A*,*f*),*Ord*)))
apply (*simp* (*no-asm*) *add*: *measure-def rvimage-def Memrel-iff*)
apply (*rule equalityI*, *auto*)
apply (*auto intro*: *Ord-in-Ord simp add*: *lt-def*)
done

lemma *wf-measure* [*iff*]: *wf*(*measure*(*A*,*f*))
by (*simp* (*no-asm*) *add*: *measure-eq-rvimage-Memrel wf-Memrel wf-rvimage*)

lemma *measure-iff* [*iff*]: $\langle x,y\rangle \in measure(A,f) \longleftrightarrow x \in A \wedge y \in A \wedge f(x)<f(y)$
by (*simp* (*no-asm*) *add*: *measure-def*)

lemma *linear-measure*:
assumes *Ordf*: $\bigwedge x.\ x \in A \Longrightarrow Ord(f(x))$
and *inj*: $\bigwedge x\ y.\ [\![x \in A;\ y \in A;\ f(x) = f(y)]\!] \Longrightarrow x=y$
shows *linear*(*A*, *measure*(*A*,*f*))
apply (*auto simp add*: *linear-def*)
apply (*rule-tac i*=*f*(*x*) **and** *j*=*f*(*y*) **in** *Ord-linear-lt*)
apply (*simp-all add*: *Ordf*)
apply (*blast intro*: *inj*)
done

lemma *wf-on-measure*: *wf*[*B*](*measure*(*A*,*f*))
by (*rule wf-imp-wf-on* [*OF wf-measure*])

lemma *well-ord-measure*:
assumes *Ordf*: $\bigwedge x.\ x \in A \Longrightarrow Ord(f(x))$
and *inj*: $\bigwedge x\ y.\ [\![x \in A;\ y \in A;\ f(x) = f(y)]\!] \Longrightarrow x=y$
shows *well-ord*(*A*, *measure*(*A*,*f*))
apply (*rule well-ordI*)
apply (*rule wf-on-measure*)
apply (*blast intro*: *linear-measure Ordf inj*)
done

lemma *measure-type*: $measure(A,f) \subseteq A*A$
by (*auto simp add*: *measure-def*)

20.5.3 Well-foundedness of Unions

lemma *wf-on-Union*:
assumes *wfA*: *wf*[*A*](*r*)
and *wfB*: $\bigwedge a.\ a\in A \Longrightarrow wf[B(a)](s)$
and *ok*: $\bigwedge a\ u\ v.\ [\![\langle u,v\rangle \in s;\ v \in B(a);\ a \in A]\!]$
$\Longrightarrow (\exists a'\in A.\ \langle a',a\rangle \in r \wedge u \in B(a')) \mid u \in B(a)$
shows $wf[\bigcup a\in A.\ B(a)](s)$
apply (*rule wf-onI2*)
apply (*erule UN-E*)
apply (*subgoal-tac* $\forall z \in B(a).\ z \in Ba$, *blast*)
apply (*rule-tac a* = *a* **in** *wf-on-induct* [*OF wfA*], *assumption*)
apply (*rule ballI*)
apply (*rule-tac a* = *z* **in** *wf-on-induct* [*OF wfB*], *assumption*, *assumption*)
apply (*rename-tac u*)
apply (*drule-tac x*=*u* **in** *bspec*, *blast*)
apply (*erule mp*, *clarify*)
apply (*frule ok*, *assumption*+, *blast*)
done

20.5.4 Bijections involving Powersets

lemma *Pow-sum-bij*:
$(\lambda Z \in Pow(A{+}B).\ \langle\{x \in A.\ Inl(x) \in Z\},\ \{y \in B.\ Inr(y) \in Z\}\rangle)$
$\in bij(Pow(A{+}B),\ Pow(A){*}Pow(B))$
apply (*rule-tac* $d = \lambda\langle X,Y\rangle.\ \{Inl\ (x).\ x \in X\} \cup \{Inr\ (y).\ y \in Y\}$
in *lam-bijective*)
apply *force+*
done

As a special case, we have $bij(Pow(A \times B),\ A \to Pow(B))$

lemma *Pow-Sigma-bij*:
$(\lambda r \in Pow(Sigma(A,B)).\ \lambda x \in A.\ r``\{x\})$
$\in bij(Pow(Sigma(A,B)),\ \prod x \in A.\ Pow(B(x)))$
apply (*rule-tac* $d = \lambda f.\ \bigcup x \in A.\ \bigcup y \in f`x.\ \{\langle x,y\rangle\}$ **in** *lam-bijective*)
apply (*blast intro*: *lam-type*)
apply (*blast dest*: *apply-type*, *simp-all*)
apply *fast*
apply (*rule fun-extension*, *auto*)
by *blast*

end

21 Order Types and Ordinal Arithmetic

theory *OrderType* **imports** *OrderArith OrdQuant Nat* **begin**

The order type of a well-ordering is the least ordinal isomorphic to it. Ordinal arithmetic is traditionally defined in terms of order types, as it is here. But a definition by transfinite recursion would be much simpler!

definition
ordermap :: $[i,i]\Rightarrow i$ **where**
$ordermap(A,r) \equiv \lambda x{\in}A.\ wfrec[A](r,\ x,\ \lambda x\ f.\ f\ ``\ pred(A,x,r))$

definition
ordertype :: $[i,i]\Rightarrow i$ **where**
$ordertype(A,r) \equiv ordermap(A,r)``A$

definition

Ord-alt :: $i \Rightarrow o$ **where**
$Ord\text{-}alt(X) \equiv well\text{-}ord(X,\ Memrel(X)) \wedge (\forall u{\in}X.\ u{=}pred(X,\ u,\ Memrel(X)))$

definition

ordify :: $i{\Rightarrow}i$ **where**
$ordify(x) \equiv if\ Ord(x)\ then\ x\ else\ 0$

definition

omult :: [*i*,*i*]⇒*i* (**infixl** ‹∗∗› *70*) **where**
i ∗∗ *j* ≡ *ordertype*(*j*∗*i*, *rmult*(*j*,*Memrel*(*j*),*i*,*Memrel*(*i*)))

definition

raw-oadd :: [*i*,*i*]⇒*i* **where**
raw-oadd(*i*,*j*) ≡ *ordertype*(*i*+*j*, *radd*(*i*,*Memrel*(*i*),*j*,*Memrel*(*j*)))

definition
oadd :: [*i*,*i*]⇒*i* (**infixl** ‹++› *65*) **where**
i ++ *j* ≡ *raw-oadd*(*ordify*(*i*),*ordify*(*j*))

definition

odiff :: [*i*,*i*]⇒*i* (**infixl** ‹−−› *65*) **where**
i −− *j* ≡ *ordertype*(*i*−*j*, *Memrel*(*i*))

21.1 Proofs needing the combination of Ordinal.thy and Order.thy

lemma *le-well-ord-Memrel*: *j* ≤ *i* ⟹ *well-ord*(*j*, *Memrel*(*i*))
apply (*rule well-ordI*)
apply (*rule wf-Memrel* [*THEN wf-imp-wf-on*])
apply (*simp add*: *ltD lt-Ord linear-def*
ltI [*THEN lt-trans2* [*of - j i*]])
apply (*intro ballI Ord-linear*)
apply (*blast intro*: *Ord-in-Ord lt-Ord*)+
done

lemmas *well-ord-Memrel* = *le-refl* [*THEN le-well-ord-Memrel*]

lemma *lt-pred-Memrel*:
j<*i* ⟹ *pred*(*i*, *j*, *Memrel*(*i*)) = *j*
apply (*simp add*: *pred-def lt-def*)
apply (*blast intro*: *Ord-trans*)
done

lemma *pred-Memrel*:
x ∈ *A* ⟹ *pred*(*A*, *x*, *Memrel*(*A*)) = *A* ∩ *x*
by (*unfold pred-def Memrel-def*, *blast*)

lemma *Ord-iso-implies-eq-lemma*:
⟦*j*<*i*; *f* ∈ *ord-iso*(*i*,*Memrel*(*i*),*j*,*Memrel*(*j*))⟧ ⟹ *R*
apply (*frule lt-pred-Memrel*)
apply (*erule ltE*)
apply (*rule well-ord-Memrel* [*THEN well-ord-iso-predE*, *of i f j*], *auto*)

unfolding *ord-iso-def*

apply (*simp* (*no-asm-simp*))
apply (*blast intro*: *bij-is-fun* [*THEN apply-type*] *Ord-trans*)
done

lemma *Ord-iso-implies-eq*:
$[\![Ord(i);\ \ Ord(j);\ \ f \in ord\text{-}iso(i,Memrel(i),j,Memrel(j))]\!]$
$\Longrightarrow i=j$
apply (*rule-tac* $i = i$ **and** $j = j$ **in** *Ord-linear-lt*)
apply (*blast intro*: *ord-iso-sym Ord-iso-implies-eq-lemma*)+
done

21.2 Ordermap and ordertype

lemma *ordermap-type*:
$ordermap(A,r) \in A \ {-}{>}\ ordertype(A,r)$
unfolding *ordermap-def ordertype-def*
apply (*rule lam-type*)
apply (*rule lamI* [*THEN imageI*], *assumption*+)
done

21.2.1 Unfolding of ordermap

lemma *ordermap-eq-image*:
$[\![wf[A](r);\ \ x \in A]\!]$
$\Longrightarrow ordermap(A,r)\ `\ x = ordermap(A,r)\ ``\ pred(A,x,r)$
unfolding *ordermap-def pred-def*
apply (*simp* (*no-asm-simp*))
apply (*erule wfrec-on* [*THEN trans*], *assumption*)
apply (*simp* (*no-asm-simp*) *add*: *subset-iff image-lam vimage-singleton-iff*)
done

lemma *ordermap-pred-unfold*:
$[\![wf[A](r);\ \ x \in A]\!]$
$\Longrightarrow ordermap(A,r)\ `\ x = \{ordermap(A,r)`y\ .\ y \in pred(A,x,r)\}$
by (*simp add*: *ordermap-eq-image pred-subset ordermap-type* [*THEN image-fun*])

lemmas *ordermap-unfold* = *ordermap-pred-unfold* [*simplified pred-def*]

21.2.2 Showing that ordermap, ordertype yield ordinals

lemma *Ord-ordermap*:
$[\![well\text{-}ord(A,r);\ \ x \in A]\!] \Longrightarrow Ord(ordermap(A,r)\ `\ x)$
apply (*unfold well-ord-def tot-ord-def part-ord-def*, *safe*)
apply (*rule-tac* $a{=}x$ **in** *wf-on-induct*, *assumption*+)
apply (*simp* (*no-asm-simp*) *add*: *ordermap-pred-unfold*)

apply (*rule OrdI* [*OF - Ord-is-Transset*])
  unfolding *pred-def Transset-def*
apply (*blast intro*: *trans-onD*
    dest!: *ordermap-unfold* [*THEN equalityD1*])+
done

lemma *Ord-ordertype*:
  well-ord(*A*,*r*) ⟹ *Ord*(*ordertype*(*A*,*r*))
  unfolding *ordertype-def*
apply (*subst image-fun* [*OF ordermap-type subset-refl*])
apply (*rule OrdI* [*OF - Ord-is-Transset*])
prefer *2* **apply** (*blast intro*: *Ord-ordermap*)
  unfolding *Transset-def well-ord-def*
apply (*blast intro*: *trans-onD*
    dest!: *ordermap-unfold* [*THEN equalityD1*])
done

21.2.3 ordermap preserves the orderings in both directions

lemma *ordermap-mono*:
  ⟦⟨*w*,*x*⟩: *r*; *wf*[*A*](*r*); *w* ∈ *A*; *x* ∈ *A*⟧
  ⟹ *ordermap*(*A*,*r*)'*w* ∈ *ordermap*(*A*,*r*)'*x*
apply (*erule-tac x1* = *x* **in** *ordermap-unfold* [*THEN ssubst*], *assumption*, *blast*)
done

lemma *converse-ordermap-mono*:
  ⟦*ordermap*(*A*,*r*)'*w* ∈ *ordermap*(*A*,*r*)'*x*; *well-ord*(*A*,*r*); *w* ∈ *A*; *x* ∈ *A*⟧
  ⟹ ⟨*w*,*x*⟩: *r*
apply (*unfold well-ord-def tot-ord-def*, *safe*)
apply (*erule-tac x*=*w* **and** *y*=*x* **in** *linearE*, *assumption*+)
apply (*blast elim*!: *mem-not-refl* [*THEN notE*])
apply (*blast dest*: *ordermap-mono intro*: *mem-asym*)
done

lemma *ordermap-surj*: *ordermap*(*A*, *r*) ∈ *surj*(*A*, *ordertype*(*A*, *r*))
  unfolding *ordertype-def*
  by (*rule surj-image*) (*rule ordermap-type*)

lemma *ordermap-bij*:
  well-ord(*A*,*r*) ⟹ *ordermap*(*A*,*r*) ∈ *bij*(*A*, *ordertype*(*A*,*r*))
  unfolding *well-ord-def tot-ord-def bij-def inj-def*
apply (*force intro*!: *ordermap-type ordermap-surj*
    elim: *linearE dest*: *ordermap-mono*
    simp add: *mem-not-refl*)
done

21.2.4 Isomorphisms involving ordertype

lemma *ordertype-ord-iso*:

well-ord(*A*,*r*)
⟹ *ordermap*(*A*,*r*) ∈ *ord-iso*(*A*,*r*, *ordertype*(*A*,*r*), *Memrel*(*ordertype*(*A*,*r*)))
unfolding *ord-iso-def*
apply (*safe elim*!: *well-ord-is-wf*
intro!: *ordermap-type* [*THEN apply-type*] *ordermap-mono ordermap-bij*)
apply (*blast dest*!: *converse-ordermap-mono*)
done

lemma *ordertype-eq*:
⟦*f* ∈ *ord-iso*(*A*,*r*,*B*,*s*); *well-ord*(*B*,*s*)⟧
⟹ *ordertype*(*A*,*r*) = *ordertype*(*B*,*s*)
apply (*frule well-ord-ord-iso*, *assumption*)
apply (*rule Ord-iso-implies-eq*, (*erule Ord-ordertype*)+)
apply (*blast intro*: *ord-iso-trans ord-iso-sym ordertype-ord-iso*)
done

lemma *ordertype-eq-imp-ord-iso*:
⟦*ordertype*(*A*,*r*) = *ordertype*(*B*,*s*); *well-ord*(*A*,*r*); *well-ord*(*B*,*s*)⟧
⟹ ∃*f*. *f* ∈ *ord-iso*(*A*,*r*,*B*,*s*)
apply (*rule exI*)
apply (*rule ordertype-ord-iso* [*THEN ord-iso-trans*], *assumption*)
apply (*erule ssubst*)
apply (*erule ordertype-ord-iso* [*THEN ord-iso-sym*])
done

21.2.5 Basic equalities for ordertype

lemma *le-ordertype-Memrel*: *j* ≤ *i* ⟹ *ordertype*(*j*,*Memrel*(*i*)) = *j*
apply (*rule Ord-iso-implies-eq* [*symmetric*])
apply (*erule ltE*, *assumption*)
apply (*blast intro*: *le-well-ord-Memrel Ord-ordertype*)
apply (*rule ord-iso-trans*)
apply (*erule-tac* [*2*] *le-well-ord-Memrel* [*THEN ordertype-ord-iso*])
apply (*rule id-bij* [*THEN ord-isoI*])
apply (*simp* (*no-asm-simp*))
apply (*fast elim*: *ltE Ord-in-Ord Ord-trans*)
done

lemmas *ordertype-Memrel* = *le-refl* [*THEN le-ordertype-Memrel*]

lemma *ordertype-0* [*simp*]: *ordertype*(*0*,*r*) = *0*
apply (*rule id-bij* [*THEN ord-isoI*, *THEN ordertype-eq*, *THEN trans*])
apply (*erule emptyE*)
apply (*rule well-ord-0*)
apply (*rule Ord-0* [*THEN ordertype-Memrel*])
done

lemmas *bij-ordertype-vimage* = *ord-iso-rvimage* [*THEN ordertype-eq*]

21.2.6 A fundamental unfolding law for ordertype.

lemma *ordermap-pred-eq-ordermap*:
⟦*well-ord*(*A*,*r*); *y* ∈ *A*; *z* ∈ *pred*(*A*,*y*,*r*)⟧
⟹ *ordermap*(*pred*(*A*,*y*,*r*), *r*) ‘ *z* = *ordermap*(*A*, *r*) ‘ *z*
apply (*frule wf-on-subset-A* [*OF well-ord-is-wf pred-subset*])
apply (*rule-tac a*=*z* **in** *wf-on-induct*, *assumption*+)
apply (*safe elim*!: *predE*)
apply (*simp* (*no-asm-simp*) *add*: *ordermap-pred-unfold well-ord-is-wf pred-iff*)

apply (*simp* (*no-asm-simp*) *add*: *pred-pred-eq*)
apply (*simp add*: *pred-def*)
apply (*rule RepFun-cong* [*OF - refl*])
apply (*drule well-ord-is-trans-on*)
apply (*fast elim*!: *trans-onD*)
done

lemma *ordertype-unfold*:
ordertype(*A*,*r*) = {*ordermap*(*A*,*r*)‘*y* . *y* ∈ *A*}
unfolding *ordertype-def*
apply (*rule image-fun* [*OF ordermap-type subset-refl*])
done

Theorems by Krzysztof Grabczewski; proofs simplified by lcp

lemma *ordertype-pred-subset*: ⟦*well-ord*(*A*,*r*); *x* ∈ *A*⟧ ⟹
ordertype(*pred*(*A*,*x*,*r*),*r*) ⊆ *ordertype*(*A*,*r*)
apply (*simp add*: *ordertype-unfold well-ord-subset* [*OF - pred-subset*])
apply (*fast intro*: *ordermap-pred-eq-ordermap elim*: *predE*)
done

lemma *ordertype-pred-lt*:
⟦*well-ord*(*A*,*r*); *x* ∈ *A*⟧
⟹ *ordertype*(*pred*(*A*,*x*,*r*),*r*) < *ordertype*(*A*,*r*)
apply (*rule ordertype-pred-subset* [*THEN subset-imp-le*, *THEN leE*])
apply (*simp-all add*: *Ord-ordertype well-ord-subset* [*OF - pred-subset*])
apply (*erule sym* [*THEN ordertype-eq-imp-ord-iso*, *THEN exE*])
apply (*erule-tac* [*3*] *well-ord-iso-predE*)
apply (*simp-all add*: *well-ord-subset* [*OF - pred-subset*])
done

lemma *ordertype-pred-unfold*:
well-ord(*A*,*r*)
⟹ *ordertype*(*A*,*r*) = {*ordertype*(*pred*(*A*,*x*,*r*),*r*). *x* ∈ *A*}
apply (*rule equalityI*)
apply (*safe intro*!: *ordertype-pred-lt* [*THEN ltD*])
apply (*auto simp add*: *ordertype-def well-ord-is-wf* [*THEN ordermap-eq-image*]

ordermap-type [*THEN image-fun*]
ordermap-pred-eq-ordermap pred-subset)
done

21.3 Alternative definition of ordinal

lemma *Ord-is-Ord-alt*: *Ord*(*i*) ⟹ *Ord-alt*(*i*)
unfolding *Ord-alt-def*
apply (*rule conjI*)
apply (*erule well-ord-Memrel*)
apply (*unfold Ord-def Transset-def pred-def Memrel-def*, *blast*)
done

lemma *Ord-alt-is-Ord*:
Ord-alt(*i*) ⟹ *Ord*(*i*)
apply (*unfold Ord-alt-def Ord-def Transset-def well-ord-def*
tot-ord-def part-ord-def trans-on-def)
apply (*simp add*: *pred-Memrel*)
apply (*blast elim*!: *equalityE*)
done

21.4 Ordinal Addition

21.4.1 Order Type calculations for radd

Addition with 0

lemma *bij-sum-0*: (λ*z*∈*A*+*0*. *case*(λ*x*. *x*, λ*y*. *y*, *z*)) ∈ *bij*(*A*+*0*, *A*)
apply (*rule-tac d* = *Inl* **in** *lam-bijective*, *safe*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *ordertype-sum-0-eq*:
well-ord(*A*,*r*) ⟹ *ordertype*(*A*+*0*, *radd*(*A*,*r*,*0*,*s*)) = *ordertype*(*A*,*r*)
apply (*rule bij-sum-0* [*THEN ord-isoI*, *THEN ordertype-eq*])
prefer *2* **apply** *assumption*
apply *force*
done

lemma *bij-0-sum*: (λ*z*∈*0*+*A*. *case*(λ*x*. *x*, λ*y*. *y*, *z*)) ∈ *bij*(*0*+*A*, *A*)
apply (*rule-tac d* = *Inr* **in** *lam-bijective*, *safe*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *ordertype-0-sum-eq*:
well-ord(*A*,*r*) ⟹ *ordertype*(*0*+*A*, *radd*(*0*,*s*,*A*,*r*)) = *ordertype*(*A*,*r*)
apply (*rule bij-0-sum* [*THEN ord-isoI*, *THEN ordertype-eq*])
prefer *2* **apply** *assumption*
apply *force*

done

Initial segments of radd. Statements by Grabczewski

lemma *pred-Inl-bij*:
$a \in A \Longrightarrow (\lambda x{\in}pred(A,a,r).\ Inl(x))$
$\in bij(pred(A,a,r),\ pred(A{+}B,\ Inl(a),\ radd(A,r,B,s)))$
unfolding *pred-def*
apply (*rule-tac d = case* ($\lambda x.\ x$, $\lambda y.\ y$) **in** *lam-bijective*)
apply *auto*
done

lemma *ordertype-pred-Inl-eq*:
$[\![a \in A;\ \ well\text{-}ord(A,r)]\!]$
$\Longrightarrow ordertype(pred(A{+}B,\ Inl(a),\ radd(A,r,B,s)),\ radd(A,r,B,s)) =$
$ordertype(pred(A,a,r),\ r)$
apply (*rule pred-Inl-bij* [*THEN ord-isoI*, *THEN ord-iso-sym*, *THEN ordertype-eq*])
apply (*simp-all add*: *well-ord-subset* [*OF - pred-subset*])
apply (*simp add*: *pred-def*)
done

lemma *pred-Inr-bij*:
$b \in B \Longrightarrow$
$id(A{+}pred(B,b,s))$
$\in bij(A{+}pred(B,b,s),\ pred(A{+}B,\ Inr(b),\ radd(A,r,B,s)))$
unfolding *pred-def id-def*
apply (*rule-tac d =* $\lambda z.\ z$ **in** *lam-bijective*, *auto*)
done

lemma *ordertype-pred-Inr-eq*:
$[\![b \in B;\ \ well\text{-}ord(A,r);\ \ well\text{-}ord(B,s)]\!]$
$\Longrightarrow ordertype(pred(A{+}B,\ Inr(b),\ radd(A,r,B,s)),\ radd(A,r,B,s)) =$
$ordertype(A{+}pred(B,b,s),\ radd(A,r,pred(B,b,s),s))$
apply (*rule pred-Inr-bij* [*THEN ord-isoI*, *THEN ord-iso-sym*, *THEN ordertype-eq*])
prefer *2* **apply** (*force simp add*: *pred-def id-def*, *assumption*)
apply (*blast intro*: *well-ord-radd well-ord-subset* [*OF - pred-subset*])
done

21.4.2 ordify: trivial coercion to an ordinal

lemma *Ord-ordify* [*iff*, *TC*]: *Ord*(*ordify*(*x*))
by (*simp add*: *ordify-def*)

lemma *ordify-idem* [*simp*]: *ordify*(*ordify*(*x*)) = *ordify*(*x*)
by (*simp add*: *ordify-def*)

21.4.3 Basic laws for ordinal addition

lemma *Ord-raw-oadd*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow Ord(raw\text{-}oadd(i,j))$
by (*simp add*: *raw-oadd-def ordify-def Ord-ordertype well-ord-radd*

well-ord-Memrel)

lemma *Ord-oadd* [*iff*,*TC*]: *Ord*(*i*++*j*)
by (*simp add*: *oadd-def Ord-raw-oadd*)

Ordinal addition with zero

lemma *raw-oadd-0*: *Ord*(*i*) ⟹ *raw-oadd*(*i*,*0*) = *i*
by (*simp add*: *raw-oadd-def ordify-def ordertype-sum-0-eq*
ordertype-Memrel well-ord-Memrel)

lemma *oadd-0* [*simp*]: *Ord*(*i*) ⟹ *i*++*0* = *i*
apply (*simp* (*no-asm-simp*) *add*: *oadd-def raw-oadd-0 ordify-def*)
done

lemma *raw-oadd-0-left*: *Ord*(*i*) ⟹ *raw-oadd*(*0*,*i*) = *i*
by (*simp add*: *raw-oadd-def ordify-def ordertype-0-sum-eq ordertype-Memrel*
well-ord-Memrel)

lemma *oadd-0-left* [*simp*]: *Ord*(*i*) ⟹ *0*++*i* = *i*
by (*simp add*: *oadd-def raw-oadd-0-left ordify-def*)

lemma *oadd-eq-if-raw-oadd*:
i++*j* = (*if Ord*(*i*) *then* (*if Ord*(*j*) *then raw-oadd*(*i*,*j*) *else i*)
else (*if Ord*(*j*) *then j else 0*))
by (*simp add*: *oadd-def ordify-def raw-oadd-0-left raw-oadd-0*)

lemma *raw-oadd-eq-oadd*: ⟦*Ord*(*i*); *Ord*(*j*)⟧ ⟹ *raw-oadd*(*i*,*j*) = *i*++*j*
by (*simp add*: *oadd-def ordify-def*)

lemma *lt-oadd1*: *k*<*i* ⟹ *k* < *i*++*j*
apply (*simp add*: *oadd-def ordify-def lt-Ord2 raw-oadd-0*, *clarify*)
apply (*simp add*: *raw-oadd-def*)
apply (*rule ltE*, *assumption*)
apply (*rule ltI*)
apply (*force simp add*: *ordertype-pred-unfold well-ord-radd well-ord-Memrel*
ordertype-pred-Inl-eq lt-pred-Memrel leI [*THEN le-ordertype-Memrel*])
apply (*blast intro*: *Ord-ordertype well-ord-radd well-ord-Memrel*)
done

lemma *oadd-le-self*: *Ord*(*i*) ⟹ *i* ≤ *i*++*j*
apply (*rule all-lt-imp-le*)
apply (*auto simp add*: *Ord-oadd lt-oadd1*)
done

Various other results

lemma *id-ord-iso-Memrel*: $A<=B \Longrightarrow id(A) \in ord\text{-}iso(A,\ Memrel(A),\ A,\ Memrel(B))$
apply (*rule id-bij* [*THEN ord-isoI*])
apply (*simp* (*no-asm-simp*))
apply *blast*
done

lemma *subset-ord-iso-Memrel*:
$[\![f \in ord\text{-}iso(A,Memrel(B),C,r);\ A<=B]\!] \Longrightarrow f \in ord\text{-}iso(A,Memrel(A),C,r)$
apply (*frule ord-iso-is-bij* [*THEN bij-is-fun*, *THEN fun-is-rel*])
apply (*frule ord-iso-trans* [*OF id-ord-iso-Memrel*], *assumption*)
apply (*simp add*: *right-comp-id*)
done

lemma *restrict-ord-iso*:
$[\![f \in ord\text{-}iso(i,\ Memrel(i),\ Order.pred(A,a,r),\ r);\ \ a \in A;\ j < i;$
$trans[A](r)]\!]$
$\Longrightarrow restrict(f,j) \in ord\text{-}iso(j,\ Memrel(j),\ Order.pred(A,f`j,r),\ r)$
apply (*frule ltD*)
apply (*frule ord-iso-is-bij* [*THEN bij-is-fun*, *THEN apply-type*], *assumption*)
apply (*frule ord-iso-restrict-pred*, *assumption*)
apply (*simp add*: *pred-iff trans-pred-pred-eq lt-pred-Memrel*)
apply (*blast intro*!: *subset-ord-iso-Memrel le-imp-subset* [*OF leI*])
done

lemma *restrict-ord-iso2*:
$[\![f \in ord\text{-}iso(Order.pred(A,a,r),\ r,\ i,\ Memrel(i));\ \ a \in A;$
$j < i;\ trans[A](r)]\!]$
$\Longrightarrow converse(restrict(converse(f),\ j))$
$\in ord\text{-}iso(Order.pred(A,\ converse(f)`j,\ r),\ r,\ j,\ Memrel(j))$
by (*blast intro*: *restrict-ord-iso ord-iso-sym ltI*)

lemma *ordertype-sum-Memrel*:
$[\![well\text{-}ord(A,r);\ \ k<j]\!]$
$\Longrightarrow ordertype(A+k,\ radd(A,\ r,\ k,\ Memrel(j))) =$
$ordertype(A+k,\ radd(A,\ r,\ k,\ Memrel(k)))$
apply (*erule ltE*)
apply (*rule ord-iso-refl* [*THEN sum-ord-iso-cong*, *THEN ordertype-eq*])
apply (*erule OrdmemD* [*THEN id-ord-iso-Memrel*, *THEN ord-iso-sym*])
apply (*simp-all add*: *well-ord-radd well-ord-Memrel*)
done

lemma *oadd-lt-mono2*: $k<j \Longrightarrow i{+}{+}k < i{+}{+}j$
apply (*simp add*: *oadd-def ordify-def raw-oadd-0-left lt-Ord lt-Ord2*, *clarify*)
apply (*simp add*: *raw-oadd-def*)
apply (*rule ltE*, *assumption*)
apply (*rule ordertype-pred-unfold* [*THEN equalityD2*, *THEN subsetD*, *THEN ltI*])
apply (*simp-all add*: *Ord-ordertype well-ord-radd well-ord-Memrel*)
apply (*rule bexI*)

apply (*erule-tac* [*2*] *InrI*)
apply (*simp add*: *ordertype-pred-Inr-eq well-ord-Memrel lt-pred-Memrel leI* [*THEN le-ordertype-Memrel*] *ordertype-sum-Memrel*)
done

lemma *oadd-lt-cancel2*: ⟦*i*++*j* < *i*++*k*; *Ord*(*j*)⟧ ⟹ *j*<*k*
apply (*simp* (*asm-lr*) *add*: *oadd-eq-if-raw-oadd split*: *split-if-asm*)
prefer *2*
apply (*frule-tac i* = *i* **and** *j* = *j* **in** *oadd-le-self*)
apply (*simp* (*asm-lr*) *add*: *oadd-def ordify-def lt-Ord not-lt-iff-le* [*THEN iff-sym*])
apply (*rule Ord-linear-lt*, *auto*)
apply (*simp-all add*: *raw-oadd-eq-oadd*)
apply (*blast dest*: *oadd-lt-mono2 elim*: *lt-irrefl lt-asym*)+
done

lemma *oadd-lt-iff2*: *Ord*(*j*) ⟹ *i*++*j* < *i*++*k* ⟷ *j*<*k*
by (*blast intro*!: *oadd-lt-mono2 dest*!: *oadd-lt-cancel2*)

lemma *oadd-inject*: ⟦*i*++*j* = *i*++*k*; *Ord*(*j*); *Ord*(*k*)⟧ ⟹ *j*=*k*
apply (*simp add*: *oadd-eq-if-raw-oadd split*: *split-if-asm*)
apply (*simp add*: *raw-oadd-eq-oadd*)
apply (*rule Ord-linear-lt*, *auto*)
apply (*force dest*: *oadd-lt-mono2* [*of concl*: *i*] *simp add*: *lt-not-refl*)+
done

lemma *lt-oadd-disj*: *k* < *i*++*j* ⟹ *k*<*i* | (∃ *l*∈*j*. *k* = *i*++*l*)
apply (*simp add*: *Ord-in-Ord′* [*of - j*] *oadd-eq-if-raw-oadd split*: *split-if-asm*)
prefer *2*
apply (*simp add*: *Ord-in-Ord′* [*of - j*] *lt-def*)
apply (*simp add*: *ordertype-pred-unfold well-ord-radd well-ord-Memrel raw-oadd-def*)
apply (*erule ltD* [*THEN RepFunE*])
apply (*force simp add*: *ordertype-pred-Inl-eq well-ord-Memrel ltI lt-pred-Memrel le-ordertype-Memrel leI ordertype-pred-Inr-eq ordertype-sum-Memrel*)
done

21.4.4 Ordinal addition with successor – via associativity!

lemma *oadd-assoc*: (*i*++*j*)++*k* = *i*++(*j*++*k*)
apply (*simp add*: *oadd-eq-if-raw-oadd Ord-raw-oadd raw-oadd-0 raw-oadd-0-left*, *clarify*)
apply (*simp add*: *raw-oadd-def*)
apply (*rule ordertype-eq* [*THEN trans*])
apply (*rule sum-ord-iso-cong* [*OF ordertype-ord-iso* [*THEN ord-iso-sym*] *ord-iso-refl*])
apply (*simp-all add*: *Ord-ordertype well-ord-radd well-ord-Memrel*)
apply (*rule sum-assoc-ord-iso* [*THEN ordertype-eq*, *THEN trans*])
apply (*rule-tac* [*2*] *ordertype-eq*)

apply (*rule-tac* [*2*] *sum-ord-iso-cong* [*OF ord-iso-refl ordertype-ord-iso*])
apply (*blast intro*: *Ord-ordertype well-ord-radd well-ord-Memrel*)+
done

lemma *oadd-unfold*: $[\![Ord(i);\ \ Ord(j)]\!] \Longrightarrow i{+}{+}j = i \cup (\bigcup k{\in}j.\ \{i{+}{+}k\})$
apply (*rule subsetI* [*THEN equalityI*])
apply (*erule ltI* [*THEN lt-oadd-disj*, *THEN disjE*])
apply (*blast intro*: *Ord-oadd*)
apply (*blast elim*!: *ltE*, *blast*)
apply (*force intro*: *lt-oadd1 oadd-lt-mono2 simp add*: *Ord-mem-iff-lt*)
done

lemma *oadd-1*: $Ord(i) \Longrightarrow i{+}{+}1 = succ(i)$
apply (*simp* (*no-asm-simp*) *add*: *oadd-unfold Ord-1 oadd-0*)
apply *blast*
done

lemma *oadd-succ* [*simp*]: $Ord(j) \Longrightarrow i{+}{+}succ(j) = succ(i{+}{+}j)$
apply (*simp add*: *oadd-eq-if-raw-oadd*, *clarify*)
apply (*simp add*: *raw-oadd-eq-oadd*)
apply (*simp add*: *oadd-1* [*of j*, *symmetric*] *oadd-1* [*of i++j*, *symmetric*]
oadd-assoc)
done

Ordinal addition with limit ordinals

lemma *oadd-UN*:
$[\![\bigwedge x.\ x \in A \Longrightarrow Ord(j(x));\ \ a \in A]\!]$
$\Longrightarrow i ++ (\bigcup x{\in}A.\ j(x)) = (\bigcup x{\in}A.\ i{+}{+}j(x))$
by (*blast intro*: *ltI Ord-UN Ord-oadd lt-oadd1* [*THEN ltD*]
oadd-lt-mono2 [*THEN ltD*]
elim!: *ltE dest*!: *ltI* [*THEN lt-oadd-disj*])

lemma *oadd-Limit*: $Limit(j) \Longrightarrow i{+}{+}j = (\bigcup k{\in}j.\ i{+}{+}k)$
apply (*frule Limit-has-0* [*THEN ltD*])
apply (*simp add*: *Limit-is-Ord* [*THEN Ord-in-Ord*] *oadd-UN* [*symmetric*]
Union-eq-UN [*symmetric*] *Limit-Union-eq*)
done

lemma *oadd-eq-0-iff*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow (i ++ j) = 0 \longleftrightarrow i{=}0 \wedge j{=}0$
apply (*erule trans-induct3* [*of j*])
apply (*simp-all add*: *oadd-Limit*)
apply (*simp add*: *Union-empty-iff Limit-def lt-def*, *blast*)
done

lemma *oadd-eq-lt-iff*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow 0 < (i ++ j) \longleftrightarrow 0{<}i \mid 0{<}j$
by (*simp add*: *Ord-0-lt-iff* [*symmetric*] *oadd-eq-0-iff*)

lemma *oadd-LimitI*: $[\![Ord(i);\ Limit(j)]\!] \Longrightarrow Limit(i ++ j)$
apply (*simp add*: *oadd-Limit*)

apply (*frule Limit-has-1* [*THEN ltD*])
apply (*rule increasing-LimitI*)
 apply (*rule Ord-0-lt*)
  apply (*blast intro*: *Ord-in-Ord* [*OF Limit-is-Ord*])
 apply (*force simp add*: *Union-empty-iff oadd-eq-0-iff*
                        Limit-is-Ord [*of j*, *THEN Ord-in-Ord*], *auto*)
apply (*rule-tac* x=*succ*(y) **in** *bexI*)
 apply (*simp add*: *ltI Limit-is-Ord* [*of j*, *THEN Ord-in-Ord*])
apply (*simp add*: *Limit-def lt-def*)
done

Order/monotonicity properties of ordinal addition

lemma *oadd-le-self2*: *Ord*(i) $\Longrightarrow$ $i \le j{+}{+}i$
proof (*induct i rule*: *trans-induct3*)
  case *0* **thus** *?case* **by** (*simp add*: *Ord-0-le*)
next
  case (*succ i*) **thus** *?case* **by** (*simp add*: *oadd-succ succ-leI*)
next
  case (*limit l*)
  hence $l = (\bigcup x{\in}l.\ x)$
    by (*simp add*: *Union-eq-UN* [*symmetric*] *Limit-Union-eq*)
  also have ... $\le (\bigcup x{\in}l.\ j{+}{+}x)$
    by (*rule le-implies-UN-le-UN*) (*rule limit.hyps*)
  finally have $l \le (\bigcup x{\in}l.\ j{+}{+}x)$ **.**
  thus *?case* **using** *limit.hyps* **by** (*simp add*: *oadd-Limit*)
qed

lemma *oadd-le-mono1*: $k \le j \Longrightarrow k{+}{+}i \le j{+}{+}i$
apply (*frule lt-Ord*)
apply (*frule le-Ord2*)
apply (*simp add*: *oadd-eq-if-raw-oadd*, *clarify*)
apply (*simp add*: *raw-oadd-eq-oadd*)
apply (*erule-tac* $i = i$ **in** *trans-induct3*)
apply (*simp* (*no-asm-simp*))
apply (*simp* (*no-asm-simp*) *add*: *oadd-succ succ-le-iff*)
apply (*simp* (*no-asm-simp*) *add*: *oadd-Limit*)
apply (*rule le-implies-UN-le-UN*, *blast*)
done

lemma *oadd-lt-mono*: $[\![i' \le i;\ j'<j]\!] \Longrightarrow i'{+}{+}j' < i{+}{+}j$
by (*blast intro*: *lt-trans1 oadd-le-mono1 oadd-lt-mono2 Ord-succD elim*: *ltE*)

lemma *oadd-le-mono*: $[\![i' \le i;\ j' \le j]\!] \Longrightarrow i'{+}{+}j' \le i{+}{+}j$
by (*simp del*: *oadd-succ add*: *oadd-succ* [*symmetric*] *le-Ord2 oadd-lt-mono*)

lemma *oadd-le-iff2*: $[\![Ord(j);\ Ord(k)]\!] \Longrightarrow i{+}{+}j \le i{+}{+}k \longleftrightarrow j \le k$
by (*simp del*: *oadd-succ add*: *oadd-lt-iff2 oadd-succ* [*symmetric*] *Ord-succ*)

lemma *oadd-lt-self*: $[\![Ord(i);\ 0<j]\!] \Longrightarrow i < i{+}{+}j$

apply (*rule lt-trans2*)
apply (*erule le-refl*)
apply (*simp only*: *lt-Ord2 oadd-1* [*of i*, *symmetric*])
apply (*blast intro*: *succ-leI oadd-le-mono*)
done

Every ordinal is exceeded by some limit ordinal.

lemma *Ord-imp-greater-Limit*: $Ord(i) \Longrightarrow \exists k.\ i<k \wedge Limit(k)$
apply (*rule-tac x=i ++ nat* **in** *exI*)
apply (*blast intro*: *oadd-LimitI oadd-lt-self Limit-nat* [*THEN Limit-has-0*])
done

lemma *Ord2-imp-greater-Limit*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow \exists k.\ i<k \wedge j<k \wedge Limit(k)$
apply (*insert Ord-Un* [*of i j*, *THEN Ord-imp-greater-Limit*])
apply (*simp add*: *Un-least-lt-iff*)
done

21.5 Ordinal Subtraction

The difference is $ordertype(j - i,\ Memrel(j))$. It's probably simpler to define the difference recursively!

lemma *bij-sum-Diff*:
$A<=B \Longrightarrow (\lambda y\in B.\ if(y \in A,\ Inl(y),\ Inr(y))) \in bij(B,\ A+(B-A))$
apply (*rule-tac d = case* ($\lambda x.\ x,\ \lambda y.\ y$) **in** *lam-bijective*)
apply (*blast intro*!: *if-type*)
apply (*fast intro*!: *case-type*)
apply (*erule-tac* [*2*] *sumE*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *ordertype-sum-Diff*:
$i \le j \Longrightarrow$
$ordertype(i+(j-i),\ radd(i,Memrel(j),j-i,Memrel(j))) =$
$ordertype(j,\ Memrel(j))$
apply (*safe dest*!: *le-subset-iff* [*THEN iffD1*])
apply (*rule bij-sum-Diff* [*THEN ord-isoI*, *THEN ord-iso-sym*, *THEN ordertype-eq*])
apply (*erule-tac* [*3*] *well-ord-Memrel*, *assumption*)
apply (*simp* (*no-asm-simp*))
apply (*frule-tac j = y* **in** *Ord-in-Ord*, *assumption*)
apply (*frule-tac j = x* **in** *Ord-in-Ord*, *assumption*)
apply (*simp* (*no-asm-simp*) *add*: *Ord-mem-iff-lt lt-Ord not-lt-iff-le*)
apply (*blast intro*: *lt-trans2 lt-trans*)
done

lemma *Ord-odiff* [*simp*,*TC*]:
$[\![Ord(i);\ Ord(j)]\!] \Longrightarrow Ord(i--j)$
unfolding *odiff-def*
apply (*blast intro*: *Ord-ordertype Diff-subset well-ord-subset well-ord-Memrel*)

done

lemma *raw-oadd-ordertype-Diff*:
  $i \le j$
  $\Longrightarrow$ *raw-oadd*$(i,j--i)$ = *ordertype*$(i+(j-i)$, *radd*$(i,$*Memrel*$(j),j-i,$*Memrel*$(j)))$
apply (*simp add*: *raw-oadd-def odiff-def*)
apply (*safe dest*!: *le-subset-iff* [*THEN iffD1*])
apply (*rule sum-ord-iso-cong* [*THEN ordertype-eq*])
apply (*erule id-ord-iso-Memrel*)
apply (*rule ordertype-ord-iso* [*THEN ord-iso-sym*])
apply (*blast intro*: *well-ord-radd Diff-subset well-ord-subset well-ord-Memrel*)+
done

lemma *oadd-odiff-inverse*: $i \le j \Longrightarrow i ++ (j--i) = j$
by (*simp add*: *lt-Ord le-Ord2 oadd-def ordify-def raw-oadd-ordertype-Diff*
  ordertype-sum-Diff ordertype-Memrel lt-Ord2 [*THEN Ord-succD*])

lemma *odiff-oadd-inverse*: $[\![$*Ord*(i); *Ord*$(j)]\!] \Longrightarrow (i{+}{+}j) -- i = j$
apply (*rule oadd-inject*)
apply (*blast intro*: *oadd-odiff-inverse oadd-le-self*)
apply (*blast intro*: *Ord-ordertype Ord-oadd Ord-odiff*)+
done

lemma *odiff-lt-mono2*: $[\![i<j;\ \ k \le i]\!] \Longrightarrow i--k < j--k$
apply (*rule-tac* $i = k$ **in** *oadd-lt-cancel2*)
apply (*simp add*: *oadd-odiff-inverse*)
apply (*subst oadd-odiff-inverse*)
apply (*blast intro*: *le-trans leI*, *assumption*)
apply (*simp* (*no-asm-simp*) *add*: *lt-Ord le-Ord2*)
done

21.6 Ordinal Multiplication

lemma *Ord-omult* [*simp*, *TC*]:
  $[\![$*Ord*(i); *Ord*$(j)]\!] \Longrightarrow$ *Ord*$(i{*}{*}j)$
 unfolding *omult-def*
apply (*blast intro*: *Ord-ordertype well-ord-rmult well-ord-Memrel*)
done

21.6.1 A useful unfolding law

lemma *pred-Pair-eq*:
$[\![a \in A;\ \ b \in B]\!] \Longrightarrow$ *pred*$(A{*}B, \langle a,b\rangle,$ *rmult*$(A,r,B,s))$ =
  pred$(A,a,r){*}B \cup (\{a\} * $ *pred*$(B,b,s))$
apply (*unfold pred-def*, *blast*)
done

lemma *ordertype-pred-Pair-eq*:

$[\![a \in A;\ \ b \in B;\ \ well\text{-}ord(A,r);\ \ well\text{-}ord(B,s)]\!] \Longrightarrow$
$ordertype(pred(A*B,\ \langle a,b\rangle,\ rmult(A,r,B,s)),\ rmult(A,r,B,s)) =$
$ordertype(pred(A,a,r)*B + pred(B,b,s),$
$radd(A*B,\ rmult(A,r,B,s),\ B,\ s))$
apply (*simp* (*no-asm-simp*) *add*: *pred-Pair-eq*)
apply (*rule ordertype-eq* [*symmetric*])
apply (*rule prod-sum-singleton-ord-iso*)
apply (*simp-all add*: *pred-subset well-ord-rmult* [*THEN well-ord-subset*])
apply (*blast intro*: *pred-subset well-ord-rmult* [*THEN well-ord-subset*]
elim!: *predE*)
done

lemma *ordertype-pred-Pair-lemma*:
$[\![i'<i;\ \ j'<j]\!]$
$\Longrightarrow ordertype(pred(i*j,\ <i',j'>,\ rmult(i,Memrel(i),j,Memrel(j))),$
$rmult(i,Memrel(i),j,Memrel(j))) =$
$raw\text{-}oadd\ (j{*}{*}i',\ j')$
unfolding *raw-oadd-def omult-def*
apply (*simp add*: *ordertype-pred-Pair-eq lt-pred-Memrel ltD lt-Ord2*
well-ord-Memrel)
apply (*rule trans*)
apply (*rule-tac* [*2*] *ordertype-ord-iso*
[*THEN sum-ord-iso-cong*, *THEN ordertype-eq*])
apply (*rule-tac* [*3*] *ord-iso-refl*)
apply (*rule id-bij* [*THEN ord-isoI*, *THEN ordertype-eq*])
apply (*elim SigmaE sumE ltE ssubst*)
apply (*simp-all add*: *well-ord-rmult well-ord-radd well-ord-Memrel*
Ord-ordertype lt-Ord lt-Ord2)
apply (*blast intro*: *Ord-trans*)+
done

lemma *lt-omult*:
$[\![Ord(i);\ \ Ord(j);\ \ k<j{*}{*}i]\!]$
$\Longrightarrow \exists j'\ i'.\ k = j{*}{*}i' \mathbin{++} j' \wedge j'<j \wedge i'<i$
unfolding *omult-def*
apply (*simp add*: *ordertype-pred-unfold well-ord-rmult well-ord-Memrel*)
apply (*safe elim*!: *ltE*)
apply (*simp add*: *ordertype-pred-Pair-lemma ltI raw-oadd-eq-oadd*
omult-def [*symmetric*] *Ord-in-Ord'* [*of - i*] *Ord-in-Ord'* [*of - j*])
apply (*blast intro*: *ltI*)
done

lemma *omult-oadd-lt*:
$[\![j'<j;\ \ i'<i]\!] \Longrightarrow j{*}{*}i' \mathbin{++} j'\ <\ j{*}{*}i$
unfolding *omult-def*
apply (*rule ltI*)
prefer *2*
apply (*simp add*: *Ord-ordertype well-ord-rmult well-ord-Memrel lt-Ord2*)
apply (*simp add*: *ordertype-pred-unfold well-ord-rmult well-ord-Memrel lt-Ord2*)

apply (*rule bexI* [*of - i'*])
apply (*rule bexI* [*of - j'*])
apply (*simp add*: *ordertype-pred-Pair-lemma ltI omult-def* [*symmetric*])
apply (*simp add*: *lt-Ord lt-Ord2 raw-oadd-eq-oadd*)
apply (*simp-all add*: *lt-def*)
done

lemma *omult-unfold*:
  $[\![Ord(i);\ \ Ord(j)]\!] \Longrightarrow j{*}{*}i = (\bigcup j'{\in}j.\ \bigcup i'{\in}i.\ \{j{*}{*}i' ++ j'\})$
apply (*rule subsetI* [*THEN equalityI*])
apply (*rule lt-omult* [*THEN exE*])
apply (*erule-tac* [*3*] *ltI*)
apply (*simp-all add*: *Ord-omult*)
apply (*blast elim*!: *ltE*)
apply (*blast intro*: *omult-oadd-lt* [*THEN ltD*] *ltI*)
done

21.6.2 Basic laws for ordinal multiplication

Ordinal multiplication by zero

lemma *omult-0* [*simp*]: $i{*}{*}0 = 0$
 unfolding *omult-def*
apply (*simp* (*no-asm-simp*))
done

lemma *omult-0-left* [*simp*]: $0{*}{*}i = 0$
 unfolding *omult-def*
apply (*simp* (*no-asm-simp*))
done

Ordinal multiplication by 1

lemma *omult-1* [*simp*]: $Ord(i) \Longrightarrow i{*}{*}1 = i$
 unfolding *omult-def*
apply (*rule-tac s1=Memrel*(*i*)
    in *ord-isoI* [*THEN ordertype-eq*, *THEN trans*])
apply (*rule-tac* $c = snd$ **and** $d = \lambda z.\langle 0,z\rangle$ **in** *lam-bijective*)
apply (*auto elim*!: *snd-type well-ord-Memrel ordertype-Memrel*)
done

lemma *omult-1-left* [*simp*]: $Ord(i) \Longrightarrow 1{*}{*}i = i$
 unfolding *omult-def*
apply (*rule-tac s1=Memrel*(*i*)
    in *ord-isoI* [*THEN ordertype-eq*, *THEN trans*])
apply (*rule-tac* $c = fst$ **and** $d = \lambda z.\langle z,0\rangle$ **in** *lam-bijective*)
apply (*auto elim*!: *fst-type well-ord-Memrel ordertype-Memrel*)
done

Distributive law for ordinal multiplication and addition

lemma *oadd-omult-distrib*:

$\llbracket Ord(i);\ \ Ord(j);\ \ Ord(k)\rrbracket \Longrightarrow i{*}{*}(j{+}{+}k) = (i{*}{*}j){+}{+}(i{*}{*}k)$
apply (*simp add*: *oadd-eq-if-raw-oadd*)
apply (*simp add*: *omult-def raw-oadd-def*)
apply (*rule ordertype-eq* [*THEN trans*])
apply (*rule prod-ord-iso-cong* [*OF ordertype-ord-iso* [*THEN ord-iso-sym*]
ord-iso-refl])
apply (*simp-all add*: *well-ord-rmult well-ord-radd well-ord-Memrel*
Ord-ordertype)
apply (*rule sum-prod-distrib-ord-iso* [*THEN ordertype-eq*, *THEN trans*])
apply (*rule-tac* [*2*] *ordertype-eq*)
apply (*rule-tac* [*2*] *sum-ord-iso-cong* [*OF ordertype-ord-iso ordertype-ord-iso*])
apply (*simp-all add*: *well-ord-rmult well-ord-radd well-ord-Memrel*
Ord-ordertype)
done

lemma *omult-succ*: $\llbracket Ord(i);\ \ Ord(j)\rrbracket \Longrightarrow i{*}{*}succ(j) = (i{*}{*}j){+}{+}i$
by (*simp del*: *oadd-succ add*: *oadd-1* [*of j*, *symmetric*] *oadd-omult-distrib*)

Associative law

lemma *omult-assoc*:
$\llbracket Ord(i);\ \ Ord(j);\ \ Ord(k)\rrbracket \Longrightarrow (i{*}{*}j){*}{*}k = i{*}{*}(j{*}{*}k)$
unfolding *omult-def*
apply (*rule ordertype-eq* [*THEN trans*])
apply (*rule prod-ord-iso-cong* [*OF ord-iso-refl*
ordertype-ord-iso [*THEN ord-iso-sym*]])
apply (*blast intro*: *well-ord-rmult well-ord-Memrel*)+
apply (*rule prod-assoc-ord-iso*
[*THEN ord-iso-sym*, *THEN ordertype-eq*, *THEN trans*])
apply (*rule-tac* [*2*] *ordertype-eq*)
apply (*rule-tac* [*2*] *prod-ord-iso-cong* [*OF ordertype-ord-iso ord-iso-refl*])
apply (*blast intro*: *well-ord-rmult well-ord-Memrel Ord-ordertype*)+
done

Ordinal multiplication with limit ordinals

lemma *omult-UN*:
$\llbracket Ord(i);\ \ \bigwedge x.\ x \in A \Longrightarrow Ord(j(x))\rrbracket$
$\Longrightarrow i\ {*}{*}\ (\bigcup x{\in}A.\ j(x)) = (\bigcup x{\in}A.\ i{*}{*}j(x))$
by (*simp* (*no-asm-simp*) *add*: *Ord-UN omult-unfold*, *blast*)

lemma *omult-Limit*: $\llbracket Ord(i);\ \ Limit(j)\rrbracket \Longrightarrow i{*}{*}j = (\bigcup k{\in}j.\ i{*}{*}k)$
by (*simp add*: *Limit-is-Ord* [*THEN Ord-in-Ord*] *omult-UN* [*symmetric*]
Union-eq-UN [*symmetric*] *Limit-Union-eq*)

21.6.3 Ordering/monotonicity properties of ordinal multiplication

lemma *lt-omult1*: $\llbracket k{<}i;\ \ 0{<}j\rrbracket \Longrightarrow k < i{*}{*}j$
apply (*safe elim*!: *ltE intro*!: *ltI Ord-omult*)
apply (*force simp add*: *omult-unfold*)

```
done

lemma omult-le-self: ⟦Ord(i);  0<j⟧ ⟹ i ≤ i**j
by (blast intro: all-lt-imp-le Ord-omult lt-omult1 lt-Ord2)

lemma omult-le-mono1:
  assumes kj: k ≤ j and i: Ord(i) shows k**i ≤ j**i
proof -
  have o: Ord(k) Ord(j) by (rule lt-Ord [OF kj] le-Ord2 [OF kj])+
  show ?thesis using i
  proof (induct i rule: trans-induct3)
    case 0 thus ?case
      by simp
  next
    case (succ i) thus ?case
      by (simp add: o kj omult-succ oadd-le-mono)
  next
    case (limit l)
    thus ?case
      by (auto simp add: o kj omult-Limit le-implies-UN-le-UN)
  qed
qed

lemma omult-lt-mono2: ⟦k<j;  0<i⟧ ⟹ i**k < i**j
apply (rule ltI)
apply (simp (no-asm-simp) add: omult-unfold lt-Ord2)
apply (safe elim!: ltE intro!: Ord-omult)
apply (force simp add: Ord-omult)
done

lemma omult-le-mono2: ⟦k ≤ j;  Ord(i)⟧ ⟹ i**k ≤ i**j
apply (rule subset-imp-le)
apply (safe elim!: ltE dest!: Ord-succD intro!: Ord-omult)
apply (simp add: omult-unfold)
apply (blast intro: Ord-trans)
done

lemma omult-le-mono: ⟦i' ≤ i;  j' ≤ j⟧ ⟹ i'**j' ≤ i**j
by (blast intro: le-trans omult-le-mono1 omult-le-mono2 Ord-succD elim: ltE)

lemma omult-lt-mono: ⟦i' ≤ i;  j'<j;  0<i⟧ ⟹ i'**j' < i**j
by (blast intro: lt-trans1 omult-le-mono1 omult-lt-mono2 Ord-succD elim: ltE)

lemma omult-le-self2:
  assumes i: Ord(i) and j: 0<j shows i ≤ j**i
proof -
  have oj: Ord(j) by (rule lt-Ord2 [OF j])
  show ?thesis using i
  proof (induct i rule: trans-induct3)
```

case *0* **thus** *?case*
by *simp*
next
case (*succ i*)
have $j ** i ++ 0 < j ** i ++ j$
by (*rule oadd-lt-mono2* [*OF j*])
with *succ.hyps* **show** *?case*
by (*simp add*: *oj j omult-succ*) (*rule lt-trans1*)
next
case (*limit l*)
hence $l = (\bigcup x\in l.\ x)$
by (*simp add*: *Union-eq-UN* [*symmetric*] *Limit-Union-eq*)
also have $... \leq (\bigcup x\in l.\ j{**}x)$
by (*rule le-implies-UN-le-UN*) (*rule limit.hyps*)
finally have $l \leq (\bigcup x\in l.\ j{**}x)$ **.**
thus *?case* **using** *limit.hyps* **by** (*simp add*: *oj omult-Limit*)
qed
qed

Further properties of ordinal multiplication

lemma *omult-inject*: $[\![i{**}j = i{**}k;\ \ 0<i;\ \ Ord(j);\ \ Ord(k)]\!] \Longrightarrow j=k$
apply (*rule Ord-linear-lt*)
prefer *4* **apply** *assumption*
apply *auto*
apply (*force dest*: *omult-lt-mono2 simp add*: *lt-not-refl*)+
done

21.7 The Relation *Lt*

lemma *wf-Lt*: *wf*(*Lt*)
apply (*rule wf-subset*)
apply (*rule wf-Memrel*)
apply (*auto simp add*: *Lt-def Memrel-def lt-def*)
done

lemma *irrefl-Lt*: *irrefl*(*A*,*Lt*)
by (*auto simp add*: *Lt-def irrefl-def*)

lemma *trans-Lt*: *trans*[*A*](*Lt*)
apply (*simp add*: *Lt-def trans-on-def*)
apply (*blast intro*: *lt-trans*)
done

lemma *part-ord-Lt*: *part-ord*(*A*,*Lt*)
by (*simp add*: *part-ord-def irrefl-Lt trans-Lt*)

lemma *linear-Lt*: *linear*(*nat*,*Lt*)
apply (*auto dest!*: *not-lt-imp-le simp add*: *Lt-def linear-def le-iff*)
apply (*drule lt-asym*, *auto*)

done

lemma *tot-ord-Lt*: *tot-ord*(*nat*,*Lt*)
by (*simp add*: *tot-ord-def linear-Lt part-ord-Lt*)

lemma *well-ord-Lt*: *well-ord*(*nat*,*Lt*)
by (*simp add*: *well-ord-def wf-Lt wf-imp-wf-on tot-ord-Lt*)

end

22 Finite Powerset Operator and Finite Function Space

theory *Finite* **imports** *Inductive Epsilon Nat* **begin**

rep-datatype
  elimination *natE*
  induction *nat-induct*
  case-eqns *nat-case-0 nat-case-succ*
  recursor-eqns *recursor-0 recursor-succ*

consts
  Fin :: *i*⇒*i*
  FiniteFun :: [*i*,*i*]⇒*i* (‹(‹notation=‹infix –||>››- –||>/ -)› [61, 60] 60)

inductive
  domains *Fin*(*A*) ⊆ *Pow*(*A*)
  intros
    emptyI: *0* ∈ *Fin*(*A*)
    consI: ⟦*a* ∈ *A*; *b* ∈ *Fin*(*A*)⟧ ⟹ *cons*(*a*,*b*) ∈ *Fin*(*A*)
  type-intros *empty-subsetI cons-subsetI PowI*
  type-elims *PowD* [*elim-format*]

inductive
  domains *FiniteFun*(*A*,*B*) ⊆ *Fin*(*A*∗*B*)
  intros
    emptyI: *0* ∈ *A* –||> *B*
    consI: ⟦*a* ∈ *A*; *b* ∈ *B*; *h* ∈ *A* –||> *B*; *a* ∉ *domain*(*h*)⟧
      ⟹ *cons*(⟨*a*,*b*⟩,*h*) ∈ *A* –||> *B*
  type-intros *Fin.intros*

22.1 Finite Powerset Operator

lemma *Fin-mono*: *A*<=*B* ⟹ *Fin*(*A*) ⊆ *Fin*(*B*)
  unfolding *Fin.defs*
apply (*rule lfp-mono*)

apply (*rule Fin.bnd-mono*)+
apply *blast*
done

lemmas *FinD* = *Fin.dom-subset* [*THEN subsetD*, *THEN PowD*]

lemma *Fin-induct* [*case-names 0 cons*, *induct set*: *Fin*]:
$\llbracket b \in Fin(A);$
$P(0);$
$\bigwedge x\ y.\ \llbracket x \in A;\ \ y \in Fin(A);\ \ x \notin y;\ \ P(y)\rrbracket \Longrightarrow P(cons(x,y))$
$\rrbracket \Longrightarrow P(b)$
apply (*erule Fin.induct*, *simp*)
apply (*case-tac* $a \in b$)
apply (*erule cons-absorb* [*THEN ssubst*], *assumption*)
apply *simp*
done

declare *Fin.intros* [*simp*]

lemma *Fin-0*: $Fin(0) = \{0\}$
by (*blast intro*: *Fin.emptyI dest*: *FinD*)

lemma *Fin-UnI* [*simp*]: $\llbracket b \in Fin(A);\ \ c \in Fin(A)\rrbracket \Longrightarrow b \cup c \in Fin(A)$
apply (*erule Fin-induct*)
apply (*simp-all add*: *Un-cons*)
done

lemma *Fin-UnionI*: $C \in Fin(Fin(A)) \Longrightarrow \bigcup(C) \in Fin(A)$
by (*erule Fin-induct*, *simp-all*)

lemma *Fin-subset-lemma* [*rule-format*]: $b \in Fin(A) \Longrightarrow \forall z.\ z<=b \longrightarrow z \in Fin(A)$
apply (*erule Fin-induct*)
apply (*simp add*: *subset-empty-iff*)
apply (*simp add*: *subset-cons-iff distrib-simps*, *safe*)
apply (*erule-tac* $b = z$ **in** *cons-Diff* [*THEN subst*], *simp*)
done

lemma *Fin-subset*: $\llbracket c<=b;\ \ b \in Fin(A)\rrbracket \Longrightarrow c \in Fin(A)$
by (*blast intro*: *Fin-subset-lemma*)

lemma *Fin-IntI1* [*intro*,*simp*]: $b \in Fin(A) \Longrightarrow b \cap c \in Fin(A)$
by (*blast intro*: *Fin-subset*)

lemma *Fin-IntI2* [*intro*,*simp*]: $c \in Fin(A) \Longrightarrow b \cap c \in Fin(A)$
by (*blast intro*: *Fin-subset*)

lemma *Fin-0-induct-lemma* [*rule-format*]:
  $[\![c \in Fin(A); \; b \in Fin(A); P(b);$
    $\bigwedge x\ y.\ [\![x \in A; \; y \in Fin(A); \; x \in y; \; P(y)]\!] \Longrightarrow P(y-\{x\})$
$]\!] \Longrightarrow c<=b \longrightarrow P(b-c)$
apply (*erule Fin-induct*, *simp*)
apply (*subst Diff-cons*)
apply (*simp add*: *cons-subset-iff Diff-subset* [*THEN Fin-subset*])
done

lemma *Fin-0-induct*:
  $[\![b \in Fin(A);$
    $P(b);$
    $\bigwedge x\ y.\ [\![x \in A; \; y \in Fin(A); \; x \in y; \; P(y)]\!] \Longrightarrow P(y-\{x\})$
$]\!] \Longrightarrow P(0)$
apply (*rule Diff-cancel* [*THEN subst*])
apply (*blast intro*: *Fin-0-induct-lemma*)
done

lemma *nat-fun-subset-Fin*: $n \in nat \Longrightarrow n->A \subseteq Fin(nat*A)$
apply (*induct-tac n*)
apply (*simp add*: *subset-iff*)
apply (*simp add*: *succ-def mem-not-refl* [*THEN cons-fun-eq*])
apply (*fast intro*!: *Fin.consI*)
done

22.2 Finite Function Space

lemma *FiniteFun-mono*:
  $[\![A<=C; \; B<=D]\!] \Longrightarrow A -||> B \; \subseteq \; C -||> D$
 unfolding *FiniteFun.defs*
apply (*rule lfp-mono*)
apply (*rule FiniteFun.bnd-mono*)+
apply (*intro Fin-mono Sigma-mono basic-monos*, *assumption*+)
done

lemma *FiniteFun-mono1*: $A<=B \Longrightarrow A -||> A \; \subseteq \; B -||> B$
by (*blast dest*: *FiniteFun-mono*)

lemma *FiniteFun-is-fun*: $h \in A -||>B \Longrightarrow h \in domain(h) -> B$
apply (*erule FiniteFun.induct*, *simp*)
apply (*simp add*: *fun-extend3*)

done

lemma *FiniteFun-domain-Fin*: $h \in A\ -||{>}B \Longrightarrow domain(h) \in Fin(A)$
by (*erule FiniteFun.induct*, *simp*, *simp*)

lemmas *FiniteFun-apply-type* = *FiniteFun-is-fun* [*THEN apply-type*]

lemma *FiniteFun-subset-lemma* [*rule-format*]:
  $b \in A{-}||{>}B \Longrightarrow \forall z.\ z{<=}b \longrightarrow z \in A{-}||{>}B$
apply (*erule FiniteFun.induct*)
apply (*simp add*: *subset-empty-iff FiniteFun.intros*)
apply (*simp add*: *subset-cons-iff distrib-simps*, *safe*)
apply (*erule-tac* $b = z$ **in** *cons-Diff* [*THEN subst*])
apply (*drule spec* [*THEN mp*], *assumption*)
apply (*fast intro*!: *FiniteFun.intros*)
done

lemma *FiniteFun-subset*: $[\![c{<=}b;\ \ b \in A{-}||{>}B]\!] \Longrightarrow c \in A{-}||{>}B$
by (*blast intro*: *FiniteFun-subset-lemma*)

lemma *fun-FiniteFunI* [*rule-format*]: $A \in Fin(X) \Longrightarrow \forall f.\ f \in A{-}{>}B \longrightarrow f \in A{-}||{>}B$
apply (*erule Fin.induct*)
 apply (*simp add*: *FiniteFun.intros*, *clarify*)
apply (*case-tac* $a \in b$)
 apply (*simp add*: *cons-absorb*)
apply (*subgoal-tac restrict* $(f,b) \in b\ -||{>}\ B$)
 prefer *2* **apply** (*blast intro*: *restrict-type2*)
apply (*subst fun-cons-restrict-eq*, *assumption*)
apply (*simp add*: *restrict-def lam-def*)
apply (*blast intro*: *apply-funtype FiniteFun.intros*
      FiniteFun-mono [*THEN* [*2*] *rev-subsetD*])
done

lemma *lam-FiniteFun*: $A \in Fin(X) \Longrightarrow (\lambda x{\in}A.\ b(x)) \in A\ -||{>}\ \{b(x).\ x \in A\}$
by (*blast intro*: *fun-FiniteFunI lam-funtype*)

lemma *FiniteFun-Collect-iff*:
  $f \in FiniteFun(A,\ \{y \in B.\ P(y)\})$
  $\longleftrightarrow f \in FiniteFun(A,B) \wedge (\forall x{\in}domain(f).\ P(f`x))$
apply *auto*
apply (*blast intro*: *FiniteFun-mono* [*THEN* [*2*] *rev-subsetD*])
apply (*blast dest*: *Pair-mem-PiD FiniteFun-is-fun*)
apply (*rule-tac* $A1{=}domain(f)$ **in**
   subset-refl [*THEN* [*2*] *FiniteFun-mono*, *THEN subsetD*])
 apply (*fast dest*: *FiniteFun-domain-Fin Fin.dom-subset* [*THEN subsetD*])

apply (*rule fun-FiniteFunI*)
apply (*erule FiniteFun-domain-Fin*)
apply (*rule-tac B = range* (*f*) **in** *fun-weaken-type*)
apply (*blast dest*: *FiniteFun-is-fun range-of-fun range-type apply-equality*)+
done

22.3 The Contents of a Singleton Set

definition
contents :: *i*⇒*i* **where**
contents(*X*) ≡ *THE x*. *X* = {*x*}

lemma *contents-eq* [*simp*]: *contents* ({*x*}) = *x*
by (*simp add*: *contents-def*)

end

23 Cardinal Numbers Without the Axiom of Choice

theory *Cardinal* **imports** *OrderType Finite Nat Sum* **begin**

definition

Least :: (*i*⇒*o*) ⇒ *i* (**binder** ‹*μ* › *10*) **where**
Least(*P*) ≡ *THE i*. *Ord*(*i*) ∧ *P*(*i*) ∧ (∀*j*. *j*<*i* ⟶ ¬*P*(*j*))

definition
eqpoll :: [*i*,*i*] ⇒ *o* (**infixl** ‹≈› *50*) **where**
A ≈ *B* ≡ ∃*f*. *f* ∈ *bij*(*A*,*B*)

definition
lepoll :: [*i*,*i*] ⇒ *o* (**infixl** ‹≲› *50*) **where**
A ≲ *B* ≡ ∃*f*. *f* ∈ *inj*(*A*,*B*)

definition
lesspoll :: [*i*,*i*] ⇒ *o* (**infixl** ‹≺› *50*) **where**
A ≺ *B* ≡ *A* ≲ *B* ∧ ¬(*A* ≈ *B*)

definition
cardinal :: *i*⇒*i* (‹(‹*open-block notation*=‹*mixfix cardinal*››|-|)›)
where |*A*| ≡ (*μ i*. *i* ≈ *A*)

definition
Finite :: *i*⇒*o* **where**
Finite(*A*) ≡ ∃*n*∈*nat*. *A* ≈ *n*

definition
Card :: *i*⇒*o* **where**
Card(*i*) ≡ (*i* = |*i*|)

23.1 The Schroeder-Bernstein Theorem

See Davey and Priestly, page 106

lemma *decomp-bnd-mono*: *bnd-mono*$(X, \lambda W.\ X - g``(Y - f``W))$
by (*rule bnd-monoI*, *blast+*)

lemma *Banach-last-equation*:
  $g \in Y{-}{>}X$
  $\Longrightarrow g``(Y - f``\ lfp(X, \lambda W.\ X - g``(Y - f``W))) =$
    $X - lfp(X, \lambda W.\ X - g``(Y - f``W))$
apply (*rule-tac* $P = \lambda u.\ v = X{-}u$ **for** v
  in *decomp-bnd-mono* [*THEN lfp-unfold*, *THEN ssubst*])
apply (*simp add*: *double-complement fun-is-rel* [*THEN image-subset*])
done

lemma *decomposition*:
  $[\![f \in X{-}{>}Y;\ g \in Y{-}{>}X]\!] \Longrightarrow$
  $\exists XA\ XB\ YA\ YB.\ (XA \cap XB = 0) \wedge (XA \cup XB = X) \wedge$
    $(YA \cap YB = 0) \wedge (YA \cup YB = Y) \wedge$
    $f``XA{=}YA \wedge g``YB{=}XB$
apply (*intro exI conjI*)
apply (*rule-tac* [*6*] *Banach-last-equation*)
apply (*rule-tac* [*5*] *refl*)
apply (*assumption* |
  rule Diff-disjoint Diff-partition fun-is-rel image-subset lfp-subset)+
done

lemma *schroeder-bernstein*:
  $[\![f \in inj(X,Y);\ g \in inj(Y,X)]\!] \Longrightarrow \exists h.\ h \in bij(X,Y)$
apply (*insert decomposition* [*of f X Y g*])
apply (*simp add*: *inj-is-fun*)
apply (*blast intro*!: *restrict-bij bij-disjoint-Un intro*: *bij-converse-bij*)

done

lemma *bij-imp-eqpoll*: $f \in bij(A,B) \Longrightarrow A \approx B$
 unfolding *eqpoll-def*
apply (*erule exI*)
done

lemmas *eqpoll-refl* = *id-bij* [*THEN bij-imp-eqpoll*, *simp*]

lemma *eqpoll-sym*: $X \approx Y \Longrightarrow Y \approx X$
 unfolding *eqpoll-def*
apply (*blast intro*: *bij-converse-bij*)

done

lemma *eqpoll-trans* [*trans*]:
$[\![X \approx Y; \;\; Y \approx Z]\!] \Longrightarrow X \approx Z$
unfolding *eqpoll-def*
apply (*blast intro*: *comp-bij*)
done

lemma *subset-imp-lepoll*: $X{<=}Y \Longrightarrow X \lesssim Y$
unfolding *lepoll-def*
apply (*rule exI*)
apply (*erule id-subset-inj*)
done

lemmas *lepoll-refl* = *subset-refl* [*THEN subset-imp-lepoll*, *simp*]

lemmas *le-imp-lepoll* = *le-imp-subset* [*THEN subset-imp-lepoll*]

lemma *eqpoll-imp-lepoll*: $X \approx Y \Longrightarrow X \lesssim Y$
by (*unfold eqpoll-def bij-def lepoll-def*, *blast*)

lemma *lepoll-trans* [*trans*]: $[\![X \lesssim Y; \;\; Y \lesssim Z]\!] \Longrightarrow X \lesssim Z$
unfolding *lepoll-def*
apply (*blast intro*: *comp-inj*)
done

lemma *eq-lepoll-trans* [*trans*]: $[\![X \approx Y; \;\; Y \lesssim Z]\!] \Longrightarrow X \lesssim Z$
by (*blast intro*: *eqpoll-imp-lepoll lepoll-trans*)

lemma *lepoll-eq-trans* [*trans*]: $[\![X \lesssim Y; \;\; Y \approx Z]\!] \Longrightarrow X \lesssim Z$
by (*blast intro*: *eqpoll-imp-lepoll lepoll-trans*)

lemma *eqpollI*: $[\![X \lesssim Y; \;\; Y \lesssim X]\!] \Longrightarrow X \approx Y$
unfolding *lepoll-def eqpoll-def*
apply (*elim exE*)
apply (*rule schroeder-bernstein*, *assumption+*)
done

lemma *eqpollE*:
$[\![X \approx Y; [\![X \lesssim Y; Y \lesssim X]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*blast intro*: *eqpoll-imp-lepoll eqpoll-sym*)

lemma *eqpoll-iff*: $X \approx Y \longleftrightarrow X \lesssim Y \wedge Y \lesssim X$
by (*blast intro*: *eqpollI elim*!: *eqpollE*)

lemma *lepoll-0-is-0*: $A \lesssim 0 \Longrightarrow A = 0$

unfolding *lepoll-def inj-def*
apply (*blast dest*: *apply-type*)
done

lemmas *empty-lepollI* = *empty-subsetI* [*THEN subset-imp-lepoll*]

lemma *lepoll-0-iff*: $A \lesssim 0 \longleftrightarrow A=0$
by (*blast intro*: *lepoll-0-is-0 lepoll-refl*)

lemma *Un-lepoll-Un*:
$[\![A \lesssim B;\ C \lesssim D;\ B \cap D = 0]\!] \Longrightarrow A \cup C \lesssim B \cup D$
unfolding *lepoll-def*
apply (*blast intro*: *inj-disjoint-Un*)
done

lemmas *eqpoll-0-is-0* = *eqpoll-imp-lepoll* [*THEN lepoll-0-is-0*]

lemma *eqpoll-0-iff*: $A \approx 0 \longleftrightarrow A=0$
by (*blast intro*: *eqpoll-0-is-0 eqpoll-refl*)

lemma *eqpoll-disjoint-Un*:
$[\![A \approx B;\ \ C \approx D;\ \ A \cap C = 0;\ \ B \cap D = 0]\!]$
$\Longrightarrow A \cup C \approx B \cup D$
unfolding *eqpoll-def*
apply (*blast intro*: *bij-disjoint-Un*)
done

23.2 lesspoll: contributions by Krzysztof Grabczewski

lemma *lesspoll-not-refl*: $\neg\ (i \prec i)$
by (*simp add*: *lesspoll-def*)

lemma *lesspoll-irrefl* [*elim*!]: $i \prec i \Longrightarrow P$
by (*simp add*: *lesspoll-def*)

lemma *lesspoll-imp-lepoll*: $A \prec B \Longrightarrow A \lesssim B$
by (*unfold lesspoll-def*, *blast*)

lemma *lepoll-well-ord*: $[\![A \lesssim B;\ well\text{-}ord(B,r)]\!] \Longrightarrow \exists\, s.\ well\text{-}ord(A,s)$
unfolding *lepoll-def*
apply (*blast intro*: *well-ord-rvimage*)
done

lemma *lepoll-iff-leqpoll*: $A \lesssim B \longleftrightarrow A \prec B \mid A \approx B$
unfolding *lesspoll-def*
apply (*blast intro*!: *eqpollI elim*!: *eqpollE*)
done

lemma *inj-not-surj-succ*:
  assumes *fi*: $f \in inj(A, succ(m))$ **and** *fns*: $f \notin surj(A, succ(m))$
  shows $\exists f.\ f \in inj(A,m)$
proof −
  from *fi* [*THEN inj-is-fun*] *fns*
  obtain *y* **where** *y*: $y \in succ(m)$ $\bigwedge x.\ x{\in}A \Longrightarrow f\ `\ x \neq y$
    by (*auto simp add*: *surj-def*)
  show *?thesis*
    proof
      show $(\lambda z{\in}A.\ if\ f`z = m\ then\ y\ else\ f`z) \in inj(A, m)$ **using** *y fi*
        by (*simp add*: *inj-def*)
          (*auto intro*!: *if-type* [*THEN lam-type*] *intro*: *Pi-type dest*: *apply-funtype*)
      qed
qed

lemma *lesspoll-trans* [*trans*]:
    $[\![X \prec Y;\ Y \prec Z]\!] \Longrightarrow X \prec Z$
  unfolding *lesspoll-def*
apply (*blast elim*!: *eqpollE intro*: *eqpollI lepoll-trans*)
done

lemma *lesspoll-trans1* [*trans*]:
    $[\![X \lesssim Y;\ Y \prec Z]\!] \Longrightarrow X \prec Z$
  unfolding *lesspoll-def*
apply (*blast elim*!: *eqpollE intro*: *eqpollI lepoll-trans*)
done

lemma *lesspoll-trans2* [*trans*]:
    $[\![X \prec Y;\ Y \lesssim Z]\!] \Longrightarrow X \prec Z$
  unfolding *lesspoll-def*
apply (*blast elim*!: *eqpollE intro*: *eqpollI lepoll-trans*)
done

lemma *eq-lesspoll-trans* [*trans*]:
    $[\![X \approx Y;\ Y \prec Z]\!] \Longrightarrow X \prec Z$
  by (*blast intro*: *eqpoll-imp-lepoll lesspoll-trans1*)

lemma *lesspoll-eq-trans* [*trans*]:
    $[\![X \prec Y;\ Y \approx Z]\!] \Longrightarrow X \prec Z$
  by (*blast intro*: *eqpoll-imp-lepoll lesspoll-trans2*)

lemma *Least-equality*:
  $[\![P(i);\ \ Ord(i);\ \ \bigwedge x.\ x{<}i \Longrightarrow \neg P(x)]\!] \Longrightarrow (\mu\ x.\ P(x)) = i$

```
  unfolding Least-def
apply (rule the-equality, blast)
apply (elim conjE)
apply (erule Ord-linear-lt, assumption, blast+)
done

lemma LeastI:
  assumes P: P(i) and i: Ord(i) shows P(μ x. P(x))
proof -
  { from i have P(i) ⟹ P(μ x. P(x))
      proof (induct i rule: trans-induct)
        case (step i)
        show ?case
          proof (cases P(μ a. P(a)))
            case True thus ?thesis .
          next
            case False
            hence ⋀x. x ∈ i ⟹ ¬P(x) using step
              by blast
            hence (μ a. P(a)) = i using step
              by (blast intro: Least-equality ltD)
            thus ?thesis using step.prems
              by simp
          qed
      qed
  }
  thus ?thesis using P .
qed
```

The proof is almost identical to the one above!

```
lemma Least-le:
  assumes P: P(i) and i: Ord(i) shows (μ x. P(x)) ≤ i
proof -
  { from i have P(i) ⟹ (μ x. P(x)) ≤ i
      proof (induct i rule: trans-induct)
        case (step i)
        show ?case
          proof (cases (μ a. P(a)) ≤ i)
            case True thus ?thesis .
          next
            case False
            hence ⋀x. x ∈ i ⟹ ¬ (μ a. P(a)) ≤ i using step
              by blast
            hence (μ a. P(a)) = i using step
              by (blast elim: ltE intro: ltI Least-equality lt-trans1)
            thus ?thesis using step
              by simp
          qed
      qed
```

}
thus *?thesis* **using** P .
qed

lemma *less-LeastE*: $[\![P(i); \ i < (\mu\ x.\ P(x))]\!] \Longrightarrow Q$
apply (*rule Least-le* [*THEN* [*2*] *lt-trans2*, *THEN lt-irrefl*], *assumption*+)
apply (*simp add*: *lt-Ord*)
done

lemma *LeastI2*:
$[\![P(i); \ Ord(i); \ \bigwedge j.\ P(j) \Longrightarrow Q(j)]\!] \Longrightarrow Q(\mu\ j.\ P(j))$
by (*blast intro*: *LeastI*)

lemma *Least-0*:
$[\![\neg (\exists i.\ Ord(i) \wedge P(i))]\!] \Longrightarrow (\mu\ x.\ P(x)) = 0$
unfolding *Least-def*
apply (*rule the-0*, *blast*)
done

lemma *Ord-Least* [*intro*,*simp*,*TC*]: $Ord(\mu\ x.\ P(x))$
proof (*cases* $\exists i.\ Ord(i) \wedge P(i)$)
case *True*
then obtain i **where** $P(i)$ $Ord(i)$ **by** *auto*
hence $(\mu\ x.\ P(x)) \le i$ **by** (*rule Least-le*)
thus *?thesis*
by (*elim ltE*)
next
case *False*
hence $(\mu\ x.\ P(x)) = 0$ **by** (*rule Least-0*)
thus *?thesis*
by *auto*
qed

23.3 Basic Properties of Cardinals

lemma *Least-cong*: $(\bigwedge y.\ P(y) \longleftrightarrow Q(y)) \Longrightarrow (\mu\ x.\ P(x)) = (\mu\ x.\ Q(x))$
by *simp*

lemma *cardinal-cong*: $X \approx Y \Longrightarrow |X| = |Y|$
unfolding *eqpoll-def cardinal-def*
apply (*rule Least-cong*)
apply (*blast intro*: *comp-bij bij-converse-bij*)
done

lemma *well-ord-cardinal-eqpoll*:
assumes *r*: *well-ord*(*A*,*r*) **shows** $|A| \approx A$
proof (*unfold cardinal-def*)
show (μ *i*. $i \approx A$) $\approx A$
by (*best intro*: *LeastI Ord-ordertype ordermap-bij bij-converse-bij bij-imp-eqpoll r*)
qed

lemmas *Ord-cardinal-eqpoll* = *well-ord-Memrel* [*THEN well-ord-cardinal-eqpoll*]

lemma *Ord-cardinal-idem*: *Ord*(*A*) $\Longrightarrow ||A|| = |A|$
by (*rule Ord-cardinal-eqpoll* [*THEN cardinal-cong*])

lemma *well-ord-cardinal-eqE*:
assumes *woX*: *well-ord*(*X*,*r*) **and** *woY*: *well-ord*(*Y*,*s*) **and** *eq*: $|X| = |Y|$
shows $X \approx Y$
proof −
have $X \approx |X|$ **by** (*blast intro*: *well-ord-cardinal-eqpoll* [*OF woX*] *eqpoll-sym*)
also have ... = $|Y|$ **by** (*rule eq*)
also have ... $\approx Y$ **by** (*rule well-ord-cardinal-eqpoll* [*OF woY*])
finally show *?thesis* **.**
qed

lemma *well-ord-cardinal-eqpoll-iff*:
$[\![$*well-ord*(*X*,*r*); *well-ord*(*Y*,*s*)$]\!] \Longrightarrow |X| = |Y| \longleftrightarrow X \approx Y$
by (*blast intro*: *cardinal-cong well-ord-cardinal-eqE*)

lemma *Ord-cardinal-le*: *Ord*(*i*) $\Longrightarrow |i| \le i$
unfolding *cardinal-def*
apply (*erule eqpoll-refl* [*THEN Least-le*])
done

lemma *Card-cardinal-eq*: *Card*(*K*) $\Longrightarrow |K| = K$
unfolding *Card-def*
apply (*erule sym*)
done

lemma *CardI*: $[\![$*Ord*(*i*); $\bigwedge j.\ j<i \Longrightarrow \neg(j \approx i)]\!] \Longrightarrow$ *Card*(*i*)
unfolding *Card-def cardinal-def*
apply (*subst Least-equality*)
apply (*blast intro*: *eqpoll-refl*)+
done

lemma *Card-is-Ord*: *Card*(*i*) $\Longrightarrow$ *Ord*(*i*)

unfolding *Card-def cardinal-def*
apply (*erule ssubst*)
apply (*rule Ord-Least*)
done

lemma *Card-cardinal-le*: $Card(K) \Longrightarrow K \leq |K|$
apply (*simp* (*no-asm-simp*) *add*: *Card-is-Ord Card-cardinal-eq*)
done

lemma *Ord-cardinal* [*simp*,*intro*!]: $Ord(|A|)$
unfolding *cardinal-def*
apply (*rule Ord-Least*)
done

The cardinals are the initial ordinals.

lemma *Card-iff-initial*: $Card(K) \longleftrightarrow Ord(K) \wedge (\forall j.\ j<K \longrightarrow \neg\ j \approx K)$
proof −
{ **fix** j
assume K: $Card(K)$ $j \approx K$
assume $j < K$
also have ... $= (\mu\ i.\ i \approx K)$ **using** K
by (*simp add*: *Card-def cardinal-def*)
finally have $j < (\mu\ i.\ i \approx K)$ **.**
hence *False* **using** K
by (*best dest*: *less-LeastE*)
}
then show *?thesis*
by (*blast intro*: *CardI Card-is-Ord*)
qed

lemma *lt-Card-imp-lesspoll*: $[\![Card(a);\ i<a]\!] \Longrightarrow i \prec a$
unfolding *lesspoll-def*
apply (*drule Card-iff-initial* [*THEN iffD1*])
apply (*blast intro*!: *leI* [*THEN le-imp-lepoll*])
done

lemma *Card-0*: $Card(0)$
apply (*rule Ord-0* [*THEN CardI*])
apply (*blast elim*!: *ltE*)
done

lemma *Card-Un*: $[\![Card(K);\ \ Card(L)]\!] \Longrightarrow Card(K \cup L)$
apply (*rule Ord-linear-le* [*of K L*])
apply (*simp-all add*: *subset-Un-iff* [*THEN iffD1*] *Card-is-Ord le-imp-subset*
subset-Un-iff2 [*THEN iffD1*])
done

lemma *Card-cardinal* [*iff*]: *Card*(|*A*|)
proof (*unfold cardinal-def*)
  show *Card*(μ *i*. *i* $\approx$ *A*)
    proof (*cases* $\exists$ *i*. *Ord* (*i*) $\wedge$ *i* $\approx$ *A*)
      case *False* **thus** *?thesis* — degenerate case
        by (*simp add*: *Least-0 Card-0*)
    next
      case *True* — real case: *A* is isomorphic to some ordinal
      then obtain *i* **where** *i*: *Ord*(*i*) *i* $\approx$ *A* **by** *blast*
      show *?thesis*
        proof (*rule CardI* [*OF Ord-Least*], *rule notI*)
          fix *j*
          assume *j*: *j* < (μ *i*. *i* $\approx$ *A*)
          assume *j* $\approx$ (μ *i*. *i* $\approx$ *A*)
          also have ... $\approx$ *A* **using** *i* **by** (*auto intro*: *LeastI*)
          finally have *j* $\approx$ *A* **.**
          thus *False*
            by (*rule less-LeastE* [*OF* - *j*])
        qed
    qed
qed

lemma *cardinal-eq-lemma*:
  assumes *i*:|*i*| $\le$ *j* **and** *j*: *j* $\le$ *i* **shows** |*j*| = |*i*|
proof (*rule eqpollI* [*THEN cardinal-cong*])
  show *j* $\lesssim$ *i* **by** (*rule le-imp-lepoll* [*OF j*])
next
  have *Oi*: *Ord*(*i*) **using** *j* **by** (*rule le-Ord2*)
  hence *i* $\approx$ |*i*|
    by (*blast intro*: *Ord-cardinal-eqpoll eqpoll-sym*)
  also have ... $\lesssim$ *j*
    by (*blast intro*: *le-imp-lepoll i*)
  finally show *i* $\lesssim$ *j* **.**
qed

lemma *cardinal-mono*:
  assumes *ij*: *i* $\le$ *j* **shows** |*i*| $\le$ |*j*|
using *Ord-cardinal* [*of i*] *Ord-cardinal* [*of j*]
proof (*cases rule*: *Ord-linear-le*)
  case *le* **thus** *?thesis* **.**
next
  case *ge*
  have *i*: *Ord*(*i*) **using** *ij*
    by (*simp add*: *lt-Ord*)
  have *ci*: |*i*| $\le$ *j*
    by (*blast intro*: *Ord-cardinal-le ij le-trans i*)
  have |*i*| = ||*i*||
    by (*auto simp add*: *Ord-cardinal-idem i*)

```
  also have ... = |j|
    by (rule cardinal-eq-lemma [OF ge ci])
  finally have |i| = |j| .
  thus ?thesis by simp
qed
```

Since we have $|succ(nat)| \leq |nat|$, the converse of *cardinal-mono* fails!

```
lemma cardinal-lt-imp-lt: ⟦|i| < |j|;  Ord(i);  Ord(j)⟧ ⟹ i < j
apply (rule Ord-linear2 [of i j], assumption+)
apply (erule lt-trans2 [THEN lt-irrefl])
apply (erule cardinal-mono)
done

lemma Card-lt-imp-lt: ⟦|i| < K;  Ord(i);  Card(K)⟧ ⟹ i < K
  by (simp (no-asm-simp) add: cardinal-lt-imp-lt Card-is-Ord Card-cardinal-eq)

lemma Card-lt-iff: ⟦Ord(i);  Card(K)⟧ ⟹ (|i| < K) ⟷ (i < K)
by (blast intro: Card-lt-imp-lt Ord-cardinal-le [THEN lt-trans1])

lemma Card-le-iff: ⟦Ord(i);  Card(K)⟧ ⟹ (K ≤ |i|) ⟷ (K ≤ i)
by (simp add: Card-lt-iff Card-is-Ord Ord-cardinal not-lt-iff-le [THEN iff-sym])


lemma well-ord-lepoll-imp-cardinal-le:
  assumes wB: well-ord(B,r) and AB: A ≲ B
  shows |A| ≤ |B|
using Ord-cardinal [of A] Ord-cardinal [of B]
proof (cases rule: Ord-linear-le)
  case le thus ?thesis .
next
  case ge
  from lepoll-well-ord [OF AB wB]
  obtain s where s: well-ord(A, s) by blast
  have B  ≈ |B| by (blast intro: wB eqpoll-sym well-ord-cardinal-eqpoll)
  also have ... ≲ |A| by (rule le-imp-lepoll [OF ge])
  also have ... ≈ A by (rule well-ord-cardinal-eqpoll [OF s])
  finally have B ≲ A .
  hence A ≈ B by (blast intro: eqpollI AB)
  hence |A| = |B| by (rule cardinal-cong)
  thus ?thesis by simp
qed

lemma lepoll-cardinal-le: ⟦A ≲ i; Ord(i)⟧ ⟹ |A| ≤ i
apply (rule le-trans)
apply (erule well-ord-Memrel [THEN well-ord-lepoll-imp-cardinal-le], assumption)
apply (erule Ord-cardinal-le)
done

lemma lepoll-Ord-imp-eqpoll: ⟦A ≲ i; Ord(i)⟧ ⟹ |A| ≈ A
```

by (*blast intro*: *lepoll-cardinal-le well-ord-Memrel well-ord-cardinal-eqpoll dest*!: *lepoll-well-ord*)

lemma *lesspoll-imp-eqpoll*: $[\![A \prec i;\ Ord(i)]\!] \Longrightarrow |A| \approx A$
  unfolding *lesspoll-def*
apply (*blast intro*: *lepoll-Ord-imp-eqpoll*)
done

lemma *cardinal-subset-Ord*: $[\![A<=i;\ Ord(i)]\!] \Longrightarrow |A| \subseteq i$
apply (*drule subset-imp-lepoll* [*THEN lepoll-cardinal-le*])
apply (*auto simp add*: *lt-def*)
apply (*blast intro*: *Ord-trans*)
done

23.4 The finite cardinals

lemma *cons-lepoll-consD*:
$[\![cons(u,A) \lesssim cons(v,B);\ \ u \notin A;\ \ v \notin B]\!] \Longrightarrow A \lesssim B$
apply (*unfold lepoll-def inj-def*, *safe*)
apply (*rule-tac* $x = \lambda x{\in}A.$ *if f'x=v then f'u else f'x* **in** *exI*)
apply (*rule CollectI*)

apply (*rule if-type* [*THEN lam-type*])
apply (*blast dest*: *apply-funtype*)
apply (*blast elim*!: *mem-irrefl dest*: *apply-funtype*)

apply (*simp* (*no-asm-simp*))
apply *blast*
done

lemma *cons-eqpoll-consD*: $[\![cons(u,A) \approx cons(v,B);\ \ u \notin A;\ \ v \notin B]\!] \Longrightarrow A \approx B$
apply (*simp add*: *eqpoll-iff*)
apply (*blast intro*: *cons-lepoll-consD*)
done

lemma *succ-lepoll-succD*: $succ(m) \lesssim succ(n) \Longrightarrow m \lesssim n$
  unfolding *succ-def*
apply (*erule cons-lepoll-consD*)
apply (*rule mem-not-refl*)+
done

lemma *nat-lepoll-imp-le*:
    $m \in nat \Longrightarrow n \in nat \Longrightarrow m \lesssim n \Longrightarrow m \le n$
proof (*induct m arbitrary*: *n rule*: *nat-induct*)
  case *0* **thus** *?case* **by** (*blast intro*!: *nat-0-le*)
next
  case (*succ m*)

show *?case* **using** ‹$n \in nat$›
proof (*cases rule*: *natE*)
case *0* **thus** *?thesis* **using** *succ*
by (*simp add*: *lepoll-def inj-def*)
next
case (*succ n'*) **thus** *?thesis* **using** *succ.hyps* ‹ $succ(m) \lesssim n$›
by (*blast intro*!: *succ-leI dest*!: *succ-lepoll-succD*)
qed
qed

lemma *nat-eqpoll-iff*: $[\![m \in nat; n \in nat]\!] \Longrightarrow m \approx n \longleftrightarrow m = n$
apply (*rule iffI*)
apply (*blast intro*: *nat-lepoll-imp-le le-anti-sym elim*!: *eqpollE*)
apply (*simp add*: *eqpoll-refl*)
done

lemma *nat-into-Card*:
assumes *n*: $n \in nat$ **shows** *Card*(n)
proof (*unfold Card-def cardinal-def*, *rule sym*)
have *Ord*(n) **using** *n* **by** *auto*
moreover
{ fix *i*
assume $i < n$ $i \approx n$
hence *False* **using** *n*
by (*auto simp add*: *lt-nat-in-nat* [*THEN nat-eqpoll-iff*])
}
ultimately show $(\mu\ i.\ i \approx n) = n$ **by** (*auto intro*!: *Least-equality*)
qed

lemmas *cardinal-0* = *nat-0I* [*THEN nat-into-Card*, *THEN Card-cardinal-eq*, *iff*]
lemmas *cardinal-1* = *nat-1I* [*THEN nat-into-Card*, *THEN Card-cardinal-eq*, *iff*]

lemma *succ-lepoll-natE*: $[\![succ(n) \lesssim n;\ n \in nat]\!] \Longrightarrow P$
by (*rule nat-lepoll-imp-le* [*THEN lt-irrefl*], *auto*)

lemma *nat-lepoll-imp-ex-eqpoll-n*:
$[\![n \in nat;\ nat \lesssim X]\!] \Longrightarrow \exists\, Y.\ Y \subseteq X \land n \approx Y$
unfolding *lepoll-def eqpoll-def*
apply (*fast del*: *subsetI subsetCE*
intro!: *subset-SIs*
dest!: *Ord-nat* [*THEN* [*2*] *OrdmemD*, *THEN* [*2*] *restrict-inj*]
elim!: *restrict-bij*
inj-is-fun [*THEN fun-is-rel*, *THEN image-subset*])
done

lemma *lepoll-succ*: $i \lesssim succ(i)$
  by (*blast intro*: *subset-imp-lepoll*)

lemma *lepoll-imp-lesspoll-succ*:
  assumes *A*: $A \lesssim m$ **and** *m*: $m \in nat$
  shows $A \prec succ(m)$
proof −
  { assume $A \approx succ(m)$
    hence $succ(m) \approx A$ **by** (*rule eqpoll-sym*)
    also have ... $\lesssim m$ **by** (*rule A*)
    finally have $succ(m) \lesssim m$ **.**
    hence *False* **by** (*rule succ-lepoll-natE*) (*rule m*) **}**
  moreover have $A \lesssim succ(m)$ **by** (*blast intro*: *lepoll-trans A lepoll-succ*)
  ultimately show *?thesis* **by** (*auto simp add*: *lesspoll-def*)
qed

lemma *lesspoll-succ-imp-lepoll*:
    $\llbracket A \prec succ(m);\ m \in nat \rrbracket \Longrightarrow A \lesssim m$
  unfolding *lesspoll-def lepoll-def eqpoll-def bij-def*
apply (*auto dest*: *inj-not-surj-succ*)
done

lemma *lesspoll-succ-iff*: $m \in nat \Longrightarrow A \prec succ(m) \longleftrightarrow A \lesssim m$
by (*blast intro*!: *lepoll-imp-lesspoll-succ lesspoll-succ-imp-lepoll*)

lemma *lepoll-succ-disj*: $\llbracket A \lesssim succ(m);\ \ m \in nat \rrbracket \Longrightarrow A \lesssim m \mid A \approx succ(m)$
apply (*rule disjCI*)
apply (*rule lesspoll-succ-imp-lepoll*)
prefer *2* **apply** *assumption*
apply (*simp* (*no-asm-simp*) *add*: *lesspoll-def*)
done

lemma *lesspoll-cardinal-lt*: $\llbracket A \prec i;\ Ord(i) \rrbracket \Longrightarrow |A| < i$
apply (*unfold lesspoll-def*, *clarify*)
apply (*frule lepoll-cardinal-le*, *assumption*)
apply (*blast intro*: *well-ord-Memrel well-ord-cardinal-eqpoll* [*THEN eqpoll-sym*]
            dest: *lepoll-well-ord* *elim*!: *leE*)
done

23.5 The first infinite cardinal: Omega, or nat

lemma *lt-not-lepoll*:
  assumes *n*: $n<i$ $n \in nat$ **shows** $\neg\ i \lesssim n$
proof −
  { assume *i*: $i \lesssim n$
    have $succ(n) \lesssim i$ **using** *n*
      by (*elim ltE*, *blast intro*: *Ord-succ-subsetI* [*THEN subset-imp-lepoll*])

also have ... $\lesssim$ *n* **by** (*rule i*)
finally have *succ*(*n*) $\lesssim$ *n* **.**
hence *False* **by** (*rule succ-lepoll-natE*) (*rule n*) **}**
thus *?thesis* **by** *auto*
qed

A slightly weaker version of *nat-eqpoll-iff*

lemma *Ord-nat-eqpoll-iff*:
assumes *i*: *Ord*(*i*) **and** *n*: *n* ∈ *nat* **shows** *i* ≈ *n* ⟷ *i*=*n*
using *i nat-into-Ord* [*OF n*]
proof (*cases rule*: *Ord-linear-lt*)
case *lt*
hence *i* ∈ *nat* **by** (*rule lt-nat-in-nat*) (*rule n*)
thus *?thesis* **by** (*simp add*: *nat-eqpoll-iff n*)
next
case *eq*
thus *?thesis* **by** (*simp add*: *eqpoll-refl*)
next
case *gt*
hence ¬ *i* $\lesssim$ *n* **using** *n* **by** (*rule lt-not-lepoll*)
hence ¬ *i* ≈ *n* **using** *n* **by** (*blast intro*: *eqpoll-imp-lepoll*)
moreover have *i* ≠ *n* **using** ‹*n*<*i*› **by** *auto*
ultimately show *?thesis* **by** *blast*
qed

lemma *Card-nat*: *Card*(*nat*)
proof −
{ fix *i*
assume *i*: *i* < *nat i* ≈ *nat*
hence ¬ *nat* $\lesssim$ *i*
by (*simp add*: *lt-def lt-not-lepoll*)
hence *False* **using** *i*
by (*simp add*: *eqpoll-iff*)
}
hence (*μ i. i* ≈ *nat*) = *nat* **by** (*blast intro*: *Least-equality eqpoll-refl*)
thus *?thesis*
by (*auto simp add*: *Card-def cardinal-def*)
qed

lemma *nat-le-cardinal*: *nat* ≤ *i* ⟹ *nat* ≤ |*i*|
apply (*rule Card-nat* [*THEN Card-cardinal-eq*, *THEN subst*])
apply (*erule cardinal-mono*)
done

lemma *n-lesspoll-nat*: *n* ∈ *nat* ⟹ *n* ≺ *nat*
by (*blast intro*: *Ord-nat Card-nat ltI lt-Card-imp-lesspoll*)

23.6 Towards Cardinal Arithmetic

lemma *cons-lepoll-cong*:
  $[\![A \lesssim B;\ \ b \notin B]\!] \Longrightarrow cons(a,A) \lesssim cons(b,B)$
apply (*unfold lepoll-def*, *safe*)
apply (*rule-tac* $x = \lambda y{\in}cons\ (a,A)\ .\ if\ y{=}a\ then\ b\ else\ f`y$ **in** *exI*)
apply (*rule-tac* $d = \lambda z.\ if\ z \in B\ then\ converse\ (f)\ `z\ else\ a$ **in** *lam-injective*)
apply (*safe elim*!: *consE'*)
  apply *simp-all*
apply (*blast intro*: *inj-is-fun* [*THEN apply-type*])+
done

lemma *cons-eqpoll-cong*:
  $[\![A \approx B;\ \ a \notin A;\ \ b \notin B]\!] \Longrightarrow cons(a,A) \approx cons(b,B)$
by (*simp add*: *eqpoll-iff cons-lepoll-cong*)

lemma *cons-lepoll-cons-iff*:
  $[\![a \notin A;\ \ b \notin B]\!] \Longrightarrow cons(a,A) \lesssim cons(b,B) \ \longleftrightarrow\ A \lesssim B$
by (*blast intro*: *cons-lepoll-cong cons-lepoll-consD*)

lemma *cons-eqpoll-cons-iff*:
  $[\![a \notin A;\ \ b \notin B]\!] \Longrightarrow cons(a,A) \approx cons(b,B) \ \longleftrightarrow\ A \approx B$
by (*blast intro*: *cons-eqpoll-cong cons-eqpoll-consD*)

lemma *singleton-eqpoll-1*: $\{a\} \approx 1$
 unfolding *succ-def*
apply (*blast intro*!: *eqpoll-refl* [*THEN cons-eqpoll-cong*])
done

lemma *cardinal-singleton*: $|\{a\}| = 1$
apply (*rule singleton-eqpoll-1* [*THEN cardinal-cong*, *THEN trans*])
apply (*simp* (*no-asm*) *add*: *nat-into-Card* [*THEN Card-cardinal-eq*])
done

lemma *not-0-is-lepoll-1*: $A \neq 0 \Longrightarrow 1 \lesssim A$
apply (*erule not-emptyE*)
apply (*rule-tac* $a = cons\ (x,\ A{-}\{x\})$ **in** *subst*)
apply (*rule-tac* [*2*] $a = cons(0,0)$ **and** $P{=}\ \lambda y.\ y \lesssim cons\ (x,\ A{-}\{x\})$ **in** *subst*)
prefer *3* **apply** (*blast intro*: *cons-lepoll-cong subset-imp-lepoll*, *auto*)
done

lemma *succ-eqpoll-cong*: $A \approx B \Longrightarrow succ(A) \approx succ(B)$
 unfolding *succ-def*
apply (*simp add*: *cons-eqpoll-cong mem-not-refl*)
done

lemma *sum-eqpoll-cong*: $[\![A \approx C;\ \ B \approx D]\!] \Longrightarrow A{+}B \approx C{+}D$
 unfolding *eqpoll-def*

apply (*blast intro*!: *sum-bij*)
done

lemma *prod-eqpoll-cong*:
$[\![A \approx C;\ \ B \approx D]\!] \Longrightarrow A{*}B \approx C{*}D$
unfolding *eqpoll-def*
apply (*blast intro*!: *prod-bij*)
done

lemma *inj-disjoint-eqpoll*:
$[\![f \in inj(A,B);\ \ A \cap B = 0]\!] \Longrightarrow A \cup (B - range(f)) \approx B$
unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule-tac* $c = \lambda x.\ if\ x \in A\ then\ f`x\ else\ x$
and $d = \lambda y.\ if\ y \in range\ (f)\ then\ converse\ (f)\ `y\ else\ y$
in *lam-bijective*)
apply (*blast intro*!: *if-type inj-is-fun* [*THEN apply-type*])
apply (*simp* (*no-asm-simp*) *add*: *inj-converse-fun* [*THEN apply-funtype*])
apply (*safe elim*!: *UnE′*)
apply (*simp-all add*: *inj-is-fun* [*THEN apply-rangeI*])
apply (*blast intro*: *inj-converse-fun* [*THEN apply-type*])+
done

23.7 Lemmas by Krzysztof Grabczewski

If A has at most $n + 1$ elements and $a \in A$ then $A - \{a\}$ has at most n.

lemma *Diff-sing-lepoll*:
$[\![a \in A;\ \ A \lesssim succ(n)]\!] \Longrightarrow A - \{a\} \lesssim n$
unfolding *succ-def*
apply (*rule cons-lepoll-consD*)
apply (*rule-tac* [*3*] *mem-not-refl*)
apply (*erule cons-Diff* [*THEN ssubst*], *safe*)
done

If A has at least $n + 1$ elements then $A - \{a\}$ has at least n.

lemma *lepoll-Diff-sing*:
assumes *A*: $succ(n) \lesssim A$ **shows** $n \lesssim A - \{a\}$
proof −
have $cons(n,n) \lesssim A$ **using** *A*
by (*unfold succ-def*)
also have ... $\lesssim cons(a,\ A{-}\{a\})$
by (*blast intro*: *subset-imp-lepoll*)
finally have $cons(n,n) \lesssim cons(a,\ A{-}\{a\})$ **.**
thus *?thesis*
by (*blast intro*: *cons-lepoll-consD mem-irrefl*)
qed

lemma *Diff-sing-eqpoll*: $[\![a \in A;\ A \approx succ(n)]\!] \Longrightarrow A - \{a\} \approx n$

by (*blast intro*!: *eqpollI*
elim!: *eqpollE*
intro: *Diff-sing-lepoll lepoll-Diff-sing*)

lemma *lepoll-1-is-sing*: $[\![A \lesssim 1;\ a \in A]\!] \Longrightarrow A = \{a\}$
apply (*frule Diff-sing-lepoll*, *assumption*)
apply (*drule lepoll-0-is-0*)
apply (*blast elim*: *equalityE*)
done

lemma *Un-lepoll-sum*: $A \cup B \lesssim A{+}B$
unfolding *lepoll-def*
apply (*rule-tac* $x = \lambda x{\in}A \cup B.$ *if* $x{\in}A$ *then Inl* (x) *else Inr* (x) **in** *exI*)
apply (*rule-tac* $d = \lambda z.$ *snd* (z) **in** *lam-injective*)
apply *force*
apply (*simp add*: *Inl-def Inr-def*)
done

lemma *well-ord-Un*:
$[\![$*well-ord*(X,R); *well-ord*$(Y,S)]\!] \Longrightarrow \exists\, T.$ *well-ord*$(X \cup Y,\ T)$
by (*erule well-ord-radd* [*THEN Un-lepoll-sum* [*THEN lepoll-well-ord*]],
assumption)

lemma *disj-Un-eqpoll-sum*: $A \cap B = 0 \Longrightarrow A \cup B \approx A + B$
unfolding *eqpoll-def*
apply (*rule-tac* $x = \lambda a{\in}A \cup B.$ *if* $a \in A$ *then Inl* (a) *else Inr* (a) **in** *exI*)
apply (*rule-tac* $d = \lambda z.$ *case* $(\lambda x.\ x,\ \lambda x.\ x,\ z)$ **in** *lam-bijective*)
apply *auto*
done

23.8 Finite and infinite sets

lemma *eqpoll-imp-Finite-iff*: $A \approx B \Longrightarrow$ *Finite*$(A) \longleftrightarrow$ *Finite*(B)
unfolding *Finite-def*
apply (*blast intro*: *eqpoll-trans eqpoll-sym*)
done

lemma *Finite-0* [*simp*]: *Finite*(*0*)
unfolding *Finite-def*
apply (*blast intro*!: *eqpoll-refl nat-0I*)
done

lemma *Finite-cons*: *Finite*$(x) \Longrightarrow$ *Finite*(*cons*(y,x))
unfolding *Finite-def*
apply (*case-tac* $y \in x$)
apply (*simp add*: *cons-absorb*)
apply (*erule bexE*)
apply (*rule bexI*)

apply (*erule-tac* [*2*] *nat-succI*)
apply (*simp* (*no-asm-simp*) *add*: *succ-def cons-eqpoll-cong mem-not-refl*)
done

lemma *Finite-succ*: *Finite*(x) $\Longrightarrow$ *Finite*(*succ*(x))
  unfolding *succ-def*
apply (*erule Finite-cons*)
done

lemma *lepoll-nat-imp-Finite*:
  assumes A: $A \lesssim n$ **and** n: $n \in nat$ **shows** *Finite*(A)
proof −
  have $A \lesssim n \Longrightarrow$ *Finite*(A) **using** n
    proof (*induct n*)
      case *0*
      hence $A = 0$ **by** (*rule lepoll-0-is-0*)
      thus *?case* **by** *simp*
    next
      case (*succ n*)
      hence $A \lesssim n \lor A \approx succ(n)$ **by** (*blast dest*: *lepoll-succ-disj*)
      thus *?case* **using** *succ* **by** (*auto simp add*: *Finite-def*)
    qed
  thus *?thesis* **using** A **.**
qed

lemma *lesspoll-nat-is-Finite*:
    $A \prec nat \Longrightarrow$ *Finite*(A)
  unfolding *Finite-def*
apply (*blast dest*: *ltD lesspoll-cardinal-lt*
          lesspoll-imp-eqpoll [*THEN eqpoll-sym*])
done

lemma *lepoll-Finite*:
  assumes Y: $Y \lesssim X$ **and** X: *Finite*(X) **shows** *Finite*(Y)
proof −
  obtain n **where** n: $n \in nat$ $X \approx n$ **using** X
    by (*auto simp add*: *Finite-def*)
  have $Y \lesssim X$ **by** (*rule Y*)
  also have ... $\approx n$ **by** (*rule n*)
  finally have $Y \lesssim n$ **.**
  thus *?thesis* **using** n **by** (*simp add*: *lepoll-nat-imp-Finite*)
qed

lemmas *subset-Finite* = *subset-imp-lepoll* [*THEN lepoll-Finite*]

lemma *Finite-cons-iff* [*iff*]: *Finite*(*cons*(y,x)) $\longleftrightarrow$ *Finite*(x)
by (*blast intro*: *Finite-cons subset-Finite*)

lemma *Finite-succ-iff* [*iff*]: *Finite*(*succ*(x)) $\longleftrightarrow$ *Finite*(x)

by (*simp add*: *succ-def*)

lemma *Finite-Int*: *Finite*(*A*) | *Finite*(*B*) $\Longrightarrow$ *Finite*(*A* $\cap$ *B*)
by (*blast intro*: *subset-Finite*)

lemmas *Finite-Diff* = *Diff-subset* [*THEN subset-Finite*]

lemma *nat-le-infinite-Ord*:
$[\![Ord(i);\ \neg\ Finite(i)]\!] \Longrightarrow nat \leq i$
unfolding *Finite-def*
apply (*erule Ord-nat* [*THEN* [*2*] *Ord-linear2*])
prefer *2* **apply** *assumption*
apply (*blast intro*!: *eqpoll-refl elim*!: *ltE*)
done

lemma *Finite-imp-well-ord*:
Finite(*A*) $\Longrightarrow \exists\, r.$ *well-ord*(*A*,*r*)
unfolding *Finite-def eqpoll-def*
apply (*blast intro*: *well-ord-rvimage bij-is-inj well-ord-Memrel nat-into-Ord*)
done

lemma *succ-lepoll-imp-not-empty*: *succ*(*x*) $\lesssim$ *y* $\Longrightarrow$ *y* $\neq$ *0*
by (*fast dest*!: *lepoll-0-is-0*)

lemma *eqpoll-succ-imp-not-empty*: *x* $\approx$ *succ*(*n*) $\Longrightarrow$ *x* $\neq$ *0*
by (*fast elim*!: *eqpoll-sym* [*THEN eqpoll-0-is-0*, *THEN succ-neq-0*])

lemma *Finite-Fin-lemma* [*rule-format*]:
n $\in$ *nat* $\Longrightarrow \forall\, A.$ (*A*$\approx$*n* $\wedge$ *A* $\subseteq$ *X*) $\longrightarrow$ *A* $\in$ *Fin*(*X*)
apply (*induct-tac n*)
apply (*rule allI*)
apply (*fast intro*!: *Fin.emptyI dest*!: *eqpoll-imp-lepoll* [*THEN lepoll-0-is-0*])
apply (*rule allI*)
apply (*rule impI*)
apply (*erule conjE*)
apply (*rule eqpoll-succ-imp-not-empty* [*THEN not-emptyE*], *assumption*)
apply (*frule Diff-sing-eqpoll*, *assumption*)
apply (*erule allE*)
apply (*erule impE*, *fast*)
apply (*drule subsetD*, *assumption*)
apply (*drule Fin.consI*, *assumption*)
apply (*simp add*: *cons-Diff*)
done

lemma *Finite-Fin*: $[\![$*Finite*(*A*); *A* $\subseteq$ *X*$]\!] \Longrightarrow$ *A* $\in$ *Fin*(*X*)
by (*unfold Finite-def*, *blast intro*: *Finite-Fin-lemma*)

lemma *Fin-lemma* [*rule-format*]: *n* $\in$ *nat* $\Longrightarrow \forall\, A.$ *A* $\approx$ *n* $\longrightarrow$ *A* $\in$ *Fin*(*A*)
apply (*induct-tac n*)

apply (*simp add*: *eqpoll-0-iff*, *clarify*)
apply (*subgoal-tac* ∃ *u*. *u* ∈ *A*)
apply (*erule exE*)
apply (*rule Diff-sing-eqpoll* [*elim-format*])
prefer *2* **apply** *assumption*
apply *assumption*
apply (*rule-tac b* = *A* **in** *cons-Diff* [*THEN subst*], *assumption*)
apply (*rule Fin.consI*, *blast*)
apply (*blast intro*: *subset-consI* [*THEN Fin-mono*, *THEN subsetD*])

unfolding *eqpoll-def*
apply (*blast intro*: *bij-converse-bij* [*THEN bij-is-fun*, *THEN apply-type*])
done

lemma *Finite-into-Fin*: *Finite*(*A*) ⟹ *A* ∈ *Fin*(*A*)
unfolding *Finite-def*
apply (*blast intro*: *Fin-lemma*)
done

lemma *Fin-into-Finite*: *A* ∈ *Fin*(*U*) ⟹ *Finite*(*A*)
by (*fast intro*!: *Finite-0 Finite-cons elim*: *Fin-induct*)

lemma *Finite-Fin-iff*: *Finite*(*A*) ⟷ *A* ∈ *Fin*(*A*)
by (*blast intro*: *Finite-into-Fin Fin-into-Finite*)

lemma *Finite-Un*: ⟦*Finite*(*A*); *Finite*(*B*)⟧ ⟹ *Finite*(*A* ∪ *B*)
by (*blast intro*!: *Fin-into-Finite Fin-UnI*
dest!: *Finite-into-Fin*
intro: *Un-upper1* [*THEN Fin-mono*, *THEN subsetD*]
Un-upper2 [*THEN Fin-mono*, *THEN subsetD*])

lemma *Finite-Un-iff* [*simp*]: *Finite*(*A* ∪ *B*) ⟷ (*Finite*(*A*) ∧ *Finite*(*B*))
by (*blast intro*: *subset-Finite Finite-Un*)

The converse must hold too.

lemma *Finite-Union*: ⟦∀ *y*∈*X*. *Finite*(*y*); *Finite*(*X*)⟧ ⟹ *Finite*(⋃(*X*))
apply (*simp add*: *Finite-Fin-iff*)
apply (*rule Fin-UnionI*)
apply (*erule Fin-induct*, *simp*)
apply (*blast intro*: *Fin.consI Fin-mono* [*THEN* [*2*] *rev-subsetD*])
done

lemma *Finite-induct* [*case-names 0 cons*, *induct set*: *Finite*]:
⟦*Finite*(*A*); *P*(*0*);
⋀*x B*. ⟦*Finite*(*B*); *x* ∉ *B*; *P*(*B*)⟧ ⟹ *P*(*cons*(*x*, *B*))⟧
⟹ *P*(*A*)
apply (*erule Finite-into-Fin* [*THEN Fin-induct*])
apply (*blast intro*: *Fin-into-Finite*)+

done

lemma *Diff-sing-Finite*: $Finite(A - \{a\}) \Longrightarrow Finite(A)$
unfolding *Finite-def*
apply (*case-tac* $a \in A$)
apply (*subgoal-tac* [*2*] $A-\{a\}=A$, *auto*)
apply (*rule-tac* $x = succ\ (n)$ **in** *bexI*)
apply (*subgoal-tac* $cons\ (a, A - \{a\}) = A \wedge cons\ (n, n) = succ\ (n)$)
apply (*drule-tac* $a = a$ **and** $b = n$ **in** *cons-eqpoll-cong*)
apply (*auto dest*: *mem-irrefl*)
done

lemma *Diff-Finite* [*rule-format*]: $Finite(B) \Longrightarrow Finite(A-B) \longrightarrow Finite(A)$
apply (*erule Finite-induct*, *auto*)
apply (*case-tac* $x \in A$)
apply (*subgoal-tac* [*2*] $A-cons\ (x, B) = A - B$)
apply (*subgoal-tac* $A - cons\ (x, B) = (A - B) - \{x\}$, *simp*)
apply (*drule Diff-sing-Finite*, *auto*)
done

lemma *Finite-RepFun*: $Finite(A) \Longrightarrow Finite(RepFun(A,f))$
by (*erule Finite-induct*, *simp-all*)

lemma *Finite-RepFun-iff-lemma* [*rule-format*]:
$[\![Finite(x); \bigwedge x\ y.\ f(x)=f(y) \Longrightarrow x=y]\!]$
$\Longrightarrow \forall A.\ x = RepFun(A,f) \longrightarrow Finite(A)$
apply (*erule Finite-induct*)
apply *clarify*
apply (*case-tac* $A=0$, *simp*)
apply (*blast del*: *allE*, *clarify*)
apply (*subgoal-tac* $\exists z{\in}A.\ x = f(z)$)
prefer *2* **apply** (*blast del*: *allE elim*: *equalityE*, *clarify*)
apply (*subgoal-tac* $B = \{f(u)\ .\ u \in A - \{z\}\}$)
apply (*blast intro*: *Diff-sing-Finite*)
apply (*thin-tac* $\forall A.\ P(A) \longrightarrow Finite(A)$ **for** P)
apply (*rule equalityI*)
apply (*blast intro*: *elim*: *equalityE*)
apply (*blast intro*: *elim*: *equalityCE*)
done

I don't know why, but if the premise is expressed using meta-connectives then the simplifier cannot prove it automatically in conditional rewriting.

lemma *Finite-RepFun-iff*:
$(\forall x\ y.\ f(x)=f(y) \longrightarrow x=y) \Longrightarrow Finite(RepFun(A,f)) \longleftrightarrow Finite(A)$
by (*blast intro*: *Finite-RepFun Finite-RepFun-iff-lemma* [*of - f*])

lemma *Finite-Pow*: $Finite(A) \Longrightarrow Finite(Pow(A))$

apply (*erule Finite-induct*)
apply (*simp-all add*: *Pow-insert Finite-Un Finite-RepFun*)
done

lemma *Finite-Pow-imp-Finite*: *Finite*(*Pow*(*A*)) $\Longrightarrow$ *Finite*(*A*)
apply (*subgoal-tac Finite*($\{\{x\} . x \in A\}$))
 apply (*simp add*: *Finite-RepFun-iff*)
apply (*blast intro*: *subset-Finite*)
done

lemma *Finite-Pow-iff* [*iff*]: *Finite*(*Pow*(*A*)) $\longleftrightarrow$ *Finite*(*A*)
by (*blast intro*: *Finite-Pow Finite-Pow-imp-Finite*)

lemma *Finite-cardinal-iff*:
  assumes *i*: *Ord*(*i*) **shows** *Finite*($|i|$) $\longleftrightarrow$ *Finite*(*i*)
 by (*auto simp add*: *Finite-def*) (*blast intro*: *eqpoll-trans eqpoll-sym Ord-cardinal-eqpoll* [*OF i*])+

lemma *nat-wf-on-converse-Memrel*: $n \in nat \Longrightarrow wf[n](converse(Memrel(n)))$
proof (*induct n rule*: *nat-induct*)
  case *0* **thus** *?case* **by** (*blast intro*: *wf-onI*)
next
  case (*succ x*)
  hence *wfx*: $\bigwedge Z.\ Z = 0 \vee (\exists z{\in}Z.\ \forall y.\ z \in y \wedge z \in x \wedge y \in x \wedge z \in x \longrightarrow y \notin Z)$
    by (*simp add*: *wf-on-def wf-def*) — not easy to erase the duplicate $z \in x$!
  show *?case*
    proof (*rule wf-onI*)
      fix *Z u*
      assume *Z*: $u \in Z\ \forall z{\in}Z.\ \exists y{\in}Z.\ \langle y, z\rangle \in converse(Memrel(succ(x)))$
      show *False*
        proof (*cases* $x \in Z$)
          case *True* **thus** *False* **using** *Z*
            by (*blast elim*: *mem-irrefl mem-asym*)
          next
          case *False* **thus** *False* **using** *wfx* [*of Z*] *Z*
            by *blast*
        qed
    qed
qed

lemma *nat-well-ord-converse-Memrel*: $n \in nat \Longrightarrow well\text{-}ord(n,converse(Memrel(n)))$
apply (*frule Ord-nat* [*THEN Ord-in-Ord*, *THEN well-ord-Memrel*])
apply (*simp add*: *well-ord-def tot-ord-converse nat-wf-on-converse-Memrel*)
done

lemma *well-ord-converse*:
  ⟦*well-ord*(*A*,*r*);
    well-ord(*ordertype*(*A*,*r*), *converse*(*Memrel*(*ordertype*(*A*, *r*))))⟧
   ⟹ *well-ord*(*A*,*converse*(*r*))
apply (*rule well-ord-Int-iff* [*THEN iffD1*])
apply (*frule ordermap-bij* [*THEN bij-is-inj*, *THEN well-ord-rvimage*], *assumption*)
apply (*simp add*: *rvimage-converse converse-Int converse-prod*
        ordertype-ord-iso [*THEN ord-iso-rvimage-eq*])
done

lemma *ordertype-eq-n*:
 assumes *r*: *well-ord*(*A*,*r*) **and** *A*: $A \approx n$ **and** *n*: $n \in nat$
 shows *ordertype*(*A*,*r*) = *n*
proof −
 have *ordertype*(*A*,*r*) ≈ *A*
  by (*blast intro*: *bij-imp-eqpoll bij-converse-bij ordermap-bij r*)
 also have ... ≈ *n* **by** (*rule A*)
 finally have *ordertype*(*A*,*r*) ≈ *n* **.**
 thus *?thesis*
  by (*simp add*: *Ord-nat-eqpoll-iff Ord-ordertype n r*)
qed

lemma *Finite-well-ord-converse*:
  ⟦*Finite*(*A*); *well-ord*(*A*,*r*)⟧ ⟹ *well-ord*(*A*,*converse*(*r*))
 unfolding *Finite-def*
apply (*rule well-ord-converse*, *assumption*)
apply (*blast dest*: *ordertype-eq-n intro*!: *nat-well-ord-converse-Memrel*)
done

lemma *nat-into-Finite*: $n \in nat$ ⟹ *Finite*(*n*)
 by (*auto simp add*: *Finite-def intro*: *eqpoll-refl*)

lemma *nat-not-Finite*: ¬ *Finite*(*nat*)
proof −
 { fix *n*
  assume *n*: $n \in nat$ $nat \approx n$
  have $n \in nat$ **by** (*rule n*)
  also have ... = *n* **using** *n*
   by (*simp add*: *Ord-nat-eqpoll-iff Ord-nat*)
  finally have $n \in n$ **.**
  hence *False*
   by (*blast elim*: *mem-irrefl*)
 }
 thus *?thesis*
  by (*auto simp add*: *Finite-def*)
qed

end

24 The Cumulative Hierarchy and a Small Universe for Recursive Types

theory *Univ* **imports** *Epsilon Cardinal* **begin**

definition
  Vfrom :: $[i,i]\Rightarrow i$ **where**
    $Vfrom(A,i) \equiv transrec(i, \lambda x\ f.\ A \cup (\bigcup y\in x.\ Pow(f`y)))$

abbreviation
  Vset :: $i\Rightarrow i$ **where**
  $Vset(x) \equiv Vfrom(0,x)$

definition
  Vrec :: $[i, [i,i]\Rightarrow i] \Rightarrow i$ **where**
    $Vrec(a,H) \equiv transrec(rank(a), \lambda x\ g.\ \lambda z\in Vset(succ(x)).$
                    $H(z, \lambda w\in Vset(x).\ g`rank(w)`w))\ `\ a$

definition
  Vrecursor :: $[[i,i]\Rightarrow i, i] \Rightarrow i$ **where**
    $Vrecursor(H,a) \equiv transrec(rank(a), \lambda x\ g.\ \lambda z\in Vset(succ(x)).$
                    $H(\lambda w\in Vset(x).\ g`rank(w)`w, z))\ `\ a$

definition
  univ :: $i\Rightarrow i$ **where**
    $univ(A) \equiv Vfrom(A,nat)$

24.1 Immediate Consequences of the Definition of $Vfrom(A, i)$

NOT SUITABLE FOR REWRITING – RECURSIVE!

lemma *Vfrom*: $Vfrom(A,i) = A \cup (\bigcup j\in i.\ Pow(Vfrom(A,j)))$
by (*subst Vfrom-def* [*THEN def-transrec*], *simp*)

24.1.1 Monotonicity

lemma *Vfrom-mono* [*rule-format*]:
  $A<=B \Longrightarrow \forall j.\ i<=j \longrightarrow Vfrom(A,i) \subseteq Vfrom(B,j)$
apply (*rule-tac a=i* **in** *eps-induct*)
apply (*rule impI* [*THEN allI*])
apply (*subst Vfrom* [*of A*])
apply (*subst Vfrom* [*of B*])
apply (*erule Un-mono*)
apply (*erule UN-mono*, *blast*)
done

lemma *VfromI*: $[\![a \in Vfrom(A,j);\ j<i]\!] \Longrightarrow a \in Vfrom(A,i)$
by (*blast dest*: *Vfrom-mono* [*OF subset-refl le-imp-subset* [*OF leI*]])

24.1.2 A fundamental equality: Vfrom does not require ordinals!

lemma *Vfrom-rank-subset1*: *Vfrom*(*A*,*x*) ⊆ *Vfrom*(*A*,*rank*(*x*))
proof (*induct x rule*: *eps-induct*)
  fix *x*
  assume ∀ *y*∈*x*. *Vfrom*(*A*,*y*) ⊆ *Vfrom*(*A*,*rank*(*y*))
  thus *Vfrom*(*A*, *x*) ⊆ *Vfrom*(*A*, *rank*(*x*))
    by (*simp add*: *Vfrom* [*of* - *x*] *Vfrom* [*of* - *rank*(*x*)],
      blast intro!: *rank-lt* [*THEN ltD*])
qed

lemma *Vfrom-rank-subset2*: *Vfrom*(*A*,*rank*(*x*)) ⊆ *Vfrom*(*A*,*x*)
apply (*rule-tac a*=*x* **in** *eps-induct*)
apply (*subst Vfrom*)
apply (*subst Vfrom*, *rule subset-refl* [*THEN Un-mono*])
apply (*rule UN-least*)

expand *rank*(*x1*) = (⋃ *y*∈*x1*. *succ*(*rank*(*y*))) in assumptions

apply (*erule rank* [*THEN equalityD1*, *THEN subsetD*, *THEN UN-E*])
apply (*rule subset-trans*)
apply (*erule-tac* [*2*] *UN-upper*)
apply (*rule subset-refl* [*THEN Vfrom-mono*, *THEN subset-trans*, *THEN Pow-mono*])
apply (*erule ltI* [*THEN le-imp-subset*])
apply (*rule Ord-rank* [*THEN Ord-succ*])
apply (*erule bspec*, *assumption*)
done

lemma *Vfrom-rank-eq*: *Vfrom*(*A*,*rank*(*x*)) = *Vfrom*(*A*,*x*)
apply (*rule equalityI*)
apply (*rule Vfrom-rank-subset2*)
apply (*rule Vfrom-rank-subset1*)
done

24.2 Basic Closure Properties

lemma *zero-in-Vfrom*: *y*:*x* ⟹ *0* ∈ *Vfrom*(*A*,*x*)
by (*subst Vfrom*, *blast*)

lemma *i-subset-Vfrom*: *i* ⊆ *Vfrom*(*A*,*i*)
apply (*rule-tac a*=*i* **in** *eps-induct*)
apply (*subst Vfrom*, *blast*)
done

lemma *A-subset-Vfrom*: *A* ⊆ *Vfrom*(*A*,*i*)
apply (*subst Vfrom*)
apply (*rule Un-upper1*)
done

lemmas *A-into-Vfrom* = *A-subset-Vfrom* [*THEN subsetD*]

lemma *subset-mem-Vfrom*: $a \subseteq Vfrom(A,i) \Longrightarrow a \in Vfrom(A,succ(i))$
by (*subst Vfrom*, *blast*)

24.2.1 Finite sets and ordered pairs

lemma *singleton-in-Vfrom*: $a \in Vfrom(A,i) \Longrightarrow \{a\} \in Vfrom(A,succ(i))$
by (*rule subset-mem-Vfrom*, *safe*)

lemma *doubleton-in-Vfrom*:
$[\![a \in Vfrom(A,i); \ b \in Vfrom(A,i)]\!] \Longrightarrow \{a,b\} \in Vfrom(A,succ(i))$
by (*rule subset-mem-Vfrom*, *safe*)

lemma *Pair-in-Vfrom*:
$[\![a \in Vfrom(A,i); \ b \in Vfrom(A,i)]\!] \Longrightarrow \langle a,b\rangle \in Vfrom(A,succ(succ(i)))$
unfolding *Pair-def*
apply (*blast intro*: *doubleton-in-Vfrom*)
done

lemma *succ-in-Vfrom*: $a \subseteq Vfrom(A,i) \Longrightarrow succ(a) \in Vfrom(A,succ(succ(i)))$
apply (*intro subset-mem-Vfrom succ-subsetI*, *assumption*)
apply (*erule subset-trans*)
apply (*rule Vfrom-mono* [*OF subset-refl subset-succI*])
done

24.3 0, Successor and Limit Equations for *Vfrom*

lemma *Vfrom-0*: $Vfrom(A,0) = A$
by (*subst Vfrom*, *blast*)

lemma *Vfrom-succ-lemma*: $Ord(i) \Longrightarrow Vfrom(A,succ(i)) = A \cup Pow(Vfrom(A,i))$
apply (*rule Vfrom* [*THEN trans*])
apply (*rule equalityI* [*THEN subst-context*,
OF - succI1 [*THEN RepFunI*, *THEN Union-upper*]])
apply (*rule UN-least*)
apply (*rule subset-refl* [*THEN Vfrom-mono*, *THEN Pow-mono*])
apply (*erule ltI* [*THEN le-imp-subset*])
apply (*erule Ord-succ*)
done

lemma *Vfrom-succ*: $Vfrom(A,succ(i)) = A \cup Pow(Vfrom(A,i))$
apply (*rule-tac x1 = succ* (*i*) **in** *Vfrom-rank-eq* [*THEN subst*])
apply (*rule-tac x1 = i* **in** *Vfrom-rank-eq* [*THEN subst*])
apply (*subst rank-succ*)
apply (*rule Ord-rank* [*THEN Vfrom-succ-lemma*])
done

lemma *Vfrom-Union*: $y{:}X \Longrightarrow Vfrom(A,\bigcup(X)) = (\bigcup y{\in}X.\ Vfrom(A,y))$
apply (*subst Vfrom*)
apply (*rule equalityI*)

first inclusion

apply (*rule Un-least*)
apply (*rule A-subset-Vfrom* [*THEN subset-trans*])
apply (*rule UN-upper*, *assumption*)
apply (*rule UN-least*)
apply (*erule UnionE*)
apply (*rule subset-trans*)
apply (*erule-tac* [*2*] *UN-upper*,
subst Vfrom, *erule subset-trans* [*OF UN-upper Un-upper2*])

opposite inclusion

apply (*rule UN-least*)
apply (*subst Vfrom*, *blast*)
done

24.4 *Vfrom* applied to Limit Ordinals

lemma *Limit-Vfrom-eq*:
$Limit(i) \Longrightarrow Vfrom(A,i) = (\bigcup y\in i.\ Vfrom(A,y))$
apply (*rule Limit-has-0* [*THEN ltD*, *THEN Vfrom-Union*, *THEN subst*], *assumption*)
apply (*simp add*: *Limit-Union-eq*)
done

lemma *Limit-VfromE*:
$[\![a \in Vfrom(A,i);\ \neg R \Longrightarrow Limit(i);$
$\bigwedge x.\ [\![x<i;\ a \in Vfrom(A,x)]\!] \Longrightarrow R$
$]\!] \Longrightarrow R$
apply (*rule classical*)
apply (*rule Limit-Vfrom-eq* [*THEN equalityD1*, *THEN subsetD*, *THEN UN-E*])
prefer *2* **apply** *assumption*
apply *blast*
apply (*blast intro*: *ltI Limit-is-Ord*)
done

lemma *singleton-in-VLimit*:
$[\![a \in Vfrom(A,i);\ Limit(i)]\!] \Longrightarrow \{a\} \in Vfrom(A,i)$
apply (*erule Limit-VfromE*, *assumption*)
apply (*erule singleton-in-Vfrom* [*THEN VfromI*])
apply (*blast intro*: *Limit-has-succ*)
done

lemmas *Vfrom-UnI1* =
Un-upper1 [*THEN subset-refl* [*THEN Vfrom-mono*, *THEN subsetD*]]
lemmas *Vfrom-UnI2* =
Un-upper2 [*THEN subset-refl* [*THEN Vfrom-mono*, *THEN subsetD*]]

Hard work is finding a single j:i such that a,b<=Vfrom(A,j)

lemma *doubleton-in-VLimit*:

⟦$a \in Vfrom(A,i)$; $b \in Vfrom(A,i)$; $Limit(i)$⟧ $\Longrightarrow \{a,b\} \in Vfrom(A,i)$
apply (*erule Limit-VfromE*, *assumption*)
apply (*erule Limit-VfromE*, *assumption*)
apply (*blast intro*: *VfromI* [*OF doubleton-in-Vfrom*]
Vfrom-UnI1 Vfrom-UnI2 Limit-has-succ Un-least-lt)
done

lemma *Pair-in-VLimit*:
⟦$a \in Vfrom(A,i)$; $b \in Vfrom(A,i)$; $Limit(i)$⟧ $\Longrightarrow \langle a,b\rangle \in Vfrom(A,i)$

Infer that a, b occur at ordinals x,xa < i.

apply (*erule Limit-VfromE*, *assumption*)
apply (*erule Limit-VfromE*, *assumption*)

Infer that $succ(succ(x \cup xa)) < i$

apply (*blast intro*: *VfromI* [*OF Pair-in-Vfrom*]
Vfrom-UnI1 Vfrom-UnI2 Limit-has-succ Un-least-lt)
done

lemma *product-VLimit*: $Limit(i) \Longrightarrow Vfrom(A,i) * Vfrom(A,i) \subseteq Vfrom(A,i)$
by (*blast intro*: *Pair-in-VLimit*)

lemmas *Sigma-subset-VLimit* =
subset-trans [*OF Sigma-mono product-VLimit*]

lemmas *nat-subset-VLimit* =
subset-trans [*OF nat-le-Limit* [*THEN le-imp-subset*] *i-subset-Vfrom*]

lemma *nat-into-VLimit*: ⟦n: nat; $Limit(i)$⟧ $\Longrightarrow n \in Vfrom(A,i)$
by (*blast intro*: *nat-subset-VLimit* [*THEN subsetD*])

24.4.1 Closure under Disjoint Union

lemmas *zero-in-VLimit* = *Limit-has-0* [*THEN ltD*, *THEN zero-in-Vfrom*]

lemma *one-in-VLimit*: $Limit(i) \Longrightarrow 1 \in Vfrom(A,i)$
by (*blast intro*: *nat-into-VLimit*)

lemma *Inl-in-VLimit*:
⟦$a \in Vfrom(A,i)$; $Limit(i)$⟧ $\Longrightarrow Inl(a) \in Vfrom(A,i)$
unfolding *Inl-def*
apply (*blast intro*: *zero-in-VLimit Pair-in-VLimit*)
done

lemma *Inr-in-VLimit*:
⟦$b \in Vfrom(A,i)$; $Limit(i)$⟧ $\Longrightarrow Inr(b) \in Vfrom(A,i)$
unfolding *Inr-def*
apply (*blast intro*: *one-in-VLimit Pair-in-VLimit*)
done

lemma *sum-VLimit*: *Limit*(*i*) $\Longrightarrow$ *Vfrom*(*C*,*i*)+*Vfrom*(*C*,*i*) $\subseteq$ *Vfrom*(*C*,*i*)
by (*blast intro*!: *Inl-in-VLimit Inr-in-VLimit*)

lemmas *sum-subset-VLimit* = *subset-trans* [*OF sum-mono sum-VLimit*]

24.5 Properties assuming *Transset*(*A*)

lemma *Transset-Vfrom*: *Transset*(*A*) $\Longrightarrow$ *Transset*(*Vfrom*(*A*,*i*))
apply (*rule-tac a*=*i* **in** *eps-induct*)
apply (*subst Vfrom*)
apply (*blast intro*!: *Transset-Union-family Transset-Un Transset-Pow*)
done

lemma *Transset-Vfrom-succ*:
  Transset(*A*) $\Longrightarrow$ *Vfrom*(*A*, *succ*(*i*)) = *Pow*(*Vfrom*(*A*,*i*))
apply (*rule Vfrom-succ* [*THEN trans*])
apply (*rule equalityI* [*OF - Un-upper2*])
apply (*rule Un-least* [*OF - subset-refl*])
apply (*rule A-subset-Vfrom* [*THEN subset-trans*])
apply (*erule Transset-Vfrom* [*THEN Transset-iff-Pow* [*THEN iffD1*]])
done

lemma *Transset-Pair-subset*: $[\![\langle a,b\rangle \subseteq C;\ Transset(C)]\!] \Longrightarrow a: C \wedge b: C$
by (*unfold Pair-def Transset-def*, *blast*)

lemma *Transset-Pair-subset-VLimit*:
  $[\![\langle a,b\rangle \subseteq Vfrom(A,i);\ \ Transset(A);\ \ Limit(i)]\!]$
   $\Longrightarrow \langle a,b\rangle \in Vfrom(A,i)$
apply (*erule Transset-Pair-subset* [*THEN conjE*])
apply (*erule Transset-Vfrom*)
apply (*blast intro*: *Pair-in-VLimit*)
done

lemma *Union-in-Vfrom*:
  $[\![X \in Vfrom(A,j);\ \ Transset(A)]\!] \Longrightarrow \bigcup(X) \in Vfrom(A,\ succ(j))$
apply (*drule Transset-Vfrom*)
apply (*rule subset-mem-Vfrom*)
apply (*unfold Transset-def*, *blast*)
done

lemma *Union-in-VLimit*:
  $[\![X \in Vfrom(A,i);\ \ Limit(i);\ \ Transset(A)]\!] \Longrightarrow \bigcup(X) \in Vfrom(A,i)$
apply (*rule Limit-VfromE*, *assumption*+)
apply (*blast intro*: *Limit-has-succ VfromI Union-in-Vfrom*)
done

General theorem for membership in Vfrom(A,i) when i is a limit ordinal

lemma *in-VLimit*:

⟦$a \in Vfrom(A,i)$; $b \in Vfrom(A,i)$; $Limit(i)$;
$\bigwedge x\ y\ j.$ ⟦$j<i$; $1{:}j$; $x \in Vfrom(A,j)$; $y \in Vfrom(A,j)$⟧
$\Longrightarrow \exists k.\ h(x,y) \in Vfrom(A,k) \wedge k<i$⟧
$\Longrightarrow h(a,b) \in Vfrom(A,i)$

Infer that a, b occur at ordinals x,xa < i.

apply (*erule Limit-VfromE*, *assumption*)
apply (*erule Limit-VfromE*, *assumption*, *atomize*)
apply (*drule-tac x=a* **in** *spec*)
apply (*drule-tac x=b* **in** *spec*)
apply (*drule-tac x=x ∪ xa ∪ 2* **in** *spec*)
apply (*simp add*: *Un-least-lt-iff lt-Ord Vfrom-UnI1 Vfrom-UnI2*)
apply (*blast intro*: *Limit-has-0 Limit-has-succ VfromI*)
done

24.5.1 Products

lemma *prod-in-Vfrom*:
⟦$a \in Vfrom(A,j)$; $b \in Vfrom(A,j)$; $Transset(A)$⟧
$\Longrightarrow a*b \in Vfrom(A, succ(succ(succ(j))))$
apply (*drule Transset-Vfrom*)
apply (*rule subset-mem-Vfrom*)
unfolding *Transset-def*
apply (*blast intro*: *Pair-in-Vfrom*)
done

lemma *prod-in-VLimit*:
⟦$a \in Vfrom(A,i)$; $b \in Vfrom(A,i)$; $Limit(i)$; $Transset(A)$⟧
$\Longrightarrow a*b \in Vfrom(A,i)$
apply (*erule in-VLimit*, *assumption+*)
apply (*blast intro*: *prod-in-Vfrom Limit-has-succ*)
done

24.5.2 Disjoint Sums, or Quine Ordered Pairs

lemma *sum-in-Vfrom*:
⟦$a \in Vfrom(A,j)$; $b \in Vfrom(A,j)$; $Transset(A)$; $1{:}j$⟧
$\Longrightarrow a+b \in Vfrom(A, succ(succ(succ(j))))$
unfolding *sum-def*
apply (*drule Transset-Vfrom*)
apply (*rule subset-mem-Vfrom*)
unfolding *Transset-def*
apply (*blast intro*: *zero-in-Vfrom Pair-in-Vfrom i-subset-Vfrom* [*THEN subsetD*])
done

lemma *sum-in-VLimit*:
⟦$a \in Vfrom(A,i)$; $b \in Vfrom(A,i)$; $Limit(i)$; $Transset(A)$⟧
$\Longrightarrow a+b \in Vfrom(A,i)$
apply (*erule in-VLimit*, *assumption+*)
apply (*blast intro*: *sum-in-Vfrom Limit-has-succ*)

done

24.5.3 Function Space!

lemma *fun-in-Vfrom*:
$[\![a \in Vfrom(A,j);\ \ b \in Vfrom(A,j);\ \ Transset(A)]\!] \Longrightarrow$
$a{-}{>}b \in Vfrom(A,\ succ(succ(succ(succ(j)))))$
unfolding *Pi-def*
apply (*drule Transset-Vfrom*)
apply (*rule subset-mem-Vfrom*)
apply (*rule Collect-subset* [*THEN subset-trans*])
apply (*subst Vfrom*)
apply (*rule subset-trans* [*THEN subset-trans*])
apply (*rule-tac* [*3*] *Un-upper2*)
apply (*rule-tac* [*2*] *succI1* [*THEN UN-upper*])
apply (*rule Pow-mono*)
unfolding *Transset-def*
apply (*blast intro*: *Pair-in-Vfrom*)
done

lemma *fun-in-VLimit*:
$[\![a \in Vfrom(A,i);\ \ b \in Vfrom(A,i);\ \ Limit(i);\ \ Transset(A)]\!]$
$\Longrightarrow a{-}{>}b \in Vfrom(A,i)$
apply (*erule in-VLimit*, *assumption+*)
apply (*blast intro*: *fun-in-Vfrom Limit-has-succ*)
done

lemma *Pow-in-Vfrom*:
$[\![a \in Vfrom(A,j);\ \ Transset(A)]\!] \Longrightarrow Pow(a) \in Vfrom(A,\ succ(succ(j)))$
apply (*drule Transset-Vfrom*)
apply (*rule subset-mem-Vfrom*)
unfolding *Transset-def*
apply (*subst Vfrom*, *blast*)
done

lemma *Pow-in-VLimit*:
$[\![a \in Vfrom(A,i);\ \ Limit(i);\ \ Transset(A)]\!] \Longrightarrow Pow(a) \in Vfrom(A,i)$
by (*blast elim*: *Limit-VfromE intro*: *Limit-has-succ Pow-in-Vfrom VfromI*)

24.6 The Set *Vset*(*i*)

lemma *Vset*: $Vset(i) = (\bigcup j{\in}i.\ Pow(Vset(j)))$
by (*subst Vfrom*, *blast*)

lemmas *Vset-succ* = *Transset-0* [*THEN Transset-Vfrom-succ*]
lemmas *Transset-Vset* = *Transset-0* [*THEN Transset-Vfrom*]

24.6.1 Characterisation of the elements of *Vset*(*i*)

lemma *VsetD* [*rule-format*]: $Ord(i) \Longrightarrow \forall\, b.\ b \in Vset(i) \longrightarrow rank(b) < i$

apply (*erule trans-induct*)
apply (*subst Vset*, *safe*)
apply (*subst rank*)
apply (*blast intro*: *ltI UN-succ-least-lt*)
done

lemma *VsetI-lemma* [*rule-format*]:
  $Ord(i) \Longrightarrow \forall b.\ rank(b) \in i \longrightarrow b \in Vset(i)$
apply (*erule trans-induct*)
apply (*rule allI*)
apply (*subst Vset*)
apply (*blast intro*!: *rank-lt* [*THEN ltD*])
done

lemma *VsetI*: $rank(x)<i \Longrightarrow x \in Vset(i)$
by (*blast intro*: *VsetI-lemma elim*: *ltE*)

Merely a lemma for the next result

lemma *Vset-Ord-rank-iff*: $Ord(i) \Longrightarrow b \in Vset(i) \longleftrightarrow rank(b) < i$
by (*blast intro*: *VsetD VsetI*)

lemma *Vset-rank-iff* [*simp*]: $b \in Vset(a) \longleftrightarrow rank(b) < rank(a)$
apply (*rule Vfrom-rank-eq* [*THEN subst*])
apply (*rule Ord-rank* [*THEN Vset-Ord-rank-iff*])
done

This is rank(rank(a)) = rank(a)

declare *Ord-rank* [*THEN rank-of-Ord*, *simp*]

lemma *rank-Vset*: $Ord(i) \Longrightarrow rank(Vset(i)) = i$
apply (*subst rank*)
apply (*rule equalityI*, *safe*)
apply (*blast intro*: *VsetD* [*THEN ltD*])
apply (*blast intro*: *VsetD* [*THEN ltD*] *Ord-trans*)
apply (*blast intro*: *i-subset-Vfrom* [*THEN subsetD*]
    Ord-in-Ord [*THEN rank-of-Ord*, *THEN ssubst*])
done

lemma *Finite-Vset*: $i \in nat \Longrightarrow Finite(Vset(i))$
apply (*erule nat-induct*)
apply (*simp add*: *Vfrom-0*)
apply (*simp add*: *Vset-succ*)
done

24.6.2 Reasoning about Sets in Terms of Their Elements' Ranks

lemma *arg-subset-Vset-rank*: $a \subseteq Vset(rank(a))$
apply (*rule subsetI*)
apply (*erule rank-lt* [*THEN VsetI*])

done

lemma *Int-Vset-subset*:
$[\![\bigwedge i.\ Ord(i) \Longrightarrow a \cap Vset(i) \subseteq b]\!] \Longrightarrow a \subseteq b$
apply (*rule subset-trans*)
apply (*rule Int-greatest* [*OF subset-refl arg-subset-Vset-rank*])
apply (*blast intro*: *Ord-rank*)
done

24.6.3 Set Up an Environment for Simplification

lemma *rank-Inl*: $rank(a) < rank(Inl(a))$
unfolding *Inl-def*
apply (*rule rank-pair2*)
done

lemma *rank-Inr*: $rank(a) < rank(Inr(a))$
unfolding *Inr-def*
apply (*rule rank-pair2*)
done

lemmas *rank-rls* = *rank-Inl rank-Inr rank-pair1 rank-pair2*

24.6.4 Recursion over Vset Levels!

NOT SUITABLE FOR REWRITING: recursive!

lemma *Vrec*: $Vrec(a,H) = H(a, \lambda x{\in}Vset(rank(a)).\ Vrec(x,H))$
unfolding *Vrec-def*
apply (*subst transrec*, *simp*)
apply (*rule refl* [*THEN lam-cong*, *THEN subst-context*], *simp add*: *lt-def*)
done

This form avoids giant explosions in proofs. NOTE the form of the premise!

lemma *def-Vrec*:
$[\![\bigwedge x.\ h(x) \equiv Vrec(x,H)]\!] \Longrightarrow$
$h(a) = H(a, \lambda x{\in}Vset(rank(a)).\ h(x))$
apply *simp*
apply (*rule Vrec*)
done

NOT SUITABLE FOR REWRITING: recursive!

lemma *Vrecursor*:
$Vrecursor(H,a) = H(\lambda x{\in}Vset(rank(a)).\ Vrecursor(H,x),\ \ a)$
unfolding *Vrecursor-def*
apply (*subst transrec*, *simp*)
apply (*rule refl* [*THEN lam-cong*, *THEN subst-context*], *simp add*: *lt-def*)
done

This form avoids giant explosions in proofs. NOTE the form of the premise!

lemma *def-Vrecursor*:
$h \equiv Vrecursor(H) \Longrightarrow h(a) = H(\lambda x{\in}Vset(rank(a)).\ h(x),\ \ a)$
apply *simp*
apply (*rule Vrecursor*)
done

24.7 The Datatype Universe: $univ(A)$

lemma *univ-mono*: $A{<=}B \Longrightarrow univ(A) \subseteq univ(B)$
unfolding *univ-def*
apply (*erule Vfrom-mono*)
apply (*rule subset-refl*)
done

lemma *Transset-univ*: $Transset(A) \Longrightarrow Transset(univ(A))$
unfolding *univ-def*
apply (*erule Transset-Vfrom*)
done

24.7.1 The Set $univ(A)$ as a Limit

lemma *univ-eq-UN*: $univ(A) = (\bigcup i{\in}nat.\ Vfrom(A,i))$
unfolding *univ-def*
apply (*rule Limit-nat* [*THEN Limit-Vfrom-eq*])
done

lemma *subset-univ-eq-Int*: $c \subseteq univ(A) \Longrightarrow c = (\bigcup i{\in}nat.\ c \cap Vfrom(A,i))$
apply (*rule subset-UN-iff-eq* [*THEN iffD1*])
apply (*erule univ-eq-UN* [*THEN subst*])
done

lemma *univ-Int-Vfrom-subset*:
$[\![a \subseteq univ(X);$
$\bigwedge i.\ i{:}nat \Longrightarrow a \cap Vfrom(X,i) \subseteq b]\!]$
$\Longrightarrow a \subseteq b$
apply (*subst subset-univ-eq-Int*, *assumption*)
apply (*rule UN-least*, *simp*)
done

lemma *univ-Int-Vfrom-eq*:
$[\![a \subseteq univ(X);\quad b \subseteq univ(X);$
$\bigwedge i.\ i{:}nat \Longrightarrow a \cap Vfrom(X,i) = b \cap Vfrom(X,i)$
$]\!] \Longrightarrow a = b$
apply (*rule equalityI*)
apply (*rule univ-Int-Vfrom-subset*, *assumption*)
apply (*blast elim*: *equalityCE*)
apply (*rule univ-Int-Vfrom-subset*, *assumption*)
apply (*blast elim*: *equalityCE*)
done

24.8 Closure Properties for *univ*(*A*)

lemma *zero-in-univ*: *0* ∈ *univ*(*A*)
  unfolding *univ-def*
apply (*rule nat-0I* [*THEN zero-in-Vfrom*])
done

lemma *zero-subset-univ*: {*0*} ⊆ *univ*(*A*)
by (*blast intro*: *zero-in-univ*)

lemma *A-subset-univ*: *A* ⊆ *univ*(*A*)
  unfolding *univ-def*
apply (*rule A-subset-Vfrom*)
done

lemmas *A-into-univ* = *A-subset-univ* [*THEN subsetD*]

24.8.1 Closure under Unordered and Ordered Pairs

lemma *singleton-in-univ*: *a*: *univ*(*A*) ⟹ {*a*} ∈ *univ*(*A*)
  unfolding *univ-def*
apply (*blast intro*: *singleton-in-VLimit Limit-nat*)
done

lemma *doubleton-in-univ*:
    ⟦*a*: *univ*(*A*); *b*: *univ*(*A*)⟧ ⟹ {*a*,*b*} ∈ *univ*(*A*)
  unfolding *univ-def*
apply (*blast intro*: *doubleton-in-VLimit Limit-nat*)
done

lemma *Pair-in-univ*:
    ⟦*a*: *univ*(*A*); *b*: *univ*(*A*)⟧ ⟹ ⟨*a*,*b*⟩ ∈ *univ*(*A*)
  unfolding *univ-def*
apply (*blast intro*: *Pair-in-VLimit Limit-nat*)
done

lemma *Union-in-univ*:
    ⟦*X*: *univ*(*A*); *Transset*(*A*)⟧ ⟹ ⋃(*X*) ∈ *univ*(*A*)
  unfolding *univ-def*
apply (*blast intro*: *Union-in-VLimit Limit-nat*)
done

lemma *product-univ*: *univ*(*A*)∗*univ*(*A*) ⊆ *univ*(*A*)
  unfolding *univ-def*
apply (*rule Limit-nat* [*THEN product-VLimit*])
done

24.8.2 The Natural Numbers

lemma *nat-subset-univ*: *nat* ⊆ *univ*(*A*)

unfolding *univ-def*
apply (*rule i-subset-Vfrom*)
done

lemma *nat-into-univ*: $n \in nat \Longrightarrow n \in univ(A)$
by (*rule nat-subset-univ* [*THEN subsetD*])

24.8.3 Instances for 1 and 2

lemma *one-in-univ*: $1 \in univ(A)$
unfolding *univ-def*
apply (*rule Limit-nat* [*THEN one-in-VLimit*])
done

unused!

lemma *two-in-univ*: $2 \in univ(A)$
by (*blast intro*: *nat-into-univ*)

lemma *bool-subset-univ*: $bool \subseteq univ(A)$
unfolding *bool-def*
apply (*blast intro*!: *zero-in-univ one-in-univ*)
done

lemmas *bool-into-univ* = *bool-subset-univ* [*THEN subsetD*]

24.8.4 Closure under Disjoint Union

lemma *Inl-in-univ*: a: $univ(A) \Longrightarrow Inl(a) \in univ(A)$
unfolding *univ-def*
apply (*erule Inl-in-VLimit* [*OF - Limit-nat*])
done

lemma *Inr-in-univ*: b: $univ(A) \Longrightarrow Inr(b) \in univ(A)$
unfolding *univ-def*
apply (*erule Inr-in-VLimit* [*OF - Limit-nat*])
done

lemma *sum-univ*: $univ(C)+univ(C) \subseteq univ(C)$
unfolding *univ-def*
apply (*rule Limit-nat* [*THEN sum-VLimit*])
done

lemmas *sum-subset-univ* = *subset-trans* [*OF sum-mono sum-univ*]

lemma *Sigma-subset-univ*:
$[\![A \subseteq univ(D); \bigwedge x.\ x \in A \Longrightarrow B(x) \subseteq univ(D)]\!] \Longrightarrow Sigma(A,B) \subseteq univ(D)$
apply (*simp add*: *univ-def*)
apply (*blast intro*: *Sigma-subset-VLimit del*: *subsetI*)
done

24.9 Finite Branching Closure Properties

24.9.1 Closure under Finite Powerset

```
lemma Fin-Vfrom-lemma:
    ⟦b: Fin(Vfrom(A,i)); Limit(i)⟧ ⟹ ∃ j. b ⊆ Vfrom(A,j) ∧ j<i
apply (erule Fin-induct)
apply (blast dest!: Limit-has-0, safe)
apply (erule Limit-VfromE, assumption)
apply (blast intro!: Un-least-lt intro: Vfrom-UnI1 Vfrom-UnI2)
done
```

```
lemma Fin-VLimit: Limit(i) ⟹ Fin(Vfrom(A,i)) ⊆ Vfrom(A,i)
apply (rule subsetI)
apply (drule Fin-Vfrom-lemma, safe)
apply (rule Vfrom [THEN ssubst])
apply (blast dest!: ltD)
done
```

```
lemmas Fin-subset-VLimit = subset-trans [OF Fin-mono Fin-VLimit]
```

```
lemma Fin-univ: Fin(univ(A)) ⊆ univ(A)
  unfolding univ-def
apply (rule Limit-nat [THEN Fin-VLimit])
done
```

24.9.2 Closure under Finite Powers: Functions from a Natural Number

```
lemma nat-fun-VLimit:
    ⟦n: nat; Limit(i)⟧ ⟹ n −> Vfrom(A,i) ⊆ Vfrom(A,i)
apply (erule nat-fun-subset-Fin [THEN subset-trans])
apply (blast del: subsetI
    intro: subset-refl Fin-subset-VLimit Sigma-subset-VLimit nat-subset-VLimit)
done
```

```
lemmas nat-fun-subset-VLimit = subset-trans [OF Pi-mono nat-fun-VLimit]
```

```
lemma nat-fun-univ: n: nat ⟹ n −> univ(A) ⊆ univ(A)
  unfolding univ-def
apply (erule nat-fun-VLimit [OF - Limit-nat])
done
```

24.9.3 Closure under Finite Function Space

General but seldom-used version; normally the domain is fixed

```
lemma FiniteFun-VLimit1:
    Limit(i) ⟹ Vfrom(A,i) −||> Vfrom(A,i) ⊆ Vfrom(A,i)
apply (rule FiniteFun.dom-subset [THEN subset-trans])
apply (blast del: subsetI
```

intro: *Fin-subset-VLimit Sigma-subset-VLimit subset-refl*)
done

lemma *FiniteFun-univ1*: *univ*(*A*) −||> *univ*(*A*) ⊆ *univ*(*A*)
unfolding *univ-def*
apply (*rule Limit-nat* [*THEN FiniteFun-VLimit1*])
done

Version for a fixed domain

lemma *FiniteFun-VLimit*:
⟦*W* ⊆ *Vfrom*(*A*,*i*); *Limit*(*i*)⟧ ⟹ *W* −||> *Vfrom*(*A*,*i*) ⊆ *Vfrom*(*A*,*i*)
apply (*rule subset-trans*)
apply (*erule FiniteFun-mono* [*OF - subset-refl*])
apply (*erule FiniteFun-VLimit1*)
done

lemma *FiniteFun-univ*:
W ⊆ *univ*(*A*) ⟹ *W* −||> *univ*(*A*) ⊆ *univ*(*A*)
unfolding *univ-def*
apply (*erule FiniteFun-VLimit* [*OF - Limit-nat*])
done

lemma *FiniteFun-in-univ*:
⟦*f*: *W* −||> *univ*(*A*); *W* ⊆ *univ*(*A*)⟧ ⟹ *f* ∈ *univ*(*A*)
by (*erule FiniteFun-univ* [*THEN subsetD*], *assumption*)

Remove ⊆ from the rule above

lemmas *FiniteFun-in-univ′* = *FiniteFun-in-univ* [*OF - subsetI*]

24.10 * For QUniv. Properties of Vfrom analogous to the "take-lemma" *

Intersecting a*b with Vfrom...

This version says a, b exist one level down, in the smaller set Vfrom(X,i)

lemma *doubleton-in-Vfrom-D*:
⟦{*a*,*b*} ∈ *Vfrom*(*X*,*succ*(*i*)); *Transset*(*X*)⟧
⟹ *a* ∈ *Vfrom*(*X*,*i*) ∧ *b* ∈ *Vfrom*(*X*,*i*)
by (*drule Transset-Vfrom-succ* [*THEN equalityD1*, *THEN subsetD*, *THEN PowD*], *assumption*, *fast*)

This weaker version says a, b exist at the same level

lemmas *Vfrom-doubleton-D* = *Transset-Vfrom* [*THEN Transset-doubleton-D*]

lemma *Pair-in-Vfrom-D*:
⟦⟨*a*,*b*⟩ ∈ *Vfrom*(*X*,*succ*(*i*)); *Transset*(*X*)⟧

$\Longrightarrow a \in Vfrom(X,i) \ \wedge \ b \in Vfrom(X,i)$
unfolding *Pair-def*
apply (*blast dest*!: *doubleton-in-Vfrom-D Vfrom-doubleton-D*)
done

lemma *product-Int-Vfrom-subset*:
$Transset(X) \Longrightarrow$
$(a{*}b) \cap Vfrom(X,\ succ(i)) \subseteq (a \cap Vfrom(X,i)) * (b \cap Vfrom(X,i))$
by (*blast dest*!: *Pair-in-Vfrom-D*)

ML
‹
val rank-ss =
simpset-of (***context*** *addsimps* [@{*thm VsetI*}]
addsimps @{*thms rank-rls*} @ (@{*thms rank-rls*} *RLN* (*2*, [@{*thm lt-trans*}])));
›

end

25 A Small Universe for Lazy Recursive Types

theory *QUniv* **imports** *Univ QPair* **begin**

rep-datatype
elimination *sumE*
induction *TrueI*
case-eqns *case-Inl case-Inr*

rep-datatype
elimination *qsumE*
induction *TrueI*
case-eqns *qcase-QInl qcase-QInr*

definition
quniv :: $i \Rightarrow i$ **where**
$quniv(A) \equiv Pow(univ(eclose(A)))$

25.1 Properties involving Transset and Sum

lemma *Transset-includes-summands*:
$[\![Transset(C);\ A{+}B \subseteq C]\!] \Longrightarrow A \subseteq C \wedge B \subseteq C$
apply (*simp add*: *sum-def Un-subset-iff*)
apply (*blast dest*: *Transset-includes-range*)
done

lemma *Transset-sum-Int-subset*:

Transset(*C*) $\Longrightarrow$ (*A*+*B*) ∩ *C* ⊆ (*A* ∩ *C*) + (*B* ∩ *C*)
apply (*simp add*: *sum-def Int-Un-distrib2*)
apply (*blast dest*: *Transset-Pair-D*)
done

25.2 Introduction and Elimination Rules

lemma *qunivI*: *X* ⊆ *univ*(*eclose*(*A*)) $\Longrightarrow$ *X* ∈ *quniv*(*A*)
by (*simp add*: *quniv-def*)

lemma *qunivD*: *X* ∈ *quniv*(*A*) $\Longrightarrow$ *X* ⊆ *univ*(*eclose*(*A*))
by (*simp add*: *quniv-def*)

lemma *quniv-mono*: *A*<=*B* $\Longrightarrow$ *quniv*(*A*) ⊆ *quniv*(*B*)
unfolding *quniv-def*
apply (*erule eclose-mono* [*THEN univ-mono*, *THEN Pow-mono*])
done

25.3 Closure Properties

lemma *univ-eclose-subset-quniv*: *univ*(*eclose*(*A*)) ⊆ *quniv*(*A*)
apply (*simp add*: *quniv-def Transset-iff-Pow* [*symmetric*])
apply (*rule Transset-eclose* [*THEN Transset-univ*])
done

lemma *univ-subset-quniv*: *univ*(*A*) ⊆ *quniv*(*A*)
apply (*rule arg-subset-eclose* [*THEN univ-mono*, *THEN subset-trans*])
apply (*rule univ-eclose-subset-quniv*)
done

lemmas *univ-into-quniv* = *univ-subset-quniv* [*THEN subsetD*]

lemma *Pow-univ-subset-quniv*: *Pow*(*univ*(*A*)) ⊆ *quniv*(*A*)
unfolding *quniv-def*
apply (*rule arg-subset-eclose* [*THEN univ-mono*, *THEN Pow-mono*])
done

lemmas *univ-subset-into-quniv* =
PowI [*THEN Pow-univ-subset-quniv* [*THEN subsetD*]]

lemmas *zero-in-quniv* = *zero-in-univ* [*THEN univ-into-quniv*]
lemmas *one-in-quniv* = *one-in-univ* [*THEN univ-into-quniv*]
lemmas *two-in-quniv* = *two-in-univ* [*THEN univ-into-quniv*]

lemmas *A-subset-quniv* = *subset-trans* [*OF A-subset-univ univ-subset-quniv*]

lemmas *A-into-quniv* = *A-subset-quniv* [*THEN subsetD*]

lemma *QPair-subset-univ*:
  ⟦*a* ⊆ *univ*(*A*); *b* ⊆ *univ*(*A*)⟧ ⟹ <*a*;*b*> ⊆ *univ*(*A*)
by (*simp add*: *QPair-def sum-subset-univ*)

25.4 Quine Disjoint Sum

lemma *QInl-subset-univ*: *a* ⊆ *univ*(*A*) ⟹ *QInl*(*a*) ⊆ *univ*(*A*)
 unfolding *QInl-def*
apply (*erule empty-subsetI* [*THEN QPair-subset-univ*])
done

lemmas *naturals-subset-nat* =
  Ord-nat [*THEN Ord-is-Transset*, *unfolded Transset-def*, *THEN bspec*]

lemmas *naturals-subset-univ* =
  subset-trans [*OF naturals-subset-nat nat-subset-univ*]

lemma *QInr-subset-univ*: *a* ⊆ *univ*(*A*) ⟹ *QInr*(*a*) ⊆ *univ*(*A*)
 unfolding *QInr-def*
apply (*erule nat-1I* [*THEN naturals-subset-univ*, *THEN QPair-subset-univ*])
done

25.5 Closure for Quine-Inspired Products and Sums

lemma *QPair-in-quniv*:
  ⟦*a*: *quniv*(*A*); *b*: *quniv*(*A*)⟧ ⟹ <*a*;*b*> ∈ *quniv*(*A*)
by (*simp add*: *quniv-def QPair-def sum-subset-univ*)

lemma *QSigma-quniv*: *quniv*(*A*) <∗> *quniv*(*A*) ⊆ *quniv*(*A*)
by (*blast intro*: *QPair-in-quniv*)

lemmas *QSigma-subset-quniv* = *subset-trans* [*OF QSigma-mono QSigma-quniv*]

lemma *quniv-QPair-D*:
  <*a*;*b*> ∈ *quniv*(*A*) ⟹ *a*: *quniv*(*A*) ∧ *b*: *quniv*(*A*)
 unfolding *quniv-def QPair-def*
apply (*rule Transset-includes-summands* [*THEN conjE*])
apply (*rule Transset-eclose* [*THEN Transset-univ*])
apply (*erule PowD*, *blast*)
done

lemmas *quniv-QPair-E* = *quniv-QPair-D* [*THEN conjE*]

lemma *quniv-QPair-iff*: <*a*;*b*> ∈ *quniv*(*A*) ⟷ *a*: *quniv*(*A*) ∧ *b*: *quniv*(*A*)
by (*blast intro*: *QPair-in-quniv dest*: *quniv-QPair-D*)

25.6 Quine Disjoint Sum

lemma *QInl-in-quniv*: *a*: *quniv*(*A*) ⟹ *QInl*(*a*) ∈ *quniv*(*A*)
by (*simp add*: *QInl-def zero-in-quniv QPair-in-quniv*)

lemma *QInr-in-quniv*: *b*: *quniv*(*A*) ⟹ *QInr*(*b*) ∈ *quniv*(*A*)
by (*simp add*: *QInr-def one-in-quniv QPair-in-quniv*)

lemma *qsum-quniv*: *quniv*(*C*) <+> *quniv*(*C*) ⊆ *quniv*(*C*)
by (*blast intro*: *QInl-in-quniv QInr-in-quniv*)

lemmas *qsum-subset-quniv* = *subset-trans* [*OF qsum-mono qsum-quniv*]

25.7 The Natural Numbers

lemmas *nat-subset-quniv* = *subset-trans* [*OF nat-subset-univ univ-subset-quniv*]

lemmas *nat-into-quniv* = *nat-subset-quniv* [*THEN subsetD*]

lemmas *bool-subset-quniv* = *subset-trans* [*OF bool-subset-univ univ-subset-quniv*]

lemmas *bool-into-quniv* = *bool-subset-quniv* [*THEN subsetD*]

lemma *QPair-Int-Vfrom-succ-subset*:
Transset(*X*) ⟹
<*a*;*b*> ∩ *Vfrom*(*X*, *succ*(*i*)) ⊆ <*a* ∩ *Vfrom*(*X*,*i*); *b* ∩ *Vfrom*(*X*,*i*)>
by (*simp add*: *QPair-def sum-def Int-Un-distrib2 Un-mono*
product-Int-Vfrom-subset [*THEN subset-trans*]
Sigma-mono [*OF Int-lower1 subset-refl*])

25.8 "Take-Lemma" Rules

lemma *QPair-Int-Vfrom-subset*:
Transset(*X*) ⟹
<*a*;*b*> ∩ *Vfrom*(*X*,*i*) ⊆ <*a* ∩ *Vfrom*(*X*,*i*); *b* ∩ *Vfrom*(*X*,*i*)>
unfolding *QPair-def*
apply (*erule Transset-Vfrom* [*THEN Transset-sum-Int-subset*])
done

lemmas *QPair-Int-Vset-subset-trans* =
subset-trans [*OF Transset-0* [*THEN QPair-Int-Vfrom-subset*] *QPair-mono*]

lemma *QPair-Int-Vset-subset-UN*:
Ord(*i*) ⟹ <*a*;*b*> ∩ *Vset*(*i*) ⊆ (⋃*j*∈*i*. <*a* ∩ *Vset*(*j*); *b* ∩ *Vset*(*j*)>)
apply (*erule Ord-cases*)

apply (*simp add*: *Vfrom-0*)

apply (*erule ssubst*)
apply (*rule Transset-0* [*THEN QPair-Int-Vfrom-succ-subset*, *THEN subset-trans*])
apply (*rule succI1* [*THEN UN-upper*])

apply (*simp del*: *UN-simps*
    add: *Limit-Vfrom-eq Int-UN-distrib UN-mono QPair-Int-Vset-subset-trans*)
done

end

26 Datatype and CoDatatype Definitions

theory *Datatype*
imports *Inductive Univ QUniv*
keywords *datatype codatatype* :: *thy-decl*
begin

ML-file ‹*Tools/datatype-package.ML*›

ML ‹
(∗*Typechecking rules for most datatypes involving univ*∗)
structure Data-Arg =
  struct
  val intrs =
      [@{*thm SigmaI*}, @{*thm InlI*}, @{*thm InrI*},
       @{*thm Pair-in-univ*}, @{*thm Inl-in-univ*}, @{*thm Inr-in-univ*},
        @{*thm zero-in-univ*}, @{*thm A-into-univ*}, @{*thm nat-into-univ*}, @{*thm UnCI*}];

  val elims = [*make-elim* @{*thm InlD*}, *make-elim* @{*thm InrD*},   (∗*for mutual recursion*∗)
              @{*thm SigmaE*}, @{*thm sumE*}];   (∗*allows* ∗ *and* + *in spec*∗)
  end;

structure Data-Package =
  Add-datatype-def-Fun
   (*structure Fp*=*Lfp and Pr*=*Standard-Prod and CP*=*Standard-CP*
    and Su=*Standard-Sum*
    and Ind-Package = *Ind-Package*
    and Datatype-Arg = *Data-Arg*
    val coind = *false*);

```
(*Typechecking rules for most codatatypes involving quniv*)
structure CoData-Arg =
  struct
  val intrs =
      [@{thm QSigmaI}, @{thm QInlI}, @{thm QInrI},
       @{thm QPair-in-quniv}, @{thm QInl-in-quniv}, @{thm QInr-in-quniv},
       @{thm zero-in-quniv}, @{thm A-into-quniv}, @{thm nat-into-quniv}, @{thm
UnCI}];

  val elims = [make-elim @{thm QInlD}, make-elim @{thm QInrD},   (*for mutual
recursion*)
               @{thm QSigmaE}, @{thm qsumE}];                     (*allows * and +
in spec*)
  end;

structure CoData-Package =
  Add-datatype-def-Fun
   (structure Fp=Gfp and Pr=Quine-Prod and CP=Quine-CP
    and Su=Quine-Sum
    and Ind-Package = CoInd-Package
    and Datatype-Arg = CoData-Arg
    val coind = true);



(* Simproc for freeness reasoning: compare datatype constructors for equality *)

structure Data-Free:
sig
  val trace: bool Config.T
  val proc: Simplifier.proc
end =
struct

val trace = Attrib.setup-config-bool binding ‹data-free-trace› (K false);

fun mk-new ([],[]) = Const ‹True›
  | mk-new (largs,rargs) =
      Balanced-Tree.make FOLogic.mk-conj
               (map FOLogic.mk-eq (ListPair.zip (largs,rargs)));

val datatype-ss = simpset-of context;

fun proc ctxt ct =
  let
    val old = Thm.term-of ct
    val thy = Proof-Context.theory-of ctxt
    val - =
     if Config.get ctxt trace then tracing (data-free: OLD = ^Syntax.string-of-term
```

```
ctxt old)
      else ()
    val (lhs,rhs) = FOLogic.dest-eq old
    val (lhead, largs) = strip-comb lhs
    and (rhead, rargs) = strip-comb rhs
    val lname = dest-Const-name lhead handle TERM - => raise Match;
    val rname = dest-Const-name rhead handle TERM - => raise Match;
    val lcon-info = the (Symtab.lookup (ConstructorsData.get thy) lname)
      handle Option.Option => raise Match;
    val rcon-info = the (Symtab.lookup (ConstructorsData.get thy) rname)
      handle Option.Option => raise Match;
    val new =
      if #big-rec-name lcon-info = #big-rec-name rcon-info
          andalso not (null (#free-iffs lcon-info)) then
        if lname = rname then mk-new (largs, rargs)
        else Const ‹False›
      else raise Match;
     val - =
       if Config.get ctxt trace then tracing (NEW = ^ Syntax.string-of-term ctxt
new)
       else ();
     val goal = Logic.mk-equals (old, new);
     val thm = Goal.prove ctxt [] [] goal
       (fn - => resolve-tac ctxt @{thms iff-reflection} 1 THEN
         simp-tac (put-simpset datatype-ss ctxt addsimps
          (map (Thm.transfer thy) (#free-iffs lcon-info))) 1)
       handle ERROR msg =>
     (warning (msg ^\ndata-free simproc:\nfailed to prove ^Syntax.string-of-term
ctxt goal);
        raise Match)
  in SOME thm end
  handle Match => NONE;

end;
›
```

simproc-setup *data-free* ((*x*::*i*) = *y*) = ‹*fn* - => *Data-Free.proc*›

end

27 Arithmetic Operators and Their Definitions

theory *Arith* **imports** *Univ* **begin**

Proofs about elementary arithmetic: addition, multiplication, etc.

definition
  pred :: *i*⇒*i* **where**
  pred(*y*) ≡ *nat-case*(*0*, *λx*. *x*, *y*)

definition
natify :: $i \Rightarrow i$ **where**
natify ≡ *Vrecursor*($\lambda f\ a.$ *if* $a = succ(pred(a))$ *then* $succ(f`pred(a))$ *else* 0)

consts
raw-add :: $[i,i] \Rightarrow i$
raw-diff :: $[i,i] \Rightarrow i$
raw-mult :: $[i,i] \Rightarrow i$

primrec
raw-add $(0, n) = n$
raw-add $(succ(m), n) = succ(raw\text{-}add(m, n))$

primrec
raw-diff-0: $raw\text{-}diff(m, 0) = m$
raw-diff-succ: $raw\text{-}diff(m, succ(n)) =$
$nat\text{-}case(0, \lambda x.\ x, raw\text{-}diff(m, n))$

primrec
$raw\text{-}mult(0, n) = 0$
$raw\text{-}mult(succ(m), n) = raw\text{-}add\ (n, raw\text{-}mult(m, n))$

definition
add :: $[i,i] \Rightarrow i$ (**infixl** ‹#+› *65*) **where**
$m\ \#{+}\ n \equiv raw\text{-}add\ (natify(m), natify(n))$

definition
diff :: $[i,i] \Rightarrow i$ (**infixl** ‹#−› *65*) **where**
$m\ \#{-}\ n \equiv raw\text{-}diff\ (natify(m), natify(n))$

definition
mult :: $[i,i] \Rightarrow i$ (**infixl** ‹#∗› *70*) **where**
$m\ \#{*}\ n \equiv raw\text{-}mult\ (natify(m), natify(n))$

definition
raw-div :: $[i,i] \Rightarrow i$ **where**
raw-div $(m, n) \equiv$
$transrec(m, \lambda j\ f.\ if\ j{<}n \mid n{=}0\ then\ 0\ else\ succ(f`(j\#{-}n)))$

definition
raw-mod :: $[i,i] \Rightarrow i$ **where**
raw-mod $(m, n) \equiv$
$transrec(m, \lambda j\ f.\ if\ j{<}n \mid n{=}0\ then\ j\ else\ f`(j\#{-}n))$

definition
div :: $[i,i] \Rightarrow i$ (**infixl** ‹*div*› *70*) **where**
$m\ div\ n \equiv raw\text{-}div\ (natify(m), natify(n))$

definition
  mod :: [*i*,*i*]⇒*i*   (**infixl** ‹*mod*› *70*) **where**
    m mod n ≡ *raw-mod* (*natify*(*m*), *natify*(*n*))

declare *rec-type* [*simp*]
        nat-0-le [*simp*]

lemma *zero-lt-lemma*: ⟦*0*<*k*; *k* ∈ *nat*⟧ ⟹ ∃ *j*∈*nat*. *k* = *succ*(*j*)
apply (*erule rev-mp*)
apply (*induct-tac k*, *auto*)
done

lemmas *zero-lt-natE* = *zero-lt-lemma* [*THEN bexE*]

27.1 *natify*, the Coercion to *nat*

lemma *pred-succ-eq* [*simp*]: *pred*(*succ*(*y*)) = *y*
by (*unfold pred-def*, *auto*)

lemma *natify-succ*: *natify*(*succ*(*x*)) = *succ*(*natify*(*x*))
by (*rule natify-def* [*THEN def-Vrecursor*, *THEN trans*], *auto*)

lemma *natify-0* [*simp*]: *natify*(*0*) = *0*
by (*rule natify-def* [*THEN def-Vrecursor*, *THEN trans*], *auto*)

lemma *natify-non-succ*: ∀ *z*. *x* ≠ *succ*(*z*) ⟹ *natify*(*x*) = *0*
by (*rule natify-def* [*THEN def-Vrecursor*, *THEN trans*], *auto*)

lemma *natify-in-nat* [*iff*,*TC*]: *natify*(*x*) ∈ *nat*
apply (*rule-tac a*=*x* **in** *eps-induct*)
apply (*case-tac* ∃ *z*. *x* = *succ*(*z*))
apply (*auto simp add*: *natify-succ natify-non-succ*)
done

lemma *natify-ident* [*simp*]: *n* ∈ *nat* ⟹ *natify*(*n*) = *n*
apply (*induct-tac n*)
apply (*auto simp add*: *natify-succ*)
done

lemma *natify-eqE*: ⟦*natify*(*x*) = *y*; *x* ∈ *nat*⟧ ⟹ *x*=*y*
by *auto*

lemma *natify-idem* [*simp*]: *natify*(*natify*(*x*)) = *natify*(*x*)
by *simp*

lemma *add-natify1* [*simp*]: *natify*(*m*) #+ *n* = *m* #+ *n*
by (*simp add*: *add-def*)

lemma *add-natify2* [*simp*]: *m* #+ *natify*(*n*) = *m* #+ *n*
by (*simp add*: *add-def*)

lemma *mult-natify1* [*simp*]: *natify*(*m*) #* *n* = *m* #* *n*
by (*simp add*: *mult-def*)

lemma *mult-natify2* [*simp*]: *m* #* *natify*(*n*) = *m* #* *n*
by (*simp add*: *mult-def*)

lemma *diff-natify1* [*simp*]: *natify*(*m*) #− *n* = *m* #− *n*
by (*simp add*: *diff-def*)

lemma *diff-natify2* [*simp*]: *m* #− *natify*(*n*) = *m* #− *n*
by (*simp add*: *diff-def*)

lemma *mod-natify1* [*simp*]: *natify*(*m*) *mod n* = *m mod n*
by (*simp add*: *mod-def*)

lemma *mod-natify2* [*simp*]: *m mod natify*(*n*) = *m mod n*
by (*simp add*: *mod-def*)

lemma *div-natify1* [*simp*]: *natify*(*m*) *div n* = *m div n*
by (*simp add*: *div-def*)

lemma *div-natify2* [*simp*]: *m div natify*(*n*) = *m div n*
by (*simp add*: *div-def*)

27.2 Typing rules

lemma *raw-add-type*: ⟦*m*∈*nat*; *n*∈*nat*⟧ ⟹ *raw-add* (*m*, *n*) ∈ *nat*
by (*induct-tac m*, *auto*)

lemma *add-type* [*iff*,*TC*]: *m* #+ *n* ∈ *nat*
by (*simp add*: *add-def raw-add-type*)

lemma *raw-mult-type*: $[\![m{\in}nat;\ \ n{\in}nat]\!] \Longrightarrow raw\text{-}mult\ (m,\ n) \in nat$
apply (*induct-tac m*)
apply (*simp-all add*: *raw-add-type*)
done

lemma *mult-type* [*iff*,*TC*]: $m\ \#{*}\ n \in nat$
by (*simp add*: *mult-def raw-mult-type*)

lemma *raw-diff-type*: $[\![m{\in}nat;\ \ n{\in}nat]\!] \Longrightarrow raw\text{-}diff\ (m,\ n) \in nat$
by (*induct-tac n*, *auto*)

lemma *diff-type* [*iff*,*TC*]: $m\ \#{-}\ n \in nat$
by (*simp add*: *diff-def raw-diff-type*)

lemma *diff-0-eq-0* [*simp*]: $0\ \#{-}\ n = 0$
 unfolding *diff-def*
apply (*rule natify-in-nat* [*THEN nat-induct*], *auto*)
done

lemma *diff-succ-succ* [*simp*]: $succ(m)\ \#{-}\ succ(n) = m\ \#{-}\ n$
apply (*simp add*: *natify-succ diff-def*)
apply (*rule-tac x1 = n* **in** *natify-in-nat* [*THEN nat-induct*], *auto*)
done

declare *raw-diff-succ* [*simp del*]

lemma *diff-0* [*simp*]: $m\ \#{-}\ 0 = natify(m)$
by (*simp add*: *diff-def*)

lemma *diff-le-self*: $m{\in}nat \Longrightarrow (m\ \#{-}\ n) \le m$
apply (*subgoal-tac* $(m\ \#{-}\ natify\ (n)) \le m$)
apply (*rule-tac* [*2*] $m = m$ **and** $n = natify\ (n)$ **in** *diff-induct*)
apply (*erule-tac* [*6*] *leE*)
apply (*simp-all add*: *le-iff*)
done

27.3 Addition

lemma *add-0-natify* [*simp*]: $0\ \#{+}\ m = natify(m)$
by (*simp add*: *add-def*)

lemma *add-succ* [*simp*]: *succ*(*m*) #+ *n* = *succ*(*m* #+ *n*)
by (*simp add*: *natify-succ add-def*)

lemma *add-0*: *m* ∈ *nat* ⟹ *0* #+ *m* = *m*
by *simp*

lemma *add-assoc*: (*m* #+ *n*) #+ *k* = *m* #+ (*n* #+ *k*)
apply (*subgoal-tac* (*natify*(*m*) #+ *natify*(*n*)) #+ *natify*(*k*) =
natify(*m*) #+ (*natify*(*n*) #+ *natify*(*k*)))
apply (*rule-tac* [*2*] *n* = *natify*(*m*) **in** *nat-induct*)
apply *auto*
done

lemma *add-0-right-natify* [*simp*]: *m* #+ *0* = *natify*(*m*)
apply (*subgoal-tac natify*(*m*) #+ *0* = *natify*(*m*))
apply (*rule-tac* [*2*] *n* = *natify*(*m*) **in** *nat-induct*)
apply *auto*
done

lemma *add-succ-right* [*simp*]: *m* #+ *succ*(*n*) = *succ*(*m* #+ *n*)
unfolding *add-def*
apply (*rule-tac n* = *natify*(*m*) **in** *nat-induct*)
apply (*auto simp add*: *natify-succ*)
done

lemma *add-0-right*: *m* ∈ *nat* ⟹ *m* #+ *0* = *m*
by *auto*

lemma *add-commute*: *m* #+ *n* = *n* #+ *m*
apply (*subgoal-tac natify*(*m*) #+ *natify*(*n*) = *natify*(*n*) #+ *natify*(*m*))
apply (*rule-tac* [*2*] *n* = *natify*(*m*) **in** *nat-induct*)
apply *auto*
done

lemma *add-left-commute*: *m*#+(*n*#+*k*)=*n*#+(*m*#+*k*)
apply (*rule add-commute* [*THEN trans*])
apply (*rule add-assoc* [*THEN trans*])
apply (*rule add-commute* [*THEN subst-context*])
done

lemmas *add-ac* = *add-assoc add-commute add-left-commute*

lemma *raw-add-left-cancel*:
  $[\![raw\text{-}add(k,\ m) = raw\text{-}add(k,\ n);\ \ k{\in}nat]\!] \Longrightarrow m{=}n$
apply (*erule rev-mp*)
apply (*induct-tac k*, *auto*)
done

lemma *add-left-cancel-natify*: $k \ \#{+}\ m = k \ \#{+}\ n \Longrightarrow natify(m) = natify(n)$
 unfolding *add-def*
apply (*drule raw-add-left-cancel*, *auto*)
done

lemma *add-left-cancel*:
  $[\![i = j;\ \ i \ \#{+}\ m = j \ \#{+}\ n;\ \ m{\in}nat;\ \ n{\in}nat]\!] \Longrightarrow m = n$
by (*force dest*!: *add-left-cancel-natify*)

lemma *add-le-elim1-natify*: $k\#{+}m \le k\#{+}n \Longrightarrow natify(m) \le natify(n)$
apply (*rule-tac* $P = natify(k)\ \#{+}m \le natify(k)\ \#{+}n$ **in** *rev-mp*)
apply (*rule-tac* [*2*] $n = natify(k)$ **in** *nat-induct*)
apply *auto*
done

lemma *add-le-elim1*: $[\![k\#{+}m \le k\#{+}n;\ m \in nat;\ n \in nat]\!] \Longrightarrow m \le n$
by (*drule add-le-elim1-natify*, *auto*)

lemma *add-lt-elim1-natify*: $k\#{+}m < k\#{+}n \Longrightarrow natify(m) < natify(n)$
apply (*rule-tac* $P = natify(k)\ \#{+}m < natify(k)\ \#{+}n$ **in** *rev-mp*)
apply (*rule-tac* [*2*] $n = natify(k)$ **in** *nat-induct*)
apply *auto*
done

lemma *add-lt-elim1*: $[\![k\#{+}m < k\#{+}n;\ m \in nat;\ n \in nat]\!] \Longrightarrow m < n$
by (*drule add-lt-elim1-natify*, *auto*)

lemma *zero-less-add*: $[\![n \in nat;\ m \in nat]\!] \Longrightarrow 0 < m \ \#{+}\ n \longleftrightarrow (0{<}m \mid 0{<}n)$
by (*induct-tac n*, *auto*)

27.4 Monotonicity of Addition

lemma *add-lt-mono1*: $[\![i{<}j;\ j{\in}nat]\!] \Longrightarrow i\#{+}k < j\#{+}k$
apply (*frule lt-nat-in-nat*, *assumption*)
apply (*erule succ-lt-induct*)
apply (*simp-all add*: *leI*)
done

strict, in second argument

lemma *add-lt-mono2*: $[\![i{<}j;\ j{\in}nat]\!] \Longrightarrow k\#{+}i < k\#{+}j$
by (*simp add*: *add-commute* [*of k*] *add-lt-mono1*)

A [clumsy] way of lifting < monotonicity to ≤ monotonicity

lemma *Ord-lt-mono-imp-le-mono*:
  assumes *lt-mono*: $\bigwedge i\ j.\ [\![i<j;\ j:k]\!] \Longrightarrow f(i) < f(j)$
    and *ford*: $\bigwedge i.\ i:k \Longrightarrow Ord(f(i))$
    and *leij*: $i \le j$
    and *jink*: $j:k$
  shows $f(i) \le f(j)$
apply (*insert leij jink*)
apply (*blast intro*!: *leCI lt-mono ford elim*!: *leE*)
done

≤ monotonicity, 1st argument

lemma *add-le-mono1*: $[\![i \le j;\ j\in nat]\!] \Longrightarrow i \#+ k \le j \#+ k$
apply (*rule-tac* $f = \lambda j.\ j\#+k$ **in** *Ord-lt-mono-imp-le-mono*, *typecheck*)
apply (*blast intro*: *add-lt-mono1 add-type* [*THEN nat-into-Ord*])+
done

≤ monotonicity, both arguments

lemma *add-le-mono*: $[\![i \le j;\ k \le l;\ j\in nat;\ l\in nat]\!] \Longrightarrow i \#+ k \le j \#+ l$
apply (*rule add-le-mono1* [*THEN le-trans*], *assumption*+)
apply (*subst add-commute*, *subst add-commute*, *rule add-le-mono1*, *assumption*+)
done

Combinations of less-than and less-than-or-equals

lemma *add-lt-le-mono*: $[\![i<j;\ k\le l;\ j\in nat;\ l\in nat]\!] \Longrightarrow i \#+ k < j \#+ l$
apply (*rule add-lt-mono1* [*THEN lt-trans2*], *assumption*+)
apply (*subst add-commute*, *subst add-commute*, *rule add-le-mono1*, *assumption*+)
done

lemma *add-le-lt-mono*: $[\![i\le j;\ k<l;\ j\in nat;\ l\in nat]\!] \Longrightarrow i \#+ k < j \#+ l$
by (*subst add-commute*, *subst add-commute*, *erule add-lt-le-mono*, *assumption*+)

Less-than: in other words, strict in both arguments

lemma *add-lt-mono*: $[\![i<j;\ k<l;\ j\in nat;\ l\in nat]\!] \Longrightarrow i \#+ k < j \#+ l$
apply (*rule add-lt-le-mono*)
apply (*auto intro*: *leI*)
done

lemma *diff-add-inverse*: $(n\#+m)\ \#-\ n = natify(m)$
apply (*subgoal-tac* $(natify(n)\ \#+\ m)\ \#-\ natify(n) = natify(m)$)
apply (*rule-tac* [*2*] $n = natify(n)$ **in** *nat-induct*)
apply *auto*
done

lemma *diff-add-inverse2*: $(m\#+n)\ \#-\ n = natify(m)$
by (*simp add*: *add-commute* [*of m*] *diff-add-inverse*)

lemma *diff-cancel*: $(k\#{+}m)\ \#{-}\ (k\#{+}n) = m\ \#{-}\ n$
apply (*subgoal-tac* $(natify(k)\ \#{+}\ natify(m))\ \#{-}\ (natify(k)\ \#{+}\ natify(n)) =$
$natify(m)\ \#{-}\ natify(n)$)
apply (*rule-tac* [*2*] $n = natify(k)$ **in** *nat-induct*)
apply *auto*
done

lemma *diff-cancel2*: $(m\#{+}k)\ \#{-}\ (n\#{+}k) = m\ \#{-}\ n$
by (*simp add*: *add-commute* [*of - k*] *diff-cancel*)

lemma *diff-add-0*: $n\ \#{-}\ (n\#{+}m) = 0$
apply (*subgoal-tac* $natify(n)\ \#{-}\ (natify(n)\ \#{+}\ natify(m)) = 0$)
apply (*rule-tac* [*2*] $n = natify(n)$ **in** *nat-induct*)
apply *auto*
done

lemma *pred-0* [*simp*]: $pred(0) = 0$
by (*simp add*: *pred-def*)

lemma *eq-succ-imp-eq-m1*: $[\![i = succ(j);\ i{\in}nat]\!] \Longrightarrow j = i\ \#{-}\ 1 \wedge j \in nat$
by *simp*

lemma *pred-Un-distrib*:
$[\![i{\in}nat;\ j{\in}nat]\!] \Longrightarrow pred(i \cup j) = pred(i) \cup pred(j)$
apply (*erule-tac* $n{=}i$ **in** *natE*, *simp*)
apply (*erule-tac* $n{=}j$ **in** *natE*, *simp*)
apply (*simp add*: *succ-Un-distrib* [*symmetric*])
done

lemma *pred-type* [*TC*,*simp*]:
$i \in nat \Longrightarrow pred(i) \in nat$
by (*simp add*: *pred-def split*: *split-nat-case*)

lemma *nat-diff-pred*: $[\![i{\in}nat;\ j{\in}nat]\!] \Longrightarrow i\ \#{-}\ succ(j) = pred(i\ \#{-}\ j)$
apply (*rule-tac* $m{=}i$ **and** $n{=}j$ **in** *diff-induct*)
apply (*auto simp add*: *pred-def nat-imp-quasinat split*: *split-nat-case*)
done

lemma *diff-succ-eq-pred*: $i\ \#{-}\ succ(j) = pred(i\ \#{-}\ j)$
apply (*insert nat-diff-pred* [*of natify*(*i*) *natify*(*j*)])
apply (*simp add*: *natify-succ* [*symmetric*])
done

lemma *nat-diff-Un-distrib*:
$[\![i{\in}nat;\ j{\in}nat;\ k{\in}nat]\!] \Longrightarrow (i \cup j)\ \#{-}\ k = (i\#{-}k) \cup (j\#{-}k)$
apply (*rule-tac* $n{=}k$ **in** *nat-induct*)
apply (*simp-all add*: *diff-succ-eq-pred pred-Un-distrib*)
done

lemma *diff-Un-distrib*:
$[\![i{\in}nat;\ j{\in}nat]\!] \Longrightarrow (i \cup j)\ \#-\ k = (i\#-k) \cup (j\#-k)$
by (*insert nat-diff-Un-distrib* [*of i j natify*(k)], *simp*)

We actually prove $i\ \#-\ j\ \#-\ k = i\ \#-\ (j\ \#+\ k)$

lemma *diff-diff-left* [*simplified*]:
$natify(i)\#-natify(j)\#-k = natify(i)\ \#-\ (natify(j)\#+k)$
by (*rule-tac* $m{=}natify(i)$ **and** $n{=}natify(j)$ **in** *diff-induct*, *auto*)

lemma *eq-add-iff*: $(u\ \#+\ m = u\ \#+\ n) \longleftrightarrow (0\ \#+\ m = natify(n))$
apply *auto*
apply (*blast dest*: *add-left-cancel-natify*)
apply (*simp add*: *add-def*)
done

lemma *less-add-iff*: $(u\ \#+\ m < u\ \#+\ n) \longleftrightarrow (0\ \#+\ m < natify(n))$
apply (*auto simp add*: *add-lt-elim1-natify*)
apply (*drule add-lt-mono1*)
apply (*auto simp add*: *add-commute* [*of u*])
done

lemma *diff-add-eq*: $((u\ \#+\ m)\ \#-\ (u\ \#+\ n)) = ((0\ \#+\ m)\ \#-\ n)$
by (*simp add*: *diff-cancel*)

lemma *eq-cong2*: $u = u' \Longrightarrow (t{\equiv}u) \equiv (t{\equiv}u')$
by *auto*

lemma *iff-cong2*: $u \longleftrightarrow u' \Longrightarrow (t{\equiv}u) \equiv (t{\equiv}u')$
by *auto*

27.5 Multiplication

lemma *mult-0* [*simp*]: $0\ \#*\ m = 0$
by (*simp add*: *mult-def*)

lemma *mult-succ* [*simp*]: $succ(m)\ \#*\ n = n\ \#+\ (m\ \#*\ n)$
by (*simp add*: *add-def mult-def natify-succ raw-mult-type*)

lemma *mult-0-right* [*simp*]: $m\ \#*\ 0 = 0$
unfolding *mult-def*
apply (*rule-tac* $n = natify(m)$ **in** *nat-induct*)
apply *auto*
done

lemma *mult-succ-right* [*simp*]: m #* *succ*(n) = m #+ (m #* n)
apply (*subgoal-tac natify*(m) #* *succ* (*natify*(n)) =
natify(m) #+ (*natify*(m) #* *natify*(n)))
apply (*simp* (*no-asm-use*) *add*: *natify-succ add-def mult-def*)
apply (*rule-tac* n = *natify*(m) **in** *nat-induct*)
apply (*simp-all add*: *add-ac*)
done

lemma *mult-1-natify* [*simp*]: *1* #* n = *natify*(n)
by *auto*

lemma *mult-1-right-natify* [*simp*]: n #* *1* = *natify*(n)
by *auto*

lemma *mult-1*: $n \in nat \Longrightarrow$ *1* #* n = n
by *simp*

lemma *mult-1-right*: $n \in nat \Longrightarrow n$ #* *1* = n
by *simp*

lemma *mult-commute*: m #* n = n #* m
apply (*subgoal-tac natify*(m) #* *natify*(n) = *natify*(n) #* *natify*(m))
apply (*rule-tac* [*2*] n = *natify*(m) **in** *nat-induct*)
apply *auto*
done

lemma *add-mult-distrib*: (m #+ n) #* k = (m #* k) #+ (n #* k)
apply (*subgoal-tac* (*natify*(m) #+ *natify*(n)) #* *natify*(k) =
(*natify*(m) #* *natify*(k)) #+ (*natify*(n) #* *natify*(k)))
apply (*rule-tac* [*2*] n = *natify*(m) **in** *nat-induct*)
apply (*simp-all add*: *add-assoc* [*symmetric*])
done

lemma *add-mult-distrib-left*: k #* (m #+ n) = (k #* m) #+ (k #* n)
apply (*subgoal-tac natify*(k) #* (*natify*(m) #+ *natify*(n)) =
(*natify*(k) #* *natify*(m)) #+ (*natify*(k) #* *natify*(n)))
apply (*rule-tac* [*2*] n = *natify*(m) **in** *nat-induct*)
apply (*simp-all add*: *add-ac*)
done

lemma *mult-assoc*: (m #* n) #* k = m #* (n #* k)
apply (*subgoal-tac* (*natify*(m) #* *natify*(n)) #* *natify*(k) =
natify(m) #* (*natify*(n) #* *natify*(k)))
apply (*rule-tac* [*2*] n = *natify*(m) **in** *nat-induct*)

apply (*simp-all add*: *add-mult-distrib*)
done

lemma *mult-left-commute*: $m \#* (n \#* k) = n \#* (m \#* k)$
apply (*rule mult-commute* [*THEN trans*])
apply (*rule mult-assoc* [*THEN trans*])
apply (*rule mult-commute* [*THEN subst-context*])
done

lemmas *mult-ac* = *mult-assoc mult-commute mult-left-commute*

lemma *lt-succ-eq-0-disj*:
$[\![m\in nat; n\in nat]\!]$
$\Longrightarrow (m < succ(n)) \longleftrightarrow (m = 0 \mid (\exists j\in nat.\ m = succ(j) \wedge j < n))$
by (*induct-tac m*, *auto*)

lemma *less-diff-conv* [*rule-format*]:
$[\![j\in nat; k\in nat]\!] \Longrightarrow \forall i\in nat.\ (i < j \#- k) \longleftrightarrow (i \#+ k < j)$
by (*erule-tac m* = *k* **in** *diff-induct*, *auto*)

lemmas *nat-typechecks* = *rec-type nat-0I nat-1I nat-succI Ord-nat*

end

28 Arithmetic with simplification

theory *ArithSimp*
imports *Arith*
begin

28.1 Arithmetic simplification

ML-file ‹~~/*src*/*Provers*/*Arith*/*cancel-numerals.ML*›
ML-file ‹~~/*src*/*Provers*/*Arith*/*combine-numerals.ML*›
ML-file ‹*arith-data.ML*›

simproc-setup *nateq-cancel-numerals*
$(l \#+ m = n \mid l = m \#+ n \mid l \#* m = n \mid l = m \#* n \mid succ(m) = n \mid m = succ(n)) =$
‹*K ArithData.nateq-cancel-numerals-proc*›

simproc-setup *natless-cancel-numerals*
$(l \#+ m < n \mid l < m \#+ n \mid l \#* m < n \mid l < m \#* n \mid succ(m) < n \mid m < succ(n)) =$
‹*K ArithData.natless-cancel-numerals-proc*›

simproc-setup *natdiff-cancel-numerals*

$((l \#+ m) \#- n \mid l \#- (m \#+ n) \mid (l \#* m) \#- n \mid l \#- (m \#* n) \mid$
$succ(m) \#- n \mid m \#- succ(n)) =$
⟨*K ArithData.natdiff-cancel-numerals-proc*⟩

28.1.1 Examples

lemma $x \#+ y = x \#+ z$ **apply** *simp* **oops**
lemma $y \#+ x = x \#+ z$ **apply** *simp* **oops**
lemma $x \#+ y \#+ z = x \#+ z$ **apply** *simp* **oops**
lemma $y \#+ (z \#+ x) = z \#+ x$ **apply** *simp* **oops**
lemma $x \#+ y \#+ z = (z \#+ y) \#+ (x \#+ w)$ **apply** *simp* **oops**
lemma $x\#*y \#+ z = (z \#+ y) \#+ (y\#*x \#+ w)$ **apply** *simp* **oops**

lemma $x \#+ succ(y) = x \#+ z$ **apply** *simp* **oops**
lemma $x \#+ succ(y) = succ(z \#+ x)$ **apply** *simp* **oops**
lemma $succ(x) \#+ succ(y) \#+ z = succ(z \#+ y) \#+ succ(x \#+ w)$ **apply** *simp* **oops**

lemma $(x \#+ y) \#- (x \#+ z) = w$ **apply** *simp* **oops**
lemma $(y \#+ x) \#- (x \#+ z) = dd$ **apply** *simp* **oops**
lemma $(x \#+ y \#+ z) \#- (x \#+ z) = dd$ **apply** *simp* **oops**
lemma $(y \#+ (z \#+ x)) \#- (z \#+ x) = dd$ **apply** *simp* **oops**
lemma $(x \#+ y \#+ z) \#- ((z \#+ y) \#+ (x \#+ w)) = dd$ **apply** *simp* **oops**
lemma $(x\#*y \#+ z) \#- ((z \#+ y) \#+ (y\#*x \#+ w)) = dd$ **apply** *simp* **oops**

lemma $(x \#+ succ(y)) \#- (x \#+ z) = dd$ **apply** *simp* **oops**

lemma $x \#* y2 \#+ y \#* x2 = y \#* x2 \#+ x \#* y2$ **apply** *simp* **oops**

lemma $(x \#+ succ(y)) \#- (succ(z \#+ x)) = dd$ **apply** *simp* **oops**
lemma $(succ(x) \#+ succ(y) \#+ z) \#- (succ(z \#+ y) \#+ succ(x \#+ w)) = dd$ **apply** *simp* **oops**

lemma $x : nat ==> x \#+ y = x$ **apply** *simp* **oops**
lemma $x : nat --> x \#+ y = x$ **apply** *simp* **oops**
lemma $x : nat ==> x \#+ y < x$ **apply** *simp* **oops**
lemma $x : nat ==> x < y\#+x$ **apply** *simp* **oops**
lemma $x : nat ==> x \leq succ(x)$ **apply** *simp* **oops**

lemma $x \#+ y = x$ **apply** *simp?* **oops**

lemma $x \#+ y < x \#+ z$ **apply** *simp* **oops**
lemma $y \#+ x < x \#+ z$ **apply** *simp* **oops**
lemma $x \#+ y \#+ z < x \#+ z$ **apply** *simp* **oops**
lemma $y \#+ z \#+ x < x \#+ z$ **apply** *simp* **oops**
lemma $y \#+ (z \#+ x) < z \#+ x$ **apply** *simp* **oops**

lemma $x \#+ y \#+ z < (z \#+ y) \#+ (x \#+ w)$ **apply** *simp* **oops**
lemma $x\#*y \#+ z < (z \#+ y) \#+ (y\#*x \#+ w)$ **apply** *simp* **oops**

lemma $x \#+ succ(y) < x \#+ z$ **apply** *simp* **oops**
lemma $x \#+ succ(y) < succ(z \#+ x)$ **apply** *simp* **oops**
lemma $succ(x) \#+ succ(y) \#+ z < succ(z \#+ y) \#+ succ(x \#+ w)$ **apply** *simp* **oops**

lemma $x \#+ succ(y) \le succ(z \#+ x)$ **apply** *simp* **oops**

28.2 Difference

lemma *diff-self-eq-0* [*simp*]: $m \#- m = 0$
apply (*subgoal-tac natify* $(m) \#-$ *natify* $(m) = 0$)
apply (*rule-tac* [*2*] *natify-in-nat* [*THEN nat-induct*], *auto*)
done

lemma *add-diff-inverse*: $[\![n \le m;\ m{:}nat]\!] \Longrightarrow n \#+ (m\#-n) = m$
apply (*frule lt-nat-in-nat*, *erule nat-succI*)
apply (*erule rev-mp*)
apply (*rule-tac* $m = m$ **and** $n = n$ **in** *diff-induct*, *auto*)
done

lemma *add-diff-inverse2*: $[\![n \le m;\ m{:}nat]\!] \Longrightarrow (m\#-n) \#+ n = m$
apply (*frule lt-nat-in-nat*, *erule nat-succI*)
apply (*simp* (*no-asm-simp*) *add*: *add-commute add-diff-inverse*)
done

lemma *diff-succ*: $[\![n \le m;\ m{:}nat]\!] \Longrightarrow succ(m) \#- n = succ(m\#-n)$
apply (*frule lt-nat-in-nat*, *erule nat-succI*)
apply (*erule rev-mp*)
apply (*rule-tac* $m = m$ **and** $n = n$ **in** *diff-induct*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *zero-less-diff* [*simp*]:
$[\![m{:}\ nat;\ n{:}\ nat]\!] \Longrightarrow 0 < (n \#- m) \longleftrightarrow m<n$
apply (*rule-tac* $m = m$ **and** $n = n$ **in** *diff-induct*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *diff-mult-distrib*: $(m \#- n) \#* k = (m \#* k) \#- (n \#* k)$

apply (*subgoal-tac* (*natify* (*m*) #− *natify* (*n*)) #∗ *natify* (*k*) = (*natify* (*m*) #∗ *natify* (*k*)) #− (*natify* (*n*) #∗ *natify* (*k*)))
apply (*rule-tac* [*2*] *m* = *natify* (*m*) **and** *n* = *natify* (*n*) **in** *diff-induct*)
apply (*simp-all add*: *diff-cancel*)
done

lemma *diff-mult-distrib2*: *k* #∗ (*m* #− *n*) = (*k* #∗ *m*) #− (*k* #∗ *n*)
apply (*simp* (*no-asm*) *add*: *mult-commute* [*of k*] *diff-mult-distrib*)
done

28.3 Remainder

lemma *div-termination*: ⟦*0*<*n*; *n* ≤ *m*; *m*:*nat*⟧ ⟹ *m* #− *n* < *m*
apply (*frule lt-nat-in-nat*, *erule nat-succI*)
apply (*erule rev-mp*)
apply (*erule rev-mp*)
apply (*rule-tac m* = *m* **and** *n* = *n* **in** *diff-induct*)
apply (*simp-all* (*no-asm-simp*) *add*: *diff-le-self*)
done

lemmas *div-rls* =
  nat-typechecks Ord-transrec-type apply-funtype
  div-termination [*THEN ltD*]
  nat-into-Ord not-lt-iff-le [*THEN iffD1*]

lemma *raw-mod-type*: ⟦*m*:*nat*; *n*:*nat*⟧ ⟹ *raw-mod* (*m*, *n*) ∈ *nat*
  unfolding *raw-mod-def*
apply (*rule Ord-transrec-type*)
apply (*auto simp add*: *nat-into-Ord* [*THEN Ord-0-lt-iff*])
apply (*blast intro*: *div-rls*)
done

lemma *mod-type* [*TC*,*iff*]: *m mod n* ∈ *nat*
  unfolding *mod-def*
apply (*simp* (*no-asm*) *add*: *mod-def raw-mod-type*)
done

lemma *DIVISION-BY-ZERO-DIV*: *a div 0* = *0*
  unfolding *div-def*
apply (*rule raw-div-def* [*THEN def-transrec*, *THEN trans*])
apply (*simp* (*no-asm-simp*))
done

lemma *DIVISION-BY-ZERO-MOD*: *a mod 0* = *natify*(*a*)
  unfolding *mod-def*

apply (*rule raw-mod-def* [*THEN def-transrec*, *THEN trans*])
apply (*simp* (*no-asm-simp*))
done

lemma *raw-mod-less*: *m*<*n* ⟹ *raw-mod* (*m*,*n*) = *m*
apply (*rule raw-mod-def* [*THEN def-transrec*, *THEN trans*])
apply (*simp* (*no-asm-simp*) *add*: *div-termination* [*THEN ltD*])
done

lemma *mod-less* [*simp*]: ⟦*m*<*n*; *n* ∈ *nat*⟧ ⟹ *m mod n* = *m*
apply (*frule lt-nat-in-nat*, *assumption*)
apply (*simp* (*no-asm-simp*) *add*: *mod-def raw-mod-less*)
done

lemma *raw-mod-geq*:
⟦*0*<*n*; *n* ≤ *m*; *m*:*nat*⟧ ⟹ *raw-mod* (*m*, *n*) = *raw-mod* (*m*#−*n*, *n*)
apply (*frule lt-nat-in-nat*, *erule nat-succI*)
apply (*rule raw-mod-def* [*THEN def-transrec*, *THEN trans*])
apply (*simp* (*no-asm-simp*) *add*: *div-termination* [*THEN ltD*] *not-lt-iff-le* [*THEN iffD2*], *blast*)
done

lemma *mod-geq*: ⟦*n* ≤ *m*; *m*:*nat*⟧ ⟹ *m mod n* = (*m*#−*n*) *mod n*
apply (*frule lt-nat-in-nat*, *erule nat-succI*)
apply (*case-tac n*=*0*)
apply (*simp add*: *DIVISION-BY-ZERO-MOD*)
apply (*simp add*: *mod-def raw-mod-geq nat-into-Ord* [*THEN Ord-0-lt-iff*])
done

28.4 Division

lemma *raw-div-type*: ⟦*m*:*nat*; *n*:*nat*⟧ ⟹ *raw-div* (*m*, *n*) ∈ *nat*
unfolding *raw-div-def*
apply (*rule Ord-transrec-type*)
apply (*auto simp add*: *nat-into-Ord* [*THEN Ord-0-lt-iff*])
apply (*blast intro*: *div-rls*)
done

lemma *div-type* [*TC*,*iff*]: *m div n* ∈ *nat*
unfolding *div-def*
apply (*simp* (*no-asm*) *add*: *div-def raw-div-type*)
done

lemma *raw-div-less*: *m*<*n* ⟹ *raw-div* (*m*,*n*) = *0*
apply (*rule raw-div-def* [*THEN def-transrec*, *THEN trans*])
apply (*simp* (*no-asm-simp*) *add*: *div-termination* [*THEN ltD*])
done

lemma *div-less* [*simp*]: ⟦*m*<*n*; *n* ∈ *nat*⟧ ⟹ *m div n* = *0*
apply (*frule lt-nat-in-nat*, *assumption*)
apply (*simp* (*no-asm-simp*) *add*: *div-def raw-div-less*)
done

lemma *raw-div-geq*: ⟦*0*<*n*; *n* ≤ *m*; *m*:*nat*⟧ ⟹ *raw-div*(*m*,*n*) = *succ*(*raw-div*(*m*#−*n*, *n*))
apply (*subgoal-tac n* ≠ *0*)
prefer *2* **apply** *blast*
apply (*frule lt-nat-in-nat*, *erule nat-succI*)
apply (*rule raw-div-def* [*THEN def-transrec*, *THEN trans*])
apply (*simp* (*no-asm-simp*) *add*: *div-termination* [*THEN ltD*] *not-lt-iff-le* [*THEN iffD2*])
done

lemma *div-geq* [*simp*]:
  ⟦*0*<*n*; *n* ≤ *m*; *m*:*nat*⟧ ⟹ *m div n* = *succ* ((*m*#−*n*) *div n*)
apply (*frule lt-nat-in-nat*, *erule nat-succI*)
apply (*simp* (*no-asm-simp*) *add*: *div-def raw-div-geq*)
done

declare *div-less* [*simp*] *div-geq* [*simp*]

lemma *mod-div-lemma*: ⟦*m*: *nat*; *n*: *nat*⟧ ⟹ (*m div n*)#∗*n* #+ *m mod n* = *m*
apply (*case-tac n*=*0*)
apply (*simp add*: *DIVISION-BY-ZERO-MOD*)
apply (*simp add*: *nat-into-Ord* [*THEN Ord-0-lt-iff*])
apply (*erule complete-induct*)
apply (*case-tac x*<*n*)

case x<n

apply (*simp* (*no-asm-simp*))

case *n* ≤ *x*

apply (*simp add*: *not-lt-iff-le add-assoc mod-geq div-termination* [*THEN ltD*] *add-diff-inverse*)
done

lemma *mod-div-equality-natify*: (*m div n*)#∗*n* #+ *m mod n* = *natify*(*m*)
apply (*subgoal-tac* (*natify* (*m*) *div natify* (*n*))#∗*natify* (*n*) #+ *natify* (*m*) *mod natify* (*n*) = *natify* (*m*))
apply *force*
apply (*subst mod-div-lemma*, *auto*)
done

lemma *mod-div-equality*: *m*: *nat* ⟹ (*m div n*)#∗*n* #+ *m mod n* = *m*
apply (*simp* (*no-asm-simp*) *add*: *mod-div-equality-natify*)
done

28.5 Further Facts about Remainder

(mainly for mutilated chess board)

lemma *mod-succ-lemma*:
  ⟦*0*<*n*; *m*:*nat*; *n*:*nat*⟧
  ⟹ *succ*(*m*) *mod n* = (*if succ*(*m mod n*) = *n then 0 else succ*(*m mod n*))
apply (*erule complete-induct*)
apply (*case-tac succ* (*x*) <*n*)

case succ(x) < n

apply (*simp* (*no-asm-simp*) *add*: *nat-le-refl* [*THEN lt-trans*] *succ-neq-self*)
apply (*simp add*: *ltD* [*THEN mem-imp-not-eq*])

case $n \leq succ(x)$

apply (*simp add*: *mod-geq not-lt-iff-le*)
apply (*erule leE*)
apply (*simp* (*no-asm-simp*) *add*: *mod-geq div-termination* [*THEN ltD*] *diff-succ*)

equality case

apply (*simp add*: *diff-self-eq-0*)
done

lemma *mod-succ*:
  n:*nat* ⟹ *succ*(*m*) *mod n* = (*if succ*(*m mod n*) = *n then 0 else succ*(*m mod n*))
apply (*case-tac n*=*0*)
apply (*simp* (*no-asm-simp*) *add*: *natify-succ DIVISION-BY-ZERO-MOD*)
apply (*subgoal-tac natify* (*succ* (*m*)) *mod n* = (*if succ* (*natify* (*m*) *mod n*) = *n then 0 else succ* (*natify* (*m*) *mod n*)))
prefer *2*
apply (*subst natify-succ*)
apply (*rule mod-succ-lemma*)
apply (*auto simp del*: *natify-succ simp add*: *nat-into-Ord* [*THEN Ord-0-lt-iff*])
done

lemma *mod-less-divisor*: ⟦*0*<*n*; *n*:*nat*⟧ ⟹ *m mod n* < *n*
apply (*subgoal-tac natify* (*m*) *mod n* < *n*)
apply (*rule-tac* [*2*] *i* = *natify* (*m*) **in** *complete-induct*)
apply (*case-tac* [*3*] *x*<*n*, *auto*)

case $n \leq x$

apply (*simp add*: *mod-geq not-lt-iff-le div-termination* [*THEN ltD*])
done

lemma *mod-1-eq* [*simp*]: *m mod 1* = *0*
by (*cut-tac n* = *1* **in** *mod-less-divisor*, *auto*)

lemma *mod2-cases*: *b*<*2* ⟹ *k mod 2* = *b* | *k mod 2* = (*if b*=*1 then 0 else 1*)
apply (*subgoal-tac k mod 2*: *2*)
prefer *2* **apply** (*simp add*: *mod-less-divisor* [*THEN ltD*])

apply (*drule ltD*, *auto*)
done

lemma *mod2-succ-succ* [*simp*]: *succ*(*succ*(*m*)) *mod 2* = *m mod 2*
apply (*subgoal-tac m mod 2*: *2*)
prefer *2* **apply** (*simp add*: *mod-less-divisor* [*THEN ltD*])
apply (*auto simp add*: *mod-succ*)
done

lemma *mod2-add-more* [*simp*]: (*m*#+*m*#+*n*) *mod 2* = *n mod 2*
apply (*subgoal-tac* (*natify* (*m*) #+*natify* (*m*) #+*n*) *mod 2* = *n mod 2*)
apply (*rule-tac* [*2*] *n* = *natify* (*m*) **in** *nat-induct*)
apply *auto*
done

lemma *mod2-add-self* [*simp*]: (*m*#+*m*) *mod 2* = *0*
by (*cut-tac n* = *0* **in** *mod2-add-more*, *auto*)

28.6 Additional theorems about ≤

lemma *add-le-self*: *m*:*nat* ⟹ *m* ≤ (*m* #+ *n*)
apply (*simp* (*no-asm-simp*))
done

lemma *add-le-self2*: *m*:*nat* ⟹ *m* ≤ (*n* #+ *m*)
apply (*simp* (*no-asm-simp*))
done

lemma *mult-le-mono1*: ⟦*i* ≤ *j*; *j*:*nat*⟧ ⟹ (*i*#∗*k*) ≤ (*j*#∗*k*)
apply (*subgoal-tac natify* (*i*) #∗*natify* (*k*) ≤ *j*#∗*natify* (*k*))
apply (*frule-tac* [*2*] *lt-nat-in-nat*)
apply (*rule-tac* [*3*] *n* = *natify* (*k*) **in** *nat-induct*)
apply (*simp-all add*: *add-le-mono*)
done

lemma *mult-le-mono*: ⟦*i* ≤ *j*; *k* ≤ *l*; *j*:*nat*; *l*:*nat*⟧ ⟹ *i*#∗*k* ≤ *j*#∗*l*
apply (*rule mult-le-mono1* [*THEN le-trans*], *assumption*+)
apply (*subst mult-commute*, *subst mult-commute*, *rule mult-le-mono1*, *assumption*+)
done

lemma *mult-lt-mono2*: ⟦*i*<*j*; *0*<*k*; *j*:*nat*; *k*:*nat*⟧ ⟹ *k*#∗*i* < *k*#∗*j*
apply (*erule zero-lt-natE*)
apply (*frule-tac* [*2*] *lt-nat-in-nat*)
apply (*simp-all* (*no-asm-simp*))

apply (*induct-tac x*)
apply (*simp-all* (*no-asm-simp*) *add*: *add-lt-mono*)
done

lemma *mult-lt-mono1*: ⟦*i*<*j*; *0*<*k*; *j*:*nat*; *k*:*nat*⟧ ⟹ *i*#∗*k* < *j*#∗*k*
apply (*simp* (*no-asm-simp*) *add*: *mult-lt-mono2 mult-commute* [*of - k*])
done

lemma *add-eq-0-iff* [*iff*]: *m*#+*n* = *0* ⟷ *natify*(*m*)=*0* ∧ *natify*(*n*)=*0*
apply (*subgoal-tac natify* (*m*) #+ *natify* (*n*) = *0* ⟷ *natify* (*m*) =*0* ∧ *natify* (*n*) =*0*)
apply (*rule-tac* [*2*] *n* = *natify* (*m*) **in** *natE*)
 apply (*rule-tac* [*4*] *n* = *natify* (*n*) **in** *natE*)
apply *auto*
done

lemma *zero-lt-mult-iff* [*iff*]: *0* < *m*#∗*n* ⟷ *0* < *natify*(*m*) ∧ *0* < *natify*(*n*)
apply (*subgoal-tac 0* < *natify* (*m*) #∗*natify* (*n*) ⟷ *0* < *natify* (*m*) ∧ *0* < *natify* (*n*))
apply (*rule-tac* [*2*] *n* = *natify* (*m*) **in** *natE*)
 apply (*rule-tac* [*4*] *n* = *natify* (*n*) **in** *natE*)
  apply (*rule-tac* [*3*] *n* = *natify* (*n*) **in** *natE*)
apply *auto*
done

lemma *mult-eq-1-iff* [*iff*]: *m*#∗*n* = *1* ⟷ *natify*(*m*)=*1* ∧ *natify*(*n*)=*1*
apply (*subgoal-tac natify* (*m*) #∗ *natify* (*n*) = *1* ⟷ *natify* (*m*) =*1* ∧ *natify* (*n*) =*1*)
apply (*rule-tac* [*2*] *n* = *natify* (*m*) **in** *natE*)
 apply (*rule-tac* [*4*] *n* = *natify* (*n*) **in** *natE*)
apply *auto*
done

lemma *mult-is-zero*: ⟦*m*: *nat*; *n*: *nat*⟧ ⟹ (*m* #∗ *n* = *0*) ⟷ (*m* = *0* | *n* = *0*)
apply *auto*
apply (*erule natE*)
apply (*erule-tac* [*2*] *natE*, *auto*)
done

lemma *mult-is-zero-natify* [*iff*]:
  (*m* #∗ *n* = *0*) ⟷ (*natify*(*m*) = *0* | *natify*(*n*) = *0*)
apply (*cut-tac m* = *natify* (*m*) **and** *n* = *natify* (*n*) **in** *mult-is-zero*)
apply *auto*
done

28.7 Cancellation Laws for Common Factors in Comparisons

lemma *mult-less-cancel-lemma*:

$[\![k: nat; m: nat; n: nat]\!] \Longrightarrow (m\#*k < n\#*k) \longleftrightarrow (0<k \wedge m<n)$
apply (*safe intro*!: *mult-lt-mono1*)
apply (*erule natE*, *auto*)
apply (*rule not-le-iff-lt* [*THEN iffD1*])
apply (*drule-tac* [*3*] *not-le-iff-lt* [*THEN* [*2*] *rev-iffD2*])
prefer *5* **apply** (*blast intro*: *mult-le-mono1*, *auto*)
done

lemma *mult-less-cancel2* [*simp*]:
$(m\#*k < n\#*k) \longleftrightarrow (0 < natify(k) \wedge natify(m) < natify(n))$
apply (*rule iff-trans*)
apply (*rule-tac* [*2*] *mult-less-cancel-lemma*, *auto*)
done

lemma *mult-less-cancel1* [*simp*]:
$(k\#*m < k\#*n) \longleftrightarrow (0 < natify(k) \wedge natify(m) < natify(n))$
apply (*simp* (*no-asm*) *add*: *mult-less-cancel2 mult-commute* [*of k*])
done

lemma *mult-le-cancel2* [*simp*]: $(m\#*k \le n\#*k) \longleftrightarrow (0 < natify(k) \longrightarrow natify(m) \le natify(n))$
apply (*simp* (*no-asm-simp*) *add*: *not-lt-iff-le* [*THEN iff-sym*])
apply *auto*
done

lemma *mult-le-cancel1* [*simp*]: $(k\#*m \le k\#*n) \longleftrightarrow (0 < natify(k) \longrightarrow natify(m) \le natify(n))$
apply (*simp* (*no-asm-simp*) *add*: *not-lt-iff-le* [*THEN iff-sym*])
apply *auto*
done

lemma *mult-le-cancel-le1*: $k \in nat \Longrightarrow k \#* m \le k \longleftrightarrow (0 < k \longrightarrow natify(m) \le 1)$
by (*cut-tac* $k = k$ **and** $m = m$ **and** $n = 1$ **in** *mult-le-cancel1*, *auto*)

lemma *Ord-eq-iff-le*: $[\![Ord(m); Ord(n)]\!] \Longrightarrow m=n \longleftrightarrow (m \le n \wedge n \le m)$
by (*blast intro*: *le-anti-sym*)

lemma *mult-cancel2-lemma*:
$[\![k: nat; m: nat; n: nat]\!] \Longrightarrow (m\#*k = n\#*k) \longleftrightarrow (m=n \mid k=0)$
apply (*simp* (*no-asm-simp*) *add*: *Ord-eq-iff-le* [*of m#*k*] *Ord-eq-iff-le* [*of m*])
apply (*auto simp add*: *Ord-0-lt-iff*)
done

lemma *mult-cancel2* [*simp*]:
$(m\#*k = n\#*k) \longleftrightarrow (natify(m) = natify(n) \mid natify(k) = 0)$
apply (*rule iff-trans*)
apply (*rule-tac* [*2*] *mult-cancel2-lemma*, *auto*)
done

lemma *mult-cancel1* [*simp*]:
  ($k\#*m = k\#*n$) $\longleftrightarrow$ ($natify(m) = natify(n)$ | $natify(k) = 0$)
apply (*simp* (*no-asm*) *add*: *mult-cancel2 mult-commute* [*of k*])
done

lemma *div-cancel-raw*:
  $[\![0<n; 0<k; k{:}nat; m{:}nat; n{:}nat]\!] \Longrightarrow (k\#*m)$ *div* $(k\#*n) = m$ *div* n
apply (*erule-tac* $i = m$ **in** *complete-induct*)
apply (*case-tac* $x<n$)
apply (*simp add*: *div-less zero-lt-mult-iff mult-lt-mono2*)
apply (*simp add*: *not-lt-iff-le zero-lt-mult-iff le-refl* [*THEN mult-le-mono*]
  div-geq diff-mult-distrib2 [*symmetric*] *div-termination* [*THEN ltD*])
done

lemma *div-cancel*:
  $[\![0 < natify(n);\ \ 0 < natify(k)]\!] \Longrightarrow (k\#*m)$ *div* $(k\#*n) = m$ *div* n
apply (*cut-tac* $k = natify\ (k)$ **and** $m = natify\ (m)$ **and** $n = natify\ (n)$
  in *div-cancel-raw*)
apply *auto*
done

28.8 More Lemmas about Remainder

lemma *mult-mod-distrib-raw*:
  $[\![k{:}nat; m{:}nat; n{:}nat]\!] \Longrightarrow (k\#*m)$ *mod* $(k\#*n) = k\ \#*\ (m$ *mod* $n)$
apply (*case-tac* $k=0$)
apply (*simp add*: *DIVISION-BY-ZERO-MOD*)
apply (*case-tac* $n=0$)
apply (*simp add*: *DIVISION-BY-ZERO-MOD*)
apply (*simp add*: *nat-into-Ord* [*THEN Ord-0-lt-iff*])
apply (*erule-tac* $i = m$ **in** *complete-induct*)
apply (*case-tac* $x<n$)
apply (*simp* (*no-asm-simp*) *add*: *mod-less zero-lt-mult-iff mult-lt-mono2*)
apply (*simp add*: *not-lt-iff-le zero-lt-mult-iff le-refl* [*THEN mult-le-mono*]
  mod-geq diff-mult-distrib2 [*symmetric*] *div-termination* [*THEN ltD*])
done

lemma *mod-mult-distrib2*: $k\ \#*\ (m$ *mod* $n) = (k\#*m)$ *mod* $(k\#*n)$
apply (*cut-tac* $k = natify\ (k)$ **and** $m = natify\ (m)$ **and** $n = natify\ (n)$
  in *mult-mod-distrib-raw*)
apply *auto*
done

lemma *mult-mod-distrib*: $(m$ *mod* $n)\ \#*\ k = (m\#*k)$ *mod* $(n\#*k)$
apply (*simp* (*no-asm*) *add*: *mult-commute mod-mult-distrib2*)

done

lemma *mod-add-self2-raw*: *n* ∈ *nat* ⟹ (*m* #+ *n*) *mod n* = *m mod n*
apply (*subgoal-tac* (*n* #+ *m*) *mod n* = (*n* #+ *m* #− *n*) *mod n*)
apply (*simp add*: *add-commute*)
apply (*subst mod-geq* [*symmetric*], *auto*)
done

lemma *mod-add-self2* [*simp*]: (*m* #+ *n*) *mod n* = *m mod n*
apply (*cut-tac n* = *natify* (*n*) **in** *mod-add-self2-raw*)
apply *auto*
done

lemma *mod-add-self1* [*simp*]: (*n*#+*m*) *mod n* = *m mod n*
apply (*simp* (*no-asm-simp*) *add*: *add-commute mod-add-self2*)
done

lemma *mod-mult-self1-raw*: *k* ∈ *nat* ⟹ (*m* #+ *k*#∗*n*) *mod n* = *m mod n*
apply (*erule nat-induct*)
apply (*simp-all* (*no-asm-simp*) *add*: *add-left-commute* [*of* - *n*])
done

lemma *mod-mult-self1* [*simp*]: (*m* #+ *k*#∗*n*) *mod n* = *m mod n*
apply (*cut-tac k* = *natify* (*k*) **in** *mod-mult-self1-raw*)
apply *auto*
done

lemma *mod-mult-self2* [*simp*]: (*m* #+ *n*#∗*k*) *mod n* = *m mod n*
apply (*simp* (*no-asm*) *add*: *mult-commute mod-mult-self1*)
done

lemma *mult-eq-self-implies-10*: *m* = *m*#∗*n* ⟹ *natify*(*n*)=*1* | *m*=*0*
apply (*subgoal-tac m*: *nat*)
 prefer *2*
 apply (*erule ssubst*)
 apply *simp*
apply (*rule disjCI*)
apply (*drule sym*)
apply (*rule Ord-linear-lt* [*of natify*(*n*) *1*])
apply *simp-all*
 apply (*subgoal-tac m* #∗ *n* = *0*, *simp*)
 apply (*subst mult-natify2* [*symmetric*])
 apply (*simp del*: *mult-natify2*)
apply (*drule nat-into-Ord* [*THEN Ord-0-lt*, *THEN* [*2*] *mult-lt-mono2*], *auto*)
done

lemma *less-imp-succ-add* [*rule-format*]:
  ⟦*m*<*n*; *n*: *nat*⟧ ⟹ ∃ *k*∈*nat*. *n* = *succ*(*m*#+*k*)

apply (*frule lt-nat-in-nat*, *assumption*)
apply (*erule rev-mp*)
apply (*induct-tac n*)
apply (*simp-all* (*no-asm*) *add*: *le-iff*)
apply (*blast elim*!: *leE intro*!: *add-0-right* [*symmetric*] *add-succ-right* [*symmetric*])
done

lemma *less-iff-succ-add*:
  $[\![m: nat; n: nat]\!] \Longrightarrow (m<n) \longleftrightarrow (\exists k\in nat.\ n = succ(m\#+k))$
by (*auto intro*: *less-imp-succ-add*)

lemma *add-lt-elim2*:
  $[\![a \#+ d = b \#+ c; a < b; b \in nat; c \in nat; d \in nat]\!] \Longrightarrow c < d$
by (*drule less-imp-succ-add*, *auto*)

lemma *add-le-elim2*:
  $[\![a \#+ d = b \#+ c; a \le b; b \in nat; c \in nat; d \in nat]\!] \Longrightarrow c \le d$
by (*drule less-imp-succ-add*, *auto*)

28.8.1 More Lemmas About Difference

lemma *diff-is-0-lemma*:
  $[\![m: nat; n: nat]\!] \Longrightarrow m \#- n = 0 \longleftrightarrow m \le n$
apply (*rule-tac* $m = m$ **and** $n = n$ **in** *diff-induct*, *simp-all*)
done

lemma *diff-is-0-iff*: $m \#- n = 0 \longleftrightarrow natify(m) \le natify(n)$
by (*simp add*: *diff-is-0-lemma* [*symmetric*])

lemma *nat-lt-imp-diff-eq-0*:
  $[\![a{:}nat; b{:}nat; a<b]\!] \Longrightarrow a \#- b = 0$
by (*simp add*: *diff-is-0-iff le-iff*)

lemma *raw-nat-diff-split*:
  $[\![a{:}nat; b{:}nat]\!] \Longrightarrow$
   $(P(a \#- b)) \longleftrightarrow ((a < b \longrightarrow P(0)) \wedge (\forall d\in nat.\ a = b \#+ d \longrightarrow P(d)))$
apply (*case-tac* $a < b$)
 apply (*force simp add*: *nat-lt-imp-diff-eq-0*)
apply (*rule iffI*, *force*, *simp*)
apply (*drule-tac* $x=a\#-b$ **in** *bspec*)
apply (*simp-all add*: *Ordinal.not-lt-iff-le add-diff-inverse*)
done

lemma *nat-diff-split*:
 $(P(a \#- b)) \longleftrightarrow$
  $(natify(a) < natify(b) \longrightarrow P(0)) \wedge (\forall d\in nat.\ natify(a) = b \#+ d \longrightarrow P(d))$
apply (*cut-tac* $P=P$ **and** $a=natify(a)$ **and** $b=natify(b)$ **in** *raw-nat-diff-split*)
apply *simp-all*
done

Difference and less-than

lemma *diff-lt-imp-lt*: ⟦$(k\#-i) < (k\#-j)$; $i\in nat$; $j\in nat$; $k\in nat$⟧ $\Longrightarrow$ $j<i$
apply (*erule rev-mp*)
apply (*simp split*: *nat-diff-split*, *auto*)
 apply (*blast intro*: *add-le-self lt-trans1*)
apply (*rule not-le-iff-lt* [*THEN iffD1*], *auto*)
apply (*subgoal-tac* i #+ $da < j$ #+ d, *force*)
apply (*blast intro*: *add-le-lt-mono*)
done

lemma *lt-imp-diff-lt*: ⟦$j<i$; $i\le k$; $k\in nat$⟧ $\Longrightarrow$ $(k\#-i) < (k\#-j)$
apply (*frule le-in-nat*, *assumption*)
apply (*frule lt-nat-in-nat*, *assumption*)
apply (*simp split*: *nat-diff-split*, *auto*)
  apply (*blast intro*: *lt-asym lt-trans2*)
 apply (*blast intro*: *lt-irrefl lt-trans2*)
apply (*rule not-le-iff-lt* [*THEN iffD1*], *auto*)
apply (*subgoal-tac* j #+ $d < i$ #+ da, *force*)
apply (*blast intro*: *add-lt-le-mono*)
done

lemma *diff-lt-iff-lt*: ⟦$i\le k$; $j\in nat$; $k\in nat$⟧ $\Longrightarrow$ $(k\#-i) < (k\#-j) \longleftrightarrow j<i$
apply (*frule le-in-nat*, *assumption*)
apply (*blast intro*: *lt-imp-diff-lt diff-lt-imp-lt*)
done

end

29 Lists in Zermelo-Fraenkel Set Theory

theory *List* **imports** *Datatype ArithSimp* **begin**

consts
  list   :: $i \Rightarrow i$

datatype
  list(*A*) = *Nil* | *Cons* ($a \in A$, $l \in$ *list*(*A*))

notation *Nil* (‹[]›)

syntax
  -List :: *is* ⇒ *i* (‹(‹*indent=1 notation=*‹*mixfix list enumeration*››[-])›)
translations
  [*x*, *xs*]   == *CONST Cons*(*x*, [*xs*])
  [*x*]   == *CONST Cons*(*x*, [])

consts

length :: *i*⇒*i*
hd :: *i*⇒*i*
tl :: *i*⇒*i*

primrec
length([]) = *0*
length(*Cons*(*a*,*l*)) = *succ*(*length*(*l*))

primrec
hd([]) = *0*
hd(*Cons*(*a*,*l*)) = *a*

primrec
tl([]) = []
tl(*Cons*(*a*,*l*)) = *l*

consts
map :: [*i*⇒*i*, *i*] ⇒ *i*
set-of-list :: *i*⇒*i*
app :: [*i*,*i*]⇒*i* (**infixr** ‹@› *60*)

primrec
map(*f*,[]) = []
map(*f*,*Cons*(*a*,*l*)) = *Cons*(*f*(*a*), *map*(*f*,*l*))

primrec
set-of-list([]) = *0*
set-of-list(*Cons*(*a*,*l*)) = *cons*(*a*, *set-of-list*(*l*))

primrec
app-Nil: [] @ *ys* = *ys*
app-Cons: (*Cons*(*a*,*l*)) @ *ys* = *Cons*(*a*, *l* @ *ys*)

consts
rev :: *i*⇒*i*
flat :: *i*⇒*i*
list-add :: *i*⇒*i*

primrec
rev([]) = []
rev(*Cons*(*a*,*l*)) = *rev*(*l*) @ [*a*]

primrec
flat([]) = []
flat(*Cons*(*l*,*ls*)) = *l* @ *flat*(*ls*)

primrec
$list\text{-}add([]) = 0$
$list\text{-}add(Cons(a,l)) = a \#+ list\text{-}add(l)$

consts
$drop :: [i,i] \Rightarrow i$

primrec
$drop\text{-}0$: $drop(0,l) = l$
$drop\text{-}succ$: $drop(succ(i), l) = tl\ (drop(i,l))$

definition

$take :: [i,i] \Rightarrow i$ **where**
$take(n, as) \equiv list\text{-}rec(\lambda n \in nat. [],$
$\lambda a\ l\ r. \lambda n \in nat. nat\text{-}case([], \lambda m. Cons(a, r`m), n), as)`n$

definition
$nth :: [i, i] \Rightarrow i$ **where**
— returns the (n+1)th element of a list, or 0 if the list is too short.
$nth(n, as) \equiv list\text{-}rec(\lambda n \in nat. 0,$
$\lambda a\ l\ r. \lambda n \in nat. nat\text{-}case(a, \lambda m. r`m, n), as)\ `\ n$

definition
$list\text{-}update :: [i, i, i] \Rightarrow i$ **where**
$list\text{-}update(xs, i, v) \equiv list\text{-}rec(\lambda n \in nat. Nil,$
$\lambda u\ us\ vs. \lambda n \in nat. nat\text{-}case(Cons(v, us), \lambda m. Cons(u, vs`m), n), xs)`i$

consts
$filter :: [i \Rightarrow o, i] \Rightarrow i$
$upt :: [i, i] \Rightarrow i$

primrec
$filter(P, Nil) = Nil$
$filter(P, Cons(x, xs)) =$
$(if\ P(x)\ then\ Cons(x, filter(P, xs))\ else\ filter(P, xs))$

primrec
$upt(i, 0) = Nil$
$upt(i, succ(j)) = (if\ i \leq j\ then\ upt(i, j)@[j]\ else\ Nil)$

definition
$min :: [i,i] \Rightarrow i$ **where**
$min(x, y) \equiv (if\ x \leq y\ then\ x\ else\ y)$

definition

max :: [*i*, *i*] ⇒*i* **where**
max(*x*, *y*) ≡ (*if* *x* ≤ *y* *then* *y* *else* *x*)

declare *list.intros* [*simp*,*TC*]

inductive-cases *ConsE*: *Cons*(*a*,*l*) ∈ *list*(*A*)

lemma *Cons-type-iff* [*simp*]: *Cons*(*a*,*l*) ∈ *list*(*A*) ⟷ *a* ∈ *A* ∧ *l* ∈ *list*(*A*)
by (*blast elim*: *ConsE*)

lemma *Cons-iff*: *Cons*(*a*,*l*)=*Cons*(*a*′,*l*′) ⟷ *a*=*a*′ ∧ *l*=*l*′
by *auto*

lemma *Nil-Cons-iff*: ¬ *Nil*=*Cons*(*a*,*l*)
by *auto*

lemma *list-unfold*: *list*(*A*) = {*0*} + (*A* ∗ *list*(*A*))
by (*blast intro*!: *list.intros* [*unfolded list.con-defs*]
elim: *list.cases* [*unfolded list.con-defs*])

lemma *list-mono*: *A*<=*B* ⟹ *list*(*A*) ⊆ *list*(*B*)
unfolding *list.defs*
apply (*rule lfp-mono*)
apply (*simp-all add*: *list.bnd-mono*)
apply (*assumption* | *rule univ-mono basic-monos*)+
done

lemma *list-univ*: *list*(*univ*(*A*)) ⊆ *univ*(*A*)
unfolding *list.defs list.con-defs*
apply (*rule lfp-lowerbound*)
apply (*rule-tac* [*2*] *A-subset-univ* [*THEN univ-mono*])
apply (*blast intro*!: *zero-in-univ Inl-in-univ Inr-in-univ Pair-in-univ*)
done

lemmas *list-subset-univ* = *subset-trans* [*OF list-mono list-univ*]

lemma *list-into-univ*: ⟦*l* ∈ *list*(*A*); *A* ⊆ *univ*(*B*)⟧ ⟹ *l* ∈ *univ*(*B*)
by (*blast intro*: *list-subset-univ* [*THEN subsetD*])

lemma *list-case-type*:

$[\![l \in list(A);$
$c \in C(Nil);$
$\bigwedge x\ y.\ [\![x \in A;\ \ y \in list(A)]\!] \Longrightarrow h(x,y) \colon C(Cons(x,y))$
$]\!] \Longrightarrow list\text{-}case(c,h,l) \in C(l)$
by (*erule list.induct*, *auto*)

lemma *list-0-triv*: $list(0) = \{Nil\}$
apply (*rule equalityI*, *auto*)
apply (*induct-tac x*, *auto*)
done

lemma *tl-type*: $l \in list(A) \Longrightarrow tl(l) \in list(A)$
apply (*induct-tac l*)
apply (*simp-all* (*no-asm-simp*) *add*: *list.intros*)
done

lemma *drop-Nil* [*simp*]: $i \in nat \Longrightarrow drop(i,\ Nil) = Nil$
apply (*induct-tac i*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *drop-succ-Cons* [*simp*]: $i \in nat \Longrightarrow drop(succ(i),\ Cons(a,l)) = drop(i,l)$
apply (*rule sym*)
apply (*induct-tac i*)
apply (*simp* (*no-asm*))
apply (*simp* (*no-asm-simp*))
done

lemma *drop-type* [*simp,TC*]: $[\![i \in nat;\ l \in list(A)]\!] \Longrightarrow drop(i,l) \in list(A)$
apply (*induct-tac i*)
apply (*simp-all* (*no-asm-simp*) *add*: *tl-type*)
done

declare *drop-succ* [*simp del*]

lemma *list-rec-type* [*TC*]:
$[\![l \in list(A);$
$c \in C(Nil);$
$\bigwedge x\ y\ r.\ [\![x \in A;\ \ y \in list(A);\ \ r \in C(y)]\!] \Longrightarrow h(x,y,r) \colon C(Cons(x,y))$
$]\!] \Longrightarrow list\text{-}rec(c,h,l) \in C(l)$
by (*induct-tac l*, *auto*)

lemma *map-type* [*TC*]:
$[\![l \in list(A);\ \bigwedge x.\ x \in A \Longrightarrow h(x){:}\ B]\!] \Longrightarrow map(h,l) \in list(B)$
apply (*simp add*: *map-list-def*)
apply (*typecheck add*: *list.intros list-rec-type*, *blast*)
done

lemma *map-type2* [*TC*]: $l \in list(A) \Longrightarrow map(h,l) \in list(\{h(u).\ u \in A\})$
apply (*erule map-type*)
apply (*erule RepFunI*)
done

lemma *length-type* [*TC*]: $l \in list(A) \Longrightarrow length(l) \in nat$
by (*simp add*: *length-list-def*)

lemma *lt-length-in-nat*:
$[\![x < length(xs);\ xs \in list(A)]\!] \Longrightarrow x \in nat$
by (*frule lt-nat-in-nat*, *typecheck*)

lemma *app-type* [*TC*]: $[\![xs{:}\ list(A);\ \ ys{:}\ list(A)]\!] \Longrightarrow xs@ys \in list(A)$
by (*simp add*: *app-list-def*)

lemma *rev-type* [*TC*]: $xs{:}\ list(A) \Longrightarrow rev(xs) \in list(A)$
by (*simp add*: *rev-list-def*)

lemma *flat-type* [*TC*]: $ls{:}\ list(list(A)) \Longrightarrow flat(ls) \in list(A)$
by (*simp add*: *flat-list-def*)

lemma *set-of-list-type* [*TC*]: $l \in list(A) \Longrightarrow set\text{-}of\text{-}list(l) \in Pow(A)$
unfolding *set-of-list-list-def*
apply (*erule list-rec-type*, *auto*)
done

lemma *set-of-list-append*:
$xs{:}\ list(A) \Longrightarrow set\text{-}of\text{-}list\ (xs@ys) = set\text{-}of\text{-}list(xs) \cup set\text{-}of\text{-}list(ys)$

apply (*erule list.induct*)
apply (*simp-all* (*no-asm-simp*) *add*: *Un-cons*)
done

lemma *list-add-type* [*TC*]: *xs*: *list*(*nat*) $\Longrightarrow$ *list-add*(*xs*) $\in$ *nat*
by (*simp add*: *list-add-list-def*)

lemma *map-ident* [*simp*]: $l \in$ *list*(*A*) $\Longrightarrow$ *map*($\lambda u.\ u$, *l*) = *l*
apply (*induct-tac l*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *map-compose*: $l \in$ *list*(*A*) $\Longrightarrow$ *map*(*h*, *map*(*j*,*l*)) = *map*($\lambda u.\ h(j(u))$, *l*)
apply (*induct-tac l*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *map-app-distrib*: *xs*: *list*(*A*) $\Longrightarrow$ *map*(*h*, *xs*@*ys*) = *map*(*h*,*xs*) @ *map*(*h*,*ys*)
apply (*induct-tac xs*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *map-flat*: *ls*: *list*(*list*(*A*)) $\Longrightarrow$ *map*(*h*, *flat*(*ls*)) = *flat*(*map*(*map*(*h*),*ls*))
apply (*induct-tac ls*)
apply (*simp-all* (*no-asm-simp*) *add*: *map-app-distrib*)
done

lemma *list-rec-map*:
  $l \in$ *list*(*A*) $\Longrightarrow$
   list-rec(*c*, *d*, *map*(*h*,*l*)) =
   list-rec(*c*, $\lambda x\ xs\ r.\ d(h(x), map(h,xs), r)$, *l*)
apply (*induct-tac l*)
apply (*simp-all* (*no-asm-simp*))
done

lemmas *list-CollectD* = *Collect-subset* [*THEN list-mono*, *THEN subsetD*]

lemma *map-list-Collect*: $l \in$ *list*($\{x \in A.\ h(x)=j(x)\}$) $\Longrightarrow$ *map*(*h*,*l*) = *map*(*j*,*l*)
apply (*induct-tac l*)
apply (*simp-all* (*no-asm-simp*))

done

lemma *length-map* [*simp*]: *xs*: *list*(A) $\Longrightarrow$ *length*(*map*(*h*,*xs*)) = *length*(*xs*)
by (*induct-tac xs*, *simp-all*)

lemma *length-app* [*simp*]:
 $\llbracket$*xs*: *list*(A); *ys*: *list*(A)$\rrbracket$
 $\Longrightarrow$ *length*(*xs*@*ys*) = *length*(*xs*) #+ *length*(*ys*)
by (*induct-tac xs*, *simp-all*)

lemma *length-rev* [*simp*]: *xs*: *list*(A) $\Longrightarrow$ *length*(*rev*(*xs*)) = *length*(*xs*)
apply (*induct-tac xs*)
apply (*simp-all* (*no-asm-simp*) *add*: *length-app*)
done

lemma *length-flat*:
 ls: *list*(*list*(A)) $\Longrightarrow$ *length*(*flat*(*ls*)) = *list-add*(*map*(*length*,*ls*))
apply (*induct-tac ls*)
apply (*simp-all* (*no-asm-simp*) *add*: *length-app*)
done

lemma *drop-length-Cons* [*rule-format*]:
 xs: *list*(A) $\Longrightarrow$
  $\forall x$. $\exists z$ *zs*. *drop*(*length*(*xs*), *Cons*(*x*,*xs*)) = *Cons*(*z*,*zs*)
by (*erule list.induct*, *simp-all*)

lemma *drop-length* [*rule-format*]:
 $l \in$ *list*(A) $\Longrightarrow \forall i \in$ *length*(l). ($\exists z$ *zs*. *drop*(i,l) = *Cons*(*z*,*zs*))
apply (*erule list.induct*, *simp-all*, *safe*)
apply (*erule drop-length-Cons*)
apply (*rule natE*)
apply (*erule Ord-trans* [*OF asm-rl length-type Ord-nat*], *assumption*, *simp-all*)
apply (*blast intro*: *succ-in-naturalD length-type*)
done

lemma *app-right-Nil* [*simp*]: *xs*: *list*(A) $\Longrightarrow$ *xs*@*Nil*=*xs*
by (*erule list.induct*, *simp-all*)

lemma *app-assoc*: *xs*: *list*(A) $\Longrightarrow$ (*xs*@*ys*)@*zs* = *xs*@(*ys*@*zs*)
by (*induct-tac xs*, *simp-all*)

lemma *flat-app-distrib*: *ls*: *list*(*list*(*A*)) $\Longrightarrow$ *flat*(*ls*@*ms*) = *flat*(*ls*)@*flat*(*ms*)
apply (*induct-tac ls*)
apply (*simp-all* (*no-asm-simp*) *add*: *app-assoc*)
done

lemma *rev-map-distrib*: $l \in$ *list*(*A*) $\Longrightarrow$ *rev*(*map*(*h*,*l*)) = *map*(*h*,*rev*(*l*))
apply (*induct-tac l*)
apply (*simp-all* (*no-asm-simp*) *add*: *map-app-distrib*)
done

lemma *rev-app-distrib*:
⟦*xs*: *list*(*A*); *ys*: *list*(*A*)⟧ $\Longrightarrow$ *rev*(*xs*@*ys*) = *rev*(*ys*)@*rev*(*xs*)
apply (*erule list.induct*)
apply (*simp-all add*: *app-assoc*)
done

lemma *rev-rev-ident* [*simp*]: $l \in$ *list*(*A*) $\Longrightarrow$ *rev*(*rev*(*l*))=*l*
apply (*induct-tac l*)
apply (*simp-all* (*no-asm-simp*) *add*: *rev-app-distrib*)
done

lemma *rev-flat*: *ls*: *list*(*list*(*A*)) $\Longrightarrow$ *rev*(*flat*(*ls*)) = *flat*(*map*(*rev*,*rev*(*ls*)))
apply (*induct-tac ls*)
apply (*simp-all add*: *map-app-distrib flat-app-distrib rev-app-distrib*)
done

lemma *list-add-app*:
⟦*xs*: *list*(*nat*); *ys*: *list*(*nat*)⟧
$\Longrightarrow$ *list-add*(*xs*@*ys*) = *list-add*(*ys*) #+ *list-add*(*xs*)
apply (*induct-tac xs*, *simp-all*)
done

lemma *list-add-rev*: $l \in$ *list*(*nat*) $\Longrightarrow$ *list-add*(*rev*(*l*)) = *list-add*(*l*)
apply (*induct-tac l*)
apply (*simp-all* (*no-asm-simp*) *add*: *list-add-app*)
done

lemma *list-add-flat*:
ls: *list*(*list*(*nat*)) $\Longrightarrow$ *list-add*(*flat*(*ls*)) = *list-add*(*map*(*list-add*,*ls*))
apply (*induct-tac ls*)
apply (*simp-all* (*no-asm-simp*) *add*: *list-add-app*)
done

lemma *list-append-induct* [*case-names Nil snoc*, *consumes 1*]:
$[\![l \in list(A);$
$P(Nil);$
$\bigwedge x\ y.\ [\![x \in A;\ \ y \in list(A);\ \ P(y)]\!] \Longrightarrow P(y\ @\ [x])$
$]\!] \Longrightarrow P(l)$
apply (*subgoal-tac* $P(rev(rev(l)))$, *simp*)
apply (*erule rev-type* [*THEN list.induct*], *simp-all*)
done

lemma *list-complete-induct-lemma* [*rule-format*]:
assumes *ih*:
$\bigwedge l.\ [\![l \in list(A);$
$\forall l' \in list(A).\ length(l') < length(l) \longrightarrow P(l')]\!]$
$\Longrightarrow P(l)$
shows $n \in nat \Longrightarrow \forall l \in list(A).\ length(l) < n \longrightarrow P(l)$
apply (*induct-tac n*, *simp*)
apply (*blast intro*: *ih elim*!: *leE*)
done

theorem *list-complete-induct*:
$[\![l \in list(A);$
$\bigwedge l.\ [\![l \in list(A);$
$\forall l' \in list(A).\ length(l') < length(l) \longrightarrow P(l')]\!]$
$\Longrightarrow P(l)$
$]\!] \Longrightarrow P(l)$
apply (*rule list-complete-induct-lemma* [*of A*])
prefer *4* **apply** (*rule le-refl*, *simp*)
apply *blast*
apply *simp*
apply *assumption*
done

lemma *min-sym*: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow min(i,j)=min(j,i)$
unfolding *min-def*
apply (*auto dest*!: *not-lt-imp-le dest*: *lt-not-sym intro*: *le-anti-sym*)
done

lemma *min-type* [*simp*,*TC*]: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow min(i,j):nat$
by (*unfold min-def*, *auto*)

lemma *min-0* [*simp*]: $i \in nat \Longrightarrow min(0,i) = 0$
unfolding *min-def*

apply (*auto dest*: *not-lt-imp-le*)
done

lemma *min-02* [*simp*]: $i \in nat \Longrightarrow min(i,\ 0) = 0$
  unfolding *min-def*
apply (*auto dest*: *not-lt-imp-le*)
done

lemma *lt-min-iff*: $[\![i \in nat;\ j \in nat;\ k \in nat]\!] \Longrightarrow i<min(j,k) \longleftrightarrow i<j \wedge i<k$
  unfolding *min-def*
apply (*auto dest*!: *not-lt-imp-le intro*: *lt-trans2 lt-trans*)
done

lemma *min-succ-succ* [*simp*]:
  $[\![i \in nat;\ j \in nat]\!] \Longrightarrow\ min(succ(i),\ succ(j))= succ(min(i,\ j))$
apply (*unfold min-def*, *auto*)
done

lemma *filter-append* [*simp*]:
  $xs{:}list(A) \Longrightarrow filter(P,\ xs@ys) = filter(P,\ xs)\ @\ filter(P,\ ys)$
by (*induct-tac xs*, *auto*)

lemma *filter-type* [*simp*,*TC*]: $xs{:}list(A) \Longrightarrow filter(P,\ xs){:}list(A)$
by (*induct-tac xs*, *auto*)

lemma *length-filter*: $xs{:}list(A) \Longrightarrow length(filter(P,\ xs)) \leq length(xs)$
apply (*induct-tac xs*, *auto*)
apply (*rule-tac j = length (l)* **in** *le-trans*)
apply (*auto simp add*: *le-iff*)
done

lemma *filter-is-subset*: $xs{:}list(A) \Longrightarrow set\text{-}of\text{-}list(filter(P,xs)) \subseteq set\text{-}of\text{-}list(xs)$
by (*induct-tac xs*, *auto*)

lemma *filter-False* [*simp*]: $xs{:}list(A) \Longrightarrow filter(\lambda p.\ False,\ xs) = Nil$
by (*induct-tac xs*, *auto*)

lemma *filter-True* [*simp*]: $xs{:}list(A) \Longrightarrow filter(\lambda p.\ True,\ xs) = xs$
by (*induct-tac xs*, *auto*)

lemma *length-is-0-iff* [*simp*]: $xs{:}list(A) \Longrightarrow length(xs)=0 \longleftrightarrow xs=Nil$
by (*erule list.induct*, *auto*)

lemma *length-is-0-iff2* [*simp*]: *xs*:*list*(*A*) $\Longrightarrow$ *0* = *length*(*xs*) $\longleftrightarrow$ *xs*=*Nil*
by (*erule list.induct*, *auto*)

lemma *length-tl* [*simp*]: *xs*:*list*(*A*) $\Longrightarrow$ *length*(*tl*(*xs*)) = *length*(*xs*) #− *1*
by (*erule list.induct*, *auto*)

lemma *length-greater-0-iff*: *xs*:*list*(*A*) $\Longrightarrow$ *0*<*length*(*xs*) $\longleftrightarrow$ *xs* $\neq$ *Nil*
by (*erule list.induct*, *auto*)

lemma *length-succ-iff*: *xs*:*list*(*A*) $\Longrightarrow$ *length*(*xs*)=*succ*(*n*) $\longleftrightarrow$ ($\exists$ *y ys*. *xs*=*Cons*(*y*, *ys*) $\wedge$ *length*(*ys*)=*n*)
by (*erule list.induct*, *auto*)

lemma *append-is-Nil-iff* [*simp*]:
xs:*list*(*A*) $\Longrightarrow$ (*xs*@*ys* = *Nil*) $\longleftrightarrow$ (*xs*=*Nil* $\wedge$ *ys* = *Nil*)
by (*erule list.induct*, *auto*)

lemma *append-is-Nil-iff2* [*simp*]:
xs:*list*(*A*) $\Longrightarrow$ (*Nil* = *xs*@*ys*) $\longleftrightarrow$ (*xs*=*Nil* $\wedge$ *ys* = *Nil*)
by (*erule list.induct*, *auto*)

lemma *append-left-is-self-iff* [*simp*]:
xs:*list*(*A*) $\Longrightarrow$ (*xs*@*ys* = *xs*) $\longleftrightarrow$ (*ys* = *Nil*)
by (*erule list.induct*, *auto*)

lemma *append-left-is-self-iff2* [*simp*]:
xs:*list*(*A*) $\Longrightarrow$ (*xs* = *xs*@*ys*) $\longleftrightarrow$ (*ys* = *Nil*)
by (*erule list.induct*, *auto*)

lemma *append-left-is-Nil-iff* [*rule-format*]:
$[\![$*xs*:*list*(*A*); *ys*:*list*(*A*); *zs*:*list*(*A*)$]\!]$ $\Longrightarrow$
length(*ys*)=*length*(*zs*) $\longrightarrow$ (*xs*@*ys*=*zs* $\longleftrightarrow$ (*xs*=*Nil* $\wedge$ *ys*=*zs*))
apply (*erule list.induct*)
apply (*auto simp add*: *length-app*)
done

lemma *append-left-is-Nil-iff2* [*rule-format*]:
$[\![$*xs*:*list*(*A*); *ys*:*list*(*A*); *zs*:*list*(*A*)$]\!]$ $\Longrightarrow$
length(*ys*)=*length*(*zs*) $\longrightarrow$ (*zs*=*ys*@*xs* $\longleftrightarrow$ (*xs*=*Nil* $\wedge$ *ys*=*zs*))
apply (*erule list.induct*)
apply (*auto simp add*: *length-app*)
done

lemma *append-eq-append-iff* [*rule-format*]:
xs:*list*(*A*) $\Longrightarrow$ $\forall$ *ys* $\in$ *list*(*A*).

$length(xs)=length(ys) \longrightarrow (xs@us = ys@vs) \longleftrightarrow (xs=ys \wedge us=vs)$
apply (*erule list.induct*)
apply (*simp* (*no-asm-simp*))
apply *clarify*
apply (*erule-tac a = ys* **in** *list.cases*, *auto*)
done
declare *append-eq-append-iff* [*simp*]

lemma *append-eq-append* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall ys \in list(A).\ \forall us \in list(A).\ \forall vs \in list(A).$
$length(us) = length(vs) \longrightarrow (xs@us = ys@vs) \longrightarrow (xs=ys \wedge us=vs)$
apply (*induct-tac xs*)
apply (*force simp add*: *length-app*, *clarify*)
apply (*erule-tac a = ys* **in** *list.cases*, *simp*)
apply (*subgoal-tac Cons* (*a*, *l*) @ *us* =*vs*)
apply (*drule rev-iffD1* [*OF* - *append-left-is-Nil-iff*], *simp-all*, *blast*)
done

lemma *append-eq-append-iff2* [*simp*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ us{:}list(A);\ vs{:}list(A);\ length(us)=length(vs)]\!]$
$\Longrightarrow\ xs@us = ys@vs \longleftrightarrow (xs=ys \wedge us=vs)$
apply (*rule iffI*)
apply (*rule append-eq-append*, *auto*)
done

lemma *append-self-iff* [*simp*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ zs{:}list(A)]\!] \Longrightarrow xs@ys=xs@zs \longleftrightarrow ys=zs$
by *simp*

lemma *append-self-iff2* [*simp*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ zs{:}list(A)]\!] \Longrightarrow ys@xs=zs@xs \longleftrightarrow ys=zs$
by *simp*

lemma *append1-eq-iff* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall ys \in list(A).\ xs@[x] = ys@[y] \longleftrightarrow (xs = ys \wedge x=y)$
apply (*erule list.induct*)
apply *clarify*
apply (*erule list.cases*)
apply *simp-all*

Inductive step

apply *clarify*
apply (*erule-tac a=ys* **in** *list.cases*, *simp-all*)
done
declare *append1-eq-iff* [*simp*]

lemma *append-right-is-self-iff* [*simp*]:

$[\![xs{:}list(A);\ ys{:}list(A)]\!] \Longrightarrow (xs@ys = ys) \longleftrightarrow (xs{=}Nil)$
by (*simp* (*no-asm-simp*) *add*: *append-left-is-Nil-iff*)

lemma *append-right-is-self-iff2* [*simp*]:
$[\![xs{:}list(A);\ ys{:}list(A)]\!] \Longrightarrow (ys = xs@ys) \longleftrightarrow (xs{=}Nil)$
apply (*rule iffI*)
apply (*drule sym*, *auto*)
done

lemma *hd-append* [*rule-format*]:
$xs{:}list(A) \Longrightarrow xs \neq Nil \longrightarrow hd(xs\ @\ ys) = hd(xs)$
by (*induct-tac xs*, *auto*)
declare *hd-append* [*simp*]

lemma *tl-append* [*rule-format*]:
$xs{:}list(A) \Longrightarrow xs{\neq}Nil \longrightarrow tl(xs\ @\ ys) = tl(xs)@ys$
by (*induct-tac xs*, *auto*)
declare *tl-append* [*simp*]

lemma *rev-is-Nil-iff* [*simp*]: $xs{:}list(A) \Longrightarrow (rev(xs) = Nil \longleftrightarrow xs = Nil)$
by (*erule list.induct*, *auto*)

lemma *Nil-is-rev-iff* [*simp*]: $xs{:}list(A) \Longrightarrow (Nil = rev(xs) \longleftrightarrow xs = Nil)$
by (*erule list.induct*, *auto*)

lemma *rev-is-rev-iff* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall\, ys \in list(A).\ rev(xs){=}rev(ys) \longleftrightarrow xs{=}ys$
apply (*erule list.induct*, *force*, *clarify*)
apply (*erule-tac a = ys* **in** *list.cases*, *auto*)
done
declare *rev-is-rev-iff* [*simp*]

lemma *rev-list-elim* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$(xs{=}Nil \longrightarrow P) \longrightarrow (\forall\, ys \in list(A).\ \forall\, y \in A.\ xs = ys@[y] \longrightarrow P) \longrightarrow P$
by (*erule list-append-induct*, *auto*)

lemma *length-drop* [*rule-format*]:
$n \in nat \Longrightarrow \forall\, xs \in list(A).\ length(drop(n,\ xs)) = length(xs)\ \#{-}\ n$
apply (*erule nat-induct*)
apply (*auto elim*: *list.cases*)
done
declare *length-drop* [*simp*]

lemma *drop-all* [*rule-format*]:

$n \in nat \Longrightarrow \forall xs \in list(A).\ length(xs) \leq n \longrightarrow drop(n,\ xs)=Nil$
apply (*erule nat-induct*)
apply (*auto elim*: *list.cases*)
done
declare *drop-all* [*simp*]

lemma *drop-append* [*rule-format*]:
$n \in nat \Longrightarrow$
$\forall xs \in list(A).\ drop(n,\ xs@ys) = drop(n,xs)\ @\ drop(n\ \#-\ length(xs),\ ys)$
apply (*induct-tac n*)
apply (*auto elim*: *list.cases*)
done

lemma *drop-drop*:
$m \in nat \Longrightarrow \forall xs \in list(A).\ \forall n \in nat.\ drop(n,\ drop(m,\ xs))=drop(n\ \#+\ m,\ xs)$
apply (*induct-tac m*)
apply (*auto elim*: *list.cases*)
done

lemma *take-0* [*simp*]: $xs{:}list(A) \Longrightarrow take(0,\ xs) = Nil$
unfolding *take-def*
apply (*erule list.induct*, *auto*)
done

lemma *take-succ-Cons* [*simp*]:
$n \in nat \Longrightarrow take(succ(n),\ Cons(a,\ xs)) = Cons(a,\ take(n,\ xs))$
by (*simp add*: *take-def*)

lemma *take-Nil* [*simp*]: $n \in nat \Longrightarrow take(n,\ Nil) = Nil$
by (*unfold take-def*, *auto*)

lemma *take-all* [*rule-format*]:
$n \in nat \Longrightarrow \forall xs \in list(A).\ length(xs) \leq n \longrightarrow take(n,\ xs) = xs$
apply (*erule nat-induct*)
apply (*auto elim*: *list.cases*)
done
declare *take-all* [*simp*]

lemma *take-type* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall n \in nat.\ take(n,\ xs){:}list(A)$
apply (*erule list.induct*, *simp*, *clarify*)
apply (*erule natE*, *auto*)
done
declare *take-type* [*simp*,*TC*]

lemma *take-append* [*rule-format*]:

xs:*list*(*A*) $\Longrightarrow$
$\forall$ *ys* $\in$ *list*(*A*). $\forall$ *n* $\in$ *nat*. *take*(*n*, *xs* @ *ys*) =
take(*n*, *xs*) @ *take*(*n* #− *length*(*xs*), *ys*)
apply (*erule list.induct*, *simp*, *clarify*)
apply (*erule natE*, *auto*)
done
declare *take-append* [*simp*]

lemma *take-take* [*rule-format*]:
m $\in$ *nat* $\Longrightarrow$
$\forall$ *xs* $\in$ *list*(*A*). $\forall$ *n* $\in$ *nat*. *take*(*n*, *take*(*m*,*xs*))= *take*(*min*(*n*, *m*), *xs*)
apply (*induct-tac m*, *auto*)
apply (*erule-tac a* = *xs* **in** *list.cases*)
apply (*auto simp add*: *take-Nil*)
apply (*erule-tac n*=*n* **in** *natE*)
apply (*auto intro*: *take-0 take-type*)
done

lemma *nth-0* [*simp*]: *nth*(*0*, *Cons*(*a*, *l*)) = *a*
by (*simp add*: *nth-def*)

lemma *nth-Cons* [*simp*]: *n* $\in$ *nat* $\Longrightarrow$ *nth*(*succ*(*n*), *Cons*(*a*,*l*)) = *nth*(*n*,*l*)
by (*simp add*: *nth-def*)

lemma *nth-empty* [*simp*]: *nth*(*n*, *Nil*) = *0*
by (*simp add*: *nth-def*)

lemma *nth-type* [*rule-format*]:
xs:*list*(*A*) $\Longrightarrow$ $\forall$ *n*. *n* < *length*(*xs*) $\longrightarrow$ *nth*(*n*,*xs*) $\in$ *A*
apply (*erule list.induct*, *simp*, *clarify*)
apply (*subgoal-tac n* $\in$ *nat*)
apply (*erule natE*, *auto dest*!: *le-in-nat*)
done
declare *nth-type* [*simp*,*TC*]

lemma *nth-eq-0* [*rule-format*]:
xs:*list*(*A*) $\Longrightarrow$ $\forall$ *n* $\in$ *nat*. *length*(*xs*) $\le$ *n* $\longrightarrow$ *nth*(*n*,*xs*) = *0*
apply (*erule list.induct*, *simp*, *clarify*)
apply (*erule natE*, *auto*)
done

lemma *nth-append* [*rule-format*]:
xs:*list*(*A*) $\Longrightarrow$
$\forall$ *n* $\in$ *nat*. *nth*(*n*, *xs* @ *ys*) = (*if n* < *length*(*xs*) *then nth*(*n*,*xs*)
else nth(*n* #− *length*(*xs*), *ys*))
apply (*induct-tac xs*, *simp*, *clarify*)
apply (*erule natE*, *auto*)

done

lemma *set-of-list-conv-nth*:
  xs:*list*(*A*)
   $\Longrightarrow$ *set-of-list*(*xs*) = {*x* ∈ *A*. ∃ *i*∈*nat*. *i*<*length*(*xs*) ∧ *x* = *nth*(*i*,*xs*)}
apply (*induct-tac xs*, *simp-all*)
apply (*rule equalityI*, *auto*)
apply (*rule-tac x* = *0* **in** *bexI*, *auto*)
apply (*erule natE*, *auto*)
done

lemma *nth-take-lemma* [*rule-format*]:
 k ∈ *nat* $\Longrightarrow$
  ∀ *xs* ∈ *list*(*A*). (∀ *ys* ∈ *list*(*A*). *k* ≤ *length*(*xs*) ⟶ *k* ≤ *length*(*ys*) ⟶
   (∀ *i* ∈ *nat*. *i*<*k* ⟶ *nth*(*i*,*xs*) = *nth*(*i*,*ys*))⟶ *take*(*k*,*xs*) = *take*(*k*,*ys*))
apply (*induct-tac k*)
apply (*simp-all* (*no-asm-simp*) *add*: *lt-succ-eq-0-disj all-conj-distrib*)
apply *clarify*

apply (*erule-tac a*=*xs* **in** *list*.*cases*, *simp*)
apply (*erule-tac a*=*ys* **in** *list*.*cases*, *clarify*)
apply (*simp* (*no-asm-use*))
apply *clarify*
apply (*simp* (*no-asm-simp*))
apply (*rule conjI*, *force*)
apply (*rename-tac y ys z zs*)
apply (*drule-tac x* = *zs* **and** *x1* = *ys* **in** *bspec* [*THEN bspec*], *auto*)
done

lemma *nth-equalityI* [*rule-format*]:
  ⟦*xs*:*list*(*A*); *ys*:*list*(*A*); *length*(*xs*) = *length*(*ys*);
   ∀ *i* ∈ *nat*. *i* < *length*(*xs*) ⟶ *nth*(*i*,*xs*) = *nth*(*i*,*ys*)⟧
  $\Longrightarrow$ *xs* = *ys*
apply (*subgoal-tac length* (*xs*) ≤ *length* (*ys*))
apply (*cut-tac k*=*length*(*xs*) **and** *xs*=*xs* **and** *ys*=*ys* **in** *nth-take-lemma*)
apply (*simp-all add*: *take-all*)
done

lemma *take-equalityI* [*rule-format*]:
  ⟦*xs*:*list*(*A*); *ys*:*list*(*A*); (∀ *i* ∈ *nat*. *take*(*i*, *xs*) = *take*(*i*,*ys*))⟧
  $\Longrightarrow$ *xs* = *ys*
apply (*case-tac length* (*xs*) ≤ *length* (*ys*))
apply (*drule-tac x* = *length* (*ys*) **in** *bspec*)
apply (*drule-tac* [*3*] *not-lt-imp-le*)
apply (*subgoal-tac* [*5*] *length* (*ys*) ≤ *length* (*xs*))

apply (*rule-tac* [*6*] *j* = *succ* (*length* (*ys*)) **in** *le-trans*)
apply (*rule-tac* [*6*] *leI*)
apply (*drule-tac* [*5*] *x* = *length* (*xs*) **in** *bspec*)
apply (*simp-all add*: *take-all*)
done

lemma *nth-drop* [*rule-format*]:
  $n \in nat \Longrightarrow \forall\, i \in nat.\ \forall\, xs \in list(A).\ nth(i,\ drop(n,\ xs)) = nth(n\ \#{+}\ i,\ xs)$
apply (*induct-tac n*, *simp-all*, *clarify*)
apply (*erule list.cases*, *auto*)
done

lemma *take-succ* [*rule-format*]:
  $xs{\in}list(A)$
  $\Longrightarrow \forall\, i.\ i < length(xs) \longrightarrow take(succ(i),\ xs) = take(i{,}xs)\ @\ [nth(i,\ xs)]$
apply (*induct-tac xs*, *auto*)
apply (*subgoal-tac i*∈*nat*)
apply (*erule natE*)
apply (*auto simp add*: *le-in-nat*)
done

lemma *take-add* [*rule-format*]:
  $[\![xs{\in}list(A);\ j{\in}nat]\!]$
  $\Longrightarrow \forall\, i{\in}nat.\ take(i\ \#{+}\ j,\ xs) = take(i{,}xs)\ @\ take(j,\ drop(i{,}xs))$
apply (*induct-tac xs*, *simp-all*, *clarify*)
apply (*erule-tac n* = *i* **in** *natE*, *simp-all*)
done

lemma *length-take*:
  $l{\in}list(A) \Longrightarrow \forall\, n{\in}nat.\ length(take(n{,}l)) = min(n,\ length(l))$
apply (*induct-tac l*, *safe*, *simp-all*)
apply (*erule natE*, *simp-all*)
done

29.1 The function zip

Crafty definition to eliminate a type argument

consts
  zip-aux :: [*i*,*i*]⇒*i*

primrec
  $zip\text{-}aux(B{,}[]) =$
    $(\lambda ys \in list(B).\ list\text{-}case([],\ \lambda y\ l.\ [],\ ys))$

  $zip\text{-}aux(B{,}Cons(x{,}l)) =$
    $(\lambda ys \in list(B).$
      $list\text{-}case(Nil,\ \lambda y\ zs.\ Cons(\langle x{,}y\rangle,\ zip\text{-}aux(B{,}l)`zs),\ ys))$

definition

zip :: [*i*, *i*]⇒*i* **where**
zip(*xs*, *ys*) ≡ *zip-aux*(*set-of-list*(*ys*),*xs*)‘*ys*

lemma *list-on-set-of-list*: *xs* ∈ *list*(*A*) ⟹ *xs* ∈ *list*(*set-of-list*(*xs*))
apply (*induct-tac xs*, *simp-all*)
apply (*blast intro*: *list-mono* [*THEN subsetD*])
done

lemma *zip-Nil* [*simp*]: *ys*:*list*(*A*) ⟹ *zip*(*Nil*, *ys*)=*Nil*
apply (*simp add*: *zip-def list-on-set-of-list* [*of - A*])
apply (*erule list.cases*, *simp-all*)
done

lemma *zip-Nil2* [*simp*]: *xs*:*list*(*A*) ⟹ *zip*(*xs*, *Nil*)=*Nil*
apply (*simp add*: *zip-def list-on-set-of-list* [*of - A*])
apply (*erule list.cases*, *simp-all*)
done

lemma *zip-aux-unique* [*rule-format*]:
⟦*B*<=*C*; *xs* ∈ *list*(*A*)⟧
⟹ ∀ *ys* ∈ *list*(*B*). *zip-aux*(*C*,*xs*) ‘ *ys* = *zip-aux*(*B*,*xs*) ‘ *ys*
apply (*induct-tac xs*)
apply *simp-all*
apply (*blast intro*: *list-mono* [*THEN subsetD*], *clarify*)
apply (*erule-tac a*=*ys* **in** *list.cases*, *auto*)
apply (*blast intro*: *list-mono* [*THEN subsetD*])
done

lemma *zip-Cons-Cons* [*simp*]:
⟦*xs*:*list*(*A*); *ys*:*list*(*B*); *x* ∈ *A*; *y* ∈ *B*⟧ ⟹
zip(*Cons*(*x*,*xs*), *Cons*(*y*, *ys*)) = *Cons*(⟨*x*,*y*⟩, *zip*(*xs*, *ys*))
apply (*simp add*: *zip-def*, *auto*)
apply (*rule zip-aux-unique*, *auto*)
apply (*simp add*: *list-on-set-of-list* [*of - B*])
apply (*blast intro*: *list-on-set-of-list list-mono* [*THEN subsetD*])
done

lemma *zip-type* [*rule-format*]:
xs:*list*(*A*) ⟹ ∀ *ys* ∈ *list*(*B*). *zip*(*xs*, *ys*):*list*(*A*∗*B*)
apply (*induct-tac xs*)
apply (*simp* (*no-asm*))
apply *clarify*
apply (*erule-tac a* = *ys* **in** *list.cases*, *auto*)
done
declare *zip-type* [*simp*,*TC*]

lemma *length-zip* [*rule-format*]:
xs:*list*(*A*) ⟹ ∀ *ys* ∈ *list*(*B*). *length*(*zip*(*xs*,*ys*)) =
min(*length*(*xs*), *length*(*ys*))
unfolding *min-def*
apply (*induct-tac xs*, *simp-all*, *clarify*)
apply (*erule-tac a* = *ys* **in** *list.cases*, *auto*)
done
declare *length-zip* [*simp*]

lemma *zip-append1* [*rule-format*]:
⟦*ys*:*list*(*A*); *zs*:*list*(*B*)⟧ ⟹
∀ *xs* ∈ *list*(*A*). *zip*(*xs* @ *ys*, *zs*) =
zip(*xs*, *take*(*length*(*xs*), *zs*)) @ *zip*(*ys*, *drop*(*length*(*xs*),*zs*))
apply (*induct-tac zs*, *force*, *clarify*)
apply (*erule-tac a* = *xs* **in** *list.cases*, *simp-all*)
done

lemma *zip-append2* [*rule-format*]:
⟦*xs*:*list*(*A*); *zs*:*list*(*B*)⟧ ⟹ ∀ *ys* ∈ *list*(*B*). *zip*(*xs*, *ys*@*zs*) =
zip(*take*(*length*(*ys*), *xs*), *ys*) @ *zip*(*drop*(*length*(*ys*), *xs*), *zs*)
apply (*induct-tac xs*, *force*, *clarify*)
apply (*erule-tac a* = *ys* **in** *list.cases*, *auto*)
done

lemma *zip-append* [*simp*]:
⟦*length*(*xs*) = *length*(*us*); *length*(*ys*) = *length*(*vs*);
xs:*list*(*A*); *us*:*list*(*B*); *ys*:*list*(*A*); *vs*:*list*(*B*)⟧
⟹ *zip*(*xs*@*ys*,*us*@*vs*) = *zip*(*xs*, *us*) @ *zip*(*ys*, *vs*)
by (*simp* (*no-asm-simp*) *add*: *zip-append1 drop-append diff-self-eq-0*)

lemma *zip-rev* [*rule-format*]:
ys:*list*(*B*) ⟹ ∀ *xs* ∈ *list*(*A*).
length(*xs*) = *length*(*ys*) ⟶ *zip*(*rev*(*xs*), *rev*(*ys*)) = *rev*(*zip*(*xs*, *ys*))
apply (*induct-tac ys*, *force*, *clarify*)
apply (*erule-tac a* = *xs* **in** *list.cases*)
apply (*auto simp add*: *length-rev*)
done
declare *zip-rev* [*simp*]

lemma *nth-zip* [*rule-format*]:
ys:*list*(*B*) ⟹ ∀ *i* ∈ *nat*. ∀ *xs* ∈ *list*(*A*).
i < *length*(*xs*) ⟶ *i* < *length*(*ys*) ⟶
nth(*i*,*zip*(*xs*, *ys*)) = <*nth*(*i*,*xs*),*nth*(*i*, *ys*)>
apply (*induct-tac ys*, *force*, *clarify*)
apply (*erule-tac a* = *xs* **in** *list.cases*, *simp*)
apply (*auto elim*: *natE*)
done

declare *nth-zip* [*simp*]

lemma *set-of-list-zip* [*rule-format*]:
$[\![xs{:}list(A);\ ys{:}list(B);\ i \in nat]\!]$
$\Longrightarrow$ *set-of-list*(*zip*(*xs*, *ys*)) =
$\{\langle x,\ y\rangle{:}A*B.\ \exists\, i{\in}nat.\ i < min(length(xs),\ length(ys))$
$\wedge\ x = nth(i,\ xs)\ \wedge\ y = nth(i,\ ys)\}$
by (*force intro*!: *Collect-cong simp add*: *lt-min-iff set-of-list-conv-nth*)

lemma *list-update-Nil* [*simp*]: $i \in nat \Longrightarrow$*list-update*(*Nil*, *i*, *v*) = *Nil*
by (*unfold list-update-def*, *auto*)

lemma *list-update-Cons-0* [*simp*]: *list-update*(*Cons*(*x*, *xs*), *0*, *v*)= *Cons*(*v*, *xs*)
by (*unfold list-update-def*, *auto*)

lemma *list-update-Cons-succ* [*simp*]:
$n \in nat \Longrightarrow$
list-update(*Cons*(*x*, *xs*), *succ*(*n*), *v*)= *Cons*(*x*, *list-update*(*xs*, *n*, *v*))
apply (*unfold list-update-def*, *auto*)
done

lemma *list-update-type* [*rule-format*]:
$[\![xs{:}list(A);\ v \in A]\!] \Longrightarrow \forall\, n \in nat.$ *list-update*(*xs*, *n*, *v*):*list*(*A*)
apply (*induct-tac xs*)
apply (*simp* (*no-asm*))
apply *clarify*
apply (*erule natE*, *auto*)
done
declare *list-update-type* [*simp*,*TC*]

lemma *length-list-update* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall\, i \in nat.$ *length*(*list-update*(*xs*, *i*, *v*))=*length*(*xs*)
apply (*induct-tac xs*)
apply (*simp* (*no-asm*))
apply *clarify*
apply (*erule natE*, *auto*)
done
declare *length-list-update* [*simp*]

lemma *nth-list-update* [*rule-format*]:
$[\![xs{:}list(A)]\!] \Longrightarrow \forall\, i \in nat.\ \forall\, j \in nat.\ i < length(xs) \longrightarrow$
nth(*j*, *list-update*(*xs*, *i*, *x*)) = (*if i*=*j then x else nth*(*j*, *xs*))
apply (*induct-tac xs*)
apply *simp-all*
apply *clarify*
apply (*rename-tac i j*)
apply (*erule-tac n*=*i* **in** *natE*)

apply (*erule-tac* [*2*] *n*=*j* **in** *natE*)
apply (*erule-tac n*=*j* **in** *natE*, *simp-all*, *force*)
done

lemma *nth-list-update-eq* [*simp*]:
$[\![i < length(xs); xs{:}list(A)]\!] \Longrightarrow nth(i, list\text{-}update(xs, i,x)) = x$
by (*simp* (*no-asm-simp*) *add*: *lt-length-in-nat nth-list-update*)

lemma *nth-list-update-neq* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall i \in nat. \forall j \in nat. i \neq j \longrightarrow nth(j, list\text{-}update(xs,i,x)) = nth(j,xs)$
apply (*induct-tac xs*)
apply (*simp* (*no-asm*))
apply *clarify*
apply (*erule natE*)
apply (*erule-tac* [*2*] *natE*, *simp-all*)
apply (*erule natE*, *simp-all*)
done
declare *nth-list-update-neq* [*simp*]

lemma *list-update-overwrite* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall i \in nat. i < length(xs)$
$\longrightarrow list\text{-}update(list\text{-}update(xs, i, x), i, y) = list\text{-}update(xs, i,y)$
apply (*induct-tac xs*)
apply (*simp* (*no-asm*))
apply *clarify*
apply (*erule natE*, *auto*)
done
declare *list-update-overwrite* [*simp*]

lemma *list-update-same-conv* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall i \in nat. i < length(xs) \longrightarrow$
$(list\text{-}update(xs, i, x) = xs) \longleftrightarrow (nth(i, xs) = x)$
apply (*induct-tac xs*)
apply (*simp* (*no-asm*))
apply *clarify*
apply (*erule natE*, *auto*)
done

lemma *update-zip* [*rule-format*]:
$ys{:}list(B) \Longrightarrow$
$\forall i \in nat. \forall xy \in A*B. \forall xs \in list(A).$
$length(xs) = length(ys) \longrightarrow$
$list\text{-}update(zip(xs, ys), i, xy) = zip(list\text{-}update(xs, i, fst(xy)),$
$list\text{-}update(ys, i, snd(xy)))$
apply (*induct-tac ys*)
apply *auto*

apply (*erule-tac a* = *xs* **in** *list.cases*)
apply (*auto elim*: *natE*)
done

lemma *set-update-subset-cons* [*rule-format*]:
xs:*list*(*A*) $\Longrightarrow$
$\forall$ *i* $\in$ *nat*. *set-of-list*(*list-update*(*xs*, *i*, *x*)) $\subseteq$ *cons*(*x*, *set-of-list*(*xs*))
apply (*induct-tac xs*)
apply *simp*
apply (*rule ballI*)
apply (*erule natE*, *simp-all*, *auto*)
done

lemma *set-of-list-update-subsetI*:
⟦*set-of-list*(*xs*) $\subseteq$ *A*; *xs*:*list*(*A*); *x* $\in$ *A*; *i* $\in$ *nat*⟧
$\Longrightarrow$ *set-of-list*(*list-update*(*xs*, *i*,*x*)) $\subseteq$ *A*
apply (*rule subset-trans*)
apply (*rule set-update-subset-cons*, *auto*)
done

lemma *upt-rec*:
j $\in$ *nat* $\Longrightarrow$ *upt*(*i*,*j*) = (*if i*<*j then Cons*(*i*, *upt*(*succ*(*i*), *j*)) *else Nil*)
apply (*induct-tac j*, *auto*)
apply (*drule not-lt-imp-le*)
apply (*auto simp*: *lt-Ord intro*: *le-anti-sym*)
done

lemma *upt-conv-Nil* [*simp*]: ⟦*j* $\leq$ *i*; *j* $\in$ *nat*⟧ $\Longrightarrow$ *upt*(*i*,*j*) = *Nil*
apply (*subst upt-rec*, *auto*)
apply (*auto simp add*: *le-iff*)
apply (*drule lt-asym* [*THEN notE*], *auto*)
done

lemma *upt-succ-append*:
⟦*i* $\leq$ *j*; *j* $\in$ *nat*⟧ $\Longrightarrow$ *upt*(*i*,*succ*(*j*)) = *upt*(*i*, *j*)@[*j*]
by *simp*

lemma *upt-conv-Cons*:
⟦*i*<*j*; *j* $\in$ *nat*⟧ $\Longrightarrow$ *upt*(*i*,*j*) = *Cons*(*i*,*upt*(*succ*(*i*),*j*))
apply (*rule trans*)
apply (*rule upt-rec*, *auto*)
done

lemma *upt-type* [*simp*,*TC*]: *j* $\in$ *nat* $\Longrightarrow$ *upt*(*i*,*j*):*list*(*nat*)
by (*induct-tac j*, *auto*)

lemma *upt-add-eq-append*:
$[\![i \le j;\ j \in nat;\ k \in nat]\!] \Longrightarrow upt(i,\ j\ \#{+}k) = upt(i,j)@upt(j,j\#{+}k)$
apply (*induct-tac k*)
apply (*auto simp add*: *app-assoc app-type*)
apply (*rule-tac j = j* **in** *le-trans*, *auto*)
done

lemma *length-upt* [*simp*]: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow length(upt(i,j)) = j\ \#{-}\ i$
apply (*induct-tac j*)
apply (*rule-tac* [*2*] *sym*)
apply (*auto dest*!: *not-lt-imp-le simp add*: *diff-succ diff-is-0-iff*)
done

lemma *nth-upt* [*simp*]:
$[\![i \in nat;\ j \in nat;\ k \in nat;\ i\ \#{+}\ k < j]\!] \Longrightarrow nth(k,\ upt(i,j)) = i\ \#{+}\ k$
apply (*rotate-tac −1*, *erule rev-mp*)
apply (*induct-tac j*, *simp*)
apply (*auto dest*!: *not-lt-imp-le*
simp add: *nth-append le-iff less-diff-conv add-commute*)
done

lemma *take-upt* [*rule-format*]:
$[\![m \in nat;\ n \in nat]\!] \Longrightarrow$
$\forall\, i \in nat.\ i\ \#{+}\ m \le n \longrightarrow take(m,\ upt(i,n)) = upt(i,i\#{+}m)$
apply (*induct-tac m*)
apply (*simp* (*no-asm-simp*) *add*: *take-0*)
apply *clarify*
apply (*subst upt-rec*, *simp*)
apply (*rule sym*)
apply (*subst upt-rec*, *simp*)
apply (*simp-all del*: *upt.simps*)
apply (*rule-tac j = succ* (*i* #+ *x*) **in** *lt-trans2*)
apply *auto*
done
declare *take-upt* [*simp*]

lemma *map-succ-upt*:
$[\![m \in nat;\ n \in nat]\!] \Longrightarrow map(succ,\ upt(m,n)) = upt(succ(m),\ succ(n))$
apply (*induct-tac n*)
apply (*auto simp add*: *map-app-distrib*)
done

lemma *nth-map* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall\, n \in nat.\ n < length(xs) \longrightarrow nth(n,\ map(f,\ xs)) = f(nth(n,\ xs))$
apply (*induct-tac xs*, *simp*)
apply (*rule ballI*)
apply (*induct-tac n*, *auto*)

done
declare *nth-map* [*simp*]

lemma *nth-map-upt* [*rule-format*]:
  $[\![m \in nat; n \in nat]\!] \Longrightarrow$
  $\forall i \in nat.\ i < n \mathbin{\#-} m \longrightarrow nth(i, map(f, upt(m,n))) = f(m \mathbin{\#+} i)$
apply (*rule-tac n = m* **and** *m = n* **in** *diff-induct*, *typecheck*, *simp*, *simp*)
apply (*subst map-succ-upt* [*symmetric*], *simp-all*, *clarify*)
apply (*subgoal-tac i < length* (*upt* (*0*, *x*)))
 prefer *2*
 apply (*simp add*: *less-diff-conv*)
 apply (*rule-tac j = succ* (*i #+ y*) **in** *lt-trans2*)
  apply *simp*
 apply *simp*
apply (*subgoal-tac i < length* (*upt* (*y*, *x*)))
 apply (*simp-all add*: *add-commute less-diff-conv*)
done

definition
  sublist :: [*i*, *i*] ⇒ *i* **where**
   sublist(*xs*, *A*)≡
    map(*fst*, (*filter*(λ*p*. *snd*(*p*): *A*, *zip*(*xs*, *upt*(*0*,*length*(*xs*))))))

lemma *sublist-0* [*simp*]: *xs*:*list*(*A*) ⟹*sublist*(*xs*, *0*) =*Nil*
by (*unfold sublist-def*, *auto*)

lemma *sublist-Nil* [*simp*]: *sublist*(*Nil*, *A*) = *Nil*
by (*unfold sublist-def*, *auto*)

lemma *sublist-shift-lemma*:
 ⟦*xs*:*list*(*B*); *i* ∈ *nat*⟧ ⟹
  map(*fst*, *filter*(λ*p*. *snd*(*p*):*A*, *zip*(*xs*, *upt*(*i*,*i* #+ *length*(*xs*))))) =
  map(*fst*, *filter*(λ*p*. *snd*(*p*):*nat* ∧ *snd*(*p*) #+ *i* ∈ *A*, *zip*(*xs*,*upt*(*0*,*length*(*xs*)))))
apply (*erule list-append-induct*)
apply (*simp* (*no-asm-simp*))
apply (*auto simp add*: *add-commute length-app filter-append map-app-distrib*)
done

lemma *sublist-type* [*simp*,*TC*]:
  xs:*list*(*B*) ⟹ *sublist*(*xs*, *A*):*list*(*B*)
 unfolding *sublist-def*
apply (*induct-tac xs*)
apply (*auto simp add*: *filter-append map-app-distrib*)
done

lemma *upt-add-eq-append2*:
  ⟦*i* ∈ *nat*; *j* ∈ *nat*⟧ ⟹ *upt*(*0*, *i* #+ *j*) = *upt*(*0*, *i*) @ *upt*(*i*, *i* #+ *j*)

by (*simp add*: *upt-add-eq-append* [*of 0*] *nat-0-le*)

lemma *sublist-append*:
⟦*xs*:*list*(*B*); *ys*:*list*(*B*)⟧ ⟹
sublist(*xs*@*ys*, *A*) = *sublist*(*xs*, *A*) @ *sublist*(*ys*, {*j* ∈ *nat*. *j* #+ *length*(*xs*): *A*})
unfolding *sublist-def*
apply (*erule-tac l* = *ys* **in** *list-append-induct*, *simp*)
apply (*simp* (*no-asm-simp*) *add*: *upt-add-eq-append2 app-assoc* [*symmetric*])
apply (*auto simp add*: *sublist-shift-lemma length-type map-app-distrib app-assoc*)
apply (*simp-all add*: *add-commute*)
done

lemma *sublist-Cons*:
⟦*xs*:*list*(*B*); *x* ∈ *B*⟧ ⟹
sublist(*Cons*(*x*, *xs*), *A*) =
(*if 0* ∈ *A then* [*x*] *else* []) @ *sublist*(*xs*, {*j* ∈ *nat*. *succ*(*j*) ∈ *A*})
apply (*erule-tac l* = *xs* **in** *list-append-induct*)
apply (*simp* (*no-asm-simp*) *add*: *sublist-def*)
apply (*simp del*: *app-Cons add*: *app-Cons* [*symmetric*] *sublist-append*, *simp*)
done

lemma *sublist-singleton* [*simp*]:
sublist([*x*], *A*) = (*if 0* ∈ *A then* [*x*] *else* [])
by (*simp add*: *sublist-Cons*)

lemma *sublist-upt-eq-take* [*rule-format*]:
xs:*list*(*A*) ⟹ ∀ *n*∈*nat*. *sublist*(*xs*,*n*) = *take*(*n*,*xs*)
apply (*erule list.induct*, *simp*)
apply (*clarify*)
apply (*erule natE*)
apply (*simp-all add*: *nat-eq-Collect-lt Ord-mem-iff-lt sublist-Cons*)
done
declare *sublist-upt-eq-take* [*simp*]

lemma *sublist-Int-eq*:
xs ∈ *list*(*B*) ⟹ *sublist*(*xs*, *A* ∩ *nat*) = *sublist*(*xs*, *A*)
apply (*erule list.induct*)
apply (*simp-all add*: *sublist-Cons*)
done

Repetition of a List Element

consts *repeat* :: [*i*,*i*]⇒*i*
primrec
repeat(*a*,*0*) = []

repeat(*a*,*succ*(*n*)) = *Cons*(*a*,*repeat*(*a*,*n*))

lemma *length-repeat*: *n* ∈ *nat* ⟹ *length*(*repeat*(*a*,*n*)) = *n*

by (*induct-tac n*, *auto*)

lemma *repeat-succ-app*: $n \in nat \Longrightarrow repeat(a,succ(n)) = repeat(a,n)$ @ $[a]$
apply (*induct-tac n*)
apply (*simp-all del*: *app-Cons add*: *app-Cons* [*symmetric*])
done

lemma *repeat-type* [*TC*]: $[\![a \in A;\ n \in nat]\!] \Longrightarrow repeat(a,n) \in list(A)$
by (*induct-tac n*, *auto*)

end

30 Equivalence Relations

theory *EquivClass* **imports** *Trancl Perm* **begin**

definition
quotient :: $[i,i]\Rightarrow i$ (**infixl** ‹'/'/› *90*) **where**
$A//r \equiv \{r``\{x\} \ .\ x \in A\}$

definition
congruent :: $[i,i\Rightarrow i]\Rightarrow o$ **where**
$congruent(r,b) \equiv \forall y\ z.\ \langle y,z\rangle{:}r \longrightarrow b(y)=b(z)$

definition
congruent2 :: $[i,i,[i,i]\Rightarrow i]\Rightarrow o$ **where**
$congruent2(r1,r2,b) \equiv \forall y1\ z1\ y2\ z2.$
$\langle y1,z1\rangle{:}r1 \longrightarrow \langle y2,z2\rangle{:}r2 \longrightarrow b(y1,y2) = b(z1,z2)$

abbreviation
RESPECTS ::$[i\Rightarrow i,\ i] \Rightarrow o$ (**infixr** ‹*respects*› *80*) **where**
f respects r $\equiv congruent(r,f)$

abbreviation
RESPECTS2 ::$[i\Rightarrow i\Rightarrow i,\ i] \Rightarrow o$ (**infixr** ‹*respects2* › *80*) **where**
f respects2 r $\equiv congruent2(r,r,f)$
— Abbreviation for the common case where the relations are identical

30.1 Suppes, Theorem 70: r is an equiv relation iff $converse(r)$ $O\ r = r$

lemma *sym-trans-comp-subset*:
$[\![sym(r);\ trans(r)]\!] \Longrightarrow converse(r)\ O\ r \subseteq r$
by (*unfold trans-def sym-def*, *blast*)

lemma *refl-comp-subset*:
$[\![refl(A,r);\ r \subseteq A*A]\!] \Longrightarrow r \subseteq converse(r)\ O\ r$
by (*unfold refl-def*, *blast*)

lemma *equiv-comp-eq*:
  $equiv(A,r) \Longrightarrow converse(r)\ O\ r = r$
 unfolding *equiv-def*
apply (*blast del*: *subsetI intro*!: *sym-trans-comp-subset refl-comp-subset*)
done

lemma *comp-equivI*:
  $[\![converse(r)\ O\ r = r;\ \ domain(r) = A]\!] \Longrightarrow equiv(A,r)$
 unfolding *equiv-def refl-def sym-def trans-def*
apply (*erule equalityE*)
apply (*subgoal-tac* $\forall x\ y.\ \langle x,y\rangle \in r \longrightarrow \langle y,x\rangle \in r$, *blast*+)
done

lemma *equiv-class-subset*:
  $[\![sym(r);\ \ trans(r);\ \ \langle a,b\rangle: r]\!] \Longrightarrow r``\{a\} \subseteq r``\{b\}$
by (*unfold trans-def sym-def*, *blast*)

lemma *equiv-class-eq*:
  $[\![equiv(A,r);\ \ \langle a,b\rangle: r]\!] \Longrightarrow r``\{a\} = r``\{b\}$
 unfolding *equiv-def*
apply (*safe del*: *subsetI intro*!: *equalityI equiv-class-subset*)
apply (*unfold sym-def*, *blast*)
done

lemma *equiv-class-self*:
  $[\![equiv(A,r);\ \ a \in A]\!] \Longrightarrow a \in r``\{a\}$
by (*unfold equiv-def refl-def*, *blast*)

lemma *subset-equiv-class*:
  $[\![equiv(A,r);\ \ r``\{b\} \subseteq r``\{a\};\ \ b \in A]\!] \Longrightarrow \langle a,b\rangle: r$
by (*unfold equiv-def refl-def*, *blast*)

lemma *eq-equiv-class*: $[\![r``\{a\} = r``\{b\};\ \ equiv(A,r);\ \ b \in A]\!] \Longrightarrow \langle a,b\rangle: r$
by (*assumption* | *rule equalityD2 subset-equiv-class*)+

lemma *equiv-class-nondisjoint*:
  $[\![equiv(A,r);\ \ x: (r``\{a\} \cap r``\{b\})]\!] \Longrightarrow \langle a,b\rangle: r$
by (*unfold equiv-def trans-def sym-def*, *blast*)

lemma *equiv-type*: $equiv(A,r) \Longrightarrow r \subseteq A*A$
by (*unfold equiv-def*, *blast*)

lemma *equiv-class-eq-iff*:

$equiv(A,r) \Longrightarrow \langle x,y\rangle: r \longleftrightarrow r``\{x\} = r``\{y\} \wedge x \in A \wedge y \in A$
by (*blast intro*: *eq-equiv-class equiv-class-eq dest*: *equiv-type*)

lemma *eq-equiv-class-iff*:
$[\![equiv(A,r);\ x \in A;\ y \in A]\!] \Longrightarrow r``\{x\} = r``\{y\} \longleftrightarrow \langle x,y\rangle: r$
by (*blast intro*: *eq-equiv-class equiv-class-eq dest*: *equiv-type*)

lemma *quotientI* [*TC*]: $x \in A \Longrightarrow r``\{x\}: A//r$
unfolding *quotient-def*
apply (*erule RepFunI*)
done

lemma *quotientE*:
$[\![X \in A//r;\ \bigwedge x.\ [\![X = r``\{x\};\ x \in A]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*unfold quotient-def*, *blast*)

lemma *Union-quotient*:
$equiv(A,r) \Longrightarrow \bigcup(A//r) = A$
by (*unfold equiv-def refl-def quotient-def*, *blast*)

lemma *quotient-disj*:
$[\![equiv(A,r);\ X \in A//r;\ Y \in A//r]\!] \Longrightarrow X{=}Y \mid (X \cap Y \subseteq 0)$
unfolding *quotient-def*
apply (*safe intro*!: *equiv-class-eq*, *assumption*)
apply (*unfold equiv-def trans-def sym-def*, *blast*)
done

30.2 Defining Unary Operations upon Equivalence Classes

lemma *UN-equiv-class*:
$[\![equiv(A,r);\ b\ respects\ r;\ a \in A]\!] \Longrightarrow (\bigcup x{\in}r``\{a\}.\ b(x)) = b(a)$
apply (*subgoal-tac* $\forall x \in r``\{a\}.\ b(x) = b(a)$)
apply *simp*
apply (*blast intro*: *equiv-class-self*)
apply (*unfold equiv-def sym-def congruent-def*, *blast*)
done

lemma *UN-equiv-class-type*:
$[\![equiv(A,r);\ b\ respects\ r;\ X \in A//r;\ \bigwedge x.\ x \in A \Longrightarrow b(x) \in B]\!]$
$\Longrightarrow (\bigcup x{\in}X.\ b(x)) \in B$
apply (*unfold quotient-def*, *safe*)
apply (*simp* (*no-asm-simp*) *add*: *UN-equiv-class*)
done

lemma *UN-equiv-class-inject*:
$[\![equiv(A,r);\ \ b\ respects\ r;$
$(\bigcup x{\in}X.\ b(x))=(\bigcup y{\in}Y.\ b(y));\ \ X \in A//r;\ \ Y \in A//r;$
$\bigwedge x\ y.\ [\![x \in A;\ y \in A;\ b(x)=b(y)]\!] \Longrightarrow \langle x,y\rangle{:}r]\!]$
$\Longrightarrow X=Y$
apply (*unfold quotient-def*, *safe*)
apply (*rule equiv-class-eq*, *assumption*)
apply (*simp add*: *UN-equiv-class* [*of A r b*])
done

30.3 Defining Binary Operations upon Equivalence Classes

lemma *congruent2-implies-congruent*:
$[\![equiv(A,r1);\ \ congruent2(r1,r2,b);\ \ a \in A]\!] \Longrightarrow congruent(r2,b(a))$
by (*unfold congruent-def congruent2-def equiv-def refl-def*, *blast*)

lemma *congruent2-implies-congruent-UN*:
$[\![equiv(A1,r1);\ \ equiv(A2,r2);\ \ congruent2(r1,r2,b);\ \ a \in A2]\!] \Longrightarrow$
$congruent(r1,\ \lambda x1.\ \bigcup x2 \in r2``\{a\}.\ b(x1,x2))$
apply (*unfold congruent-def*, *safe*)
apply (*frule equiv-type* [*THEN subsetD*], *assumption*)
apply *clarify*
apply (*simp add*: *UN-equiv-class congruent2-implies-congruent*)
apply (*unfold congruent2-def equiv-def refl-def*, *blast*)
done

lemma *UN-equiv-class2*:
$[\![equiv(A1,r1);\ \ equiv(A2,r2);\ \ congruent2(r1,r2,b);\ \ a1{:}\ A1;\ \ a2{:}\ A2]\!]$
$\Longrightarrow (\bigcup x1 \in r1``\{a1\}.\ \bigcup x2 \in r2``\{a2\}.\ b(x1,x2)) = b(a1,a2)$
by (*simp add*: *UN-equiv-class congruent2-implies-congruent*
congruent2-implies-congruent-UN)

lemma *UN-equiv-class-type2*:
$[\![equiv(A,r);\ \ b\ respects2\ r;$
$X1{:}\ A//r;\ \ X2{:}\ A//r;$
$\bigwedge x1\ x2.\ \ [\![x1{:}\ A;\ x2{:}\ A]\!] \Longrightarrow b(x1,x2) \in B$
$]\!] \Longrightarrow (\bigcup x1{\in}X1.\ \bigcup x2{\in}X2.\ b(x1,x2)) \in B$
apply (*unfold quotient-def*, *safe*)
apply (*blast intro*: *UN-equiv-class-type congruent2-implies-congruent-UN*
congruent2-implies-congruent quotientI)
done

lemma *congruent2I*:
$[\![equiv(A1,r1);\ \ equiv(A2,r2);$
$\bigwedge y\ z\ w.\ [\![w \in A2;\ \ \langle y,z\rangle \in r1]\!] \Longrightarrow b(y,w) = b(z,w);$

$\bigwedge y\ z\ w.\ [\![w \in A1;\ \ \langle y,z\rangle \in r2]\!] \Longrightarrow b(w,y) = b(w,z)$
$]\!] \Longrightarrow congruent2(r1,r2,b)$
apply (*unfold congruent2-def equiv-def refl-def*, *safe*)
apply (*blast intro*: *trans*)
done

lemma *congruent2-commuteI*:
assumes *equivA*: *equiv*(*A*,*r*)
and *commute*: $\bigwedge y\ z.\ [\![y \in A;\ \ z \in A]\!] \Longrightarrow b(y,z) = b(z,y)$
and *congt*: $\bigwedge y\ z\ w.\ [\![w \in A;\ \ \langle y,z\rangle: r]\!] \Longrightarrow b(w,y) = b(w,z)$
shows *b respects2 r*
apply (*insert equivA* [*THEN equiv-type*, *THEN subsetD*])
apply (*rule congruent2I* [*OF equivA equivA*])
apply (*rule commute* [*THEN trans*])
apply (*rule-tac* [*3*] *commute* [*THEN trans*, *symmetric*])
apply (*rule-tac* [*5*] *sym*)
apply (*blast intro*: *congt*)+
done

lemma *congruent-commuteI*:
$[\![equiv(A,r);\ \ Z \in A//r;$
$\bigwedge w.\ [\![w \in A]\!] \Longrightarrow congruent(r,\ \lambda z.\ b(w,z));$
$\bigwedge x\ y.\ [\![x \in A;\ \ y \in A]\!] \Longrightarrow b(y,x) = b(x,y)$
$]\!] \Longrightarrow congruent(r,\ \lambda w.\ \bigcup z{\in}Z.\ b(w,z))$
apply (*simp* (*no-asm*) *add*: *congruent-def*)
apply (*safe elim*!: *quotientE*)
apply (*frule equiv-type* [*THEN subsetD*], *assumption*)
apply (*simp add*: *UN-equiv-class* [*of A r*])
apply (*simp add*: *congruent-def*)
done

end

31 The Integers as Equivalence Classes Over Pairs of Natural Numbers

theory *Int* **imports** *EquivClass ArithSimp* **begin**

definition
intrel :: *i* **where**
$intrel \equiv \{p \in (nat*nat)*(nat*nat).$
$\exists x1\ y1\ x2\ y2.\ p{=}{<}\langle x1,y1\rangle,\langle x2,y2\rangle{>} \wedge x1 \#{+} y2 = x2 \#{+} y1\}$

definition
int :: *i* **where**
$int \equiv (nat*nat)//intrel$

definition
int-of :: *i*⇒*i* — coercion from nat to int (‹(‹*open-block notation*=‹*prefix* $#››$# -)› [*80*] *80*)
where $# *m* ≡ *intrel* ‘‘ {<*natify*(*m*), *0*>}

definition
intify :: *i*⇒*i* — coercion from ANYTHING to int **where**
intify(*m*) ≡ *if m* ∈ *int then m else* $#*0*

definition
raw-zminus :: *i*⇒*i* **where**
raw-zminus(*z*) ≡ ⋃⟨*x*,*y*⟩∈*z*. *intrel*‘‘{⟨*y*,*x*⟩}

definition
zminus :: *i*⇒*i* (‹(‹*open-block notation*=‹*prefix* $−››$− -)› [*80*] *80*)
where $− *z* ≡ *raw-zminus* (*intify*(*z*))

definition
znegative :: *i*⇒*o* **where**
znegative(*z*) ≡ ∃ *x y*. *x*<*y* ∧ *y*∈*nat* ∧ ⟨*x*,*y*⟩∈*z*

definition
iszero :: *i*⇒*o* **where**
iszero(*z*) ≡ *z* = $# *0*

definition
raw-nat-of :: *i*⇒*i* **where**
raw-nat-of(*z*) ≡ *natify* (⋃⟨*x*,*y*⟩∈*z*. *x*#−*y*)

definition
nat-of :: *i*⇒*i* **where**
nat-of(*z*) ≡ *raw-nat-of* (*intify*(*z*))

definition
zmagnitude :: *i*⇒*i* **where**
— could be replaced by an absolute value function from int to int?
zmagnitude(*z*) ≡
THE m. *m*∈*nat* ∧ ((¬ *znegative*(*z*) ∧ *z* = $# *m*) |
(*znegative*(*z*) ∧ $− *z* = $# *m*))

definition
raw-zmult :: [*i*,*i*]⇒*i* **where**

raw-zmult(*z1*,*z2*) ≡
⋃*p1*∈*z1*. ⋃*p2*∈*z2*. *split*(λ*x1 y1*. *split*(λ*x2 y2*.
intrel‘‘{<*x1*#∗*x2* #+ *y1*#∗*y2*, *x1*#∗*y2* #+ *y1*#∗*x2*>}, *p2*), *p1*)

definition
zmult :: [*i*,*i*]⇒*i* (**infixl** ‹$∗› *70*) **where**

z1 $∗ z2 ≡ *raw-zmult* (*intify*(*z1*),*intify*(*z2*))

definition
raw-zadd :: [*i*,*i*]⇒*i* **where**
raw-zadd (*z1*, *z2*) ≡
⋃*z1*∈*z1*. ⋃*z2*∈*z2*. *let* ⟨*x1*,*y1*⟩=*z1*; ⟨*x2*,*y2*⟩=*z2*
in intrel‘‘{<*x1*#+*x2*, *y1*#+*y2*>}

definition
zadd :: [*i*,*i*]⇒*i* (**infixl** ‹$+› *65*) **where**
z1 $+ z2 ≡ *raw-zadd* (*intify*(*z1*),*intify*(*z2*))

definition
zdiff :: [*i*,*i*]⇒*i* (**infixl** ‹$−› *65*) **where**
z1 $− z2 ≡ *z1 $+ zminus*(*z2*)

definition
zless :: [*i*,*i*]⇒*o* (**infixl** ‹$<› *50*) **where**
z1 $< z2 ≡ *znegative*(*z1 $− z2*)

definition
zle :: [*i*,*i*]⇒*o* (**infixl** ‹$≤› *50*) **where**
z1 $≤ z2 ≡ *z1 $< z2* | *intify*(*z1*)=*intify*(*z2*)

declare *quotientE* [*elim*!]

31.1 Proving that *intrel* is an equivalence relation

lemma *intrel-iff* [*simp*]:
<⟨*x1*,*y1*⟩,⟨*x2*,*y2*⟩>: *intrel* ⟷
x1∈*nat* ∧ *y1*∈*nat* ∧ *x2*∈*nat* ∧ *y2*∈*nat* ∧ *x1*#+*y2* = *x2*#+*y1*
by (*simp add*: *intrel-def*)

lemma *intrelI* [*intro*!]:
⟦*x1*#+*y2* = *x2*#+*y1*; *x1*∈*nat*; *y1*∈*nat*; *x2*∈*nat*; *y2*∈*nat*⟧
⟹ <⟨*x1*,*y1*⟩,⟨*x2*,*y2*⟩>: *intrel*
by (*simp add*: *intrel-def*)

lemma *intrelE* [*elim*!]:
⟦*p* ∈ *intrel*;
⋀*x1 y1 x2 y2*. ⟦*p* = <⟨*x1*,*y1*⟩,⟨*x2*,*y2*⟩>; *x1*#+*y2* = *x2*#+*y1*;
x1∈*nat*; *y1*∈*nat*; *x2*∈*nat*; *y2*∈*nat*⟧ ⟹ *Q*⟧
⟹ *Q*
by (*simp add*: *intrel-def*, *blast*)

lemma *int-trans-lemma*:
⟦*x1* #+ *y2* = *x2* #+ *y1*; *x2* #+ *y3* = *x3* #+ *y2*⟧ ⟹ *x1* #+ *y3* = *x3* #+
y1

apply (*rule sym*)
apply (*erule add-left-cancel*)+
apply (*simp-all* (*no-asm-simp*))
done

lemma *equiv-intrel*: *equiv*(*nat*∗*nat*, *intrel*)
apply (*simp add*: *equiv-def refl-def sym-def trans-def*)
apply (*fast elim*!: *sym int-trans-lemma*)
done

lemma *image-intrel-int*: ⟦*m*∈*nat*; *n*∈*nat*⟧ ⟹ *intrel* ‘‘ {⟨*m*,*n*⟩} ∈ *int*
by (*simp add*: *int-def*)

declare *equiv-intrel* [*THEN eq-equiv-class-iff*, *simp*]
declare *conj-cong* [*cong*]

lemmas *eq-intrelD* = *eq-equiv-class* [*OF - equiv-intrel*]

lemma *int-of-type* [*simp*,*TC*]: $#*m* ∈ *int*
by (*simp add*: *int-def quotient-def int-of-def*, *auto*)

lemma *int-of-eq* [*iff*]: ($# *m* = $# *n*) ⟷ *natify*(*m*)=*natify*(*n*)
by (*simp add*: *int-of-def*)

lemma *int-of-inject*: ⟦$#*m* = $#*n*; *m*∈*nat*; *n*∈*nat*⟧ ⟹ *m*=*n*
by (*drule int-of-eq* [*THEN iffD1*], *auto*)

lemma *intify-in-int* [*iff*,*TC*]: *intify*(*x*) ∈ *int*
by (*simp add*: *intify-def*)

lemma *intify-ident* [*simp*]: *n* ∈ *int* ⟹ *intify*(*n*) = *n*
by (*simp add*: *intify-def*)

31.2 Collapsing rules: to remove *intify* from arithmetic expressions

lemma *intify-idem* [*simp*]: *intify*(*intify*(*x*)) = *intify*(*x*)
by *simp*

lemma *int-of-natify* [*simp*]: $# (*natify*(*m*)) = $# *m*
by (*simp add*: *int-of-def*)

lemma *zminus-intify* [*simp*]: $− (*intify*(*m*)) = $− *m*
by (*simp add*: *zminus-def*)

lemma *zadd-intify1* [*simp*]: *intify*(x) \$+ y = x \$+ y
by (*simp add*: *zadd-def*)

lemma *zadd-intify2* [*simp*]: x \$+ *intify*(y) = x \$+ y
by (*simp add*: *zadd-def*)

lemma *zdiff-intify1* [*simp*]:*intify*(x) \$− y = x \$− y
by (*simp add*: *zdiff-def*)

lemma *zdiff-intify2* [*simp*]:x \$− *intify*(y) = x \$− y
by (*simp add*: *zdiff-def*)

lemma *zmult-intify1* [*simp*]:*intify*(x) \$∗ y = x \$∗ y
by (*simp add*: *zmult-def*)

lemma *zmult-intify2* [*simp*]:x \$∗ *intify*(y) = x \$∗ y
by (*simp add*: *zmult-def*)

lemma *zless-intify1* [*simp*]:*intify*(x) \$< y ⟷ x \$< y
by (*simp add*: *zless-def*)

lemma *zless-intify2* [*simp*]:x \$< *intify*(y) ⟷ x \$< y
by (*simp add*: *zless-def*)

lemma *zle-intify1* [*simp*]:*intify*(x) \$≤ y ⟷ x \$≤ y
by (*simp add*: *zle-def*)

lemma *zle-intify2* [*simp*]:x \$≤ *intify*(y) ⟷ x \$≤ y
by (*simp add*: *zle-def*)

31.3 *zminus*: unary negation on *int*

lemma *zminus-congruent*: (λ⟨*x*,*y*⟩. *intrel*``{⟨*y*,*x*⟩}) *respects intrel*
by (*auto simp add*: *congruent-def add-ac*)

lemma *raw-zminus-type*: $z \in$ *int* ⟹ *raw-zminus*(z) ∈ *int*
apply (*simp add*: *int-def raw-zminus-def*)
apply (*typecheck add*: *UN-equiv-class-type* [*OF equiv-intrel zminus-congruent*])
done

lemma *zminus-type* [*TC*,*iff*]: $\$-z \in int$
by (*simp add*: *zminus-def raw-zminus-type*)

lemma *raw-zminus-inject*:
  $[\![raw\text{-}zminus(z) = raw\text{-}zminus(w);\ z \in int;\ w \in int]\!] \Longrightarrow z=w$
apply (*simp add*: *int-def raw-zminus-def*)
apply (*erule UN-equiv-class-inject* [*OF equiv-intrel zminus-congruent*], *safe*)
apply (*auto dest*: *eq-intrelD simp add*: *add-ac*)
done

lemma *zminus-inject-intify* [*dest*!]: $\$-z = \$-w \Longrightarrow intify(z) = intify(w)$
apply (*simp add*: *zminus-def*)
apply (*blast dest*!: *raw-zminus-inject*)
done

lemma *zminus-inject*: $[\![\$-z = \$-w;\ z \in int;\ w \in int]\!] \Longrightarrow z=w$
by *auto*

lemma *raw-zminus*:
  $[\![x\in nat;\ y\in nat]\!] \Longrightarrow raw\text{-}zminus(intrel``\{\langle x,y\rangle\}) = intrel `` \{\langle y,x\rangle\}$
apply (*simp add*: *raw-zminus-def UN-equiv-class* [*OF equiv-intrel zminus-congruent*])
done

lemma *zminus*:
  $[\![x\in nat;\ y\in nat]\!]$
  $\Longrightarrow \$-\ (intrel``\{\langle x,y\rangle\}) = intrel `` \{\langle y,x\rangle\}$
by (*simp add*: *zminus-def raw-zminus image-intrel-int*)

lemma *raw-zminus-zminus*: $z \in int \Longrightarrow raw\text{-}zminus\ (raw\text{-}zminus(z)) = z$
by (*auto simp add*: *int-def raw-zminus*)

lemma *zminus-zminus-intify* [*simp*]: $\$-\ (\$-\ z) = intify(z)$
by (*simp add*: *zminus-def raw-zminus-type raw-zminus-zminus*)

lemma *zminus-int0* [*simp*]: $\$-\ (\$\#0) = \$\#0$
by (*simp add*: *int-of-def zminus*)

lemma *zminus-zminus*: $z \in int \Longrightarrow \$-\ (\$-\ z) = z$
by *simp*

31.4 *znegative*: the test for negative integers

lemma *znegative*: $[\![x\in nat;\ y\in nat]\!] \Longrightarrow znegative(intrel``\{\langle x,y\rangle\}) \longleftrightarrow x<y$
apply (*cases x<y*)
apply (*auto simp add*: *znegative-def not-lt-iff-le*)
apply (*subgoal-tac y #+ x2 < x #+ y2*, *force*)
apply (*rule add-le-lt-mono*, *auto*)
done

lemma *not-znegative-int-of* [*iff*]: ¬ *znegative*($\$\#$ *n*)
by (*simp add*: *znegative int-of-def*)

lemma *znegative-zminus-int-of* [*simp*]: *znegative*($\$-$ $\$\#$ *succ*(*n*))
by (*simp add*: *znegative int-of-def zminus natify-succ*)

lemma *not-znegative-imp-zero*: ¬ *znegative*($\$-$ $\$\#$ *n*) $\Longrightarrow$ *natify*(*n*)=*0*
by (*simp add*: *znegative int-of-def zminus Ord-0-lt-iff* [*THEN iff-sym*])

31.5 *nat-of*: Coercion of an Integer to a Natural Number

lemma *nat-of-intify* [*simp*]: *nat-of*(*intify*(*z*)) = *nat-of*(*z*)
by (*simp add*: *nat-of-def*)

lemma *nat-of-congruent*: (λx. ($\lambda\langle x,y\rangle$. x #− y)(x)) *respects intrel*
by (*auto simp add*: *congruent-def split*: *nat-diff-split*)

lemma *raw-nat-of*:
  $[\![x\in nat;\ y\in nat]\!] \Longrightarrow$ *raw-nat-of*(*intrel*``{⟨*x*,*y*⟩}) = *x*#−*y*
by (*simp add*: *raw-nat-of-def UN-equiv-class* [*OF equiv-intrel nat-of-congruent*])

lemma *raw-nat-of-int-of*: *raw-nat-of*($\$\#$ *n*) = *natify*(*n*)
by (*simp add*: *int-of-def raw-nat-of*)

lemma *nat-of-int-of* [*simp*]: *nat-of*($\$\#$ *n*) = *natify*(*n*)
by (*simp add*: *raw-nat-of-int-of nat-of-def*)

lemma *raw-nat-of-type*: *raw-nat-of*(*z*) ∈ *nat*
by (*simp add*: *raw-nat-of-def*)

lemma *nat-of-type* [*iff*,*TC*]: *nat-of*(*z*) ∈ *nat*
by (*simp add*: *nat-of-def raw-nat-of-type*)

31.6 zmagnitude: magnitide of an integer, as a natural number

lemma *zmagnitude-int-of* [*simp*]: *zmagnitude*($\$\#$ *n*) = *natify*(*n*)
by (*auto simp add*: *zmagnitude-def int-of-eq*)

lemma *natify-int-of-eq*: *natify*(*x*)=*n* $\Longrightarrow$ $\$\#x$ = $\$\#$ *n*
apply (*drule sym*)
apply (*simp* (*no-asm-simp*) *add*: *int-of-eq*)
done

lemma *zmagnitude-zminus-int-of* [*simp*]: *zmagnitude*($\$-$ $\$\#$ *n*) = *natify*(*n*)
apply (*simp add*: *zmagnitude-def*)
apply (*rule the-equality*)
apply (*auto dest*!: *not-znegative-imp-zero natify-int-of-eq*

iff del: *int-of-eq*, *auto*)
done

lemma *zmagnitude-type* [*iff*,*TC*]: *zmagnitude*(*z*)∈*nat*
apply (*simp add*: *zmagnitude-def*)
apply (*rule theI2*, *auto*)
done

lemma *not-zneg-int-of*:
⟦*z* ∈ *int*; ¬ *znegative*(*z*)⟧ ⟹ ∃ *n*∈*nat*. *z* = $# *n*
apply (*auto simp add*: *int-def znegative int-of-def not-lt-iff-le*)
apply (*rename-tac x y*)
apply (*rule-tac x*=*x*#−*y* **in** *bexI*)
apply (*auto simp add*: *add-diff-inverse2*)
done

lemma *not-zneg-mag* [*simp*]:
⟦*z* ∈ *int*; ¬ *znegative*(*z*)⟧ ⟹ $# (*zmagnitude*(*z*)) = *z*
by (*drule not-zneg-int-of*, *auto*)

lemma *zneg-int-of*:
⟦*znegative*(*z*); *z* ∈ *int*⟧ ⟹ ∃ *n*∈*nat*. *z* = $− ($# *succ*(*n*))
by (*auto simp add*: *int-def znegative zminus int-of-def dest*!: *less-imp-succ-add*)

lemma *zneg-mag* [*simp*]:
⟦*znegative*(*z*); *z* ∈ *int*⟧ ⟹ $# (*zmagnitude*(*z*)) = $− *z*
by (*drule zneg-int-of*, *auto*)

lemma *int-cases*: *z* ∈ *int* ⟹ ∃ *n*∈*nat*. *z* = $# *n* | *z* = $− ($# *succ*(*n*))
apply (*case-tac znegative* (*z*))
prefer *2* **apply** (*blast dest*: *not-zneg-mag sym*)
apply (*blast dest*: *zneg-int-of*)
done

lemma *not-zneg-raw-nat-of*:
⟦¬ *znegative*(*z*); *z* ∈ *int*⟧ ⟹ $# (*raw-nat-of*(*z*)) = *z*
apply (*drule not-zneg-int-of*)
apply (*auto simp add*: *raw-nat-of-type raw-nat-of-int-of*)
done

lemma *not-zneg-nat-of-intify*:
¬ *znegative*(*intify*(*z*)) ⟹ $# (*nat-of*(*z*)) = *intify*(*z*)
by (*simp* (*no-asm-simp*) *add*: *nat-of-def not-zneg-raw-nat-of*)

lemma *not-zneg-nat-of*: ⟦¬ *znegative*(*z*); *z* ∈ *int*⟧ ⟹ $# (*nat-of*(*z*)) = *z*
apply (*simp* (*no-asm-simp*) *add*: *not-zneg-nat-of-intify*)
done

lemma *zneg-nat-of* [*simp*]: *znegative*(*intify*(*z*)) ⟹ *nat-of*(*z*) = *0*

apply (*subgoal-tac intify*(*z*) ∈ *int*)
apply (*simp add*: *int-def*)
apply (*auto simp add*: *znegative nat-of-def raw-nat-of*
split: *nat-diff-split*)
done

31.7 ($+): addition on int

Congruence Property for Addition

lemma *zadd-congruent2*:
(λ*z1 z2*. *let* ⟨*x1*,*y1*⟩=*z1*; ⟨*x2*,*y2*⟩=*z2*
in intrel‘‘{<*x1*#+*x2*, *y1*#+*y2*>})
respects2 intrel
apply (*simp add*: *congruent2-def*)

apply *safe*
apply (*simp* (*no-asm-simp*) *add*: *add-assoc Let-def*)

apply (*rule-tac m1* = *x1a* **in** *add-left-commute* [*THEN ssubst*])
apply (*rule-tac m1* = *x2a* **in** *add-left-commute* [*THEN ssubst*])
apply (*simp* (*no-asm-simp*) *add*: *add-assoc* [*symmetric*])
done

lemma *raw-zadd-type*: ⟦*z* ∈ *int*; *w* ∈ *int*⟧ ⟹ *raw-zadd*(*z*,*w*) ∈ *int*
apply (*simp add*: *int-def raw-zadd-def*)
apply (*rule UN-equiv-class-type2* [*OF equiv-intrel zadd-congruent2*], *assumption*+)
apply (*simp add*: *Let-def*)
done

lemma *zadd-type* [*iff*,*TC*]: *z* $+ *w* ∈ *int*
by (*simp add*: *zadd-def raw-zadd-type*)

lemma *raw-zadd*:
⟦*x1*∈*nat*; *y1*∈*nat*; *x2*∈*nat*; *y2*∈*nat*⟧
⟹ *raw-zadd* (*intrel*‘‘{⟨*x1*,*y1*⟩}, *intrel*‘‘{⟨*x2*,*y2*⟩}) =
intrel ‘‘ {<*x1*#+*x2*, *y1*#+*y2*>}
apply (*simp add*: *raw-zadd-def*
UN-equiv-class2 [*OF equiv-intrel equiv-intrel zadd-congruent2*])
apply (*simp add*: *Let-def*)
done

lemma *zadd*:
⟦*x1*∈*nat*; *y1*∈*nat*; *x2*∈*nat*; *y2*∈*nat*⟧
⟹ (*intrel*‘‘{⟨*x1*,*y1*⟩}) $+ (*intrel*‘‘{⟨*x2*,*y2*⟩}) =
intrel ‘‘ {<*x1*#+*x2*, *y1*#+*y2*>}
by (*simp add*: *zadd-def raw-zadd image-intrel-int*)

lemma *raw-zadd-int0*: *z* ∈ *int* ⟹ *raw-zadd* ($#*0*,*z*) = *z*
by (*auto simp add*: *int-def int-of-def raw-zadd*)

lemma *zadd-int0-intify* [*simp*]: $#*0* $+ *z* = *intify*(*z*)
by (*simp add*: *zadd-def raw-zadd-int0*)

lemma *zadd-int0*: *z* ∈ *int* ⟹ $#*0* $+ *z* = *z*
by *simp*

lemma *raw-zminus-zadd-distrib*:
⟦*z* ∈ *int*; *w* ∈ *int*⟧ ⟹ $− *raw-zadd*(*z*,*w*) = *raw-zadd*($− *z*, $− *w*)
by (*auto simp add*: *zminus raw-zadd int-def*)

lemma *zminus-zadd-distrib* [*simp*]: $− (*z* $+ *w*) = $− *z* $+ $− *w*
by (*simp add*: *zadd-def raw-zminus-zadd-distrib*)

lemma *raw-zadd-commute*:
⟦*z* ∈ *int*; *w* ∈ *int*⟧ ⟹ *raw-zadd*(*z*,*w*) = *raw-zadd*(*w*,*z*)
by (*auto simp add*: *raw-zadd add-ac int-def*)

lemma *zadd-commute*: *z* $+ *w* = *w* $+ *z*
by (*simp add*: *zadd-def raw-zadd-commute*)

lemma *raw-zadd-assoc*:
⟦*z1*: *int*; *z2*: *int*; *z3*: *int*⟧
⟹ *raw-zadd* (*raw-zadd*(*z1*,*z2*),*z3*) = *raw-zadd*(*z1*,*raw-zadd*(*z2*,*z3*))
by (*auto simp add*: *int-def raw-zadd add-assoc*)

lemma *zadd-assoc*: (*z1* $+ *z2*) $+ *z3* = *z1* $+ (*z2* $+ *z3*)
by (*simp add*: *zadd-def raw-zadd-type raw-zadd-assoc*)

lemma *zadd-left-commute*: *z1*$+(*z2*$+*z3*) = *z2*$+(*z1*$+*z3*)
apply (*simp add*: *zadd-assoc* [*symmetric*])
apply (*simp add*: *zadd-commute*)
done

lemmas *zadd-ac* = *zadd-assoc zadd-commute zadd-left-commute*

lemma *int-of-add*: $# (*m* #+ *n*) = ($#*m*) $+ ($#*n*)
by (*simp add*: *int-of-def zadd*)

lemma *int-succ-int-1*: $# *succ*(*m*) = $# *1* $+ ($# *m*)
by (*simp add*: *int-of-add* [*symmetric*] *natify-succ*)

lemma *int-of-diff*:
⟦*m*∈*nat*; *n* ≤ *m*⟧ ⟹ $# (*m* #− *n*) = ($#*m*) $− ($#*n*)
apply (*simp add*: *int-of-def zdiff-def*)
apply (*frule lt-nat-in-nat*)
apply (*simp-all add*: *zadd zminus add-diff-inverse2*)

done

lemma *raw-zadd-zminus-inverse*: *z* ∈ *int* ⟹ *raw-zadd* (*z*, $− *z*) = $#*0*
by (*auto simp add*: *int-def int-of-def zminus raw-zadd add-commute*)

lemma *zadd-zminus-inverse* [*simp*]: *z* $+ ($− *z*) = $#*0*
apply (*simp add*: *zadd-def*)
apply (*subst zminus-intify* [*symmetric*])
apply (*rule intify-in-int* [*THEN raw-zadd-zminus-inverse*])
done

lemma *zadd-zminus-inverse2* [*simp*]: ($− *z*) $+ *z* = $#*0*
by (*simp add*: *zadd-commute zadd-zminus-inverse*)

lemma *zadd-int0-right-intify* [*simp*]: *z* $+ $#*0* = *intify*(*z*)
by (*rule trans* [*OF zadd-commute zadd-int0-intify*])

lemma *zadd-int0-right*: *z* ∈ *int* ⟹ *z* $+ $#*0* = *z*
by *simp*

31.8 ($∗): Integer Multiplication

Congruence property for multiplication

lemma *zmult-congruent2*:
(λ*p1 p2*. *split*(λ*x1 y1*. *split*(λ*x2 y2*.
intrel``{<*x1*#∗*x2* #+ *y1*#∗*y2*, *x1*#∗*y2* #+ *y1*#∗*x2*>}, *p2*), *p1*))
respects2 intrel
apply (*rule equiv-intrel* [*THEN congruent2-commuteI*], *auto*)

apply (*rename-tac x y*)
apply (*frule-tac t* = λ*u*. *x*#∗*u* **in** *sym* [*THEN subst-context*])
apply (*drule-tac t* = λ*u*. *y*#∗*u* **in** *subst-context*)
apply (*erule add-left-cancel*)+
apply (*simp-all add*: *add-mult-distrib-left*)
done

lemma *raw-zmult-type*: ⟦*z* ∈ *int*; *w* ∈ *int*⟧ ⟹ *raw-zmult*(*z*,*w*) ∈ *int*
apply (*simp add*: *int-def raw-zmult-def*)
apply (*rule UN-equiv-class-type2* [*OF equiv-intrel zmult-congruent2*], *assumption*+)
apply (*simp add*: *Let-def*)
done

lemma *zmult-type* [*iff*,*TC*]: *z* $∗ *w* ∈ *int*
by (*simp add*: *zmult-def raw-zmult-type*)

lemma *raw-zmult*:
⟦*x1*∈*nat*; *y1*∈*nat*; *x2*∈*nat*; *y2*∈*nat*⟧
⟹ *raw-zmult*(*intrel*``{⟨*x1*,*y1*⟩}, *intrel*``{⟨*x2*,*y2*⟩}) =

intrel `` {<*x1*#**x2* #+ *y1*#**y2*, *x1*#**y2* #+ *y1*#**x2*>}
by (*simp add*: *raw-zmult-def*
UN-equiv-class2 [*OF equiv-intrel equiv-intrel zmult-congruent2*])

lemma *zmult*:
⟦*x1*∈*nat*; *y1*∈*nat*; *x2*∈*nat*; *y2*∈*nat*⟧
⟹ (*intrel*``{⟨*x1*,*y1*⟩}) $\* (*intrel*``{⟨*x2*,*y2*⟩}) =
intrel `` {<*x1*#**x2* #+ *y1*#**y2*, *x1*#**y2* #+ *y1*#**x2*>}
by (*simp add*: *zmult-def raw-zmult image-intrel-int*)

lemma *raw-zmult-int0*: *z* ∈ *int* ⟹ *raw-zmult* ($#*0*,*z*) = $#*0*
by (*auto simp add*: *int-def int-of-def raw-zmult*)

lemma *zmult-int0* [*simp*]: $#*0* $* *z* = $#*0*
by (*simp add*: *zmult-def raw-zmult-int0*)

lemma *raw-zmult-int1*: *z* ∈ *int* ⟹ *raw-zmult* ($#*1*,*z*) = *z*
by (*auto simp add*: *int-def int-of-def raw-zmult*)

lemma *zmult-int1-intify* [*simp*]: $#*1* $* *z* = *intify*(*z*)
by (*simp add*: *zmult-def raw-zmult-int1*)

lemma *zmult-int1*: *z* ∈ *int* ⟹ $#*1* $* *z* = *z*
by *simp*

lemma *raw-zmult-commute*:
⟦*z* ∈ *int*; *w* ∈ *int*⟧ ⟹ *raw-zmult*(*z*,*w*) = *raw-zmult*(*w*,*z*)
by (*auto simp add*: *int-def raw-zmult add-ac mult-ac*)

lemma *zmult-commute*: *z* $* *w* = *w* $* *z*
by (*simp add*: *zmult-def raw-zmult-commute*)

lemma *raw-zmult-zminus*:
⟦*z* ∈ *int*; *w* ∈ *int*⟧ ⟹ *raw-zmult*($− *z*, *w*) = $− *raw-zmult*(*z*, *w*)
by (*auto simp add*: *int-def zminus raw-zmult add-ac*)

lemma *zmult-zminus* [*simp*]: ($− *z*) $* *w* = $− (*z* $* *w*)
apply (*simp add*: *zmult-def raw-zmult-zminus*)
apply (*subst zminus-intify* [*symmetric*], *rule raw-zmult-zminus*, *auto*)
done

lemma *zmult-zminus-right* [*simp*]: *w* $* ($− *z*) = $− (*w* $* *z*)
by (*simp add*: *zmult-commute* [*of w*])

lemma *raw-zmult-assoc*:
⟦*z1*: *int*; *z2*: *int*; *z3*: *int*⟧
⟹ *raw-zmult* (*raw-zmult*(*z1*,*z2*),*z3*) = *raw-zmult*(*z1*,*raw-zmult*(*z2*,*z3*))
by (*auto simp add*: *int-def raw-zmult add-mult-distrib-left add-ac mult-ac*)

lemma *zmult-assoc*: (*z1* \$∗ *z2*) \$∗ *z3* = *z1* \$∗ (*z2* \$∗ *z3*)
by (*simp add*: *zmult-def raw-zmult-type raw-zmult-assoc*)

lemma *zmult-left-commute*: *z1*\$∗(*z2*\$∗*z3*) = *z2*\$∗(*z1*\$∗*z3*)
apply (*simp add*: *zmult-assoc* [*symmetric*])
apply (*simp add*: *zmult-commute*)
done

lemmas *zmult-ac* = *zmult-assoc zmult-commute zmult-left-commute*

lemma *raw-zadd-zmult-distrib*:
  ⟦*z1*: *int*; *z2*: *int*; *w* ∈ *int*⟧
  ⟹ *raw-zmult*(*raw-zadd*(*z1*,*z2*), *w*) =
    raw-zadd (*raw-zmult*(*z1*,*w*), *raw-zmult*(*z2*,*w*))
by (*auto simp add*: *int-def raw-zadd raw-zmult add-mult-distrib-left add-ac mult-ac*)

lemma *zadd-zmult-distrib*: (*z1* \$+ *z2*) \$∗ *w* = (*z1* \$∗ *w*) \$+ (*z2* \$∗ *w*)
by (*simp add*: *zmult-def zadd-def raw-zadd-type raw-zmult-type*
    raw-zadd-zmult-distrib)

lemma *zadd-zmult-distrib2*: *w* \$∗ (*z1* \$+ *z2*) = (*w* \$∗ *z1*) \$+ (*w* \$∗ *z2*)
by (*simp add*: *zmult-commute* [*of w*] *zadd-zmult-distrib*)

lemmas *int-typechecks* =
  int-of-type zminus-type zmagnitude-type zadd-type zmult-type

lemma *zdiff-type* [*iff*,*TC*]: *z* \$− *w* ∈ *int*
by (*simp add*: *zdiff-def*)

lemma *zminus-zdiff-eq* [*simp*]: \$− (*z* \$− *y*) = *y* \$− *z*
by (*simp add*: *zdiff-def zadd-commute*)

lemma *zdiff-zmult-distrib*: (*z1* \$− *z2*) \$∗ *w* = (*z1* \$∗ *w*) \$− (*z2* \$∗ *w*)
apply (*simp add*: *zdiff-def*)
apply (*subst zadd-zmult-distrib*)
apply (*simp add*: *zmult-zminus*)
done

lemma *zdiff-zmult-distrib2*: *w* \$∗ (*z1* \$− *z2*) = (*w* \$∗ *z1*) \$− (*w* \$∗ *z2*)
by (*simp add*: *zmult-commute* [*of w*] *zdiff-zmult-distrib*)

lemma *zadd-zdiff-eq*: *x* \$+ (*y* \$− *z*) = (*x* \$+ *y*) \$− *z*
by (*simp add*: *zdiff-def zadd-ac*)

lemma *zdiff-zadd-eq*: (*x* $− *y*) $+ *z* = (*x* $+ *z*) $− *y*
by (*simp add*: *zdiff-def zadd-ac*)

31.9 The "Less Than" Relation

lemma *zless-linear-lemma*:
⟦*z* ∈ *int*; *w* ∈ *int*⟧ ⟹ *z*$<*w* | *z*=*w* | *w*$<*z*
apply (*simp add*: *int-def zless-def znegative-def zdiff-def*, *auto*)
apply (*simp add*: *zadd zminus image-iff Bex-def*)
apply (*rule-tac i* = *xb*#+*ya* **and** *j* = *xc* #+ *y* **in** *Ord-linear-lt*)
apply (*force dest*!: *spec simp add*: *add-ac*)+
done

lemma *zless-linear*: *z*$<*w* | *intify*(*z*)=*intify*(*w*) | *w*$<*z*
apply (*cut-tac z* = *intify* (*z*) **and** *w* = *intify* (*w*) **in** *zless-linear-lemma*)
apply *auto*
done

lemma *zless-not-refl* [*iff*]: ¬ (*z*$<*z*)
by (*auto simp add*: *zless-def znegative-def int-of-def zdiff-def*)

lemma *neq-iff-zless*: ⟦*x* ∈ *int*; *y* ∈ *int*⟧ ⟹ (*x* ≠ *y*) ⟷ (*x* $< *y* | *y* $< *x*)
by (*cut-tac z* = *x* **and** *w* = *y* **in** *zless-linear*, *auto*)

lemma *zless-imp-intify-neq*: *w* $< *z* ⟹ *intify*(*w*) ≠ *intify*(*z*)
apply *auto*
apply (*subgoal-tac* ¬ (*intify* (*w*) $< *intify* (*z*)))
apply (*erule-tac* [*2*] *ssubst*)
apply (*simp* (*no-asm-use*))
apply *auto*
done

lemma *zless-imp-succ-zadd-lemma*:
⟦*w* $< *z*; *w* ∈ *int*; *z* ∈ *int*⟧ ⟹ (∃ *n*∈*nat*. *z* = *w* $+ $#(*succ*(*n*)))
apply (*simp add*: *zless-def znegative-def zdiff-def int-def*)
apply (*auto dest*!: *less-imp-succ-add simp add*: *zadd zminus int-of-def*)
apply (*rule-tac x* = *k* **in** *bexI*)
apply (*erule-tac i*=*succ* (*v*) **for** *v* **in** *add-left-cancel*, *auto*)
done

lemma *zless-imp-succ-zadd*:
w $< *z* ⟹ (∃ *n*∈*nat*. *w* $+ $#(*succ*(*n*)) = *intify*(*z*))
apply (*subgoal-tac intify* (*w*) $< *intify* (*z*))
apply (*drule-tac w* = *intify* (*w*) **in** *zless-imp-succ-zadd-lemma*)
apply *auto*
done

lemma *zless-succ-zadd-lemma*:

w ∈ *int* ⟹ *w* $< *w* $+ $# *succ*(*n*)
apply (*simp add*: *zless-def znegative-def zdiff-def int-def*)
apply (*auto simp add*: *zadd zminus int-of-def image-iff*)
apply (*rule-tac x* = *0* **in** *exI*, *auto*)
done

lemma *zless-succ-zadd*: *w* $< *w* $+ $# *succ*(*n*)
by (*cut-tac intify-in-int* [*THEN zless-succ-zadd-lemma*], *auto*)

lemma *zless-iff-succ-zadd*:
w $< *z* ⟷ (∃ *n*∈*nat*. *w* $+ $#(*succ*(*n*)) = *intify*(*z*))
apply (*rule iffI*)
apply (*erule zless-imp-succ-zadd*, *auto*)
apply (*rename-tac n*)
apply (*cut-tac w* = *w* **and** *n* = *n* **in** *zless-succ-zadd*, *auto*)
done

lemma *zless-int-of* [*simp*]: ⟦*m*∈*nat*; *n*∈*nat*⟧ ⟹ ($#*m* $< $#*n*) ⟷ (*m*<*n*)
apply (*simp add*: *less-iff-succ-add zless-iff-succ-zadd int-of-add* [*symmetric*])
apply (*blast intro*: *sym*)
done

lemma *zless-trans-lemma*:
⟦*x* $< *y*; *y* $< *z*; *x* ∈ *int*; *y* ∈ *int*; *z* ∈ *int*⟧ ⟹ *x* $< *z*
apply (*simp add*: *zless-def znegative-def zdiff-def int-def*)
apply (*auto simp add*: *zadd zminus image-iff*)
apply (*rename-tac x1 x2 y1 y2*)
apply (*rule-tac x* = *x1*#+*x2* **in** *exI*)
apply (*rule-tac x* = *y1*#+*y2* **in** *exI*)
apply (*auto simp add*: *add-lt-mono*)
apply (*rule sym*)
apply *hypsubst-thin*
apply (*erule add-left-cancel*)+
apply *auto*
done

lemma *zless-trans* [*trans*]: ⟦*x* $< *y*; *y* $< *z*⟧ ⟹ *x* $< *z*
apply (*subgoal-tac intify* (*x*) $< *intify* (*z*))
apply (*rule-tac* [*2*] *y* = *intify* (*y*) **in** *zless-trans-lemma*)
apply *auto*
done

lemma *zless-not-sym*: *z* $< *w* ⟹ ¬ (*w* $< *z*)
by (*blast dest*: *zless-trans*)

lemmas *zless-asym* = *zless-not-sym* [*THEN swap*]

lemma *zless-imp-zle*: *z* $< *w* ⟹ *z* $≤ *w*

by (*simp add*: *zle-def*)

lemma *zle-linear*: $z \ \$\leq\ w \mid w \ \$\leq\ z$
apply (*simp add*: *zle-def*)
apply (*cut-tac zless-linear*, *blast*)
done

31.10 Less Than or Equals

lemma *zle-refl*: $z \ \$\leq\ z$
by (*simp add*: *zle-def*)

lemma *zle-eq-refl*: $x{=}y \Longrightarrow x \ \$\leq\ y$
by (*simp add*: *zle-refl*)

lemma *zle-anti-sym-intify*: $[\![x \ \$\leq\ y;\ y \ \$\leq\ x]\!] \Longrightarrow intify(x) = intify(y)$
apply (*simp add*: *zle-def*, *auto*)
apply (*blast dest*: *zless-trans*)
done

lemma *zle-anti-sym*: $[\![x \ \$\leq\ y;\ y \ \$\leq\ x;\ x \in int;\ y \in int]\!] \Longrightarrow x{=}y$
by (*drule zle-anti-sym-intify*, *auto*)

lemma *zle-trans-lemma*:
$[\![x \in int;\ y \in int;\ z \in int;\ x \ \$\leq\ y;\ y \ \$\leq\ z]\!] \Longrightarrow x \ \$\leq\ z$
apply (*simp add*: *zle-def*, *auto*)
apply (*blast intro*: *zless-trans*)
done

lemma *zle-trans* [*trans*]: $[\![x \ \$\leq\ y;\ y \ \$\leq\ z]\!] \Longrightarrow x \ \$\leq\ z$
apply (*subgoal-tac intify* (*x*) $\$\leq$ *intify* (*z*))
apply (*rule-tac* [*2*] *y* = *intify* (*y*) **in** *zle-trans-lemma*)
apply *auto*
done

lemma *zle-zless-trans* [*trans*]: $[\![i \ \$\leq\ j;\ j \ \$<\ k]\!] \Longrightarrow i \ \$<\ k$
apply (*auto simp add*: *zle-def*)
apply (*blast intro*: *zless-trans*)
apply (*simp add*: *zless-def zdiff-def zadd-def*)
done

lemma *zless-zle-trans* [*trans*]: $[\![i \ \$<\ j;\ j \ \$\leq\ k]\!] \Longrightarrow i \ \$<\ k$
apply (*auto simp add*: *zle-def*)
apply (*blast intro*: *zless-trans*)
apply (*simp add*: *zless-def zdiff-def zminus-def*)
done

lemma *not-zless-iff-zle*: $\neg\ (z \ \$<\ w) \longleftrightarrow (w \ \$\leq\ z)$
apply (*cut-tac z* = *z* **and** *w* = *w* **in** *zless-linear*)

apply (*auto dest*: *zless-trans simp add*: *zle-def*)
apply (*auto dest*!: *zless-imp-intify-neq*)
done

lemma *not-zle-iff-zless*: ¬ (*z* $≤ *w*) ⟷ (*w* $< *z*)
by (*simp add*: *not-zless-iff-zle* [*THEN iff-sym*])

31.11 More subtraction laws (for *zcompare-rls*)

lemma *zdiff-zdiff-eq*: (*x* $− *y*) $− *z* = *x* $− (*y* $+ *z*)
by (*simp add*: *zdiff-def zadd-ac*)

lemma *zdiff-zdiff-eq2*: *x* $− (*y* $− *z*) = (*x* $+ *z*) $− *y*
by (*simp add*: *zdiff-def zadd-ac*)

lemma *zdiff-zless-iff*: (*x*$−*y* $< *z*) ⟷ (*x* $< *z* $+ *y*)
by (*simp add*: *zless-def zdiff-def zadd-ac*)

lemma *zless-zdiff-iff*: (*x* $< *z*$−*y*) ⟷ (*x* $+ *y* $< *z*)
by (*simp add*: *zless-def zdiff-def zadd-ac*)

lemma *zdiff-eq-iff*: ⟦*x* ∈ *int*; *z* ∈ *int*⟧ ⟹ (*x*$−*y* = *z*) ⟷ (*x* = *z* $+ *y*)
by (*auto simp add*: *zdiff-def zadd-assoc*)

lemma *eq-zdiff-iff*: ⟦*x* ∈ *int*; *z* ∈ *int*⟧ ⟹ (*x* = *z*$−*y*) ⟷ (*x* $+ *y* = *z*)
by (*auto simp add*: *zdiff-def zadd-assoc*)

lemma *zdiff-zle-iff-lemma*:
⟦*x* ∈ *int*; *z* ∈ *int*⟧ ⟹ (*x*$−*y* $≤ *z*) ⟷ (*x* $≤ *z* $+ *y*)
by (*auto simp add*: *zle-def zdiff-eq-iff zdiff-zless-iff*)

lemma *zdiff-zle-iff*: (*x*$−*y* $≤ *z*) ⟷ (*x* $≤ *z* $+ *y*)
by (*cut-tac zdiff-zle-iff-lemma* [*OF intify-in-int intify-in-int*], *simp*)

lemma *zle-zdiff-iff-lemma*:
⟦*x* ∈ *int*; *z* ∈ *int*⟧ ⟹(*x* $≤ *z*$−*y*) ⟷ (*x* $+ *y* $≤ *z*)
apply (*auto simp add*: *zle-def zdiff-eq-iff zless-zdiff-iff*)
apply (*auto simp add*: *zdiff-def zadd-assoc*)
done

lemma *zle-zdiff-iff*: (*x* $≤ *z*$−*y*) ⟷ (*x* $+ *y* $≤ *z*)
by (*cut-tac zle-zdiff-iff-lemma* [*OF intify-in-int intify-in-int*], *simp*)

This list of rewrites simplifies (in)equalities by bringing subtractions to the top and then moving negative terms to the other side. Use with *zadd-ac*

lemmas *zcompare-rls* =
zdiff-def [*symmetric*]
zadd-zdiff-eq zdiff-zadd-eq zdiff-zdiff-eq zdiff-zdiff-eq2
zdiff-zless-iff zless-zdiff-iff zdiff-zle-iff zle-zdiff-iff

zdiff-eq-iff eq-zdiff-iff

31.12 Monotonicity and Cancellation Results for Instantiation of the CancelNumerals Simprocs

lemma *zadd-left-cancel*:
$[\![w \in int; w': int]\!] \Longrightarrow (z \$+ w' = z \$+ w) \longleftrightarrow (w' = w)$
apply *safe*
apply (*drule-tac t* = $\lambda x.\ x$ \$+ (\$−*z*) **in** *subst-context*)
apply (*simp add*: *zadd-ac*)
done

lemma *zadd-left-cancel-intify* [*simp*]:
$(z \$+ w' = z \$+ w) \longleftrightarrow intify(w') = intify(w)$
apply (*rule iff-trans*)
apply (*rule-tac* [*2*] *zadd-left-cancel*, *auto*)
done

lemma *zadd-right-cancel*:
$[\![w \in int; w': int]\!] \Longrightarrow (w' \$+ z = w \$+ z) \longleftrightarrow (w' = w)$
apply *safe*
apply (*drule-tac t* = $\lambda x.\ x$ \$+ (\$−*z*) **in** *subst-context*)
apply (*simp add*: *zadd-ac*)
done

lemma *zadd-right-cancel-intify* [*simp*]:
$(w' \$+ z = w \$+ z) \longleftrightarrow intify(w') = intify(w)$
apply (*rule iff-trans*)
apply (*rule-tac* [*2*] *zadd-right-cancel*, *auto*)
done

lemma *zadd-right-cancel-zless* [*simp*]: $(w' \$+ z \$< w \$+ z) \longleftrightarrow (w' \$< w)$
by (*simp add*: *zdiff-zless-iff* [*THEN iff-sym*] *zdiff-def zadd-assoc*)

lemma *zadd-left-cancel-zless* [*simp*]: $(z \$+ w' \$< z \$+ w) \longleftrightarrow (w' \$< w)$
by (*simp add*: *zadd-commute* [*of z*] *zadd-right-cancel-zless*)

lemma *zadd-right-cancel-zle* [*simp*]: $(w' \$+ z \$\le w \$+ z) \longleftrightarrow w' \$\le w$
by (*simp add*: *zle-def*)

lemma *zadd-left-cancel-zle* [*simp*]: $(z \$+ w' \$\le z \$+ w) \longleftrightarrow w' \$\le w$
by (*simp add*: *zadd-commute* [*of z*] *zadd-right-cancel-zle*)

lemmas *zadd-zless-mono1* = *zadd-right-cancel-zless* [*THEN iffD2*]

lemmas *zadd-zless-mono2* = *zadd-left-cancel-zless* [*THEN iffD2*]

lemmas *zadd-zle-mono1* = *zadd-right-cancel-zle* [*THEN iffD2*]

lemmas *zadd-zle-mono2* = *zadd-left-cancel-zle* [*THEN iffD2*]

lemma *zadd-zle-mono*: $[\![w' \$\leq w; z' \$\leq z]\!] \Longrightarrow w' \$+ z' \$\leq w \$+ z$
by (*erule zadd-zle-mono1* [*THEN zle-trans*], *simp*)

lemma *zadd-zless-mono*: $[\![w' \$< w; z' \$\leq z]\!] \Longrightarrow w' \$+ z' \$< w \$+ z$
by (*erule zadd-zless-mono1* [*THEN zless-zle-trans*], *simp*)

31.13 Comparison laws

lemma *zminus-zless-zminus* [*simp*]: $(\$- x \$< \$- y) \longleftrightarrow (y \$< x)$
by (*simp add*: *zless-def zdiff-def zadd-ac*)

lemma *zminus-zle-zminus* [*simp*]: $(\$- x \$\leq \$- y) \longleftrightarrow (y \$\leq x)$
by (*simp add*: *not-zless-iff-zle* [*THEN iff-sym*])

31.13.1 More inequality lemmas

lemma *equation-zminus*: $[\![x \in int;\ y \in int]\!] \Longrightarrow (x = \$- y) \longleftrightarrow (y = \$- x)$
by *auto*

lemma *zminus-equation*: $[\![x \in int;\ y \in int]\!] \Longrightarrow (\$- x = y) \longleftrightarrow (\$- y = x)$
by *auto*

lemma *equation-zminus-intify*: $(intify(x) = \$- y) \longleftrightarrow (intify(y) = \$- x)$
apply (*cut-tac* $x = intify\ (x)$ **and** $y = intify\ (y)$ **in** *equation-zminus*)
apply *auto*
done

lemma *zminus-equation-intify*: $(\$- x = intify(y)) \longleftrightarrow (\$- y = intify(x))$
apply (*cut-tac* $x = intify\ (x)$ **and** $y = intify\ (y)$ **in** *zminus-equation*)
apply *auto*
done

31.13.2 The next several equations are permutative: watch out!

lemma *zless-zminus*: $(x \$< \$- y) \longleftrightarrow (y \$< \$- x)$
by (*simp add*: *zless-def zdiff-def zadd-ac*)

lemma *zminus-zless*: $(\$- x \$< y) \longleftrightarrow (\$- y \$< x)$
by (*simp add*: *zless-def zdiff-def zadd-ac*)

lemma *zle-zminus*: $(x \$\leq \$- y) \longleftrightarrow (y \$\leq \$- x)$
by (*simp add*: *not-zless-iff-zle* [*THEN iff-sym*] *zminus-zless*)

lemma *zminus-zle*: ($− x $≤ y) ⟷ ($− y $≤ x)
by (*simp add*: *not-zless-iff-zle* [*THEN iff-sym*] *zless-zminus*)

end

32 Arithmetic on Binary Integers

theory *Bin*
imports *Int Datatype*
begin

consts *bin* :: *i*
datatype
  bin = *Pls*
    | *Min*
    | *Bit* (*w* ∈ *bin*, *b* ∈ *bool*)   (**infixl** ‹*BIT*› *90*)

consts
  integ-of :: *i*⇒*i*
  NCons :: [*i*,*i*]⇒*i*
  bin-succ :: *i*⇒*i*
  bin-pred :: *i*⇒*i*
  bin-minus :: *i*⇒*i*
  bin-adder :: *i*⇒*i*
  bin-mult :: [*i*,*i*]⇒*i*

primrec
  integ-of-Pls: *integ-of* (*Pls*) = $# *0*
  integ-of-Min: *integ-of* (*Min*) = $−($#*1*)
  integ-of-BIT: *integ-of* (*w BIT b*) = $#*b* $+ *integ-of*(*w*) $+ *integ-of*(*w*)

primrec
  NCons-Pls: *NCons* (*Pls*,*b*) = *cond*(*b*,*Pls BIT b*,*Pls*)
  NCons-Min: *NCons* (*Min*,*b*) = *cond*(*b*,*Min*,*Min BIT b*)
  NCons-BIT: *NCons* (*w BIT c*,*b*) = *w BIT c BIT b*

primrec
  bin-succ-Pls: *bin-succ* (*Pls*) = *Pls BIT 1*
  bin-succ-Min: *bin-succ* (*Min*) = *Pls*
  bin-succ-BIT: *bin-succ* (*w BIT b*) = *cond*(*b*, *bin-succ*(*w*) *BIT 0*, *NCons*(*w*,*1*))

primrec
  bin-pred-Pls: *bin-pred* (*Pls*) = *Min*
  bin-pred-Min: *bin-pred* (*Min*) = *Min BIT 0*
  bin-pred-BIT: *bin-pred* (*w BIT b*) = *cond*(*b*, *NCons*(*w*,*0*), *bin-pred*(*w*) *BIT 1*)

primrec

bin-minus-Pls:
  bin-minus (*Pls*) = *Pls*
bin-minus-Min:
  bin-minus (*Min*) = *Pls BIT 1*
bin-minus-BIT:
  bin-minus (*w BIT b*) = *cond*(*b*, *bin-pred*(*NCons*(*bin-minus*(*w*),*0*)),
  bin-minus(*w*) *BIT 0*)

primrec
bin-adder-Pls:
  bin-adder (*Pls*) = ($\lambda w \in bin.\ w$)
bin-adder-Min:
  bin-adder (*Min*) = ($\lambda w \in bin.$ *bin-pred*(*w*))
bin-adder-BIT:
  bin-adder (*v BIT x*) =
    ($\lambda w \in bin.$
      bin-case (*v BIT x*, *bin-pred*(*v BIT x*),
        $\lambda w\ y.$ *NCons*(*bin-adder* (*v*) ‘ *cond*(*x and y*, *bin-succ*(*w*), *w*),
          x xor y),
        w))

definition
bin-add :: [*i*,*i*]⇒*i* **where**
  bin-add(*v*,*w*) ≡ *bin-adder*(*v*)‘*w*

primrec
bin-mult-Pls:
  bin-mult (*Pls*,*w*) = *Pls*
bin-mult-Min:
  bin-mult (*Min*,*w*) = *bin-minus*(*w*)
bin-mult-BIT:
  bin-mult (*v BIT b*,*w*) = *cond*(*b*, *bin-add*(*NCons*(*bin-mult*(*v*,*w*),*0*),*w*),
  NCons(*bin-mult*(*v*,*w*),*0*))

syntax
-Int0 :: *i* (‹#()*0*›)
-Int1 :: *i* (‹#()*1*›)
-Int2 :: *i* (‹#()*2*›)
-Neg-Int1 :: *i* (‹#−()*1*›)
-Neg-Int2 :: *i* (‹#−()*2*›)
translations
#0 ⇌ *CONST integ-of*(*CONST Pls*)
#1 ⇌ *CONST integ-of*(*CONST Pls BIT 1*)
#2 ⇌ *CONST integ-of*(*CONST Pls BIT 1 BIT 0*)
#−1 ⇌ *CONST integ-of*(*CONST Min*)
#−2 ⇌ *CONST integ-of*(*CONST Min BIT 0*)

syntax
-Int :: *num-token* ⇒ *i* (‹(‹*open-block notation*=‹*literal number*››#-)› *1000*)
-Neg-Int :: *num-token* ⇒ *i* (‹(‹*open-block notation*=‹*literal number*››#−-)› *1000*)

syntax-consts
-Int0 -Int1 -Int2 -Int -Neg-Int1 -Neg-Int2 -Neg-Int ⇌ *integ-of*

ML-file ‹*Tools/numeral-syntax.ML*›

declare *bin.intros* [*simp*, *TC*]

lemma *NCons-Pls-0*: *NCons*(*Pls*,*0*) = *Pls*
by *simp*

lemma *NCons-Pls-1*: *NCons*(*Pls*,*1*) = *Pls BIT 1*
by *simp*

lemma *NCons-Min-0*: *NCons*(*Min*,*0*) = *Min BIT 0*
by *simp*

lemma *NCons-Min-1*: *NCons*(*Min*,*1*) = *Min*
by *simp*

lemma *NCons-BIT*: *NCons*(*w BIT x*,*b*) = *w BIT x BIT b*
by (*simp add*: *bin.case-eqns*)

lemmas *NCons-simps* [*simp*] =
NCons-Pls-0 NCons-Pls-1 NCons-Min-0 NCons-Min-1 NCons-BIT

lemma *integ-of-type* [*TC*]: *w* ∈ *bin* ⟹ *integ-of*(*w*) ∈ *int*
apply (*induct-tac w*)
apply (*simp-all add*: *bool-into-nat*)
done

lemma *NCons-type* [*TC*]: ⟦*w* ∈ *bin*; *b* ∈ *bool*⟧ ⟹ *NCons*(*w*,*b*) ∈ *bin*
by (*induct-tac w*, *auto*)

lemma *bin-succ-type* [*TC*]: *w* ∈ *bin* ⟹ *bin-succ*(*w*) ∈ *bin*
by (*induct-tac w*, *auto*)

lemma *bin-pred-type* [*TC*]: *w* ∈ *bin* ⟹ *bin-pred*(*w*) ∈ *bin*
by (*induct-tac w*, *auto*)

lemma *bin-minus-type* [*TC*]: $w \in bin \Longrightarrow bin\text{-}minus(w) \in bin$
by (*induct-tac w*, *auto*)

lemma *bin-add-type* [*rule-format*]:
$v \in bin \Longrightarrow \forall w \in bin.\ bin\text{-}add(v,w) \in bin$
unfolding *bin-add-def*
apply (*induct-tac v*)
apply (*rule-tac* [*3*] *ballI*)
apply (*rename-tac* [*3*] *w'*)
apply (*induct-tac* [*3*] *w'*)
apply (*simp-all add*: *NCons-type*)
done

declare *bin-add-type* [*TC*]

lemma *bin-mult-type* [*TC*]: $[\![v \in bin;\ w \in bin]\!] \Longrightarrow bin\text{-}mult(v,w) \in bin$
by (*induct-tac v*, *auto*)

32.0.1 The Carry and Borrow Functions, *bin-succ* and *bin-pred*

lemma *integ-of-NCons* [*simp*]:
$[\![w \in bin;\ b \in bool]\!] \Longrightarrow integ\text{-}of(NCons(w,b)) = integ\text{-}of(w\ BIT\ b)$
apply (*erule bin.cases*)
apply (*auto elim*!: *boolE*)
done

lemma *integ-of-succ* [*simp*]:
$w \in bin \Longrightarrow integ\text{-}of(bin\text{-}succ(w)) = \$\#1\ \$+\ integ\text{-}of(w)$
apply (*erule bin.induct*)
apply (*auto simp add*: *zadd-ac elim*!: *boolE*)
done

lemma *integ-of-pred* [*simp*]:
$w \in bin \Longrightarrow integ\text{-}of(bin\text{-}pred(w)) = \$-\ (\$\#1)\ \$+\ integ\text{-}of(w)$
apply (*erule bin.induct*)
apply (*auto simp add*: *zadd-ac elim*!: *boolE*)
done

32.0.2 *bin-minus*: Unary Negation of Binary Integers

lemma *integ-of-minus*: $w \in bin \Longrightarrow integ\text{-}of(bin\text{-}minus(w)) = \$-\ integ\text{-}of(w)$
apply (*erule bin.induct*)
apply (*auto simp add*: *zadd-ac zminus-zadd-distrib elim*!: *boolE*)
done

32.0.3 *bin-add*: Binary Addition

lemma *bin-add-Pls* [*simp*]: $w \in bin \Longrightarrow bin\text{-}add(Pls,w) = w$
by (*unfold bin-add-def*, *simp*)

lemma *bin-add-Pls-right*: $w \in bin \Longrightarrow bin\text{-}add(w,Pls) = w$
  unfolding *bin-add-def*
apply (*erule bin.induct*, *auto*)
done

lemma *bin-add-Min* [*simp*]: $w \in bin \Longrightarrow bin\text{-}add(Min,w) = bin\text{-}pred(w)$
by (*unfold bin-add-def*, *simp*)

lemma *bin-add-Min-right*: $w \in bin \Longrightarrow bin\text{-}add(w,Min) = bin\text{-}pred(w)$
  unfolding *bin-add-def*
apply (*erule bin.induct*, *auto*)
done

lemma *bin-add-BIT-Pls* [*simp*]: $bin\text{-}add(v\ BIT\ x,Pls) = v\ BIT\ x$
by (*unfold bin-add-def*, *simp*)

lemma *bin-add-BIT-Min* [*simp*]: $bin\text{-}add(v\ BIT\ x,Min) = bin\text{-}pred(v\ BIT\ x)$
by (*unfold bin-add-def*, *simp*)

lemma *bin-add-BIT-BIT* [*simp*]:
  $[\![w \in bin;\ y \in bool]\!]$
  $\Longrightarrow bin\text{-}add(v\ BIT\ x,\ w\ BIT\ y) =$
    $NCons(bin\text{-}add(v,\ cond(x\ and\ y,\ bin\text{-}succ(w),\ w)),\ x\ xor\ y)$
by (*unfold bin-add-def*, *simp*)

lemma *integ-of-add* [*rule-format*]:
  $v \in bin \Longrightarrow$
    $\forall w{\in}bin.\ integ\text{-}of(bin\text{-}add(v,w)) = integ\text{-}of(v)\ \$+\ integ\text{-}of(w)$
apply (*erule bin.induct*, *simp*, *simp*)
apply (*rule ballI*)
apply (*induct-tac wa*)
apply (*auto simp add*: *zadd-ac elim*!: *boolE*)
done

lemma *diff-integ-of-eq*:
  $[\![v \in bin;\ w \in bin]\!]$
  $\Longrightarrow integ\text{-}of(v)\ \$-\ integ\text{-}of(w) = integ\text{-}of(bin\text{-}add\ (v,\ bin\text{-}minus(w)))$
  unfolding *zdiff-def*
apply (*simp add*: *integ-of-add integ-of-minus*)
done

32.0.4 *bin-mult*: Binary Multiplication

lemma *integ-of-mult*:
  $[\![v \in bin;\ w \in bin]\!]$
  $\Longrightarrow integ\text{-}of(bin\text{-}mult(v,w)) = integ\text{-}of(v)\ \$*\ integ\text{-}of(w)$
apply (*induct-tac v*, *simp*)

apply (*simp add*: *integ-of-minus*)
apply (*auto simp add*: *zadd-ac integ-of-add zadd-zmult-distrib elim*!: *boolE*)
done

32.1 Computations

lemma *bin-succ-1*: *bin-succ*(*w BIT 1*) = *bin-succ*(*w*) *BIT 0*
by *simp*

lemma *bin-succ-0*: *bin-succ*(*w BIT 0*) = *NCons*(*w*,*1*)
by *simp*

lemma *bin-pred-1*: *bin-pred*(*w BIT 1*) = *NCons*(*w*,*0*)
by *simp*

lemma *bin-pred-0*: *bin-pred*(*w BIT 0*) = *bin-pred*(*w*) *BIT 1*
by *simp*

lemma *bin-minus-1*: *bin-minus*(*w BIT 1*) = *bin-pred*(*NCons*(*bin-minus*(*w*), *0*))
by *simp*

lemma *bin-minus-0*: *bin-minus*(*w BIT 0*) = *bin-minus*(*w*) *BIT 0*
by *simp*

lemma *bin-add-BIT-11*: *w* ∈ *bin* ⟹ *bin-add*(*v BIT 1*, *w BIT 1*) =
NCons(*bin-add*(*v*, *bin-succ*(*w*)), *0*)
by *simp*

lemma *bin-add-BIT-10*: *w* ∈ *bin* ⟹ *bin-add*(*v BIT 1*, *w BIT 0*) =
NCons(*bin-add*(*v*,*w*), *1*)
by *simp*

lemma *bin-add-BIT-0*: ⟦*w* ∈ *bin*; *y* ∈ *bool*⟧
⟹ *bin-add*(*v BIT 0*, *w BIT y*) = *NCons*(*bin-add*(*v*,*w*), *y*)
by *simp*

lemma *bin-mult-1*: *bin-mult*(*v BIT 1*, *w*) = *bin-add*(*NCons*(*bin-mult*(*v*,*w*),*0*), *w*)
by *simp*

lemma *bin-mult-0*: *bin-mult*(*v BIT 0*, *w*) = *NCons*(*bin-mult*(*v*,*w*),*0*)
by *simp*

lemma *int-of-0*: $#*0* = #*0*
by *simp*

lemma *int-of-succ*: $# *succ*(*n*) = #*1* $+ $#*n*
by (*simp add*: *int-of-add* [*symmetric*] *natify-succ*)

lemma *zminus-0* [*simp*]: $− #*0* = #*0*
by *simp*

lemma *zadd-0-intify* [*simp*]: #*0* $+ *z* = *intify*(*z*)
by *simp*

lemma *zadd-0-right-intify* [*simp*]: *z* $+ #*0* = *intify*(*z*)
by *simp*

lemma *zmult-1-intify* [*simp*]: #*1* $∗ *z* = *intify*(*z*)
by *simp*

lemma *zmult-1-right-intify* [*simp*]: *z* $∗ #*1* = *intify*(*z*)
by (*subst zmult-commute*, *simp*)

lemma *zmult-0* [*simp*]: #*0* $∗ *z* = #*0*
by *simp*

lemma *zmult-0-right* [*simp*]: *z* $∗ #*0* = #*0*
by (*subst zmult-commute*, *simp*)

lemma *zmult-minus1* [*simp*]: #−*1* $∗ *z* = $−*z*
by (*simp add*: *zcompare-rls*)

lemma *zmult-minus1-right* [*simp*]: *z* $∗ #−*1* = $−*z*
apply (*subst zmult-commute*)
apply (*rule zmult-minus1*)
done

32.2 Simplification Rules for Comparison of Binary Numbers

Thanks to Norbert Voelker

lemma *eq-integ-of-eq*:
  [[*v* ∈ *bin*; *w* ∈ *bin*]]
  =⇒ ((*integ-of*(*v*)) = *integ-of*(*w*)) ←→
    iszero (*integ-of* (*bin-add* (*v*, *bin-minus*(*w*))))
 unfolding *iszero-def*
apply (*simp add*: *zcompare-rls integ-of-add integ-of-minus*)
done

lemma *iszero-integ-of-Pls*: *iszero* (*integ-of*(*Pls*))
by (*unfold iszero-def*, *simp*)

lemma *nonzero-integ-of-Min*: ¬ *iszero* (*integ-of*(*Min*))
unfolding *iszero-def*
apply (*simp add*: *zminus-equation*)
done

lemma *iszero-integ-of-BIT*:
⟦*w* ∈ *bin*; *x* ∈ *bool*⟧
⟹ *iszero* (*integ-of* (*w BIT x*)) ⟷ (*x=0* ∧ *iszero* (*integ-of*(*w*)))
apply (*unfold iszero-def*, *simp*)
apply (*subgoal-tac integ-of* (*w*) ∈ *int*)
apply *typecheck*
apply (*drule int-cases*)
apply (*safe elim*!: *boolE*)
apply (*simp-all* (*asm-lr*) *add*: *zcompare-rls zminus-zadd-distrib* [*symmetric*]
int-of-add [*symmetric*])
done

lemma *iszero-integ-of-0*:
w ∈ *bin* ⟹ *iszero* (*integ-of* (*w BIT 0*)) ⟷ *iszero* (*integ-of*(*w*))
by (*simp only*: *iszero-integ-of-BIT*, *blast*)

lemma *iszero-integ-of-1*: *w* ∈ *bin* ⟹ ¬ *iszero* (*integ-of* (*w BIT 1*))
by (*simp only*: *iszero-integ-of-BIT*, *blast*)

lemma *less-integ-of-eq-neg*:
⟦*v* ∈ *bin*; *w* ∈ *bin*⟧
⟹ *integ-of*(*v*) $< *integ-of*(*w*)
⟷ *znegative* (*integ-of* (*bin-add* (*v*, *bin-minus*(*w*))))
unfolding *zless-def zdiff-def*
apply (*simp add*: *integ-of-minus integ-of-add*)
done

lemma *not-neg-integ-of-Pls*: ¬ *znegative* (*integ-of*(*Pls*))
by *simp*

lemma *neg-integ-of-Min*: *znegative* (*integ-of*(*Min*))
by *simp*

lemma *neg-integ-of-BIT*:
⟦*w* ∈ *bin*; *x* ∈ *bool*⟧
⟹ *znegative* (*integ-of* (*w BIT x*)) ⟷ *znegative* (*integ-of*(*w*))

apply *simp*
apply (*subgoal-tac integ-of* (*w*) ∈ *int*)
apply *typecheck*
apply (*drule int-cases*)
apply (*auto elim*!: *boolE simp add*: *int-of-add* [*symmetric*] *zcompare-rls*)
apply (*simp-all add*: *zminus-zadd-distrib* [*symmetric*] *zdiff-def*
int-of-add [*symmetric*])
apply (*subgoal-tac* $#*1* $− $# *succ* (*succ* (*n* #+ *n*)) = $− $# *succ* (*n* #+ *n*))
apply (*simp add*: *zdiff-def*)
apply (*simp add*: *equation-zminus int-of-diff* [*symmetric*])
done

lemma *le-integ-of-eq-not-less*:
(*integ-of*(*x*) $≤ (*integ-of*(*w*))) ⟷ ¬ (*integ-of*(*w*) $< (*integ-of*(*x*)))
by (*simp add*: *not-zless-iff-zle* [*THEN iff-sym*])

declare *bin-succ-BIT* [*simp del*]
bin-pred-BIT [*simp del*]
bin-minus-BIT [*simp del*]
NCons-Pls [*simp del*]
NCons-Min [*simp del*]
bin-adder-BIT [*simp del*]
bin-mult-BIT [*simp del*]

declare *integ-of-Pls* [*simp del*] *integ-of-Min* [*simp del*] *integ-of-BIT* [*simp del*]

lemmas *bin-arith-extra-simps* =
integ-of-add [*symmetric*]
integ-of-minus [*symmetric*]
integ-of-mult [*symmetric*]
bin-succ-1 bin-succ-0
bin-pred-1 bin-pred-0
bin-minus-1 bin-minus-0
bin-add-Pls-right bin-add-Min-right
bin-add-BIT-0 bin-add-BIT-10 bin-add-BIT-11
diff-integ-of-eq
bin-mult-1 bin-mult-0 NCons-simps

lemmas *bin-arith-simps* =
bin-pred-Pls bin-pred-Min
bin-succ-Pls bin-succ-Min

bin-add-Pls bin-add-Min
bin-minus-Pls bin-minus-Min
bin-mult-Pls bin-mult-Min
bin-arith-extra-simps

lemmas *bin-rel-simps* =
eq-integ-of-eq iszero-integ-of-Pls nonzero-integ-of-Min
iszero-integ-of-0 iszero-integ-of-1
less-integ-of-eq-neg
not-neg-integ-of-Pls neg-integ-of-Min neg-integ-of-BIT
le-integ-of-eq-not-less

declare *bin-arith-simps* [*simp*]
declare *bin-rel-simps* [*simp*]

lemma *add-integ-of-left* [*simp*]:
⟦*v* ∈ *bin*; *w* ∈ *bin*⟧
⟹ *integ-of*(*v*) $+ (*integ-of*(*w*) $+ *z*) = (*integ-of*(*bin-add*(*v*,*w*)) $+ *z*)
by (*simp add*: *zadd-assoc* [*symmetric*])

lemma *mult-integ-of-left* [*simp*]:
⟦*v* ∈ *bin*; *w* ∈ *bin*⟧
⟹ *integ-of*(*v*) $∗ (*integ-of*(*w*) $∗ *z*) = (*integ-of*(*bin-mult*(*v*,*w*)) $∗ *z*)
by (*simp add*: *zmult-assoc* [*symmetric*])

lemma *add-integ-of-diff1* [*simp*]:
⟦*v* ∈ *bin*; *w* ∈ *bin*⟧
⟹ *integ-of*(*v*) $+ (*integ-of*(*w*) $− *c*) = *integ-of*(*bin-add*(*v*,*w*)) $− (*c*)
unfolding *zdiff-def*
apply (*rule add-integ-of-left*, *auto*)
done

lemma *add-integ-of-diff2* [*simp*]:
⟦*v* ∈ *bin*; *w* ∈ *bin*⟧
⟹ *integ-of*(*v*) $+ (*c* $− *integ-of*(*w*)) =
integ-of (*bin-add* (*v*, *bin-minus*(*w*))) $+ (*c*)
apply (*subst diff-integ-of-eq* [*symmetric*])
apply (*simp-all add*: *zdiff-def zadd-ac*)
done

declare *int-of-0* [*simp*] *int-of-succ* [*simp*]

lemma *zdiff0* [*simp*]: *#0* $\$-$ *x* = $\$-x$
by (*simp add*: *zdiff-def*)

lemma *zdiff0-right* [*simp*]: *x* $\$-$ *#0* = *intify*(*x*)
by (*simp add*: *zdiff-def*)

lemma *zdiff-self* [*simp*]: *x* $\$-$ *x* = *#0*
by (*simp add*: *zdiff-def*)

lemma *znegative-iff-zless-0*: *k* ∈ *int* ⟹ *znegative*(*k*) ⟷ *k* $< *#0*
by (*simp add*: *zless-def*)

lemma *zero-zless-imp-znegative-zminus*: ⟦*#0* $< *k*; *k* ∈ *int*⟧ ⟹ *znegative*($\$-k$)
by (*simp add*: *zless-def*)

lemma *zero-zle-int-of* [*simp*]: *#0* $≤ $# *n*
by (*simp add*: *not-zless-iff-zle* [*THEN iff-sym*] *znegative-iff-zless-0* [*THEN iff-sym*])

lemma *nat-of-0* [*simp*]: *nat-of*(*#0*) = *0*
by (*simp only*: *natify-0 int-of-0* [*symmetric*] *nat-of-int-of*)

lemma *nat-le-int0-lemma*: ⟦*z* $≤ $*#0*; *z* ∈ *int*⟧ ⟹ *nat-of*(*z*) = *0*
by (*auto simp add*: *znegative-iff-zless-0* [*THEN iff-sym*] *zle-def zneg-nat-of*)

lemma *nat-le-int0*: *z* $≤ $*#0* ⟹ *nat-of*(*z*) = *0*
apply (*subgoal-tac nat-of* (*intify* (*z*)) = *0*)
apply (*rule-tac* [*2*] *nat-le-int0-lemma*, *auto*)
done

lemma *int-of-eq-0-imp-natify-eq-0*: $# *n* = *#0* ⟹ *natify*(*n*) = *0*
by (*rule not-znegative-imp-zero*, *auto*)

lemma *nat-of-zminus-int-of*: *nat-of*($\$-$ $# *n*) = *0*
by (*simp add*: *nat-of-def int-of-def raw-nat-of zminus image-intrel-int*)

lemma *int-of-nat-of*: *#0* $≤ *z* ⟹ $# *nat-of*(*z*) = *intify*(*z*)
apply (*rule not-zneg-nat-of-intify*)
apply (*simp add*: *znegative-iff-zless-0 not-zless-iff-zle*)
done

declare *int-of-nat-of* [*simp*] *nat-of-zminus-int-of* [*simp*]

lemma *int-of-nat-of-if*: $# *nat-of*(*z*) = (*if #0* $≤ *z then intify*(*z*) *else #0*)
by (*simp add*: *int-of-nat-of znegative-iff-zless-0 not-zle-iff-zless*)

lemma *zless-nat-iff-int-zless*: ⟦*m* ∈ *nat*; *z* ∈ *int*⟧ ⟹ (*m* < *nat-of*(*z*)) ⟷ ($#*m*
$< *z*)
apply (*case-tac znegative* (*z*))
apply (*erule-tac* [*2*] *not-zneg-nat-of* [*THEN subst*])

apply (*auto dest*: *zless-trans dest*!: *zero-zle-int-of* [*THEN zle-zless-trans*]
simp add: *znegative-iff-zless-0*)
done

lemma *zless-nat-conj-lemma*: $#*0* $< *z* ⟹ (*nat-of*(*w*) < *nat-of*(*z*)) ⟷ (*w* $< *z*)
apply (*rule iff-trans*)
apply (*rule zless-int-of* [*THEN iff-sym*])
apply (*auto simp add*: *int-of-nat-of-if simp del*: *zless-int-of*)
apply (*auto elim*: *zless-asym simp add*: *not-zle-iff-zless*)
apply (*blast intro*: *zless-zle-trans*)
done

lemma *zless-nat-conj*: (*nat-of*(*w*) < *nat-of*(*z*)) ⟷ ($#*0* $< *z* ∧ *w* $< *z*)
apply (*case-tac* $#*0* $< *z*)
apply (*auto simp add*: *zless-nat-conj-lemma nat-le-int0 not-zless-iff-zle*)
done

lemma *integ-of-minus-reorient* [*simp*]:
(*integ-of*(*w*) = $− *x*) ⟷ ($− *x* = *integ-of*(*w*))
by *auto*

lemma *integ-of-add-reorient* [*simp*]:
(*integ-of*(*w*) = *x* $+ *y*) ⟷ (*x* $+ *y* = *integ-of*(*w*))
by *auto*

lemma *integ-of-diff-reorient* [*simp*]:
(*integ-of*(*w*) = *x* $− *y*) ⟷ (*x* $− *y* = *integ-of*(*w*))
by *auto*

lemma *integ-of-mult-reorient* [*simp*]:
(*integ-of*(*w*) = *x* $∗ *y*) ⟷ (*x* $∗ *y* = *integ-of*(*w*))
by *auto*

lemmas [*simp*] =
zminus-equation [**where** *y* = *integ-of*(*w*)]
equation-zminus [**where** *x* = *integ-of*(*w*)]
for *w*

lemmas [*iff*] =
zminus-zless [**where** *y* = *integ-of*(*w*)]

zless-zminus [**where** $x = integ\text{-}of(w)$]
for w

lemmas [*iff*] =
zminus-zle [**where** $y = integ\text{-}of(w)$]
zle-zminus [**where** $x = integ\text{-}of(w)$]
for w

lemmas [*simp*] =
Let-def [**where** $s = integ\text{-}of(w)$] **for** w

lemma *zless-iff-zdiff-zless-0*: $(x \ \$< \ y) \longleftrightarrow (x\$-y \ \$< \ \#0)$
by (*simp add*: *zcompare-rls*)

lemma *eq-iff-zdiff-eq-0*: $[\![x \in int;\ y \in int]\!] \Longrightarrow (x = y) \longleftrightarrow (x\$-y = \#0)$
by (*simp add*: *zcompare-rls*)

lemma *zle-iff-zdiff-zle-0*: $(x \ \$\leq \ y) \longleftrightarrow (x\$-y \ \$\leq \ \#0)$
by (*simp add*: *zcompare-rls*)

lemma *left-zadd-zmult-distrib*: $i\$*u \ \$+ \ (j\$*u \ \$+ \ k) = (i\$+j)\$*u \ \$+ \ k$
by (*simp add*: *zadd-zmult-distrib zadd-ac*)

lemma *eq-add-iff1*: $(i\$*u \ \$+ \ m = j\$*u \ \$+ \ n) \longleftrightarrow ((i\$-j)\$*u \ \$+ \ m = intify(n))$
apply (*simp add*: *zdiff-def zadd-zmult-distrib*)
apply (*simp add*: *zcompare-rls*)
apply (*simp add*: *zadd-ac*)
done

lemma *eq-add-iff2*: $(i\$*u \ \$+ \ m = j\$*u \ \$+ \ n) \longleftrightarrow (intify(m) = (j\$-i)\$*u \ \$+ \ n)$
apply (*simp add*: *zdiff-def zadd-zmult-distrib*)
apply (*simp add*: *zcompare-rls*)
apply (*simp add*: *zadd-ac*)
done

context fixes $n :: i$
begin

lemmas *rel-iff-rel-0-rls* =
zless-iff-zdiff-zless-0 [**where** $y = u \$+ v$]
eq-iff-zdiff-eq-0 [**where** $y = u \$+ v$]
zle-iff-zdiff-zle-0 [**where** $y = u \$+ v$]
zless-iff-zdiff-zless-0 [**where** $y = n$]
eq-iff-zdiff-eq-0 [**where** $y = n$]
zle-iff-zdiff-zle-0 [**where** $y = n$]
for *u v*

lemma *less-add-iff1*: $(i\$*u \$+ m \$< j\$*u \$+ n) \longleftrightarrow ((i\$-j)\$*u \$+ m \$< n)$
apply (*simp add*: *zdiff-def zadd-zmult-distrib zadd-ac rel-iff-rel-0-rls*)
done

lemma *less-add-iff2*: $(i\$*u \$+ m \$< j\$*u \$+ n) \longleftrightarrow (m \$< (j\$-i)\$*u \$+ n)$
apply (*simp add*: *zdiff-def zadd-zmult-distrib zadd-ac rel-iff-rel-0-rls*)
done

end

lemma *le-add-iff1*: $(i\$*u \$+ m \$\le j\$*u \$+ n) \longleftrightarrow ((i\$-j)\$*u \$+ m \$\le n)$
apply (*simp add*: *zdiff-def zadd-zmult-distrib*)
apply (*simp add*: *zcompare-rls*)
apply (*simp add*: *zadd-ac*)
done

lemma *le-add-iff2*: $(i\$*u \$+ m \$\le j\$*u \$+ n) \longleftrightarrow (m \$\le (j\$-i)\$*u \$+ n)$
apply (*simp add*: *zdiff-def zadd-zmult-distrib*)
apply (*simp add*: *zcompare-rls*)
apply (*simp add*: *zadd-ac*)
done

ML-file ‹*int-arith.ML*›

simproc-setup *inteq-cancel-numerals*
$(l \$+ m = n \mid l = m \$+ n \mid l \$- m = n \mid l = m \$- n \mid l \$* m = n \mid l = m \$* n) =$
‹*K Int-Numeral-Simprocs.inteq-cancel-numerals-proc*›

simproc-setup *intless-cancel-numerals*
$(l \$+ m \$< n \mid l \$< m \$+ n \mid l \$- m \$< n \mid l \$< m \$- n \mid l \$* m \$< n \mid l \$< m \$* n) =$
‹*K Int-Numeral-Simprocs.intless-cancel-numerals-proc*›

simproc-setup *intle-cancel-numerals*
$(l \$+ m \$\le n \mid l \$\le m \$+ n \mid l \$- m \$\le n \mid l \$\le m \$- n \mid l \$* m \$\le n \mid l \$\le m \$* n) =$
‹*K Int-Numeral-Simprocs.intle-cancel-numerals-proc*›

simproc-setup *int-combine-numerals* $(i \$+ j \mid i \$- j) =$

‹K Int-Numeral-Simprocs.int-combine-numerals-proc›

simproc-setup *int-combine-numerals-prod* (i $∗ j) =
‹K Int-Numeral-Simprocs.int-combine-numerals-prod-proc›

32.2.1 Examples

combine-numerals-prod (products of separate literals)

lemma #5 $∗ x $∗ #3 = y **apply** *simp* **oops**

schematic-goal y2 $+ ?x42 = y $+ y2 **apply** *simp* **oops**

lemma oo : int ⟹ l $+ (l $+ #2) $+ oo = oo **apply** *simp* **oops**

lemma #9$∗x $+ y = x$∗#23 $+ z **apply** *simp* **oops**
lemma y $+ x = x $+ z **apply** *simp* **oops**

lemma x : int ⟹ x $+ y $+ z = x $+ z **apply** *simp* **oops**
lemma x : int ⟹ y $+ (z $+ x) = z $+ x **apply** *simp* **oops**
lemma z : int ⟹ x $+ y $+ z = (z $+ y) $+ (x $+ w) **apply** *simp* **oops**
lemma z : int ⟹ x$∗y $+ z = (z $+ y) $+ (y$∗x $+ w) **apply** *simp* **oops**

lemma #−3 $∗ x $+ y $≤ x $∗ #2 $+ z **apply** *simp* **oops**
lemma y $+ x $≤ x $+ z **apply** *simp* **oops**
lemma x $+ y $+ z $≤ x $+ z **apply** *simp* **oops**

lemma y $+ (z $+ x) $< z $+ x **apply** *simp* **oops**
lemma x $+ y $+ z $< (z $+ y) $+ (x $+ w) **apply** *simp* **oops**
lemma x$∗y $+ z $< (z $+ y) $+ (y$∗x $+ w) **apply** *simp* **oops**

lemma l $+ #2 $+ #2 $+ #2 $+ (l $+ #2) $+ (oo $+ #2) = uu **apply** *simp* **oops**
lemma u : int ⟹ #2 $∗ u = u **apply** *simp* **oops**
lemma (i $+ j $+ #12 $+ k) $− #15 = y **apply** *simp* **oops**
lemma (i $+ j $+ #12 $+ k) $− #5 = y **apply** *simp* **oops**

lemma y $− b $< b **apply** *simp* **oops**
lemma y $− (#3 $∗ b $+ c) $< b $− #2 $∗ c **apply** *simp* **oops**

lemma (#2 $∗ x $− (u $∗ v) $+ y) $− v $∗ #3 $∗ u = w **apply** *simp* **oops**
lemma (#2 $∗ x $∗ u $∗ v $+ (u $∗ v) $∗ #4 $+ y) $− v $∗ u $∗ #4 = w **apply** *simp* **oops**
lemma (#2 $∗ x $∗ u $∗ v $+ (u $∗ v) $∗ #4 $+ y) $− v $∗ u = w **apply** *simp* **oops**
lemma u $∗ v $− (x $∗ u $∗ v $+ (u $∗ v) $∗ #4 $+ y) = w **apply** *simp* **oops**

lemma (i $+ j $+ #12 $+ k) = u $+ #15 $+ y **apply** *simp* **oops**
lemma (i $+ j $∗ #2 $+ #12 $+ k) = j $+ #5 $+ y **apply** *simp* **oops**

lemma #2 $∗ y $+ #3 $∗ z $+ #6 $∗ w $+ #2 $∗ y $+ #3 $∗ z $+ #2 $∗ u = #2 $∗ y′ $+ #3 $∗ z′ $+ #6 $∗ w′ $+ #2 $∗ y′ $+ #3 $∗ z′ $+ u $+ vv
apply *simp* **oops**

lemma a $+ $−(b$+c) $+ b = d **apply** *simp* **oops**
lemma a $+ $−(b$+c) $− b = d **apply** *simp* **oops**

negative numerals

lemma (i $+ j $+ #−2 $+ k) $− (u $+ #5 $+ y) = zz **apply** *simp* **oops**
lemma (i $+ j $+ #−3 $+ k) $< u $+ #5 $+ y **apply** *simp* **oops**
lemma (i $+ j $+ #3 $+ k) $< u $+ #−6 $+ y **apply** *simp* **oops**
lemma (i $+ j $+ #−12 $+ k) $− #15 = y **apply** *simp* **oops**
lemma (i $+ j $+ #12 $+ k) $− #−15 = y **apply** *simp* **oops**
lemma (i $+ j $+ #−12 $+ k) $− #−15 = y **apply** *simp* **oops**

Multiplying separated numerals

lemma #6 $∗ ($# x $∗ #2) = uu **apply** *simp* **oops**
lemma #4 $∗ ($# x $∗ $# x) $∗ (#2 $∗ $# x) = uu **apply** *simp* **oops**

end

33 The Division Operators Div and Mod

theory *IntDiv*
imports *Bin OrderArith*
begin

definition
quorem :: [i,i] ⇒ o **where**
quorem ≡ λ⟨a,b⟩ ⟨q,r⟩.
a = b$∗q $+ r ∧
(#0$<b ∧ #0$≤r ∧ r$<b | ¬(#0$<b) ∧ b$<r ∧ r $≤ #0)

definition
adjust :: [i,i] ⇒ i **where**
adjust(b) ≡ λ⟨q,r⟩. if #0 $≤ r$−b then <#2$∗q $+ #1,r$−b>
else <#2$∗q,r>

definition
posDivAlg :: i ⇒ i **where**

posDivAlg(ab) ≡
wfrec(*measure*(int∗int, λ⟨a,b⟩. nat-of (a $− b $+ #1)),
ab,

λ⟨a,b⟩ f. if (a$<b | b$≤#0) then <#0,a>
else adjust(b, f ‘ <a,#2$*b>))

definition
negDivAlg :: i ⇒ i **where**

negDivAlg(ab) ≡
wfrec(measure(int*int, λ⟨a,b⟩. nat-of ($− a $− b)),
ab,
λ⟨a,b⟩ f. if (#0 $≤ a$+b | b$≤#0) then <#−1,a$+b>
else adjust(b, f ‘ <a,#2$*b>))

definition
negateSnd :: i ⇒ i **where**
negateSnd ≡ λ⟨q,r⟩. <q, $−r>

definition
divAlg :: i ⇒ i **where**
divAlg ≡
λ⟨a,b⟩. if #0 $≤ a then
if #0 $≤ b then posDivAlg (⟨a,b⟩)
else if a=#0 then <#0,#0>
else negateSnd (negDivAlg (<$−a,$−b>))
else
if #0$<b then negDivAlg (⟨a,b⟩)
else negateSnd (posDivAlg (<$−a,$−b>))

definition
zdiv :: [i,i]⇒i (**infixl** ‹zdiv› 70) **where**
a zdiv b ≡ fst (divAlg (<intify(a), intify(b)>))

definition
zmod :: [i,i]⇒i (**infixl** ‹zmod› 70) **where**
a zmod b ≡ snd (divAlg (<intify(a), intify(b)>))

lemma zspos-add-zspos-imp-zspos: ⟦#0 $< x; #0 $< y⟧ ⟹ #0 $< x $+ y
apply (rule-tac y = y **in** zless-trans)
apply (rule-tac [2] zdiff-zless-iff [THEN iffD1])
apply auto
done

lemma *zpos-add-zpos-imp-zpos*: $[\![\#0\ \$\leq\ x;\ \ \#0\ \$\leq\ y]\!] \Longrightarrow \#0\ \$\leq\ x\ \$+\ y$
apply (*rule-tac* $y = y$ **in** *zle-trans*)
apply (*rule-tac* [*2*] *zdiff-zle-iff* [*THEN iffD1*])
apply *auto*
done

lemma *zneg-add-zneg-imp-zneg*: $[\![x\ \$<\ \#0;\ \ y\ \$<\ \#0]\!] \Longrightarrow x\ \$+\ y\ \$<\ \#0$
apply (*rule-tac* $y = y$ **in** *zless-trans*)
apply (*rule zless-zdiff-iff* [*THEN iffD1*])
apply *auto*
done

lemma *zneg-or-0-add-zneg-or-0-imp-zneg-or-0*:
$[\![x\ \$\leq\ \#0;\ \ y\ \$\leq\ \#0]\!] \Longrightarrow x\ \$+\ y\ \$\leq\ \#0$
apply (*rule-tac* $y = y$ **in** *zle-trans*)
apply (*rule zle-zdiff-iff* [*THEN iffD1*])
apply *auto*
done

lemma *zero-lt-zmagnitude*: $[\![\#0\ \$<\ k;\ k \in int]\!] \Longrightarrow 0 < zmagnitude(k)$
apply (*drule zero-zless-imp-znegative-zminus*)
apply (*drule-tac* [*2*] *zneg-int-of*)
apply (*auto simp add*: *zminus-equation* [*of k*])
apply (*subgoal-tac* $0 < zmagnitude\ (\$\#\ succ\ (n))$)
apply *simp*
apply (*simp only*: *zmagnitude-int-of*)
apply *simp*
done

lemma *zless-add-succ-iff*:
$(w\ \$<\ z\ \$+\ \$\#\ succ(m)) \longleftrightarrow (w\ \$<\ z\ \$+\ \$\#m \mid intify(w) = z\ \$+\ \$\#m)$
apply (*auto simp add*: *zless-iff-succ-zadd zadd-assoc int-of-add* [*symmetric*])
apply (*rule-tac* [*3*] $x = 0$ **in** *bexI*)
apply (*cut-tac* $m = m$ **in** *int-succ-int-1*)
apply (*cut-tac* $m = n$ **in** *int-succ-int-1*)
apply *simp*
apply (*erule natE*)
apply *auto*
apply (*rule-tac* $x = succ\ (n)$ **in** *bexI*)
apply *auto*
done

lemma *zadd-succ-lemma*:
$z \in int \Longrightarrow (w\ \$+\ \$\#\ succ(m)\ \$\leq\ z) \longleftrightarrow (w\ \$+\ \$\#m\ \$<\ z)$
apply (*simp only*: *not-zless-iff-zle* [*THEN iff-sym*] *zless-add-succ-iff*)

apply (*auto intro*: *zle-anti-sym elim*: *zless-asym*
simp add: *zless-imp-zle not-zless-iff-zle*)
done

lemma *zadd-succ-zle-iff*: $(w \ \$+ \ \$\# \ succ(m) \ \$\leq \ z) \longleftrightarrow (w \ \$+ \ \$\#m \ \$< \ z)$
apply (*cut-tac* $z = intify\ (z)$ **in** *zadd-succ-lemma*)
apply *auto*
done

lemma *zless-add1-iff-zle*: $(w \ \$< \ z \ \$+ \ \#1) \longleftrightarrow (w\$\leq z)$
apply (*subgoal-tac* $\#1 = \$\# \ 1$)
apply (*simp only*: *zless-add-succ-iff zle-def*)
apply *auto*
done

lemma *add1-zle-iff*: $(w \ \$+ \ \#1 \ \$\leq \ z) \longleftrightarrow (w \ \$< \ z)$
apply (*subgoal-tac* $\#1 = \$\# \ 1$)
apply (*simp only*: *zadd-succ-zle-iff*)
apply *auto*
done

lemma *add1-left-zle-iff*: $(\#1 \ \$+ \ w \ \$\leq \ z) \longleftrightarrow (w \ \$< \ z)$
apply (*subst zadd-commute*)
apply (*rule add1-zle-iff*)
done

lemma *zmult-mono-lemma*: $k \in nat \Longrightarrow i \ \$\leq \ j \Longrightarrow i \ \$* \ \$\#k \ \$\leq \ j \ \$* \ \$\#k$
apply (*induct-tac k*)
prefer *2* **apply** (*subst int-succ-int-1*)
apply (*simp-all* (*no-asm-simp*) *add*: *zadd-zmult-distrib2 zadd-zle-mono*)
done

lemma *zmult-zle-mono1*: $[\![i \ \$\leq \ j; \ \ \#0 \ \$\leq \ k]\!] \Longrightarrow i\$*k \ \$\leq \ j\$*k$
apply (*subgoal-tac* $i \ \$* \ intify\ (k) \ \$\leq \ j \ \$* \ intify\ (k)$)
apply (*simp* (*no-asm-use*))
apply (*rule-tac* $b = intify\ (k)$ **in** *not-zneg-mag* [*THEN subst*])
apply (*rule-tac* [*3*] *zmult-mono-lemma*)
apply *auto*
apply (*simp add*: *znegative-iff-zless-0 not-zless-iff-zle* [*THEN iff-sym*])
done

lemma *zmult-zle-mono1-neg*: $[\![i \ \$\leq \ j; \ \ k \ \$\leq \ \#0]\!] \Longrightarrow j\$*k \ \$\leq \ i\$*k$
apply (*rule zminus-zle-zminus* [*THEN iffD1*])
apply (*simp del*: *zmult-zminus-right*

add: *zmult-zminus-right* [*symmetric*] *zmult-zle-mono1* *zle-zminus*)
done

lemma *zmult-zle-mono2*: ⟦*i* \$≤ *j*; #*0* \$≤ *k*⟧ ⟹ *k*\$∗*i* \$≤ *k*\$∗*j*
apply (*drule zmult-zle-mono1*)
apply (*simp-all add*: *zmult-commute*)
done

lemma *zmult-zle-mono2-neg*: ⟦*i* \$≤ *j*; *k* \$≤ #*0*⟧ ⟹ *k*\$∗*j* \$≤ *k*\$∗*i*
apply (*drule zmult-zle-mono1-neg*)
apply (*simp-all add*: *zmult-commute*)
done

lemma *zmult-zle-mono*:
⟦*i* \$≤ *j*; *k* \$≤ *l*; #*0* \$≤ *j*; #*0* \$≤ *k*⟧ ⟹ *i*\$∗*k* \$≤ *j*\$∗*l*
apply (*erule zmult-zle-mono1* [*THEN zle-trans*])
apply *assumption*
apply (*erule zmult-zle-mono2*)
apply *assumption*
done

lemma *zmult-zless-mono2-lemma* [*rule-format*]:
⟦*i*\$<*j*; *k* ∈ *nat*⟧ ⟹ *0*<*k* ⟶ \$#*k* \$∗ *i* \$< \$#*k* \$∗ *j*
apply (*induct-tac k*)
prefer *2*
apply (*subst int-succ-int-1*)
apply (*erule natE*)
apply (*simp-all add*: *zadd-zmult-distrib zadd-zless-mono zle-def*)
apply (*frule nat-0-le*)
apply (*subgoal-tac i* \$+ (*i* \$+ \$# *xa* \$∗ *i*) \$< *j* \$+ (*j* \$+ \$# *xa* \$∗ *j*))
apply (*simp* (*no-asm-use*))
apply (*rule zadd-zless-mono*)
apply (*simp-all* (*no-asm-simp*) *add*: *zle-def*)
done

lemma *zmult-zless-mono2*: ⟦*i*\$<*j*; #*0* \$< *k*⟧ ⟹ *k*\$∗*i* \$< *k*\$∗*j*
apply (*subgoal-tac intify* (*k*) \$∗ *i* \$< *intify* (*k*) \$∗ *j*)
apply (*simp* (*no-asm-use*))
apply (*rule-tac b* = *intify* (*k*) **in** *not-zneg-mag* [*THEN subst*])
apply (*rule-tac* [*3*] *zmult-zless-mono2-lemma*)
apply *auto*
apply (*simp add*: *znegative-iff-zless-0*)
apply (*drule zless-trans*, *assumption*)
apply (*auto simp add*: *zero-lt-zmagnitude*)
done

```
lemma zmult-zless-mono1: ⟦i$<j;  #0 $< k⟧ ⟹ i$*k $< j$*k
apply (drule zmult-zless-mono2)
apply (simp-all add: zmult-commute)
done
```

```
lemma zmult-zless-mono:
    ⟦i $< j;  k $< l;  #0 $< j;  #0 $< k⟧ ⟹ i$*k $< j$*l
apply (erule zmult-zless-mono1 [THEN zless-trans])
apply assumption
apply (erule zmult-zless-mono2)
apply assumption
done
```

```
lemma zmult-zless-mono1-neg: ⟦i $< j;  k $< #0⟧ ⟹ j$*k $< i$*k
apply (rule zminus-zless-zminus [THEN iffD1])
apply (simp del: zmult-zminus-right
           add: zmult-zminus-right [symmetric] zmult-zless-mono1 zless-zminus)
done
```

```
lemma zmult-zless-mono2-neg: ⟦i $< j;  k $< #0⟧ ⟹ k$*j $< k$*i
apply (rule zminus-zless-zminus [THEN iffD1])
apply (simp del: zmult-zminus
           add: zmult-zminus [symmetric] zmult-zless-mono2 zless-zminus)
done
```

```
lemma zmult-eq-lemma:
    ⟦m ∈ int; n ∈ int⟧ ⟹ (m = #0 | n = #0) ⟷ (m$*n = #0)
apply (case-tac m $< #0)
apply (auto simp add: not-zless-iff-zle zle-def neq-iff-zless)
apply (force dest: zmult-zless-mono1-neg zmult-zless-mono1)+
done
```

```
lemma zmult-eq-0-iff [iff]: (m$*n = #0) ⟷ (intify(m) = #0 | intify(n) = #0)
apply (simp add: zmult-eq-lemma)
done
```

```
lemma zmult-zless-lemma:
    ⟦k ∈ int; m ∈ int; n ∈ int⟧
     ⟹ (m$*k $< n$*k) ⟷ ((#0 $< k ∧ m$<n) | (k $< #0 ∧ n$<m))
apply (case-tac k = #0)
apply (auto simp add: neq-iff-zless zmult-zless-mono1 zmult-zless-mono1-neg)
```

apply (*auto simp add*: *not-zless-iff-zle*
not-zle-iff-zless [*THEN iff-sym*, *of m*$\$*$*k*]
not-zle-iff-zless [*THEN iff-sym*, *of m*])
apply (*auto elim*: *notE*
simp add: *zless-imp-zle zmult-zle-mono1 zmult-zle-mono1-neg*)
done

lemma *zmult-zless-cancel2*:
$(m\$*k\ \$<\ n\$*k) \longleftrightarrow ((\#0\ \$<\ k \wedge m\$<n) \mid (k\ \$<\ \#0 \wedge n\$<m))$
apply (*cut-tac k* = *intify* (*k*) **and** *m* = *intify* (*m*) **and** *n* = *intify* (*n*)
in *zmult-zless-lemma*)
apply *auto*
done

lemma *zmult-zless-cancel1*:
$(k\$*m\ \$<\ k\$*n) \longleftrightarrow ((\#0\ \$<\ k \wedge m\$<n) \mid (k\ \$<\ \#0 \wedge n\$<m))$
by (*simp add*: *zmult-commute* [*of k*] *zmult-zless-cancel2*)

lemma *zmult-zle-cancel2*:
$(m\$*k\ \$\leq\ n\$*k) \longleftrightarrow ((\#0\ \$<\ k \longrightarrow m\$\leq n) \wedge (k\ \$<\ \#0 \longrightarrow n\$\leq m))$
by (*auto simp add*: *not-zless-iff-zle* [*THEN iff-sym*] *zmult-zless-cancel2*)

lemma *zmult-zle-cancel1*:
$(k\$*m\ \$\leq\ k\$*n) \longleftrightarrow ((\#0\ \$<\ k \longrightarrow m\$\leq n) \wedge (k\ \$<\ \#0 \longrightarrow n\$\leq m))$
by (*auto simp add*: *not-zless-iff-zle* [*THEN iff-sym*] *zmult-zless-cancel1*)

lemma *int-eq-iff-zle*: $[\![m \in int;\ n \in int]\!] \Longrightarrow m{=}n \longleftrightarrow (m\ \$\leq\ n \wedge n\ \$\leq\ m)$
apply (*blast intro*: *zle-refl zle-anti-sym*)
done

lemma *zmult-cancel2-lemma*:
$[\![k \in int;\ m \in int;\ n \in int]\!] \Longrightarrow (m\$*k = n\$*k) \longleftrightarrow (k{=}\#0 \mid m{=}n)$
apply (*simp add*: *int-eq-iff-zle* [*of m*$\$*$*k*] *int-eq-iff-zle* [*of m*])
apply (*auto simp add*: *zmult-zle-cancel2 neq-iff-zless*)
done

lemma *zmult-cancel2* [*simp*]:
$(m\$*k = n\$*k) \longleftrightarrow (intify(k) = \#0 \mid intify(m) = intify(n))$
apply (*rule iff-trans*)
apply (*rule-tac* [*2*] *zmult-cancel2-lemma*)
apply *auto*
done

lemma *zmult-cancel1* [*simp*]:
$(k\$*m = k\$*n) \longleftrightarrow (intify(k) = \#0 \mid intify(m) = intify(n))$
by (*simp add*: *zmult-commute* [*of k*] *zmult-cancel2*)

33.1 Uniqueness and monotonicity of quotients and remainders

lemma *unique-quotient-lemma*:
 $[\![b\$*q' \$+ r' \$\leq b\$*q \$+ r; \ \#0 \$\leq r'; \ \#0 \$< b; \ r \$< b]\!]$
 $\Longrightarrow q' \$\leq q$
apply (*subgoal-tac* $r' \$+ b \$* (q'\$-q) \$\leq r$)
prefer *2* **apply** (*simp add*: *zdiff-zmult-distrib2 zadd-ac zcompare-rls*)
apply (*subgoal-tac* $\#0 \$< b \$* (\#1 \$+ q \$- q')$)
prefer *2*
apply (*erule zle-zless-trans*)
apply (*simp add*: *zdiff-zmult-distrib2 zadd-zmult-distrib2 zadd-ac zcompare-rls*)
apply (*erule zle-zless-trans*)
apply *simp*
apply (*subgoal-tac* $b \$* q' \$< b \$* (\#1 \$+ q)$)
prefer *2*
apply (*simp add*: *zdiff-zmult-distrib2 zadd-zmult-distrib2 zadd-ac zcompare-rls*)
apply (*auto elim*: *zless-asym*
 simp add: *zmult-zless-cancel1 zless-add1-iff-zle zadd-ac zcompare-rls*)
done

lemma *unique-quotient-lemma-neg*:
 $[\![b\$*q' \$+ r' \$\leq b\$*q \$+ r; \ r \$\leq \#0; \ b \$< \#0; \ b \$< r']\!]$
 $\Longrightarrow q \$\leq q'$
apply (*rule-tac* $b = \$-b$ **and** $r = \$-r'$ **and** $r' = \$-r$
 in *unique-quotient-lemma*)
apply (*auto simp del*: *zminus-zadd-distrib*
 simp add: *zminus-zadd-distrib* [*symmetric*] *zle-zminus zless-zminus*)
done

lemma *unique-quotient*:
 $[\![quorem (\langle a,b\rangle, \langle q,r\rangle); \ quorem (\langle a,b\rangle, <q',r'>); \ b \in int; \ b \neq \#0;$
 $q \in int; \ q' \in int]\!] \Longrightarrow q = q'$
apply (*simp add*: *split-ifs quorem-def neq-iff-zless*)
apply *safe*
apply *simp-all*
apply (*blast intro*: *zle-anti-sym*
 dest: *zle-eq-refl* [*THEN unique-quotient-lemma*]
 zle-eq-refl [*THEN unique-quotient-lemma-neg*] *sym*)+
done

lemma *unique-remainder*:
 $[\![quorem (\langle a,b\rangle, \langle q,r\rangle); \ quorem (\langle a,b\rangle, <q',r'>); \ b \in int; \ b \neq \#0;$
 $q \in int; \ q' \in int;$
 $r \in int; \ r' \in int]\!] \Longrightarrow r = r'$
apply (*subgoal-tac* $q = q'$)
prefer *2* **apply** (*blast intro*: *unique-quotient*)
apply (*simp add*: *quorem-def*)
done

33.2 Correctness of posDivAlg, the Division Algorithm for $a \geq 0$ and $b > 0$

```
lemma adjust-eq [simp]:
    adjust(b, ⟨q,r⟩) = (let diff = r$−b in
                        if #0 $≤ diff then <#2$∗q $+ #1,diff>
                                      else <#2$∗q,r>)
by (simp add: Let-def adjust-def)


lemma posDivAlg-termination:
    ⟦#0 $< b; ¬ a $< b⟧
    ⟹ nat-of(a $− #2 $∗ b $+ #1) < nat-of(a $− b $+ #1)
apply (simp (no-asm) add: zless-nat-conj)
apply (simp add: not-zless-iff-zle zless-add1-iff-zle zcompare-rls)
done

lemmas posDivAlg-unfold = def-wfrec [OF posDivAlg-def wf-measure]

lemma posDivAlg-eqn:
    ⟦#0 $< b; a ∈ int; b ∈ int⟧ ⟹
     posDivAlg(⟨a,b⟩) =
      (if a$<b then <#0,a> else adjust(b, posDivAlg (<a, #2$∗b>)))
apply (rule posDivAlg-unfold [THEN trans])
apply (simp add: vimage-iff not-zless-iff-zle [THEN iff-sym])
apply (blast intro: posDivAlg-termination)
done

lemma posDivAlg-induct-lemma [rule-format]:
  assumes prem:
        ⋀a b. ⟦a ∈ int; b ∈ int;
                  ¬ (a $< b | b $≤ #0) ⟶ P(<a, #2 $∗ b>)⟧ ⟹ P(⟨a,b⟩)
  shows ⟨u,v⟩ ∈ int∗int ⟹ P(⟨u,v⟩)
using wf-measure [where A = int∗int and f = λ⟨a,b⟩.nat-of (a $− b $+ #1)]
proof (induct ⟨u,v⟩ arbitrary: u v rule: wf-induct)
  case (step x)
  hence uv: u ∈ int v ∈ int by auto
  thus ?case
    apply (rule prem)
    apply (rule impI)
    apply (rule step)
    apply (auto simp add: step uv not-zle-iff-zless posDivAlg-termination)
    done
qed


lemma posDivAlg-induct [consumes 2]:
  assumes u-int: u ∈ int
      and v-int: v ∈ int
      and ih: ⋀a b. ⟦a ∈ int; b ∈ int;
```

$\neg$ (*a* \$< *b* | *b* \$≤ *#0*) ⟶ *P*(*a*, *#2* \$∗ *b*)⟧ ⟹ *P*(*a*,*b*)
shows *P*(*u*,*v*)
apply (*subgoal-tac* (λ⟨*x*,*y*⟩. *P* (*x*,*y*)) (⟨*u*,*v*⟩))
apply *simp*
apply (*rule posDivAlg-induct-lemma*)
apply (*simp* (*no-asm-use*))
apply (*rule ih*)
apply (*auto simp add*: *u-int v-int*)
done

lemma *intify-eq-0-iff-zle*: *intify*(*m*) = *#0* ⟷ (*m* \$≤ *#0* ∧ *#0* \$≤ *m*)
by (*simp add*: *int-eq-iff-zle*)

33.3 Some convenient biconditionals for products of signs

lemma *zmult-pos*: ⟦*#0* \$< *i*; *#0* \$< *j*⟧ ⟹ *#0* \$< *i* \$∗ *j*
by (*drule zmult-zless-mono1*, *auto*)

lemma *zmult-neg*: ⟦*i* \$< *#0*; *j* \$< *#0*⟧ ⟹ *#0* \$< *i* \$∗ *j*
by (*drule zmult-zless-mono1-neg*, *auto*)

lemma *zmult-pos-neg*: ⟦*#0* \$< *i*; *j* \$< *#0*⟧ ⟹ *i* \$∗ *j* \$< *#0*
by (*drule zmult-zless-mono1-neg*, *auto*)

lemma *int-0-less-lemma*:
⟦*x* ∈ *int*; *y* ∈ *int*⟧
⟹ (*#0* \$< *x* \$∗ *y*) ⟷ (*#0* \$< *x* ∧ *#0* \$< *y* | *x* \$< *#0* ∧ *y* \$< *#0*)
apply (*auto simp add*: *zle-def not-zless-iff-zle zmult-pos zmult-neg*)
apply (*rule ccontr*)
apply (*rule-tac* [*2*] *ccontr*)
apply (*auto simp add*: *zle-def not-zless-iff-zle*)
apply (*erule-tac P* = *#0*\$< *x*\$∗ *y* **in** *rev-mp*)
apply (*erule-tac* [*2*] *P* = *#0*\$< *x*\$∗ *y* **in** *rev-mp*)
apply (*drule zmult-pos-neg*, *assumption*)
prefer *2*
apply (*drule zmult-pos-neg*, *assumption*)
apply (*auto dest*: *zless-not-sym simp add*: *zmult-commute*)
done

lemma *int-0-less-mult-iff*:
(*#0* \$< *x* \$∗ *y*) ⟷ (*#0* \$< *x* ∧ *#0* \$< *y* | *x* \$< *#0* ∧ *y* \$< *#0*)
apply (*cut-tac x* = *intify* (*x*) **and** *y* = *intify* (*y*) **in** *int-0-less-lemma*)
apply *auto*
done

lemma *int-0-le-lemma*:
  $[\![x \in int; y \in int]\!]$
   $\Longrightarrow (\#0 \$\leq x \$* y) \longleftrightarrow (\#0 \$\leq x \wedge \#0 \$\leq y \mid x \$\leq \#0 \wedge y \$\leq \#0)$
by (*auto simp add*: *zle-def not-zless-iff-zle int-0-less-mult-iff*)

lemma *int-0-le-mult-iff*:
  $(\#0 \$\leq x \$* y) \longleftrightarrow ((\#0 \$\leq x \wedge \#0 \$\leq y) \mid (x \$\leq \#0 \wedge y \$\leq \#0))$
apply (*cut-tac* $x = intify\ (x)$ **and** $y = intify\ (y)$ **in** *int-0-le-lemma*)
apply *auto*
done

lemma *zmult-less-0-iff*:
  $(x \$* y \$< \#0) \longleftrightarrow (\#0 \$< x \wedge y \$< \#0 \mid x \$< \#0 \wedge \#0 \$< y)$
apply (*auto simp add*: *int-0-le-mult-iff not-zle-iff-zless* [*THEN iff-sym*])
apply (*auto dest*: *zless-not-sym simp add*: *not-zle-iff-zless*)
done

lemma *zmult-le-0-iff*:
  $(x \$* y \$\leq \#0) \longleftrightarrow (\#0 \$\leq x \wedge y \$\leq \#0 \mid x \$\leq \#0 \wedge \#0 \$\leq y)$
by (*auto dest*: *zless-not-sym*
    simp add: *int-0-less-mult-iff not-zless-iff-zle* [*THEN iff-sym*])

lemma *posDivAlg-type* [*rule-format*]:
  $[\![a \in int; b \in int]\!] \Longrightarrow posDivAlg(\langle a,b\rangle) \in int * int$
apply (*rule-tac* $u = a$ **and** $v = b$ **in** *posDivAlg-induct*)
apply *assumption+*
apply (*case-tac* $\#0 \$< ba$)
 apply (*simp add*: *posDivAlg-eqn adjust-def integ-of-type*
      split: *split-if-asm*)
 apply *clarify*
 apply (*simp add*: *int-0-less-mult-iff not-zle-iff-zless*)
apply (*simp add*: *not-zless-iff-zle*)
apply (*subst posDivAlg-unfold*)
apply *simp*
done

lemma *posDivAlg-correct* [*rule-format*]:
  $[\![a \in int; b \in int]\!]$
   $\Longrightarrow \#0 \$\leq a \longrightarrow \#0 \$< b \longrightarrow quorem\ (\langle a,b\rangle, posDivAlg(\langle a,b\rangle))$
apply (*rule-tac* $u = a$ **and** $v = b$ **in** *posDivAlg-induct*)
apply *auto*
  apply (*simp-all add*: *quorem-def*)

base case: a<b

  apply (*simp add*: *posDivAlg-eqn*)
 apply (*simp add*: *not-zless-iff-zle* [*THEN iff-sym*])

apply (*simp add*: *int-0-less-mult-iff*)

main argument

apply (*subst posDivAlg-eqn*)
apply (*simp-all* (*no-asm-simp*))
apply (*erule splitE*)
apply (*rule posDivAlg-type*)
apply (*simp-all add*: *int-0-less-mult-iff*)
apply (*auto simp add*: *zadd-zmult-distrib2 Let-def*)

now just linear arithmetic

apply (*simp add*: *not-zle-iff-zless zdiff-zless-iff*)
done

33.4 Correctness of negDivAlg, the division algorithm for a<0 and b>0

lemma *negDivAlg-termination*:
  ⟦#0 $< b; a $+ b $< #0⟧
  ⟹ nat-of($− a $− #2 $* b) < nat-of($− a $− b)
apply (*simp* (*no-asm*) *add*: *zless-nat-conj*)
apply (*simp add*: *zcompare-rls not-zle-iff-zless zless-zdiff-iff* [*THEN iff-sym*]
    zless-zminus)
done

lemmas *negDivAlg-unfold* = *def-wfrec* [*OF negDivAlg-def wf-measure*]

lemma *negDivAlg-eqn*:
  ⟦#0 $< b; a ∈ int; b ∈ int⟧ ⟹
   negDivAlg(⟨a,b⟩) =
    (if #0 $≤ a$+b then <#−1,a$+b>
         else adjust(b, negDivAlg (<a, #2$*b>)))
apply (*rule negDivAlg-unfold* [*THEN trans*])
apply (*simp* (*no-asm-simp*) *add*: *vimage-iff not-zless-iff-zle* [*THEN iff-sym*])
apply (*blast intro*: *negDivAlg-termination*)
done

lemma *negDivAlg-induct-lemma* [*rule-format*]:
 assumes *prem*:
   ⋀a b. ⟦a ∈ int; b ∈ int;
      ¬ (#0 $≤ a $+ b | b $≤ #0) ⟶ P(<a, #2 $* b>)⟧
     ⟹ P(⟨a,b⟩)
 shows ⟨u,v⟩ ∈ int*int ⟹ P(⟨u,v⟩)
using *wf-measure* [**where** A = int*int **and** f = λ⟨a,b⟩.nat-of ($− a $− b)]
proof (*induct* ⟨u,v⟩ *arbitrary*: *u v rule*: *wf-induct*)
 case (*step x*)
 hence *uv*: u ∈ int v ∈ int **by** *auto*
 thus *?case*
  apply (*rule prem*)

apply (*rule impI*)
apply (*rule step*)
apply (*auto simp add*: *step uv not-zle-iff-zless negDivAlg-termination*)
done
qed

lemma *negDivAlg-induct* [*consumes 2*]:
assumes *u-int*: $u \in int$
and *v-int*: $v \in int$
and *ih*: $\bigwedge a\ b.\ [\![a \in int;\ b \in int;$
$\neg\ (\#0\ \$\leq\ a\ \$+\ b\ |\ b\ \$\leq\ \#0) \longrightarrow P(a,\ \#2\ \$*\ b)]\!]$
$\Longrightarrow P(a,b)$
shows $P(u,v)$
apply (*subgoal-tac* ($\lambda\langle x,y\rangle.\ P\ (x,y)$) ($\langle u,v\rangle$))
apply *simp*
apply (*rule negDivAlg-induct-lemma*)
apply (*simp* (*no-asm-use*))
apply (*rule ih*)
apply (*auto simp add*: *u-int v-int*)
done

lemma *negDivAlg-type*:
$[\![a \in int;\ b \in int]\!] \Longrightarrow negDivAlg(\langle a,b\rangle) \in int * int$
apply (*rule-tac* $u = a$ **and** $v = b$ **in** *negDivAlg-induct*)
apply *assumption+*
apply (*case-tac* $\#0\ \$<\ ba$)
apply (*simp add*: *negDivAlg-eqn adjust-def integ-of-type*
split: *split-if-asm*)
apply *clarify*
apply (*simp add*: *int-0-less-mult-iff not-zle-iff-zless*)
apply (*simp add*: *not-zless-iff-zle*)
apply (*subst negDivAlg-unfold*)
apply *simp*
done

lemma *negDivAlg-correct* [*rule-format*]:
$[\![a \in int;\ b \in int]\!]$
$\Longrightarrow a\ \$<\ \#0 \longrightarrow \#0\ \$<\ b \longrightarrow quorem\ (\langle a,b\rangle,\ negDivAlg(\langle a,b\rangle))$
apply (*rule-tac* $u = a$ **and** $v = b$ **in** *negDivAlg-induct*)
apply *auto*
apply (*simp-all add*: *quorem-def*)

base case: $0\ \$\leq\ a\ \$+\ b$

apply (*simp add*: *negDivAlg-eqn*)
apply (*simp add*: *not-zless-iff-zle* [*THEN iff-sym*])

apply (*simp add*: *int-0-less-mult-iff*)

main argument

apply (*subst negDivAlg-eqn*)
apply (*simp-all* (*no-asm-simp*))
apply (*erule splitE*)
apply (*rule negDivAlg-type*)
apply (*simp-all add*: *int-0-less-mult-iff*)
apply (*auto simp add*: *zadd-zmult-distrib2 Let-def*)

now just linear arithmetic

apply (*simp add*: *not-zle-iff-zless zdiff-zless-iff*)
done

33.5 Existence shown by proving the division algorithm to be correct

lemma *quorem-0*: ⟦*b* ≠ #*0*; *b* ∈ *int*⟧ ⟹ *quorem* (<#*0*,*b*>, <#*0*,#*0*>)
by (*force simp add*: *quorem-def neq-iff-zless*)

lemma *posDivAlg-zero-divisor*: *posDivAlg*(<*a*,#*0*>) = <#*0*,*a*>
apply (*subst posDivAlg-unfold*)
apply *simp*
done

lemma *posDivAlg-0* [*simp*]: *posDivAlg* (<#*0*,*b*>) = <#*0*,#*0*>
apply (*subst posDivAlg-unfold*)
apply (*simp add*: *not-zle-iff-zless*)
done

lemma *linear-arith-lemma*: ¬ (#*0* $≤ #−*1* $+ *b*) ⟹ (*b* $≤ #*0*)
apply (*simp add*: *not-zle-iff-zless*)
apply (*drule zminus-zless-zminus* [*THEN iffD2*])
apply (*simp add*: *zadd-commute zless-add1-iff-zle zle-zminus*)
done

lemma *negDivAlg-minus1* [*simp*]: *negDivAlg* (<#−*1*,*b*>) = <#−*1*, *b*$−#*1*>
apply (*subst negDivAlg-unfold*)
apply (*simp add*: *linear-arith-lemma integ-of-type vimage-iff*)
done

lemma *negateSnd-eq* [*simp*]: *negateSnd* (⟨*q*,*r*⟩) = <*q*, $−*r*>
unfolding *negateSnd-def*
apply *auto*
done

lemma *negateSnd-type*: *qr* ∈ *int* ∗ *int* ⟹ *negateSnd* (*qr*) ∈ *int* ∗ *int*

unfolding *negateSnd-def*
apply *auto*
done

lemma *quorem-neg*:
$[\![quorem\ (<\$-a,\$-b>,\ qr);\ \ a \in int;\ \ b \in int;\ \ qr \in int * int]\!]$
$\Longrightarrow quorem\ (\langle a,b\rangle,\ negateSnd(qr))$
apply *clarify*
apply (*auto elim*: *zless-asym simp add*: *quorem-def zless-zminus*)

linear arithmetic from here on

apply (*simp-all add*: *zminus-equation* [*of a*] *zminus-zless*)
apply (*cut-tac* [*2*] *z* = *b* **and** *w* = *#0* **in** *zless-linear*)
apply (*cut-tac* [*1*] *z* = *b* **and** *w* = *#0* **in** *zless-linear*)
apply *auto*
apply (*blast dest*: *zle-zless-trans*)+
done

lemma *divAlg-correct*:
$[\![b \neq \#0;\ \ a \in int;\ \ b \in int]\!] \Longrightarrow quorem\ (\langle a,b\rangle,\ divAlg(\langle a,b\rangle))$
apply (*auto simp add*: *quorem-0 divAlg-def*)
apply (*safe intro*!: *quorem-neg posDivAlg-correct negDivAlg-correct*
posDivAlg-type negDivAlg-type)
apply (*auto simp add*: *quorem-def neq-iff-zless*)

linear arithmetic from here on

apply (*auto simp add*: *zle-def*)
done

lemma *divAlg-type*: $[\![a \in int;\ \ b \in int]\!] \Longrightarrow divAlg(\langle a,b\rangle) \in int * int$
apply (*auto simp add*: *divAlg-def*)
apply (*auto simp add*: *posDivAlg-type negDivAlg-type negateSnd-type*)
done

lemma *zdiv-intify1* [*simp*]: *intify*(*x*) *zdiv y* = *x zdiv y*
by (*simp add*: *zdiv-def*)

lemma *zdiv-intify2* [*simp*]: *x zdiv intify*(*y*) = *x zdiv y*
by (*simp add*: *zdiv-def*)

lemma *zdiv-type* [*iff*,*TC*]: *z zdiv w* ∈ *int*
unfolding *zdiv-def*
apply (*blast intro*: *fst-type divAlg-type*)
done

lemma *zmod-intify1* [*simp*]: *intify*(*x*) *zmod y* = *x zmod y*

by (*simp add*: *zmod-def*)

lemma *zmod-intify2* [*simp*]: x *zmod* $intify(y) = x$ *zmod* y
by (*simp add*: *zmod-def*)

lemma *zmod-type* [*iff*,*TC*]: z *zmod* $w \in int$
unfolding *zmod-def*
apply (*rule snd-type*)
apply (*blast intro*: *divAlg-type*)
done

lemma *DIVISION-BY-ZERO-ZDIV*: a *zdiv* $\#0 = \#0$
by (*simp add*: *zdiv-def divAlg-def posDivAlg-zero-divisor*)

lemma *DIVISION-BY-ZERO-ZMOD*: a *zmod* $\#0 = intify(a)$
by (*simp add*: *zmod-def divAlg-def posDivAlg-zero-divisor*)

lemma *raw-zmod-zdiv-equality*:
$[\![a \in int; b \in int]\!] \Longrightarrow a = b \$* (a \text{ zdiv } b) \$+ (a \text{ zmod } b)$
apply (*case-tac* $b = \#0$)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *divAlg-correct*)
apply (*auto simp add*: *quorem-def zdiv-def zmod-def split-def*)
done

lemma *zmod-zdiv-equality*: $intify(a) = b \$* (a \text{ zdiv } b) \$+ (a \text{ zmod } b)$
apply (*rule trans*)
apply (*rule-tac* $b = intify\ (b)$ **in** *raw-zmod-zdiv-equality*)
apply *auto*
done

lemma *pos-mod*: $\#0 \$< b \Longrightarrow \#0 \$\leq a \text{ zmod } b \wedge a \text{ zmod } b \$< b$
apply (*cut-tac* $a = intify\ (a)$ **and** $b = intify\ (b)$ **in** *divAlg-correct*)
apply (*auto simp add*: *intify-eq-0-iff-zle quorem-def zmod-def split-def*)
apply (*blast dest*: *zle-zless-trans*)+
done

lemmas *pos-mod-sign* = *pos-mod* [*THEN conjunct1*]
and *pos-mod-bound* = *pos-mod* [*THEN conjunct2*]

lemma *neg-mod*: $b \$< \#0 \Longrightarrow a \text{ zmod } b \$\leq \#0 \wedge b \$< a \text{ zmod } b$
apply (*cut-tac* $a = intify\ (a)$ **and** $b = intify\ (b)$ **in** *divAlg-correct*)
apply (*auto simp add*: *intify-eq-0-iff-zle quorem-def zmod-def split-def*)

apply (*blast dest*: *zle-zless-trans*)
apply (*blast dest*: *zless-trans*)+
done

lemmas *neg-mod-sign* = *neg-mod* [*THEN conjunct1*]
  and *neg-mod-bound* = *neg-mod* [*THEN conjunct2*]

lemma *quorem-div-mod*:
    ⟦*b* ≠ *#0*; *a* ∈ *int*; *b* ∈ *int*⟧
    ⟹ *quorem* (⟨*a*,*b*⟩, <*a zdiv b*, *a zmod b*>)
apply (*cut-tac a* = *a* **and** *b* = *b* **in** *zmod-zdiv-equality*)
apply (*auto simp add*: *quorem-def neq-iff-zless pos-mod-sign pos-mod-bound*
            neg-mod-sign neg-mod-bound)
done

lemma *quorem-div*:
    ⟦*quorem*(⟨*a*,*b*⟩,⟨*q*,*r*⟩); *b* ≠ *#0*; *a* ∈ *int*; *b* ∈ *int*; *q* ∈ *int*⟧
    ⟹ *a zdiv b* = *q*
by (*blast intro*: *quorem-div-mod* [*THEN unique-quotient*])

lemma *quorem-mod*:
    ⟦*quorem*(⟨*a*,*b*⟩,⟨*q*,*r*⟩); *b* ≠ *#0*; *a* ∈ *int*; *b* ∈ *int*; *q* ∈ *int*; *r* ∈ *int*⟧
    ⟹ *a zmod b* = *r*
by (*blast intro*: *quorem-div-mod* [*THEN unique-remainder*])

lemma *zdiv-pos-pos-trivial-raw*:
    ⟦*a* ∈ *int*; *b* ∈ *int*; *#0* $≤ *a*; *a* $< *b*⟧ ⟹ *a zdiv b* = *#0*
apply (*rule quorem-div*)
apply (*auto simp add*: *quorem-def*)

apply (*blast dest*: *zle-zless-trans*)+
done

lemma *zdiv-pos-pos-trivial*: ⟦*#0* $≤ *a*; *a* $< *b*⟧ ⟹ *a zdiv b* = *#0*
apply (*cut-tac a* = *intify* (*a*) **and** *b* = *intify* (*b*)
    in *zdiv-pos-pos-trivial-raw*)
apply *auto*
done

lemma *zdiv-neg-neg-trivial-raw*:
    ⟦*a* ∈ *int*; *b* ∈ *int*; *a* $≤ *#0*; *b* $< *a*⟧ ⟹ *a zdiv b* = *#0*
apply (*rule-tac r* = *a* **in** *quorem-div*)
apply (*auto simp add*: *quorem-def*)

apply (*blast dest*: *zle-zless-trans zless-trans*)+

done

lemma *zdiv-neg-neg-trivial*: $[\![a \ \$\leq \ \#0; \ \ b \ \$< \ a]\!] \Longrightarrow a \ zdiv \ b = \#0$
apply (*cut-tac* $a = intify\ (a)$ **and** $b = intify\ (b)$
  in *zdiv-neg-neg-trivial-raw*)
apply *auto*
done

lemma *zadd-le-0-lemma*: $[\![a\$+b \ \$\leq \ \#0; \ \ \#0 \ \$< \ a; \ \ \#0 \ \$< \ b]\!] \Longrightarrow False$
apply (*drule-tac* $z' = \#0$ **and** $z = b$ **in** *zadd-zless-mono*)
apply (*auto simp add*: *zle-def*)
apply (*blast dest*: *zless-trans*)
done

lemma *zdiv-pos-neg-trivial-raw*:
  $[\![a \in int; \ \ b \in int; \ \ \#0 \ \$< \ a; \ \ a\$+b \ \$\leq \ \#0]\!] \Longrightarrow a \ zdiv \ b = \#{-1}$
apply (*rule-tac* $r = a \ \$+ \ b$ **in** *quorem-div*)
apply (*auto simp add*: *quorem-def*)

apply (*blast dest*: *zadd-le-0-lemma zle-zless-trans*)+
done

lemma *zdiv-pos-neg-trivial*: $[\![\#0 \ \$< \ a; \ \ a\$+b \ \$\leq \ \#0]\!] \Longrightarrow a \ zdiv \ b = \#{-1}$
apply (*cut-tac* $a = intify\ (a)$ **and** $b = intify\ (b)$
  in *zdiv-pos-neg-trivial-raw*)
apply *auto*
done

lemma *zmod-pos-pos-trivial-raw*:
  $[\![a \in int; \ \ b \in int; \ \ \#0 \ \$\leq \ a; \ \ a \ \$< \ b]\!] \Longrightarrow a \ zmod \ b = a$
apply (*rule-tac* $q = \#0$ **in** *quorem-mod*)
apply (*auto simp add*: *quorem-def*)

apply (*blast dest*: *zle-zless-trans*)+
done

lemma *zmod-pos-pos-trivial*: $[\![\#0 \ \$\leq \ a; \ \ a \ \$< \ b]\!] \Longrightarrow a \ zmod \ b = intify(a)$
apply (*cut-tac* $a = intify\ (a)$ **and** $b = intify\ (b)$
  in *zmod-pos-pos-trivial-raw*)
apply *auto*
done

lemma *zmod-neg-neg-trivial-raw*:
  $[\![a \in int; \ \ b \in int; \ \ a \ \$\leq \ \#0; \ \ b \ \$< \ a]\!] \Longrightarrow a \ zmod \ b = a$
apply (*rule-tac* $q = \#0$ **in** *quorem-mod*)
apply (*auto simp add*: *quorem-def*)

apply (*blast dest*: *zle-zless-trans zless-trans*)+
done

lemma *zmod-neg-neg-trivial*: $[\![a \ \$\leq \ \#0; \ \ b \ \$< \ a]\!] \Longrightarrow a \ zmod \ b = intify(a)$
apply (*cut-tac* $a = intify\ (a)$ **and** $b = intify\ (b)$
  in *zmod-neg-neg-trivial-raw*)
apply *auto*
done

lemma *zmod-pos-neg-trivial-raw*:
  $[\![a \in int; \ \ b \in int; \ \ \#0 \ \$< \ a; \ \ a\$+b \ \$\leq \ \#0]\!] \Longrightarrow a \ zmod \ b = a\$+b$
apply (*rule-tac* $q = \#{-1}$ **in** *quorem-mod*)
apply (*auto simp add*: *quorem-def*)

apply (*blast dest*: *zadd-le-0-lemma zle-zless-trans*)+
done

lemma *zmod-pos-neg-trivial*: $[\![\#0 \ \$< \ a; \ \ a\$+b \ \$\leq \ \#0]\!] \Longrightarrow a \ zmod \ b = a\$+b$
apply (*cut-tac* $a = intify\ (a)$ **and** $b = intify\ (b)$
  in *zmod-pos-neg-trivial-raw*)
apply *auto*
done

lemma *zdiv-zminus-zminus-raw*:
  $[\![a \in int; \ \ b \in int]\!] \Longrightarrow (\$-a) \ zdiv \ (\$-b) = a \ zdiv \ b$
apply (*case-tac* $b = \#0$)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*subst quorem-div-mod* [*THEN quorem-neg*, *simplified*, *THEN quorem-div*])
apply *auto*
done

lemma *zdiv-zminus-zminus* [*simp*]: $(\$-a) \ zdiv \ (\$-b) = a \ zdiv \ b$
apply (*cut-tac* $a = intify\ (a)$ **and** $b = intify\ (b)$ **in** *zdiv-zminus-zminus-raw*)
apply *auto*
done

lemma *zmod-zminus-zminus-raw*:
  $[\![a \in int; \ \ b \in int]\!] \Longrightarrow (\$-a) \ zmod \ (\$-b) = \$- \ (a \ zmod \ b)$
apply (*case-tac* $b = \#0$)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*subst quorem-div-mod* [*THEN quorem-neg*, *simplified*, *THEN quorem-mod*])
apply *auto*

done

lemma *zmod-zminus-zminus* [*simp*]: ($-$a$) *zmod* ($-b) = $- ($a$ *zmod* b)
apply (*cut-tac* a = *intify* (a) **and** b = *intify* (b) **in** *zmod-zminus-zminus-raw*)
apply *auto*
done

33.6 division of a number by itself

lemma *self-quotient-aux1*: ⟦#0 $< a; a = r $+ a$*q; r $< a⟧ ⟹ #1 $≤ q
apply (*subgoal-tac* #0 $< a$*q)
apply (*cut-tac* w = #0 **and** z = q **in** *add1-zle-iff*)
apply (*simp add*: *int-0-less-mult-iff*)
apply (*blast dest*: *zless-trans*)

apply (*drule-tac* t = λx. x $- r **in** *subst-context*)
apply (*drule sym*)
apply (*simp add*: *zcompare-rls*)
done

lemma *self-quotient-aux2*: ⟦#0 $< a; a = r $+ a$*q; #0 $≤ r⟧ ⟹ q $≤ #1
apply (*subgoal-tac* #0 $≤ a$* (#1$-q))
 apply (*simp add*: *int-0-le-mult-iff zcompare-rls*)
 apply (*blast dest*: *zle-zless-trans*)
apply (*simp add*: *zdiff-zmult-distrib2*)
apply (*drule-tac* t = λx. x $- a $* q **in** *subst-context*)
apply (*simp add*: *zcompare-rls*)
done

lemma *self-quotient*:
  ⟦quorem(⟨a,a⟩,⟨q,r⟩); a ∈ int; q ∈ int; a ≠ #0⟧ ⟹ q = #1
apply (*simp add*: *split-ifs quorem-def neq-iff-zless*)
apply (*rule zle-anti-sym*)
apply *safe*
apply *auto*
prefer *4* **apply** (*blast dest*: *zless-trans*)
apply (*blast dest*: *zless-trans*)
apply (*rule-tac* [*3*] a = $-a **and** r = $-r **in** *self-quotient-aux1*)
apply (*rule-tac* a = $-a **and** r = $-r **in** *self-quotient-aux2*)
apply (*rule-tac* [*6*] *zminus-equation* [*THEN iffD1*])
apply (*rule-tac* [*2*] *zminus-equation* [*THEN iffD1*])
apply (*force intro*: *self-quotient-aux1 self-quotient-aux2*
  simp add: *zadd-commute zmult-zminus*)+
done

lemma *self-remainder*:
  ⟦quorem(⟨a,a⟩,⟨q,r⟩); a ∈ int; q ∈ int; r ∈ int; a ≠ #0⟧ ⟹ r = #0
apply (*frule self-quotient*)
apply (*auto simp add*: *quorem-def*)

done

lemma *zdiv-self-raw*: $[\![a \neq \#0;\ a \in int]\!] \Longrightarrow a\ zdiv\ a = \#1$
apply (*blast intro*: *quorem-div-mod* [*THEN self-quotient*])
done

lemma *zdiv-self* [*simp*]: $intify(a) \neq \#0 \Longrightarrow a\ zdiv\ a = \#1$
apply (*drule zdiv-self-raw*)
apply *auto*
done

lemma *zmod-self-raw*: $a \in int \Longrightarrow a\ zmod\ a = \#0$
apply (*case-tac* $a = \#0$)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*blast intro*: *quorem-div-mod* [*THEN self-remainder*])
done

lemma *zmod-self* [*simp*]: $a\ zmod\ a = \#0$
apply (*cut-tac* $a = intify\ (a)$ **in** *zmod-self-raw*)
apply *auto*
done

33.7 Computation of division and remainder

lemma *zdiv-zero* [*simp*]: $\#0\ zdiv\ b = \#0$
by (*simp add*: *zdiv-def divAlg-def*)

lemma *zdiv-eq-minus1*: $\#0\ \$<\ b \Longrightarrow \#{-1}\ zdiv\ b = \#{-1}$
by (*simp* (*no-asm-simp*) *add*: *zdiv-def divAlg-def*)

lemma *zmod-zero* [*simp*]: $\#0\ zmod\ b = \#0$
by (*simp add*: *zmod-def divAlg-def*)

lemma *zdiv-minus1*: $\#0\ \$<\ b \Longrightarrow \#{-1}\ zdiv\ b = \#{-1}$
by (*simp add*: *zdiv-def divAlg-def*)

lemma *zmod-minus1*: $\#0\ \$<\ b \Longrightarrow \#{-1}\ zmod\ b = b\ \$-\ \#1$
by (*simp add*: *zmod-def divAlg-def*)

lemma *zdiv-pos-pos*: $[\![\#0\ \$<\ a;\ \ \#0\ \$\leq\ b]\!]$
$\Longrightarrow a\ zdiv\ b = fst\ (posDivAlg(<intify(a),\ intify(b)>))$
apply (*simp* (*no-asm-simp*) *add*: *zdiv-def divAlg-def*)
apply (*auto simp add*: *zle-def*)
done

lemma *zmod-pos-pos*:

⟦#0 $< a; #0 $≤ b⟧
⟹ a zmod b = snd (posDivAlg(<intify(a), intify(b)>))
apply (*simp* (*no-asm-simp*) *add*: *zmod-def divAlg-def*)
apply (*auto simp add*: *zle-def*)
done

lemma *zdiv-neg-pos*:
⟦a $< #0; #0 $< b⟧
⟹ a zdiv b = fst (negDivAlg(<intify(a), intify(b)>))
apply (*simp* (*no-asm-simp*) *add*: *zdiv-def divAlg-def*)
apply (*blast dest*: *zle-zless-trans*)
done

lemma *zmod-neg-pos*:
⟦a $< #0; #0 $< b⟧
⟹ a zmod b = snd (negDivAlg(<intify(a), intify(b)>))
apply (*simp* (*no-asm-simp*) *add*: *zmod-def divAlg-def*)
apply (*blast dest*: *zle-zless-trans*)
done

lemma *zdiv-pos-neg*:
⟦#0 $< a; b $< #0⟧
⟹ a zdiv b = fst (negateSnd(negDivAlg (<$−a, $−b>)))
apply (*simp* (*no-asm-simp*) *add*: *zdiv-def divAlg-def intify-eq-0-iff-zle*)
apply *auto*
apply (*blast dest*: *zle-zless-trans*)+
apply (*blast dest*: *zless-trans*)
apply (*blast intro*: *zless-imp-zle*)
done

lemma *zmod-pos-neg*:
⟦#0 $< a; b $< #0⟧
⟹ a zmod b = snd (negateSnd(negDivAlg (<$−a, $−b>)))
apply (*simp* (*no-asm-simp*) *add*: *zmod-def divAlg-def intify-eq-0-iff-zle*)
apply *auto*
apply (*blast dest*: *zle-zless-trans*)+
apply (*blast dest*: *zless-trans*)
apply (*blast intro*: *zless-imp-zle*)
done

lemma *zdiv-neg-neg*:
⟦a $< #0; b $≤ #0⟧
⟹ a zdiv b = fst (negateSnd(posDivAlg(<$−a, $−b>)))

apply (*simp* (*no-asm-simp*) *add*: *zdiv-def divAlg-def*)
apply *auto*
apply (*blast dest*!: *zle-zless-trans*)+
done

lemma *zmod-neg-neg*:
⟦*a* $< #*0*; *b* $≤ #*0*⟧
⟹ *a zmod b* = *snd* (*negateSnd*(*posDivAlg*(<$−*a*, $−*b*>)))
apply (*simp* (*no-asm-simp*) *add*: *zmod-def divAlg-def*)
apply *auto*
apply (*blast dest*!: *zle-zless-trans*)+
done

declare *zdiv-pos-pos* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zdiv-neg-pos* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zdiv-pos-neg* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zdiv-neg-neg* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zmod-pos-pos* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zmod-neg-pos* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zmod-pos-neg* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zmod-neg-neg* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *posDivAlg-eqn* [*of concl*: *integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *negDivAlg-eqn* [*of concl*: *integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*

lemma *zmod-1* [*simp*]: *a zmod* #*1* = #*0*
apply (*cut-tac a* = *a* **and** *b* = #*1* **in** *pos-mod-sign*)
apply (*cut-tac* [*2*] *a* = *a* **and** *b* = #*1* **in** *pos-mod-bound*)
apply *auto*

apply (*drule add1-zle-iff* [*THEN iffD2*])
apply (*rule zle-anti-sym*)
apply *auto*
done

lemma *zdiv-1* [*simp*]: *a zdiv* #*1* = *intify*(*a*)
apply (*cut-tac a* = *a* **and** *b* = #*1* **in** *zmod-zdiv-equality*)
apply *auto*
done

lemma *zmod-minus1-right* [*simp*]: *a zmod* #−*1* = #*0*
apply (*cut-tac a* = *a* **and** *b* = #−*1* **in** *neg-mod-sign*)
apply (*cut-tac* [*2*] *a* = *a* **and** *b* = #−*1* **in** *neg-mod-bound*)
apply *auto*

apply (*drule add1-zle-iff* [*THEN iffD2*])
apply (*rule zle-anti-sym*)

apply *auto*
done

lemma *zdiv-minus1-right-raw*: $a \in int \Longrightarrow a\ zdiv\ \#-1 = \$-a$
apply (*cut-tac* $a = a$ **and** $b = \#-1$ **in** *zmod-zdiv-equality*)
apply *auto*
apply (*rule equation-zminus* [*THEN iffD2*])
apply *auto*
done

lemma *zdiv-minus1-right*: $a\ zdiv\ \#-1 = \$-a$
apply (*cut-tac* $a = intify\ (a)$ **in** *zdiv-minus1-right-raw*)
apply *auto*
done
declare *zdiv-minus1-right* [*simp*]

33.8 Monotonicity in the first argument (divisor)

lemma *zdiv-mono1*: $[\![a \ \$\leq\ a';\ \ \#0\ \$<\ b]\!] \Longrightarrow a\ zdiv\ b\ \$\leq\ a'\ zdiv\ b$
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *zmod-zdiv-equality*)
apply (*cut-tac* $a = a'$ **and** $b = b$ **in** *zmod-zdiv-equality*)
apply (*rule unique-quotient-lemma*)
apply (*erule subst*)
apply (*erule subst*)
apply (*simp-all* (*no-asm-simp*) *add*: *pos-mod-sign pos-mod-bound*)
done

lemma *zdiv-mono1-neg*: $[\![a\ \$\leq\ a';\ \ b\ \$<\ \#0]\!] \Longrightarrow a'\ zdiv\ b\ \$\leq\ a\ zdiv\ b$
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *zmod-zdiv-equality*)
apply (*cut-tac* $a = a'$ **and** $b = b$ **in** *zmod-zdiv-equality*)
apply (*rule unique-quotient-lemma-neg*)
apply (*erule subst*)
apply (*erule subst*)
apply (*simp-all* (*no-asm-simp*) *add*: *neg-mod-sign neg-mod-bound*)
done

33.9 Monotonicity in the second argument (dividend)

lemma *q-pos-lemma*:
$[\![\#0\ \$\leq\ b'\$*q'\ \$+\ r';\ r'\ \$<\ b';\ \ \#0\ \$<\ b']\!] \Longrightarrow \#0\ \$\leq\ q'$
apply (*subgoal-tac* $\#0\ \$<\ b'\$*\ (q'\ \$+\ \#1)$)
apply (*simp add*: *int-0-less-mult-iff*)
apply (*blast dest*: *zless-trans intro*: *zless-add1-iff-zle* [*THEN iffD1*])
apply (*simp add*: *zadd-zmult-distrib2*)
apply (*erule zle-zless-trans*)
apply (*erule zadd-zless-mono2*)
done

lemma *zdiv-mono2-lemma*:
$[\![b\$*q\ \$+\ r = b'\$*q'\ \$+\ r';\ \ \#0\ \$\leq\ b'\$*q'\ \$+\ r';$

r' $< b'; #0 $≤ r; #0 $< b'; b' $≤ b⟧
⟹ q $≤ q'
apply (*frule q-pos-lemma*, *assumption*+)
apply (*subgoal-tac b$∗q $< b$∗ (q' $+ #1)*)
apply (*simp add*: *zmult-zless-cancel1*)
apply (*force dest*: *zless-add1-iff-zle* [*THEN iffD1*] *zless-trans zless-zle-trans*)
apply (*subgoal-tac b$∗q = r' $− r $+ b'$∗q'*)
prefer *2* **apply** (*simp add*: *zcompare-rls*)
apply (*simp* (*no-asm-simp*) *add*: *zadd-zmult-distrib2*)
apply (*subst zadd-commute* [*of b $∗ q'*], *rule zadd-zless-mono*)
prefer *2* **apply** (*blast intro*: *zmult-zle-mono1*)
apply (*subgoal-tac r' $+ #0 $< b $+ r*)
apply (*simp add*: *zcompare-rls*)
apply (*rule zadd-zless-mono*)
apply *auto*
apply (*blast dest*: *zless-zle-trans*)
done

lemma *zdiv-mono2-raw*:
⟦#0 $≤ *a*; #0 $< *b'*; *b'* $≤ *b*; *a* ∈ *int*⟧
⟹ *a zdiv b* $≤ *a zdiv b'*
apply (*subgoal-tac #0 $< b*)
prefer *2* **apply** (*blast dest*: *zless-zle-trans*)
apply (*cut-tac a = a* **and** *b = b* **in** *zmod-zdiv-equality*)
apply (*cut-tac a = a* **and** *b = b'* **in** *zmod-zdiv-equality*)
apply (*rule zdiv-mono2-lemma*)
apply (*erule subst*)
apply (*erule subst*)
apply (*simp-all add*: *pos-mod-sign pos-mod-bound*)
done

lemma *zdiv-mono2*:
⟦#0 $≤ *a*; #0 $< *b'*; *b'* $≤ *b*⟧
⟹ *a zdiv b* $≤ *a zdiv b'*
apply (*cut-tac a = intify* (*a*) **in** *zdiv-mono2-raw*)
apply *auto*
done

lemma *q-neg-lemma*:
⟦*b'*$∗*q'* $+ *r'* $< #0; #0 $≤ *r'*; #0 $< *b'*⟧ ⟹ *q'* $< #0
apply (*subgoal-tac b'$∗q' $< #0*)
prefer *2* **apply** (*force intro*: *zle-zless-trans*)
apply (*simp add*: *zmult-less-0-iff*)
apply (*blast dest*: *zless-trans*)
done

lemma *zdiv-mono2-neg-lemma*:
  ⟦b$*q $+ r = b'$*q' $+ r'; b'$*q' $+ r' $< #0;
    r $< b; #0 $≤ r'; #0 $< b'; b' $≤ b⟧
  ⟹ q' $≤ q

```
apply (subgoal-tac #0 $< b)
 prefer 2 apply (blast dest: zless-zle-trans)
apply (frule q-neg-lemma, assumption+)
apply (subgoal-tac b$*q' $< b$* (q $+ #1))
 apply (simp add: zmult-zless-cancel1)
 apply (blast dest: zless-trans zless-add1-iff-zle [THEN iffD1])
apply (simp (no-asm-simp) add: zadd-zmult-distrib2)
apply (subgoal-tac b$*q' $≤ b'$*q')
 prefer 2
 apply (simp add: zmult-zle-cancel2)
 apply (blast dest: zless-trans)
apply (subgoal-tac b'$*q' $+ r $< b $+ (b$*q $+ r))
 prefer 2
 apply (erule ssubst)
 apply simp
 apply (drule-tac w' = r and z' = #0 in zadd-zless-mono)
  apply (assumption)
 apply simp
apply (simp (no-asm-use) add: zadd-commute)
apply (rule zle-zless-trans)
 prefer 2 apply (assumption)
apply (simp (no-asm-simp) add: zmult-zle-cancel2)
apply (blast dest: zless-trans)
done
```

lemma *zdiv-mono2-neg-raw*:
  ⟦a $< #0; #0 $< b'; b' $≤ b; a ∈ int⟧
  ⟹ a zdiv b' $≤ a zdiv b

```
apply (subgoal-tac #0 $< b)
 prefer 2 apply (blast dest: zless-zle-trans)
apply (cut-tac a = a and b = b in zmod-zdiv-equality)
apply (cut-tac a = a and b = b' in zmod-zdiv-equality)
apply (rule zdiv-mono2-neg-lemma)
apply (erule subst)
apply (erule subst)
apply (simp-all add: pos-mod-sign pos-mod-bound)
done
```

lemma *zdiv-mono2-neg*: ⟦a $< #0; #0 $< b'; b' $≤ b⟧
  ⟹ a zdiv b' $≤ a zdiv b

```
apply (cut-tac a = intify (a) in zdiv-mono2-neg-raw)
apply auto
done
```

33.10 More algebraic laws for zdiv and zmod

lemma *zmult1-lemma*:
  ⟦*quorem*(⟨*b*,*c*⟩, ⟨*q*,*r*⟩); *c* ∈ *int*; *c* ≠ #*0*⟧
  ⟹ *quorem* (<*a*$∗*b*, *c*>, <*a*$∗*q* $+ (*a*$∗*r*) *zdiv c*, (*a*$∗*r*) *zmod c*>)
apply (*auto simp add*: *split-ifs quorem-def neq-iff-zless zadd-zmult-distrib2*
      pos-mod-sign pos-mod-bound neg-mod-sign neg-mod-bound)
apply (*auto intro*: *raw-zmod-zdiv-equality*)
done

lemma *zdiv-zmult1-eq-raw*:
  ⟦*b* ∈ *int*; *c* ∈ *int*⟧
  ⟹ (*a*$∗*b*) *zdiv c* = *a*$∗(*b zdiv c*) $+ *a*$∗(*b zmod c*) *zdiv c*
apply (*case-tac c* = #*0*)
 apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*rule quorem-div-mod* [*THEN zmult1-lemma*, *THEN quorem-div*])
apply *auto*
done

lemma *zdiv-zmult1-eq*: (*a*$∗*b*) *zdiv c* = *a*$∗(*b zdiv c*) $+ *a*$∗(*b zmod c*) *zdiv c*
apply (*cut-tac b* = *intify* (*b*) **and** *c* = *intify* (*c*) **in** *zdiv-zmult1-eq-raw*)
apply *auto*
done

lemma *zmod-zmult1-eq-raw*:
  ⟦*b* ∈ *int*; *c* ∈ *int*⟧ ⟹ (*a*$∗*b*) *zmod c* = *a*$∗(*b zmod c*) *zmod c*
apply (*case-tac c* = #*0*)
 apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*rule quorem-div-mod* [*THEN zmult1-lemma*, *THEN quorem-mod*])
apply *auto*
done

lemma *zmod-zmult1-eq*: (*a*$∗*b*) *zmod c* = *a*$∗(*b zmod c*) *zmod c*
apply (*cut-tac b* = *intify* (*b*) **and** *c* = *intify* (*c*) **in** *zmod-zmult1-eq-raw*)
apply *auto*
done

lemma *zmod-zmult1-eq'*: (*a*$∗*b*) *zmod c* = ((*a zmod c*) $∗ *b*) *zmod c*
apply (*rule trans*)
apply (*rule-tac b* = (*b* $∗ *a*) *zmod c* **in** *trans*)
apply (*rule-tac* [*2*] *zmod-zmult1-eq*)
apply (*simp-all* (*no-asm*) *add*: *zmult-commute*)
done

lemma *zmod-zmult-distrib*: (*a*$∗*b*) *zmod c* = ((*a zmod c*) $∗ (*b zmod c*)) *zmod c*
apply (*rule zmod-zmult1-eq'* [*THEN trans*])
apply (*rule zmod-zmult1-eq*)
done

lemma *zdiv-zmult-self1* [*simp*]: *intify*(*b*) ≠ #*0* ⟹ (*a*$∗*b*) *zdiv b* = *intify*(*a*)

by (*simp add*: *zdiv-zmult1-eq*)

lemma *zdiv-zmult-self2* [*simp*]: *intify*(*b*) ≠ *#0* ⟹ (*b*$∗*a*) *zdiv b* = *intify*(*a*)
by (*simp add*: *zmult-commute*)

lemma *zmod-zmult-self1* [*simp*]: (*a*$∗*b*) *zmod b* = *#0*
by (*simp add*: *zmod-zmult1-eq*)

lemma *zmod-zmult-self2* [*simp*]: (*b*$∗*a*) *zmod b* = *#0*
by (*simp add*: *zmult-commute zmod-zmult1-eq*)

lemma *zadd1-lemma*:
⟦*quorem*(⟨*a*,*c*⟩, ⟨*aq*,*ar*⟩); *quorem*(⟨*b*,*c*⟩, ⟨*bq*,*br*⟩);
c ∈ *int*; *c* ≠ *#0*⟧
⟹ *quorem* (<*a*$+*b*, *c*>, <*aq* $+ *bq* $+ (*ar*$+*br*) *zdiv c*, (*ar*$+*br*) *zmod c*>)
apply (*auto simp add*: *split-ifs quorem-def neq-iff-zless zadd-zmult-distrib2*
pos-mod-sign pos-mod-bound neg-mod-sign neg-mod-bound)
apply (*auto intro*: *raw-zmod-zdiv-equality*)
done

lemma *zdiv-zadd1-eq-raw*:
⟦*a* ∈ *int*; *b* ∈ *int*; *c* ∈ *int*⟧ ⟹
(*a*$+*b*) *zdiv c* = *a zdiv c* $+ *b zdiv c* $+ ((*a zmod c* $+ *b zmod c*) *zdiv c*)
apply (*case-tac c* = *#0*)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*blast intro*: *zadd1-lemma* [*OF quorem-div-mod quorem-div-mod*,
THEN quorem-div])
done

lemma *zdiv-zadd1-eq*:
(*a*$+*b*) *zdiv c* = *a zdiv c* $+ *b zdiv c* $+ ((*a zmod c* $+ *b zmod c*) *zdiv c*)
apply (*cut-tac a* = *intify* (*a*) **and** *b* = *intify* (*b*) **and** *c* = *intify* (*c*)
in *zdiv-zadd1-eq-raw*)
apply *auto*
done

lemma *zmod-zadd1-eq-raw*:
⟦*a* ∈ *int*; *b* ∈ *int*; *c* ∈ *int*⟧
⟹ (*a*$+*b*) *zmod c* = (*a zmod c* $+ *b zmod c*) *zmod c*
apply (*case-tac c* = *#0*)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*blast intro*: *zadd1-lemma* [*OF quorem-div-mod quorem-div-mod*,
THEN quorem-mod])
done

lemma *zmod-zadd1-eq*: (*a*$+*b*) *zmod c* = (*a zmod c* $+ *b zmod c*) *zmod c*
apply (*cut-tac a* = *intify* (*a*) **and** *b* = *intify* (*b*) **and** *c* = *intify* (*c*)
  in *zmod-zadd1-eq-raw*)
apply *auto*
done

lemma *zmod-div-trivial-raw*:
 ⟦*a* ∈ *int*; *b* ∈ *int*⟧ ⟹ (*a zmod b*) *zdiv b* = *#0*
apply (*case-tac b* = *#0*)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*auto simp add*: *neq-iff-zless pos-mod-sign pos-mod-bound*
  zdiv-pos-pos-trivial neg-mod-sign neg-mod-bound zdiv-neg-neg-trivial)
done

lemma *zmod-div-trivial* [*simp*]: (*a zmod b*) *zdiv b* = *#0*
apply (*cut-tac a* = *intify* (*a*) **and** *b* = *intify* (*b*) **in** *zmod-div-trivial-raw*)
apply *auto*
done

lemma *zmod-mod-trivial-raw*:
 ⟦*a* ∈ *int*; *b* ∈ *int*⟧ ⟹ (*a zmod b*) *zmod b* = *a zmod b*
apply (*case-tac b* = *#0*)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*auto simp add*: *neq-iff-zless pos-mod-sign pos-mod-bound*
  zmod-pos-pos-trivial neg-mod-sign neg-mod-bound zmod-neg-neg-trivial)
done

lemma *zmod-mod-trivial* [*simp*]: (*a zmod b*) *zmod b* = *a zmod b*
apply (*cut-tac a* = *intify* (*a*) **and** *b* = *intify* (*b*) **in** *zmod-mod-trivial-raw*)
apply *auto*
done

lemma *zmod-zadd-left-eq*: (*a*$+*b*) *zmod c* = ((*a zmod c*) $+ *b*) *zmod c*
apply (*rule trans* [*symmetric*])
apply (*rule zmod-zadd1-eq*)
apply (*simp* (*no-asm*))
apply (*rule zmod-zadd1-eq* [*symmetric*])
done

lemma *zmod-zadd-right-eq*: (*a*$+*b*) *zmod c* = (*a* $+ (*b zmod c*)) *zmod c*
apply (*rule trans* [*symmetric*])
apply (*rule zmod-zadd1-eq*)
apply (*simp* (*no-asm*))
apply (*rule zmod-zadd1-eq* [*symmetric*])
done

lemma *zdiv-zadd-self1* [*simp*]:
 intify(*a*) ≠ *#0* ⟹ (*a*$+*b*) *zdiv a* = *b zdiv a* $+ *#1*

by (*simp* (*no-asm-simp*) *add*: *zdiv-zadd1-eq*)

lemma *zdiv-zadd-self2* [*simp*]:
intify(*a*) ≠ *#0* ⟹ (*b*$+*a*) *zdiv a* = *b zdiv a* $+ *#1*
by (*simp* (*no-asm-simp*) *add*: *zdiv-zadd1-eq*)

lemma *zmod-zadd-self1* [*simp*]: (*a*$+*b*) *zmod a* = *b zmod a*
apply (*case-tac a* = *#0*)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*simp* (*no-asm-simp*) *add*: *zmod-zadd1-eq*)
done

lemma *zmod-zadd-self2* [*simp*]: (*b*$+*a*) *zmod a* = *b zmod a*
apply (*case-tac a* = *#0*)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*simp* (*no-asm-simp*) *add*: *zmod-zadd1-eq*)
done

33.11 proving a zdiv (b*c) = (a zdiv b) zdiv c

lemma *zdiv-zmult2-aux1*:
⟦*#0* $< *c*; *b* $< *r*; *r* $≤ *#0*⟧ ⟹ *b*$∗*c* $< *b*$∗(*q zmod c*) $+ *r*
apply (*subgoal-tac b* $∗ (*c* $− *q zmod c*) $< *r* $∗ *#1*)
apply (*simp add*: *zdiff-zmult-distrib2 zadd-commute zcompare-rls*)
apply (*rule zle-zless-trans*)
apply (*erule-tac* [*2*] *zmult-zless-mono1*)
apply (*rule zmult-zle-mono2-neg*)
apply (*auto simp add*: *zcompare-rls zadd-commute add1-zle-iff pos-mod-bound*)
apply (*blast intro*: *zless-imp-zle dest*: *zless-zle-trans*)
done

lemma *zdiv-zmult2-aux2*:
⟦*#0* $< *c*; *b* $< *r*; *r* $≤ *#0*⟧ ⟹ *b* $∗ (*q zmod c*) $+ *r* $≤ *#0*
apply (*subgoal-tac b* $∗ (*q zmod c*) $≤ *#0*)
prefer *2*
apply (*simp add*: *zmult-le-0-iff pos-mod-sign*)
apply (*blast intro*: *zless-imp-zle dest*: *zless-zle-trans*)

apply (*drule zadd-zle-mono*)
apply *assumption*
apply (*simp add*: *zadd-commute*)
done

lemma *zdiv-zmult2-aux3*:
⟦*#0* $< *c*; *#0* $≤ *r*; *r* $< *b*⟧ ⟹ *#0* $≤ *b* $∗ (*q zmod c*) $+ *r*
apply (*subgoal-tac #0* $≤ *b* $∗ (*q zmod c*))
prefer *2*
apply (*simp add*: *int-0-le-mult-iff pos-mod-sign*)
apply (*blast intro*: *zless-imp-zle dest*: *zle-zless-trans*)

apply (*drule zadd-zle-mono*)
apply *assumption*
apply (*simp add*: *zadd-commute*)
done

lemma *zdiv-zmult2-aux4*:
⟦*#0* $< *c*; *#0* $≤ *r*; *r* $< *b*⟧ ⟹ *b* $∗ (*q zmod c*) $+ *r* $< *b* $∗ *c*
apply (*subgoal-tac r* $∗ *#1* $< *b* $∗ (*c* $− *q zmod c*))
apply (*simp add*: *zdiff-zmult-distrib2 zadd-commute zcompare-rls*)
apply (*rule zless-zle-trans*)
apply (*erule zmult-zless-mono1*)
apply (*rule-tac* [*2*] *zmult-zle-mono2*)
apply (*auto simp add*: *zcompare-rls zadd-commute add1-zle-iff pos-mod-bound*)
apply (*blast intro*: *zless-imp-zle dest*: *zle-zless-trans*)
done

lemma *zdiv-zmult2-lemma*:
⟦*quorem* (⟨*a*,*b*⟩, ⟨*q*,*r*⟩); *a* ∈ *int*; *b* ∈ *int*; *b* ≠ *#0*; *#0* $< *c*⟧
⟹ *quorem* (<*a*,*b*$∗*c*>, <*q zdiv c*, *b*$∗(*q zmod c*) $+ *r*>)
apply (*auto simp add*: *zmult-ac zmod-zdiv-equality* [*symmetric*] *quorem-def*
neq-iff-zless int-0-less-mult-iff
zadd-zmult-distrib2 [*symmetric*] *zdiv-zmult2-aux1 zdiv-zmult2-aux2*
zdiv-zmult2-aux3 zdiv-zmult2-aux4)
apply (*blast dest*: *zless-trans*)+
done

lemma *zdiv-zmult2-eq-raw*:
⟦*#0* $< *c*; *a* ∈ *int*; *b* ∈ *int*⟧ ⟹ *a zdiv* (*b*$∗*c*) = (*a zdiv b*) *zdiv c*
apply (*case-tac b* = *#0*)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*rule quorem-div-mod* [*THEN zdiv-zmult2-lemma*, *THEN quorem-div*])
apply (*auto simp add*: *intify-eq-0-iff-zle*)
apply (*blast dest*: *zle-zless-trans*)
done

lemma *zdiv-zmult2-eq*: *#0* $< *c* ⟹ *a zdiv* (*b*$∗*c*) = (*a zdiv b*) *zdiv c*
apply (*cut-tac a* = *intify* (*a*) **and** *b* = *intify* (*b*) **in** *zdiv-zmult2-eq-raw*)
apply *auto*
done

lemma *zmod-zmult2-eq-raw*:
⟦*#0* $< *c*; *a* ∈ *int*; *b* ∈ *int*⟧
⟹ *a zmod* (*b*$∗*c*) = *b*$∗(*a zdiv b zmod c*) $+ *a zmod b*
apply (*case-tac b* = *#0*)
apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*rule quorem-div-mod* [*THEN zdiv-zmult2-lemma*, *THEN quorem-mod*])
apply (*auto simp add*: *intify-eq-0-iff-zle*)
apply (*blast dest*: *zle-zless-trans*)

done

lemma *zmod-zmult2-eq*:
  $\#0 \ \$< \ c \Longrightarrow a \ zmod \ (b\$*c) = b\$*(a \ zdiv \ b \ zmod \ c) \ \$+ \ a \ zmod \ b$
apply (*cut-tac a = intify (a)* **and** *b = intify (b)* **in** *zmod-zmult2-eq-raw*)
apply *auto*
done

33.12 Cancellation of common factors in "zdiv"

lemma *zdiv-zmult-zmult1-aux1*:
  $[\![\#0 \ \$< \ b; \ intify(c) \neq \#0]\!] \Longrightarrow (c\$*a) \ zdiv \ (c\$*b) = a \ zdiv \ b$
apply (*subst zdiv-zmult2-eq*)
apply *auto*
done

lemma *zdiv-zmult-zmult1-aux2*:
  $[\![b \ \$< \ \#0; \ intify(c) \neq \#0]\!] \Longrightarrow (c\$*a) \ zdiv \ (c\$*b) = a \ zdiv \ b$
apply (*subgoal-tac* $(c \ \$* \ (\$-a)) \ zdiv \ (c \ \$* \ (\$-b)) = (\$-a) \ zdiv \ (\$-b)$)
apply (*rule-tac* [*2*] *zdiv-zmult-zmult1-aux1*)
apply *auto*
done

lemma *zdiv-zmult-zmult1-raw*:
  $[\![intify(c) \neq \#0; \ b \in int]\!] \Longrightarrow (c\$*a) \ zdiv \ (c\$*b) = a \ zdiv \ b$
apply (*case-tac b = #0*)
 apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*auto simp add*: *neq-iff-zless* [*of b*]
 zdiv-zmult-zmult1-aux1 zdiv-zmult-zmult1-aux2)
done

lemma *zdiv-zmult-zmult1*: $intify(c) \neq \#0 \Longrightarrow (c\$*a) \ zdiv \ (c\$*b) = a \ zdiv \ b$
apply (*cut-tac b = intify (b)* **in** *zdiv-zmult-zmult1-raw*)
apply *auto*
done

lemma *zdiv-zmult-zmult2*: $intify(c) \neq \#0 \Longrightarrow (a\$*c) \ zdiv \ (b\$*c) = a \ zdiv \ b$
apply (*drule zdiv-zmult-zmult1*)
apply (*auto simp add*: *zmult-commute*)
done

33.13 Distribution of factors over "zmod"

lemma *zmod-zmult-zmult1-aux1*:
  $[\![\#0 \ \$< \ b; \ intify(c) \neq \#0]\!]$
  $\Longrightarrow (c\$*a) \ zmod \ (c\$*b) = c \ \$* \ (a \ zmod \ b)$
apply (*subst zmod-zmult2-eq*)
apply *auto*
done

lemma *zmod-zmult-zmult1-aux2*:
  $[\![b \; \$< \; \#0; \;\; intify(c) \neq \#0]\!]$
  $\Longrightarrow (c\$*a) \; zmod \; (c\$*b) = c \; \$* \; (a \; zmod \; b)$
apply $(subgoal\text{-}tac \;\; (c \; \$* \; (\$-a)) \; zmod \; (c \; \$* \; (\$-b)) = c \; \$* \; ((\$-a) \; zmod \; (\$-b)))$
apply (*rule-tac* [*2*] *zmod-zmult-zmult1-aux1*)
apply *auto*
done

lemma *zmod-zmult-zmult1-raw*:
  $[\![b \in int; \; c \in int]\!] \Longrightarrow (c\$*a) \; zmod \; (c\$*b) = c \; \$* \; (a \; zmod \; b)$
apply $(case\text{-}tac \; b = \#0)$
 apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply $(case\text{-}tac \; c = \#0)$
 apply (*simp add*: *DIVISION-BY-ZERO-ZDIV DIVISION-BY-ZERO-ZMOD*)
apply (*auto simp add*: *neq-iff-zless* [*of b*]
  zmod-zmult-zmult1-aux1 zmod-zmult-zmult1-aux2)
done

lemma *zmod-zmult-zmult1*: $(c\$*a) \; zmod \; (c\$*b) = c \; \$* \; (a \; zmod \; b)$
apply (*cut-tac* $b = intify \; (b)$ **and** $c = intify \; (c)$ **in** *zmod-zmult-zmult1-raw*)
apply *auto*
done

lemma *zmod-zmult-zmult2*: $(a\$*c) \; zmod \; (b\$*c) = (a \; zmod \; b) \; \$* \; c$
apply (*cut-tac* $c = c$ **in** *zmod-zmult-zmult1*)
apply (*auto simp add*: *zmult-commute*)
done

lemma *zdiv-neg-pos-less0*: $[\![a \; \$< \; \#0; \;\; \#0 \; \$< \; b]\!] \Longrightarrow a \; zdiv \; b \; \$< \; \#0$
apply $(subgoal\text{-}tac \; a \; zdiv \; b \; \$\leq \; \#-1)$
apply (*erule zle-zless-trans*)
apply (*simp* (*no-asm*))
apply (*rule zle-trans*)
apply (*rule-tac* $a' = \#-1$ **in** *zdiv-mono1*)
apply (*rule zless-add1-iff-zle* [*THEN iffD1*])
apply (*simp* (*no-asm*))
apply (*auto simp add*: *zdiv-minus1*)
done

lemma *zdiv-nonneg-neg-le0*: $[\![\#0 \; \$\leq \; a; \;\; b \; \$< \; \#0]\!] \Longrightarrow a \; zdiv \; b \; \$\leq \; \#0$
apply (*drule zdiv-mono1-neg*)
apply *auto*
done

lemma *pos-imp-zdiv-nonneg-iff*: $\#0 \; \$< \; b \Longrightarrow (\#0 \; \$\leq \; a \; zdiv \; b) \longleftrightarrow (\#0 \; \$\leq \; a)$
apply *auto*

apply (*drule-tac* [*2*] *zdiv-mono1*)
apply (*auto simp add*: *neq-iff-zless*)
apply (*simp* (*no-asm-use*) *add*: *not-zless-iff-zle* [*THEN iff-sym*])
apply (*blast intro*: *zdiv-neg-pos-less0*)
done

lemma *neg-imp-zdiv-nonneg-iff*: *b* $< *#0* ⟹ (*#0* $≤ *a zdiv b*) ⟷ (*a* $≤ *#0*)
apply (*subst zdiv-zminus-zminus* [*symmetric*])
apply (*rule iff-trans*)
apply (*rule pos-imp-zdiv-nonneg-iff*)
apply *auto*
done

lemma *pos-imp-zdiv-neg-iff*: *#0* $< *b* ⟹ (*a zdiv b* $< *#0*) ⟷ (*a* $< *#0*)
apply (*simp* (*no-asm-simp*) *add*: *not-zle-iff-zless* [*THEN iff-sym*])
apply (*erule pos-imp-zdiv-nonneg-iff*)
done

lemma *neg-imp-zdiv-neg-iff*: *b* $< *#0* ⟹ (*a zdiv b* $< *#0*) ⟷ (*#0* $< *a*)
apply (*simp* (*no-asm-simp*) *add*: *not-zle-iff-zless* [*THEN iff-sym*])
apply (*erule neg-imp-zdiv-nonneg-iff*)
done

end

34 Cardinal Arithmetic Without the Axiom of Choice

theory *CardinalArith* **imports** *Cardinal OrderArith ArithSimp Finite* **begin**

definition
  InfCard :: *i*⇒*o* **where**
    InfCard(*i*) ≡ *Card*(*i*) ∧ *nat* ≤ *i*

definition
  cmult :: [*i*,*i*]⇒*i* (**infixl** ‹⊗› *70*) **where**
    i ⊗ *j* ≡ |*i*∗*j*|

definition
  cadd :: [*i*,*i*]⇒*i* (**infixl** ‹⊕› *65*) **where**
    i ⊕ *j* ≡ |*i*+*j*|

definition
  csquare-rel :: *i*⇒*i* **where**
    csquare-rel(*K*) ≡
        rvimage(*K*∗*K*,

lam ⟨*x*,*y*⟩:*K*∗*K*. <*x* ∪ *y*, *x*, *y*>,
rmult(*K*,*Memrel*(*K*), *K*∗*K*, *rmult*(*K*,*Memrel*(*K*), *K*,*Memrel*(*K*))))

definition
jump-cardinal :: *i*⇒*i* **where**
— This definition is more complex than Kunen's but it more easily proved to be a cardinal
jump-cardinal(*K*) ≡
⋃*X*∈*Pow*(*K*). {*z*. *r* ∈ *Pow*(*K*∗*K*), *well-ord*(*X*,*r*) ∧ *z* = *ordertype*(*X*,*r*)}

definition
csucc :: *i*⇒*i* **where**
— needed because *jump-cardinal*(*K*) might not be the successor of *K*
csucc(*K*) ≡ *μ* *L*. *Card*(*L*) ∧ *K*<*L*

lemma *Card-Union* [*simp*,*intro*,*TC*]:
assumes *A*: ⋀*x*. *x*∈*A* ⟹ *Card*(*x*) **shows** *Card*(⋃(*A*))
proof (*rule CardI*)
show *Ord*(⋃*A*) **using** *A*
by (*simp add*: *Card-is-Ord*)
next
fix *j*
assume *j*: *j* < ⋃*A*
hence ∃ *c*∈*A*. *j* < *c* ∧ *Card*(*c*) **using** *A*
by (*auto simp add*: *lt-def intro*: *Card-is-Ord*)
then obtain *c* **where** *c*: *c*∈*A* *j* < *c* *Card*(*c*)
by *blast*
hence *jls*: *j* ≺ *c*
by (*simp add*: *lt-Card-imp-lesspoll*)
{ assume *eqp*: *j* ≈ ⋃*A*
have *c* ≲ ⋃*A* **using** *c*
by (*blast intro*: *subset-imp-lepoll*)
also have ... ≈ *j* **by** (*rule eqpoll-sym* [*OF eqp*])
also have ... ≺ *c* **by** (*rule jls*)
finally have *c* ≺ *c* **.**
hence *False*
by *auto*
} thus ¬ *j* ≈ ⋃*A* **by** *blast*
qed

lemma *Card-UN*: (⋀*x*. *x* ∈ *A* ⟹ *Card*(*K*(*x*))) ⟹ *Card*(⋃*x*∈*A*. *K*(*x*))
by *blast*

lemma *Card-OUN* [*simp*,*intro*,*TC*]:
(⋀*x*. *x* ∈ *A* ⟹ *Card*(*K*(*x*))) ⟹ *Card*(⋃*x*<*A*. *K*(*x*))
by (*auto simp add*: *OUnion-def Card-0*)

lemma *in-Card-imp-lesspoll*: ⟦*Card*(*K*); *b* ∈ *K*⟧ ⟹ *b* ≺ *K*

unfolding *lesspoll-def*
apply (*simp add*: *Card-iff-initial*)
apply (*fast intro*!: *le-imp-lepoll ltI leI*)
done

34.1 Cardinal addition

Note: Could omit proving the algebraic laws for cardinal addition and multiplication. On finite cardinals these operations coincide with addition and multiplication of natural numbers; on infinite cardinals they coincide with union (maximum). Either way we get most laws for free.

34.1.1 Cardinal addition is commutative

lemma *sum-commute-eqpoll*: $A+B \approx B+A$
proof (*unfold eqpoll-def*, *rule exI*)
show $(\lambda z\in A+B.\ case(Inr,Inl,z)) \in bij(A+B,\ B+A)$
by (*auto intro*: *lam-bijective* [**where** $d = case(Inr,Inl)$])
qed

lemma *cadd-commute*: $i \oplus j = j \oplus i$
unfolding *cadd-def*
apply (*rule sum-commute-eqpoll* [*THEN cardinal-cong*])
done

34.1.2 Cardinal addition is associative

lemma *sum-assoc-eqpoll*: $(A+B)+C \approx A+(B+C)$
unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule sum-assoc-bij*)
done

Unconditional version requires AC

lemma *well-ord-cadd-assoc*:
assumes *i*: *well-ord*(*i*,*ri*) **and** *j*: *well-ord*(*j*,*rj*) **and** *k*: *well-ord*(*k*,*rk*)
shows $(i \oplus j) \oplus k = i \oplus (j \oplus k)$
proof (*unfold cadd-def*, *rule cardinal-cong*)
have $|i + j| + k \approx (i + j) + k$
by (*blast intro*: *sum-eqpoll-cong well-ord-cardinal-eqpoll eqpoll-refl well-ord-radd i j*)
also have ... $\approx i + (j + k)$
by (*rule sum-assoc-eqpoll*)
also have ... $\approx i + |j + k|$
by (*blast intro*: *sum-eqpoll-cong well-ord-cardinal-eqpoll eqpoll-refl well-ord-radd j k eqpoll-sym*)
finally show $|i + j| + k \approx i + |j + k|$ **.**
qed

34.1.3 0 is the identity for addition

lemma *sum-0-eqpoll*: $0+A \approx A$
 unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule bij-0-sum*)
done

lemma *cadd-0* [*simp*]: $Card(K) \Longrightarrow 0 \oplus K = K$
 unfolding *cadd-def*
apply (*simp add*: *sum-0-eqpoll* [*THEN cardinal-cong*] *Card-cardinal-eq*)
done

34.1.4 Addition by another cardinal

lemma *sum-lepoll-self*: $A \lesssim A+B$
proof (*unfold lepoll-def*, *rule exI*)
 show $(\lambda x \in A.\ Inl\ (x)) \in inj(A,\ A + B)$
  by (*simp add*: *inj-def*)
qed

lemma *cadd-le-self*:
 assumes K: $Card(K)$ **and** L: $Ord(L)$ **shows** $K \le (K \oplus L)$
proof (*unfold cadd-def*)
 have $K \le |K|$
  by (*rule Card-cardinal-le* [*OF K*])
 moreover have $|K| \le |K + L|$ **using** $K\ L$
  by (*blast intro*: *well-ord-lepoll-imp-cardinal-le sum-lepoll-self*
   well-ord-radd well-ord-Memrel Card-is-Ord)
 ultimately show $K \le |K + L|$
  by (*blast intro*: *le-trans*)
qed

34.1.5 Monotonicity of addition

lemma *sum-lepoll-mono*:
  $[\![A \lesssim C;\ \ B \lesssim D]\!] \Longrightarrow A + B \lesssim C + D$
 unfolding *lepoll-def*
apply (*elim exE*)
apply (*rule-tac* $x = \lambda z \in A+B.\ case\ (\lambda w.\ Inl(f`w),\ \lambda y.\ Inr(fa`y),\ z)$ **in** *exI*)
apply (*rule-tac* $d = case\ (\lambda w.\ Inl(converse(f)\ `w),\ \lambda y.\ Inr(converse(fa)\ `\ y))$
  in *lam-injective*)
apply (*typecheck add*: *inj-is-fun*, *auto*)
done

lemma *cadd-le-mono*:
  $[\![K' \le K;\ \ L' \le L]\!] \Longrightarrow (K' \oplus L') \le (K \oplus L)$
 unfolding *cadd-def*

apply (*safe dest*!: *le-subset-iff* [*THEN iffD1*])
apply (*rule well-ord-lepoll-imp-cardinal-le*)
apply (*blast intro*: *well-ord-radd well-ord-Memrel*)
apply (*blast intro*: *sum-lepoll-mono subset-imp-lepoll*)
done

34.1.6 Addition of finite cardinals is "ordinary" addition

lemma *sum-succ-eqpoll*: *succ*(*A*)+*B* ≈ *succ*(*A*+*B*)
  unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule-tac c* = λ*z*. *if z*=*Inl* (*A*) *then A*+*B else z*
      and *d* = λ*z*. *if z*=*A*+*B then Inl* (*A*) *else z* **in** *lam-bijective*)
   apply *simp-all*
apply (*blast dest*: *sym* [*THEN eq-imp-not-mem*] *elim*: *mem-irrefl*)+
done

lemma *cadd-succ-lemma*:
  assumes *Ord*(*m*) *Ord*(*n*) **shows** *succ*(*m*) ⊕ *n* = |*succ*(*m* ⊕ *n*)|
proof (*unfold cadd-def*)
  have [*intro*]: *m* + *n* ≈ |*m* + *n*| **using** *assms*
   by (*blast intro*: *eqpoll-sym well-ord-cardinal-eqpoll well-ord-radd well-ord-Memrel*)

  have |*succ*(*m*) + *n*| = |*succ*(*m* + *n*)|
    by (*rule sum-succ-eqpoll* [*THEN cardinal-cong*])
  also have ... = |*succ*(|*m* + *n*|)|
    by (*blast intro*: *succ-eqpoll-cong cardinal-cong*)
  finally show |*succ*(*m*) + *n*| = |*succ*(|*m* + *n*|)| **.**
qed

lemma *nat-cadd-eq-add*:
  assumes *m*: *m* ∈ *nat* **and** [*simp*]: *n* ∈ *nat* **shows***m* ⊕ *n* = *m* #+ *n*
using *m*
proof (*induct m*)
  case *0* **thus** *?case* **by** (*simp add*: *nat-into-Card cadd-0*)
next
 case (*succ m*) **thus** *?case* **by** (*simp add*: *cadd-succ-lemma nat-into-Card Card-cardinal-eq*)
qed

34.2 Cardinal multiplication

34.2.1 Cardinal multiplication is commutative

lemma *prod-commute-eqpoll*: *A*∗*B* ≈ *B*∗*A*
  unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule-tac c* = λ⟨*x*,*y*⟩.⟨*y*,*x*⟩ **and** *d* = λ⟨*x*,*y*⟩.⟨*y*,*x*⟩ **in** *lam-bijective*,
    auto)

done

lemma *cmult-commute*: $i \otimes j = j \otimes i$
unfolding *cmult-def*
apply (*rule prod-commute-eqpoll* [*THEN cardinal-cong*])
done

34.2.2 Cardinal multiplication is associative

lemma *prod-assoc-eqpoll*: $(A*B)*C \approx A*(B*C)$
unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule prod-assoc-bij*)
done

Unconditional version requires AC

lemma *well-ord-cmult-assoc*:
assumes *i*: *well-ord*(*i*,*ri*) **and** *j*: *well-ord*(*j*,*rj*) **and** *k*: *well-ord*(*k*,*rk*)
shows $(i \otimes j) \otimes k = i \otimes (j \otimes k)$
proof (*unfold cmult-def*, *rule cardinal-cong*)
have $|i * j| * k \approx (i * j) * k$
by (*blast intro*: *prod-eqpoll-cong well-ord-cardinal-eqpoll eqpoll-refl well-ord-rmult i j*)
also have ... $\approx i * (j * k)$
by (*rule prod-assoc-eqpoll*)
also have ... $\approx i * |j * k|$
by (*blast intro*: *prod-eqpoll-cong well-ord-cardinal-eqpoll eqpoll-refl well-ord-rmult j k eqpoll-sym*)
finally show $|i * j| * k \approx i * |j * k|$ **.**
qed

34.2.3 Cardinal multiplication distributes over addition

lemma *sum-prod-distrib-eqpoll*: $(A+B)*C \approx (A*C)+(B*C)$
unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule sum-prod-distrib-bij*)
done

lemma *well-ord-cadd-cmult-distrib*:
assumes *i*: *well-ord*(*i*,*ri*) **and** *j*: *well-ord*(*j*,*rj*) **and** *k*: *well-ord*(*k*,*rk*)
shows $(i \oplus j) \otimes k = (i \otimes k) \oplus (j \otimes k)$
proof (*unfold cadd-def cmult-def*, *rule cardinal-cong*)
have $|i + j| * k \approx (i + j) * k$
by (*blast intro*: *prod-eqpoll-cong well-ord-cardinal-eqpoll eqpoll-refl well-ord-radd i j*)
also have ... $\approx i * k + j * k$
by (*rule sum-prod-distrib-eqpoll*)
also have ... $\approx |i * k| + |j * k|$

by (*blast intro*: *sum-eqpoll-cong well-ord-cardinal-eqpoll well-ord-rmult i j k eqpoll-sym*)
finally show $|i + j| * k \approx |i * k| + |j * k|$ **.**
qed

34.2.4 Multiplication by 0 yields 0

lemma *prod-0-eqpoll*: $0{*}A \approx 0$
unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule lam-bijective*, *safe*)
done

lemma *cmult-0* [*simp*]: $0 \otimes i = 0$
by (*simp add*: *cmult-def prod-0-eqpoll* [*THEN cardinal-cong*])

34.2.5 1 is the identity for multiplication

lemma *prod-singleton-eqpoll*: $\{x\}{*}A \approx A$
unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule singleton-prod-bij* [*THEN bij-converse-bij*])
done

lemma *cmult-1* [*simp*]: $Card(K) \Longrightarrow 1 \otimes K = K$
unfolding *cmult-def succ-def*
apply (*simp add*: *prod-singleton-eqpoll* [*THEN cardinal-cong*] *Card-cardinal-eq*)
done

34.3 Some inequalities for multiplication

lemma *prod-square-lepoll*: $A \lesssim A{*}A$
unfolding *lepoll-def inj-def*
apply (*rule-tac* $x = \lambda x{\in}A.\ \langle x,x\rangle$ **in** *exI*, *simp*)
done

lemma *cmult-square-le*: $Card(K) \Longrightarrow K \le K \otimes K$
unfolding *cmult-def*
apply (*rule le-trans*)
apply (*rule-tac* [*2*] *well-ord-lepoll-imp-cardinal-le*)
apply (*rule-tac* [*3*] *prod-square-lepoll*)
apply (*simp add*: *le-refl Card-is-Ord Card-cardinal-eq*)
apply (*blast intro*: *well-ord-rmult well-ord-Memrel Card-is-Ord*)
done

34.3.1 Multiplication by a non-zero cardinal

lemma *prod-lepoll-self*: $b \in B \Longrightarrow A \lesssim A{*}B$
unfolding *lepoll-def inj-def*

apply (*rule-tac* $x = \lambda x{\in}A.\ \langle x,b\rangle$ **in** *exI*, *simp*)
done

lemma *cmult-le-self*:
  $[\![Card(K);\ \ Ord(L);\ \ 0<L]\!] \Longrightarrow K \le (K \otimes L)$
 unfolding *cmult-def*
apply (*rule le-trans* [*OF Card-cardinal-le well-ord-lepoll-imp-cardinal-le*])
 apply *assumption*
 apply (*blast intro*: *well-ord-rmult well-ord-Memrel Card-is-Ord*)
apply (*blast intro*: *prod-lepoll-self ltD*)
done

34.3.2 Monotonicity of multiplication

lemma *prod-lepoll-mono*:
  $[\![A \lesssim C;\ \ B \lesssim D]\!] \Longrightarrow A * B\ \lesssim\ C * D$
 unfolding *lepoll-def*
apply (*elim exE*)
apply (*rule-tac* $x = lam\ \langle w,y\rangle{:}A{*}B.\ {<}f`w,\ fa`y{>}$ **in** *exI*)
apply (*rule-tac* $d = \lambda\langle w,y\rangle.\ {<}converse\ (f)\ `w,\ converse\ (fa)\ `y{>}$
  in *lam-injective*)
apply (*typecheck add*: *inj-is-fun*, *auto*)
done

lemma *cmult-le-mono*:
  $[\![K' \le K;\ \ L' \le L]\!] \Longrightarrow (K' \otimes L') \le (K \otimes L)$
 unfolding *cmult-def*
apply (*safe dest*!: *le-subset-iff* [*THEN iffD1*])
apply (*rule well-ord-lepoll-imp-cardinal-le*)
 apply (*blast intro*: *well-ord-rmult well-ord-Memrel*)
apply (*blast intro*: *prod-lepoll-mono subset-imp-lepoll*)
done

34.4 Multiplication of finite cardinals is "ordinary" multiplication

lemma *prod-succ-eqpoll*: $succ(A){*}B \approx B + A{*}B$
 unfolding *eqpoll-def*
apply (*rule exI*)
apply (*rule-tac* $c = \lambda\langle x,y\rangle.\ if\ x{=}A\ then\ Inl\ (y)\ else\ Inr\ (\langle x,y\rangle)$
  and $d = case\ (\lambda y.\ \langle A,y\rangle,\ \lambda z.\ z)$ **in** *lam-bijective*)
apply *safe*
apply (*simp-all add*: *succI2 if-type mem-imp-not-eq*)
done

lemma *cmult-succ-lemma*:
  $[\![Ord(m);\ \ Ord(n)]\!] \Longrightarrow succ(m) \otimes n = n \oplus (m \otimes n)$
 unfolding *cmult-def cadd-def*

apply (*rule prod-succ-eqpoll* [*THEN cardinal-cong*, *THEN trans*])
apply (*rule cardinal-cong* [*symmetric*])
apply (*rule sum-eqpoll-cong* [*OF eqpoll-refl well-ord-cardinal-eqpoll*])
apply (*blast intro*: *well-ord-rmult well-ord-Memrel*)
done

lemma *nat-cmult-eq-mult*: $[\![m \in nat; \ n \in nat]\!] \Longrightarrow m \otimes n = m\#*n$
apply (*induct-tac m*)
apply (*simp-all add*: *cmult-succ-lemma nat-cadd-eq-add*)
done

lemma *cmult-2*: $Card(n) \Longrightarrow 2 \otimes n = n \oplus n$
by (*simp add*: *cmult-succ-lemma Card-is-Ord cadd-commute* [*of - 0*])

lemma *sum-lepoll-prod*:
assumes *C*: $2 \lesssim C$ **shows** $B+B \lesssim C*B$
proof −
have $B+B \lesssim 2*B$
by (*simp add*: *sum-eq-2-times*)
also have ... $\lesssim C*B$
by (*blast intro*: *prod-lepoll-mono lepoll-refl C*)
finally show $B+B \lesssim C*B$ **.**
qed

lemma *lepoll-imp-sum-lepoll-prod*: $[\![A \lesssim B; 2 \lesssim A]\!] \Longrightarrow A+B \lesssim A*B$
by (*blast intro*: *sum-lepoll-mono sum-lepoll-prod lepoll-trans lepoll-refl*)

34.5 Infinite Cardinals are Limit Ordinals

lemma *nat-cons-lepoll*: $nat \lesssim A \Longrightarrow cons(u,A) \lesssim A$
unfolding *lepoll-def*
apply (*erule exE*)
apply (*rule-tac x* =
$\lambda z \in cons\ (u,A).$
if $z=u$ *then* $f`0$
else if $z \in range\ (f)$ *then* $f`succ\ (converse\ (f)\ `z)$ *else* z
in *exI*)
apply (*rule-tac d* =
$\lambda y.$ *if* $y \in range(f)$ *then* $nat\text{-}case\ (u, \lambda z.\ f`z, converse(f)\ `y)$
else y
in *lam-injective*)
apply (*fast intro*!: *if-type apply-type intro*: *inj-is-fun inj-converse-fun*)
apply (*simp add*: *inj-is-fun* [*THEN apply-rangeI*]
inj-converse-fun [*THEN apply-rangeI*]
inj-converse-fun [*THEN apply-funtype*])
done

lemma *nat-cons-eqpoll*: $nat \lesssim A \Longrightarrow cons(u,A) \approx A$
apply (*erule nat-cons-lepoll* [*THEN eqpollI*])

apply (*rule subset-consI* [*THEN subset-imp-lepoll*])
done

lemma *nat-succ-eqpoll*: *nat* ⊆ *A* ⟹ *succ*(*A*) ≈ *A*
  unfolding *succ-def*
apply (*erule subset-imp-lepoll* [*THEN nat-cons-eqpoll*])
done

lemma *InfCard-nat*: *InfCard*(*nat*)
  unfolding *InfCard-def*
apply (*blast intro*: *Card-nat le-refl Card-is-Ord*)
done

lemma *InfCard-is-Card*: *InfCard*(*K*) ⟹ *Card*(*K*)
  unfolding *InfCard-def*
apply (*erule conjunct1*)
done

lemma *InfCard-Un*:
    ⟦*InfCard*(*K*); *Card*(*L*)⟧ ⟹ *InfCard*(*K* ∪ *L*)
  unfolding *InfCard-def*
apply (*simp add*: *Card-Un Un-upper1-le* [*THEN* [*2*] *le-trans*] *Card-is-Ord*)
done

lemma *InfCard-is-Limit*: *InfCard*(*K*) ⟹ *Limit*(*K*)
  unfolding *InfCard-def*
apply (*erule conjE*)
apply (*frule Card-is-Ord*)
apply (*rule ltI* [*THEN non-succ-LimitI*])
apply (*erule le-imp-subset* [*THEN subsetD*])
apply (*safe dest*!: *Limit-nat* [*THEN Limit-le-succD*])
  unfolding *Card-def*
apply (*drule trans*)
apply (*erule le-imp-subset* [*THEN nat-succ-eqpoll*, *THEN cardinal-cong*])
apply (*erule Ord-cardinal-le* [*THEN lt-trans2*, *THEN lt-irrefl*])
apply (*rule le-eqI*, *assumption*)
apply (*rule Ord-cardinal*)
done

lemma *ordermap-eqpoll-pred*:
    ⟦*well-ord*(*A*,*r*); *x* ∈ *A*⟧ ⟹ *ordermap*(*A*,*r*)‘*x* ≈ *Order*.*pred*(*A*,*x*,*r*)
  unfolding *eqpoll-def*
apply (*rule exI*)

apply (*simp add*: *ordermap-eq-image well-ord-is-wf*)
apply (*erule ordermap-bij* [*THEN bij-is-inj*, *THEN restrict-bij*,
THEN bij-converse-bij])
apply (*rule pred-subset*)
done

34.5.1 Establishing the well-ordering

lemma *well-ord-csquare*:
assumes *K*: *Ord*(*K*) **shows** *well-ord*(*K*∗*K*, *csquare-rel*(*K*))
proof (*unfold csquare-rel-def*, *rule well-ord-rvimage*)
show (λ⟨*x*,*y*⟩∈*K* × *K*. ⟨*x* ∪ *y*, *x*, *y*⟩) ∈ *inj*(*K* × *K*, *K* × *K* × *K*) **using** *K*
by (*force simp add*: *inj-def intro*: *lam-type Un-least-lt* [*THEN ltD*] *ltI*)
next
show *well-ord*(*K* × *K* × *K*, *rmult*(*K*, *Memrel*(*K*), *K* × *K*, *rmult*(*K*, *Memrel*(*K*), *K*, *Memrel*(*K*))))
using *K* **by** (*blast intro*: *well-ord-rmult well-ord-Memrel*)
qed

34.5.2 Characterising initial segments of the well-ordering

lemma *csquareD*:
⟦<⟨*x*,*y*⟩, ⟨*z*,*z*⟩> ∈ *csquare-rel*(*K*); *x*<*K*; *y*<*K*; *z*<*K*⟧ ⟹ *x* ≤ *z* ∧ *y* ≤ *z*
unfolding *csquare-rel-def*
apply (*erule rev-mp*)
apply (*elim ltE*)
apply (*simp add*: *rvimage-iff Un-absorb Un-least-mem-iff ltD*)
apply (*safe elim*!: *mem-irrefl intro*!: *Un-upper1-le Un-upper2-le*)
apply (*simp-all add*: *lt-def succI2*)
done

lemma *pred-csquare-subset*:
z<*K* ⟹ *Order.pred*(*K*∗*K*, ⟨*z*,*z*⟩, *csquare-rel*(*K*)) ⊆ *succ*(*z*)∗*succ*(*z*)
unfolding *Order.pred-def*
apply (*safe del*: *SigmaI dest*!: *csquareD*)
apply (*unfold lt-def*, *auto*)
done

lemma *csquare-ltI*:
⟦*x*<*z*; *y*<*z*; *z*<*K*⟧ ⟹ <⟨*x*,*y*⟩, ⟨*z*,*z*⟩> ∈ *csquare-rel*(*K*)
unfolding *csquare-rel-def*
apply (*subgoal-tac x*<*K* ∧ *y*<*K*)
prefer *2* **apply** (*blast intro*: *lt-trans*)
apply (*elim ltE*)
apply (*simp add*: *rvimage-iff Un-absorb Un-least-mem-iff ltD*)
done

lemma *csquare-or-eqI*:
⟦*x* ≤ *z*; *y* ≤ *z*; *z*<*K*⟧ ⟹ <⟨*x*,*y*⟩, ⟨*z*,*z*⟩> ∈ *csquare-rel*(*K*) | *x*=*z* ∧ *y*=*z*

unfolding *csquare-rel-def*
apply (*subgoal-tac* *x*<*K* ∧ *y*<*K*)
prefer *2* **apply** (*blast intro*: *lt-trans1*)
apply (*elim ltE*)
apply (*simp add*: *rvimage-iff Un-absorb Un-least-mem-iff ltD*)
apply (*elim succE*)
apply (*simp-all add*: *subset-Un-iff* [*THEN iff-sym*]
subset-Un-iff2 [*THEN iff-sym*] *OrdmemD*)
done

34.5.3 The cardinality of initial segments

lemma *ordermap-z-lt*:
⟦*Limit*(*K*); *x*<*K*; *y*<*K*; *z*=*succ*(*x* ∪ *y*)⟧ ⟹
ordermap(*K*∗*K*, *csquare-rel*(*K*)) ‘ ⟨*x*,*y*⟩ <
ordermap(*K*∗*K*, *csquare-rel*(*K*)) ‘ ⟨*z*,*z*⟩
apply (*subgoal-tac* *z*<*K* ∧ *well-ord* (*K*∗*K*, *csquare-rel* (*K*)))
prefer *2* **apply** (*blast intro*!: *Un-least-lt Limit-has-succ*
Limit-is-Ord [*THEN well-ord-csquare*], *clarify*)
apply (*rule csquare-ltI* [*THEN ordermap-mono*, *THEN ltI*])
apply (*erule-tac* [*4*] *well-ord-is-wf*)
apply (*blast intro*!: *Un-upper1-le Un-upper2-le Ord-ordermap elim*!: *ltE*)+
done

Kunen: "each ⟨*x*, *y*⟩ ∈ *K* × *K* has no more than *z* × *z* predecessors..." (page 29)

lemma *ordermap-csquare-le*:
assumes *K*: *Limit*(*K*) **and** *x*: *x*<*K* **and** *y*: *y*<*K*
defines *z* ≡ *succ*(*x* ∪ *y*)
shows |*ordermap*(*K* × *K*, *csquare-rel*(*K*)) ‘ ⟨*x*,*y*⟩| ≤ |*succ*(*z*)| ⊗ |*succ*(*z*)|
proof (*unfold cmult-def*, *rule well-ord-lepoll-imp-cardinal-le*)
show *well-ord*(|*succ*(*z*)| × |*succ*(*z*)|,
rmult(|*succ*(*z*)|, *Memrel*(|*succ*(*z*)|), |*succ*(*z*)|, *Memrel*(|*succ*(*z*)|)))
by (*blast intro*: *Ord-cardinal well-ord-Memrel well-ord-rmult*)
next
have *zK*: *z*<*K* **using** *x y K z-def*
by (*blast intro*: *Un-least-lt Limit-has-succ*)
hence *oz*: *Ord*(*z*) **by** (*elim ltE*)
have *ordermap*(*K* × *K*, *csquare-rel*(*K*)) ‘ ⟨*x*,*y*⟩ ≲ *ordermap*(*K* × *K*, *csquare-rel*(*K*)) ‘ ⟨*z*,*z*⟩
using *z-def*
by (*blast intro*: *ordermap-z-lt leI le-imp-lepoll K x y*)
also have ... ≈ *Order*.*pred*(*K* × *K*, ⟨*z*,*z*⟩, *csquare-rel*(*K*))
proof (*rule ordermap-eqpoll-pred*)
show *well-ord*(*K* × *K*, *csquare-rel*(*K*)) **using** *K*
by (*rule Limit-is-Ord* [*THEN well-ord-csquare*])
next
show ⟨*z*, *z*⟩ ∈ *K* × *K* **using** *zK*
by (*blast intro*: *ltD*)

qed
also have ... $\lesssim$ *succ*(z) $\times$ *succ*(z) **using** *zK*
by (*rule pred-csquare-subset* [*THEN subset-imp-lepoll*])
also have ... $\approx$ |*succ*(z)| $\times$ |*succ*(z)| **using** *oz*
by (*blast intro*: *prod-eqpoll-cong Ord-succ Ord-cardinal-eqpoll eqpoll-sym*)
finally show *ordermap*($K \times K$, *csquare-rel*(K)) ‘ $\langle x,y \rangle \lesssim$ |*succ*(z)| $\times$ |*succ*(z)| **.**
qed

Kunen: "... so the order type is $\leq$ K"

lemma *ordertype-csquare-le*:
assumes *IK*: *InfCard*(K) **and** *eq*: $\bigwedge y.\ y \in K \Longrightarrow$ *InfCard*(y) $\Longrightarrow y \otimes y = y$
shows *ordertype*($K*K$, *csquare-rel*(K)) $\leq K$
proof −
have *CK*: *Card*(K) **using** *IK* **by** (*rule InfCard-is-Card*)
hence *OK*: *Ord*(K) **by** (*rule Card-is-Ord*)
moreover have *Ord*(*ordertype*($K \times K$, *csquare-rel*(K))) **using** *OK*
by (*rule well-ord-csquare* [*THEN Ord-ordertype*])
ultimately show *?thesis*
proof (*rule all-lt-imp-le*)
fix *i*
assume *i*: $i <$ *ordertype*($K \times K$, *csquare-rel*(K))
hence *Oi*: *Ord*(i) **by** (*elim ltE*)
obtain *x y* **where** *x*: $x \in K$ **and** *y*: $y \in K$
and *ieq*: $i =$ *ordermap*($K \times K$, *csquare-rel*(K)) ‘ $\langle x,y \rangle$
using *i* **by** (*auto simp add*: *ordertype-unfold elim*: *ltE*)
hence *xy*: *Ord*(x) *Ord*(y) $x < K$ $y < K$ **using** *OK*
by (*blast intro*: *Ord-in-Ord ltI*)+
hence *ou*: *Ord*($x \cup y$)
by (*simp add*: *Ord-Un*)
show $i < K$
proof (*rule Card-lt-imp-lt* [*OF - Oi CK*])
have $|i| \leq$ |*succ*(*succ*($x \cup y$))| $\otimes$ |*succ*(*succ*($x \cup y$))| **using** *IK xy*
by (*auto simp add*: *ieq intro*: *InfCard-is-Limit* [*THEN ordermap-csquare-le*])
moreover have |*succ*(*succ*($x \cup y$))| $\otimes$ |*succ*(*succ*($x \cup y$))| $< K$
proof (*cases rule*: *Ord-linear2* [*OF ou Ord-nat*])
assume $x \cup y <$ *nat*
hence |*succ*(*succ*($x \cup y$))| $\otimes$ |*succ*(*succ*($x \cup y$))| $\in$ *nat*
by (*simp add*: *lt-def nat-cmult-eq-mult nat-succI mult-type*
nat-into-Card [*THEN Card-cardinal-eq*] *Ord-nat*)
also have ... $\subseteq K$ **using** *IK*
by (*simp add*: *InfCard-def le-imp-subset*)
finally show |*succ*(*succ*($x \cup y$))| $\otimes$ |*succ*(*succ*($x \cup y$))| $< K$
by (*simp add*: *ltI OK*)
next
assume *natxy*: *nat* $\leq x \cup y$
hence *seq*: |*succ*(*succ*($x \cup y$))| $=$ $|x \cup y|$ **using** *xy*
by (*simp add*: *le-imp-subset nat-succ-eqpoll* [*THEN cardinal-cong*]
le-succ-iff)
also have ... $< K$ **using** *xy*

by (*simp add*: *Un-least-lt Ord-cardinal-le* [*THEN lt-trans1*])
finally have $|succ(succ(x \cup y))| < K$ **.**
moreover have *InfCard*($|succ(succ(x \cup y))|$) **using** *xy natxy*
by (*simp add*: *seq InfCard-def Card-cardinal nat-le-cardinal*)
ultimately show *?thesis* **by** (*simp add*: *eq ltD*)
qed
ultimately show $|i| < K$ **by** (*blast intro*: *lt-trans1*)
qed
qed
qed

lemma *InfCard-csquare-eq*:
assumes *IK*: *InfCard*(*K*) **shows** $K \otimes K = K$
proof −
have *OK*: *Ord*(*K*) **using** *IK* **by** (*simp add*: *Card-is-Ord InfCard-is-Card*)
show $K \otimes K = K$ **using** *OK IK*
proof (*induct rule*: *trans-induct*)
case (*step i*)
show $i \otimes i = i$
proof (*rule le-anti-sym*)
have $|i \times i| = |ordertype(i \times i, csquare\text{-}rel(i))|$
by (*rule cardinal-cong*,
simp add: *step.hyps well-ord-csquare* [*THEN ordermap-bij*, *THEN bij-imp-eqpoll*])
hence $i \otimes i \le ordertype(i \times i, csquare\text{-}rel(i))$
by (*simp add*: *step.hyps cmult-def Ord-cardinal-le well-ord-csquare* [*THEN Ord-ordertype*])
moreover
have $ordertype(i \times i, csquare\text{-}rel(i)) \le i$ **using** *step*
by (*simp add*: *ordertype-csquare-le*)
ultimately show $i \otimes i \le i$ **by** (*rule le-trans*)
next
show $i \le i \otimes i$ **using** *step*
by (*blast intro*: *cmult-square-le InfCard-is-Card*)
qed
qed
qed

lemma *well-ord-InfCard-square-eq*:
assumes *r*: *well-ord*(*A*,*r*) **and** *I*: *InfCard*($|A|$) **shows** $A \times A \approx A$
proof −
have $A \times A \approx |A| \times |A|$
by (*blast intro*: *prod-eqpoll-cong well-ord-cardinal-eqpoll eqpoll-sym r*)
also have ... $\approx A$
proof (*rule well-ord-cardinal-eqE* [*OF - r*])
show *well-ord*($|A| \times |A|$, *rmult*($|A|$, *Memrel*($|A|$), $|A|$, *Memrel*($|A|$)))
by (*blast intro*: *Ord-cardinal well-ord-rmult well-ord-Memrel r*)
next

show $||A| \times |A|| = |A|$ **using** *InfCard-csquare-eq I*
by (*simp add*: *cmult-def*)
qed
finally show *?thesis* **.**
qed

lemma *InfCard-square-eqpoll*: $InfCard(K) \Longrightarrow K \times K \approx K$
apply (*rule well-ord-InfCard-square-eq*)
apply (*erule InfCard-is-Card* [*THEN Card-is-Ord*, *THEN well-ord-Memrel*])
apply (*simp add*: *InfCard-is-Card* [*THEN Card-cardinal-eq*])
done

lemma *Inf-Card-is-InfCard*: $[\![Card(i); \neg Finite(i)]\!] \Longrightarrow InfCard(i)$
by (*simp add*: *InfCard-def Card-is-Ord* [*THEN nat-le-infinite-Ord*])

34.5.4 Toward's Kunen's Corollary 10.13 (1)

lemma *InfCard-le-cmult-eq*: $[\![InfCard(K);\ L \le K;\ 0<L]\!] \Longrightarrow K \otimes L = K$
apply (*rule le-anti-sym*)
prefer *2*
apply (*erule ltE*, *blast intro*: *cmult-le-self InfCard-is-Card*)
apply (*frule InfCard-is-Card* [*THEN Card-is-Ord*, *THEN le-refl*])
apply (*rule cmult-le-mono* [*THEN le-trans*], *assumption+*)
apply (*simp add*: *InfCard-csquare-eq*)
done

lemma *InfCard-cmult-eq*: $[\![InfCard(K);\ InfCard(L)]\!] \Longrightarrow K \otimes L = K \cup L$
apply (*rule-tac* $i = K$ **and** $j = L$ **in** *Ord-linear-le*)
apply (*typecheck add*: *InfCard-is-Card Card-is-Ord*)
apply (*rule cmult-commute* [*THEN ssubst*])
apply (*rule Un-commute* [*THEN ssubst*])
apply (*simp-all add*: *InfCard-is-Limit* [*THEN Limit-has-0*] *InfCard-le-cmult-eq*
subset-Un-iff2 [*THEN iffD1*] *le-imp-subset*)
done

lemma *InfCard-cdouble-eq*: $InfCard(K) \Longrightarrow K \oplus K = K$
apply (*simp add*: *cmult-2* [*symmetric*] *InfCard-is-Card cmult-commute*)
apply (*simp add*: *InfCard-le-cmult-eq InfCard-is-Limit Limit-has-0 Limit-has-succ*)
done

lemma *InfCard-le-cadd-eq*: $[\![InfCard(K);\ L \le K]\!] \Longrightarrow K \oplus L = K$
apply (*rule le-anti-sym*)
prefer *2*
apply (*erule ltE*, *blast intro*: *cadd-le-self InfCard-is-Card*)
apply (*frule InfCard-is-Card* [*THEN Card-is-Ord*, *THEN le-refl*])
apply (*rule cadd-le-mono* [*THEN le-trans*], *assumption+*)
apply (*simp add*: *InfCard-cdouble-eq*)

done

lemma *InfCard-cadd-eq*: $[\![InfCard(K); \; InfCard(L)]\!] \Longrightarrow K \oplus L = K \cup L$
apply (*rule-tac* $i = K$ **and** $j = L$ **in** *Ord-linear-le*)
apply (*typecheck add*: *InfCard-is-Card Card-is-Ord*)
apply (*rule cadd-commute* [*THEN ssubst*])
apply (*rule Un-commute* [*THEN ssubst*])
apply (*simp-all add*: *InfCard-le-cadd-eq subset-Un-iff2* [*THEN iffD1*] *le-imp-subset*)
done

34.6 For Every Cardinal Number There Exists A Greater One

This result is Kunen's Theorem 10.16, which would be trivial using AC

lemma *Ord-jump-cardinal*: *Ord*(*jump-cardinal*(*K*))
unfolding *jump-cardinal-def*
apply (*rule Ord-is-Transset* [*THEN* [*2*] *OrdI*])
prefer *2* **apply** (*blast intro*!: *Ord-ordertype*)
unfolding *Transset-def*
apply (*safe del*: *subsetI*)
apply (*simp add*: *ordertype-pred-unfold*, *safe*)
apply (*rule UN-I*)
apply (*rule-tac* [*2*] *ReplaceI*)
prefer *4* **apply** (*blast intro*: *well-ord-subset elim*!: *predE*)+
done

lemma *jump-cardinal-iff*:
$i \in jump\text{-}cardinal(K) \longleftrightarrow$
$(\exists\, r\; X.\; r \subseteq K{*}K \wedge X \subseteq K \wedge well\text{-}ord(X,r) \wedge i = ordertype(X,r))$
unfolding *jump-cardinal-def*
apply (*blast del*: *subsetI*)
done

lemma *K-lt-jump-cardinal*: $Ord(K) \Longrightarrow K < jump\text{-}cardinal(K)$
apply (*rule Ord-jump-cardinal* [*THEN* [*2*] *ltI*])
apply (*rule jump-cardinal-iff* [*THEN iffD2*])
apply (*rule-tac* *x*=*Memrel*(*K*) **in** *exI*)
apply (*rule-tac* *x*=*K* **in** *exI*)
apply (*simp add*: *ordertype-Memrel well-ord-Memrel*)
apply (*simp add*: *Memrel-def subset-iff*)
done

lemma *Card-jump-cardinal-lemma*:
$[\![well\text{-}ord(X,r); \; r \subseteq K * K; \; X \subseteq K;$
$f \in bij(ordertype(X,r),\, jump\text{-}cardinal(K))]\!]$
$\Longrightarrow jump\text{-}cardinal(K) \in jump\text{-}cardinal(K)$

apply (*subgoal-tac f O ordermap* (*X*,*r*) ∈ *bij* (*X*, *jump-cardinal* (*K*)))
prefer *2* **apply** (*blast intro*: *comp-bij ordermap-bij*)
apply (*rule jump-cardinal-iff* [*THEN iffD2*])
apply (*intro exI conjI*)
apply (*rule subset-trans* [*OF rvimage-type Sigma-mono*], *assumption*+)
apply (*erule bij-is-inj* [*THEN well-ord-rvimage*])
apply (*rule Ord-jump-cardinal* [*THEN well-ord-Memrel*])
apply (*simp add*: *well-ord-Memrel* [*THEN* [*2*] *bij-ordertype-vimage*]
ordertype-Memrel Ord-jump-cardinal)
done

lemma *Card-jump-cardinal*: *Card*(*jump-cardinal*(*K*))
apply (*rule Ord-jump-cardinal* [*THEN CardI*])
unfolding *eqpoll-def*
apply (*safe dest*!: *ltD jump-cardinal-iff* [*THEN iffD1*])
apply (*blast intro*: *Card-jump-cardinal-lemma* [*THEN mem-irrefl*])
done

34.7 Basic Properties of Successor Cardinals

lemma *csucc-basic*: *Ord*(*K*) ⟹ *Card*(*csucc*(*K*)) ∧ *K* < *csucc*(*K*)
unfolding *csucc-def*
apply (*rule LeastI*)
apply (*blast intro*: *Card-jump-cardinal K-lt-jump-cardinal Ord-jump-cardinal*)+
done

lemmas *Card-csucc* = *csucc-basic* [*THEN conjunct1*]

lemmas *lt-csucc* = *csucc-basic* [*THEN conjunct2*]

lemma *Ord-0-lt-csucc*: *Ord*(*K*) ⟹ *0* < *csucc*(*K*)
by (*blast intro*: *Ord-0-le lt-csucc lt-trans1*)

lemma *csucc-le*: ⟦*Card*(*L*); *K*<*L*⟧ ⟹ *csucc*(*K*) ≤ *L*
unfolding *csucc-def*
apply (*rule Least-le*)
apply (*blast intro*: *Card-is-Ord*)+
done

lemma *lt-csucc-iff*: ⟦*Ord*(*i*); *Card*(*K*)⟧ ⟹ *i* < *csucc*(*K*) ⟷ |*i*| ≤ *K*
apply (*rule iffI*)
apply (*rule-tac* [*2*] *Card-lt-imp-lt*)
apply (*erule-tac* [*2*] *lt-trans1*)
apply (*simp-all add*: *lt-csucc Card-csucc Card-is-Ord*)
apply (*rule notI* [*THEN not-lt-imp-le*])
apply (*rule Card-cardinal* [*THEN csucc-le*, *THEN lt-trans1*, *THEN lt-irrefl*], *assumption*)
apply (*rule Ord-cardinal-le* [*THEN lt-trans1*])

apply (*simp-all add*: *Ord-cardinal Card-is-Ord*)
done

lemma *Card-lt-csucc-iff*:
  $[\![Card(K'); Card(K)]\!] \Longrightarrow K' < csucc(K) \longleftrightarrow K' \le K$
by (*simp add*: *lt-csucc-iff Card-cardinal-eq Card-is-Ord*)

lemma *InfCard-csucc*: $InfCard(K) \Longrightarrow InfCard(csucc(K))$
by (*simp add*: *InfCard-def Card-csucc Card-is-Ord*
  lt-csucc [*THEN leI*, *THEN* [*2*] *le-trans*])

34.7.1 Removing elements from a finite set decreases its cardinality

lemma *Finite-imp-cardinal-cons* [*simp*]:
 assumes *FA*: $Finite(A)$ **and** *a*: $a \notin A$ **shows** $|cons(a,A)| = succ(|A|)$
proof −
 { **fix** *X*
  have $Finite(X) \Longrightarrow a \notin X \Longrightarrow cons(a,X) \lesssim X \Longrightarrow False$
   proof (*induct X rule*: *Finite-induct*)
    case *0* **thus** *False* **by** (*simp add*: *lepoll-0-iff*)
   next
    case (*cons x Y*)
    hence $cons(x, cons(a, Y)) \lesssim cons(x, Y)$ **by** (*simp add*: *cons-commute*)
    hence $cons(a, Y) \lesssim Y$ **using** *cons* **by** (*blast dest*: *cons-lepoll-consD*)
    thus *False* **using** *cons* **by** *auto*
   qed
 }
 hence [*simp*]: $\neg\, cons(a,A) \lesssim A$ **using** *a FA* **by** *auto*
 have [*simp*]: $|A| \approx A$ **using** *Finite-imp-well-ord* [*OF FA*]
  by (*blast intro*: *well-ord-cardinal-eqpoll*)
 have $(\mu\ i.\ i \approx cons(a, A)) = succ(|A|)$
  proof (*rule Least-equality* [*OF - - notI*])
   show $succ(|A|) \approx cons(a, A)$
    by (*simp add*: *succ-def cons-eqpoll-cong mem-not-refl a*)
  next
   show $Ord(succ(|A|))$ **by** *simp*
  next
   fix *i*
   assume *i*: $i \le |A|$ $i \approx cons(a, A)$
   have $cons(a, A) \approx i$ **by** (*rule eqpoll-sym*) (*rule i*)
   also have ... $\lesssim |A|$ **by** (*rule le-imp-lepoll*) (*rule i*)
   also have ... $\approx A$ **by** *simp*
   finally have $cons(a, A) \lesssim A$ **.**
   thus *False* **by** *simp*
  qed
 thus *?thesis* **by** (*simp add*: *cardinal-def*)
qed

lemma *Finite-imp-succ-cardinal-Diff*:
  $[\![Finite(A);\ \ a \in A]\!] \Longrightarrow succ(|A-\{a\}|) = |A|$
apply (*rule-tac* $b = A$ **in** *cons-Diff* [*THEN subst*], *assumption*)
apply (*simp add*: *Finite-imp-cardinal-cons Diff-subset* [*THEN subset-Finite*])
apply (*simp add*: *cons-Diff*)
done

lemma *Finite-imp-cardinal-Diff*: $[\![Finite(A);\ \ a \in A]\!] \Longrightarrow |A-\{a\}| < |A|$
apply (*rule succ-leE*)
apply (*simp add*: *Finite-imp-succ-cardinal-Diff*)
done

lemma *Finite-cardinal-in-nat* [*simp*]: $Finite(A) \Longrightarrow |A| \in nat$
proof (*induct rule*: *Finite-induct*)
 case *0* **thus** *?case* **by** (*simp add*: *cardinal-0*)
next
 case (*cons x A*) **thus** *?case* **by** (*simp add*: *Finite-imp-cardinal-cons*)
qed

lemma *card-Un-Int*:
  $[\![Finite(A);\ Finite(B)]\!] \Longrightarrow |A|\ \#+\ |B| = |A \cup B|\ \#+\ |A \cap B|$
apply (*erule Finite-induct*, *simp*)
apply (*simp add*: *Finite-Int cons-absorb Un-cons Int-cons-left*)
done

lemma *card-Un-disjoint*:
  $[\![Finite(A);\ Finite(B);\ A \cap B = 0]\!] \Longrightarrow |A \cup B| = |A|\ \#+\ |B|$
by (*simp add*: *Finite-Un card-Un-Int*)

lemma *card-partition*:
 assumes *FC*: *Finite*(*C*)
 shows
  $Finite\ (\bigcup\ C) \Longrightarrow$
   $(\forall\, c{\in}C.\ |c| = k) \Longrightarrow$
   $(\forall\, c1 \in C.\ \forall\, c2 \in C.\ c1 \neq c2 \longrightarrow c1 \cap c2 = 0) \Longrightarrow$
   $k\ \#*\ |C| = |\bigcup\ C|$
using *FC*
proof (*induct rule*: *Finite-induct*)
 case *0* **thus** *?case* **by** *simp*
next
 case (*cons x B*)
 hence $x \cap \bigcup B = 0$ **by** *auto*
 thus *?case* **using** *cons*
  by (*auto simp add*: *card-Un-disjoint*)
qed

34.7.2 Theorems by Krzysztof Grabczewski, proofs by lcp

lemmas *nat-implies-well-ord* = *nat-into-Ord* [*THEN well-ord-Memrel*]

lemma *nat-sum-eqpoll-sum*:
  assumes *m*: *m* ∈ *nat* **and** *n*: *n* ∈ *nat* **shows** *m* + *n* ≈ *m* #+ *n*
proof −
  have *m* + *n* ≈ |*m*+*n*| **using** *m n*
   by (*blast intro*: *nat-implies-well-ord well-ord-radd well-ord-cardinal-eqpoll eqpoll-sym*)
  also have ... = *m* #+ *n* **using** *m n*
    by (*simp add*: *nat-cadd-eq-add* [*symmetric*] *cadd-def*)
  finally show *?thesis* **.**
qed

lemma *Ord-subset-natD* [*rule-format*]: *Ord*(*i*) ⟹ *i* ⊆ *nat* ⟹ *i* ∈ *nat* | *i*=*nat*
proof (*induct i rule*: *trans-induct3*)
  case *0* **thus** *?case* **by** *auto*
next
  case (*succ i*) **thus** *?case* **by** *auto*
next
  case (*limit l*) **thus** *?case*
    by (*blast dest*: *nat-le-Limit le-imp-subset*)
qed

lemma *Ord-nat-subset-into-Card*: ⟦*Ord*(*i*); *i* ⊆ *nat*⟧ ⟹ *Card*(*i*)
  by (*blast dest*: *Ord-subset-natD intro*: *Card-nat nat-into-Card*)

end

35 Main ZF Theory: Everything Except AC

theory *ZF* **imports** *List IntDiv CardinalArith* **begin**

35.1 Iteration of the function *F*

consts *iterates* :: [*i*⇒*i*,*i*,*i*] ⇒ *i* (‹(‹*notation*=‹*mixfix iterates*››-^- ′(-′))› [*60*,*1000*,*1000*] *60*)

primrec
    F^*0* (*x*) = *x*
    F^(*succ*(*n*)) (*x*) = *F*(*F*^*n* (*x*))

definition
   iterates-omega :: [*i*⇒*i*,*i*] ⇒ *i* (‹(‹*notation*=‹*mixfix iterates-omega*››-^ω ′(-′))› [*60*,*1000*] *60*) **where**
    F^ω (*x*) ≡ ⋃*n*∈*nat*. *F*^*n* (*x*)

lemma *iterates-triv*:
    ⟦*n*∈*nat*; *F*(*x*) = *x*⟧ ⟹ *F*^*n* (*x*) = *x*
by (*induct n rule*: *nat-induct*, *simp-all*)

lemma *iterates-type* [*TC*]:

$[\![n \in nat;\ \ a \in A;\ \bigwedge x.\ x \in A \Longrightarrow F(x) \in A]\!]$
$\Longrightarrow F\hat{\ }n\ (a) \in A$
by (*induct n rule*: *nat-induct*, *simp-all*)

lemma *iterates-omega-triv*:
$F(x) = x \Longrightarrow F\hat{\ }\omega\ (x) = x$
by (*simp add*: *iterates-omega-def iterates-triv*)

lemma *Ord-iterates* [*simp*]:
$[\![n{\in}nat;\ \ \bigwedge i.\ Ord(i) \Longrightarrow Ord(F(i));\ \ Ord(x)]\!]$
$\Longrightarrow Ord(F\hat{\ }n\ (x))$
by (*induct n rule*: *nat-induct*, *simp-all*)

lemma *iterates-commute*: $n \in nat \Longrightarrow F(F\hat{\ }n\ (x)) = F\hat{\ }n\ (F(x))$
by (*induct-tac n*, *simp-all*)

35.2 Transfinite Recursion

Transfinite recursion for definitions based on the three cases of ordinals

definition
transrec3 :: [*i*, *i*, [*i*,*i*]⇒*i*, [*i*,*i*]⇒*i*] ⇒*i* **where**
transrec3(*k*, *a*, *b*, *c*) ≡
transrec(*k*, λ*x r*.
if x=0 then a
else if Limit(*x*) *then c*(*x*, λ*y*∈*x*. *r*‘*y*)
else b(*Arith.pred*(*x*), *r* ‘ *Arith.pred*(*x*)))

lemma *transrec3-0* [*simp*]: *transrec3*(*0*,*a*,*b*,*c*) = *a*
by (*rule transrec3-def* [*THEN def-transrec*, *THEN trans*], *simp*)

lemma *transrec3-succ* [*simp*]:
transrec3(*succ*(*i*),*a*,*b*,*c*) = *b*(*i*, *transrec3*(*i*,*a*,*b*,*c*))
by (*rule transrec3-def* [*THEN def-transrec*, *THEN trans*], *simp*)

lemma *transrec3-Limit*:
Limit(*i*) ⟹
transrec3(*i*,*a*,*b*,*c*) = *c*(*i*, λ*j*∈*i*. *transrec3*(*j*,*a*,*b*,*c*))
by (*rule transrec3-def* [*THEN def-transrec*, *THEN trans*], *force*)

declaration ‹*fn - =>*
Simplifier.map-ss (*Simplifier.set-mksimps* (*fn ctxt =>*
map mk-eq o Ord-atomize o Variable.gen-all ctxt))
›

end

36 The Axiom of Choice

theory *AC* **imports** *ZF* **begin**

This definition comes from Halmos (1960), page 59.

axiomatization where
  AC: $[\![a \in A;\ \bigwedge x.\ x \in A \Longrightarrow (\exists y.\ y \in B(x))]\!] \Longrightarrow \exists z.\ z \in Pi(A,B)$

lemma *AC-Pi*: $[\![\bigwedge x.\ x \in A \Longrightarrow (\exists y.\ y \in B(x))]\!] \Longrightarrow \exists z.\ z \in Pi(A,B)$
apply (*case-tac A=0*)
apply (*simp add*: *Pi-empty1*)

apply (*blast intro*: *AC*)
done

lemma *AC-ball-Pi*: $\forall x \in A.\ \exists y.\ y \in B(x) \Longrightarrow \exists y.\ y \in Pi(A,B)$
apply (*rule AC-Pi*)
apply (*erule bspec*, *assumption*)
done

lemma *AC-Pi-Pow*: $\exists f.\ f \in (\prod X \in Pow(C)-\{0\}.\ X)$
apply (*rule-tac B1 = λx. x* **in** *AC-Pi* [*THEN exE*])
apply (*erule-tac* [*2*] *exI*, *blast*)
done

lemma *AC-func*:
  $[\![\bigwedge x.\ x \in A \Longrightarrow (\exists y.\ y \in x)]\!] \Longrightarrow \exists f \in A{-}{>}\bigcup(A).\ \forall x \in A.\ f`x \in x$
apply (*rule-tac B1 = λx. x* **in** *AC-Pi* [*THEN exE*])
prefer *2* **apply** (*blast dest*: *apply-type intro*: *Pi-type*, *blast*)
done

lemma *non-empty-family*: $[\![0 \notin A;\ x \in A]\!] \Longrightarrow \exists y.\ y \in x$
by (*subgoal-tac* $x \neq 0$, *blast+*)

lemma *AC-func0*: $0 \notin A \Longrightarrow \exists f \in A{-}{>}\bigcup(A).\ \forall x \in A.\ f`x \in x$
apply (*rule AC-func*)
apply (*simp-all add*: *non-empty-family*)
done

lemma *AC-func-Pow*: $\exists f \in (Pow(C)-\{0\}) {-}{>} C.\ \forall x \in Pow(C)-\{0\}.\ f`x \in x$
apply (*rule AC-func0* [*THEN bexE*])
apply (*rule-tac* [*2*] *bexI*)
prefer *2* **apply** *assumption*
apply (*erule-tac* [*2*] *fun-weaken-type*, *blast+*)
done

lemma *AC-Pi0*: $0 \notin A \Longrightarrow \exists f.\ f \in (\prod x \in A.\ x)$

apply (*rule AC-Pi*)
apply (*simp-all add*: *non-empty-family*)
done

end

37 Zorn's Lemma

theory *Zorn* **imports** *OrderArith AC Inductive* **begin**

Based upon the unpublished article "Towards the Mechanization of the Proofs of Some Classical Theorems of Set Theory," by Abrial and Laffitte.

definition
Subset-rel :: $i \Rightarrow i$ **where**
Subset-rel$(A) \equiv \{z \in A*A \ . \ \exists x\ y.\ z=\langle x,y\rangle \wedge x<=y \wedge x\neq y\}$

definition
chain :: $i \Rightarrow i$ **where**
chain$(A) \equiv \{F \in Pow(A).\ \forall X\in F.\ \forall Y\in F.\ X<=Y \mid Y<=X\}$

definition
super :: $[i,i] \Rightarrow i$ **where**
super$(A,c) \equiv \{d \in chain(A).\ c<=d \wedge c\neq d\}$

definition
maxchain :: $i \Rightarrow i$ **where**
maxchain$(A) \equiv \{c \in chain(A).\ super(A,c)=0\}$

definition
increasing :: $i \Rightarrow i$ **where**
increasing$(A) \equiv \{f \in Pow(A)->Pow(A).\ \forall x.\ x<=A \longrightarrow x<=f`x\}$

Lemma for the inductive definition below

lemma *Union-in-Pow*: $Y \in Pow(Pow(A)) \Longrightarrow \bigcup(Y) \in Pow(A)$
by *blast*

We could make the inductive definition conditional on $next \in increasing(S)$ but instead we make this a side-condition of an introduction rule. Thus the induction rule lets us assume that condition! Many inductive proofs are therefore unconditional.

consts
TFin :: $[i,i] \Rightarrow i$

inductive
domains $TFin(S,next) \subseteq Pow(S)$
intros
nextI: $[\![x \in TFin(S,next);\ \ next \in increasing(S)]\!]$
$\Longrightarrow next`x \in TFin(S,next)$

Pow-UnionI: $Y \in Pow(TFin(S,next)) \Longrightarrow \bigcup(Y) \in TFin(S,next)$

monos *Pow-mono*
con-defs *increasing-def*
type-intros *CollectD1* [*THEN apply-funtype*] *Union-in-Pow*

37.1 Mathematical Preamble

lemma *Union-lemma0*: $(\forall x{\in}C.\ x{<=}A \mid B{<=}x) \Longrightarrow \bigcup(C){<=}A \mid B{<=}\bigcup(C)$
by *blast*

lemma *Inter-lemma0*:
$[\![c \in C;\ \forall x{\in}C.\ A{<=}x \mid x{<=}B]\!] \Longrightarrow A \subseteq \bigcap(C) \mid \bigcap(C) \subseteq B$
by *blast*

37.2 The Transfinite Construction

lemma *increasingD1*: $f \in increasing(A) \Longrightarrow f \in Pow(A){-}{>}Pow(A)$
unfolding *increasing-def*
apply (*erule CollectD1*)
done

lemma *increasingD2*: $[\![f \in increasing(A);\ x{<=}A]\!] \Longrightarrow x \subseteq f`x$
by (*unfold increasing-def*, *blast*)

lemmas *TFin-UnionI* = *PowI* [*THEN TFin.Pow-UnionI*]

lemmas *TFin-is-subset* = *TFin.dom-subset* [*THEN subsetD*, *THEN PowD*]

Structural induction on *TFin*(*S*, *next*)

lemma *TFin-induct*:
$[\![n \in TFin(S,next);$
$\bigwedge x.\ [\![x \in TFin(S,next);\ P(x);\ next \in increasing(S)]\!] \Longrightarrow P(next`x);$
$\bigwedge Y.\ [\![Y \subseteq TFin(S,next);\ \forall y{\in}Y.\ P(y)]\!] \Longrightarrow P(\bigcup(Y))$
$]\!] \Longrightarrow P(n)$
by (*erule TFin.induct*, *blast*+)

37.3 Some Properties of the Transfinite Construction

lemmas *increasing-trans* = *subset-trans* [*OF - increasingD2*,
OF - - TFin-is-subset]

Lemma 1 of section 3.1

lemma *TFin-linear-lemma1*:
$[\![n \in TFin(S,next);\ m \in TFin(S,next);$
$\forall x \in TFin(S,next)\ .\ x{<=}m \longrightarrow x{=}m \mid next`x{<=}m]\!]$
$\Longrightarrow n{<=}m \mid next`m{<=}n$
apply (*erule TFin-induct*)

apply (*erule-tac* [*2*] *Union-lemma0*)

apply (*blast dest*: *increasing-trans*)
done

Lemma 2 of section 3.2. Interesting in its own right! Requires *next* ∈ *increasing*(*S*) in the second induction step.

lemma *TFin-linear-lemma2*:
⟦*m* ∈ *TFin*(*S*,*next*); *next* ∈ *increasing*(*S*)⟧
⟹ ∀ *n* ∈ *TFin*(*S*,*next*). *n*<=*m* ⟶ *n*=*m* | *next*‘*n* ⊆ *m*
apply (*erule TFin-induct*)
apply (*rule impI* [*THEN ballI*])

case split using *TFin-linear-lemma1*

apply (*rule-tac n1* = *n* **and** *m1* = *x* **in** *TFin-linear-lemma1* [*THEN disjE*], *assumption*+)
apply (*blast del*: *subsetI*
intro: *increasing-trans subsetI*, *blast*)

second induction step

apply (*rule impI* [*THEN ballI*])
apply (*rule Union-lemma0* [*THEN disjE*])
apply (*erule-tac* [*3*] *disjI2*)
prefer *2* **apply** *blast*
apply (*rule ballI*)
apply (*drule bspec*, *assumption*)
apply (*drule subsetD*, *assumption*)
apply (*rule-tac n1* = *n* **and** *m1* = *x* **in** *TFin-linear-lemma1* [*THEN disjE*], *assumption*+, *blast*)
apply (*erule increasingD2* [*THEN subset-trans*, *THEN disjI1*])
apply (*blast dest*: *TFin-is-subset*)+
done

a more convenient form for Lemma 2

lemma *TFin-subsetD*:
⟦*n*<=*m*; *m* ∈ *TFin*(*S*,*next*); *n* ∈ *TFin*(*S*,*next*); *next* ∈ *increasing*(*S*)⟧
⟹ *n*=*m* | *next*‘*n* ⊆ *m*
by (*blast dest*: *TFin-linear-lemma2* [*rule-format*])

Consequences from section 3.3 – Property 3.2, the ordering is total

lemma *TFin-subset-linear*:
⟦*m* ∈ *TFin*(*S*,*next*); *n* ∈ *TFin*(*S*,*next*); *next* ∈ *increasing*(*S*)⟧
⟹ *n* ⊆ *m* | *m*<=*n*
apply (*rule disjE*)
apply (*rule TFin-linear-lemma1* [*OF - -TFin-linear-lemma2*])
apply (*assumption*+, *erule disjI2*)
apply (*blast del*: *subsetI*
intro: *subsetI increasingD2* [*THEN subset-trans*] *TFin-is-subset*)

done

Lemma 3 of section 3.3

lemma *equal-next-upper*:
  $[\![n \in TFin(S,next);\ \ m \in TFin(S,next);\ \ m = next`m]\!] \Longrightarrow n \subseteq m$
apply (*erule TFin-induct*)
apply (*drule TFin-subsetD*)
apply (*assumption+*, *force*, *blast*)
done

Property 3.3 of section 3.3

lemma *equal-next-Union*:
  $[\![m \in TFin(S,next);\ \ next \in increasing(S)]\!]$
  $\Longrightarrow m = next`m <-> m = \bigcup(TFin(S,next))$
apply (*rule iffI*)
apply (*rule Union-upper* [*THEN equalityI*])
apply (*rule-tac* [*2*] *equal-next-upper* [*THEN Union-least*])
apply (*assumption+*)
apply (*erule ssubst*)
apply (*rule increasingD2* [*THEN equalityI*], *assumption*)
apply (*blast del*: *subsetI*
    intro: *subsetI TFin-UnionI TFin.nextI TFin-is-subset*)+
done

37.4 Hausdorff's Theorem: Every Set Contains a Maximal Chain

NOTE: We assume the partial ordering is ⊆, the subset relation!

* Defining the "next" operation for Hausdorff's Theorem *

lemma *chain-subset-Pow*: $chain(A) \subseteq Pow(A)$
 unfolding *chain-def*
apply (*rule Collect-subset*)
done

lemma *super-subset-chain*: $super(A,c) \subseteq chain(A)$
 unfolding *super-def*
apply (*rule Collect-subset*)
done

lemma *maxchain-subset-chain*: $maxchain(A) \subseteq chain(A)$
 unfolding *maxchain-def*
apply (*rule Collect-subset*)
done

lemma *choice-super*:
  $[\![ch \in (\prod X \in Pow(chain(S)) - \{0\}.\ X);\ X \in chain(S);\ \ X \notin maxchain(S)]\!]$
  $\Longrightarrow ch\ `\ super(S,X) \in super(S,X)$

apply (*erule apply-type*)
apply (*unfold super-def maxchain-def*, *blast*)
done

lemma *choice-not-equals*:
  $[\![ch \in (\prod X \in Pow(chain(S)) - \{0\}.\ X);\ X \in chain(S);\ \ X \notin maxchain(S)]\!]$
  $\Longrightarrow ch\ `\ super(S,X) \neq X$
apply (*rule notI*)
apply (*drule choice-super*, *assumption*, *assumption*)
apply (*simp add*: *super-def*)
done

This justifies Definition 4.4

lemma *Hausdorff-next-exists*:
  $ch \in (\prod X \in Pow(chain(S)) - \{0\}.\ X) \Longrightarrow$
  $\exists\, next \in increasing(S).\ \forall\, X \in Pow(S).$
    $next`X = if(X \in chain(S) - maxchain(S),\ ch`super(S,X),\ X)$
apply (*rule-tac* $x{=}\lambda X{\in}Pow(S).$
    if $X \in chain(S) - maxchain(S)$ *then* $ch\ `\ super(S,\ X)$ *else* X
  in *bexI*)
apply *force*
  unfolding *increasing-def*
apply (*rule CollectI*)
apply (*rule lam-type*)
apply (*simp* (*no-asm-simp*))
apply (*blast dest*: *super-subset-chain* [*THEN subsetD*]
    chain-subset-Pow [*THEN subsetD*] *choice-super*)

Now, verify that it increases

apply (*simp* (*no-asm-simp*) *add*: *Pow-iff subset-refl*)
apply *safe*
apply (*drule choice-super*)
apply (*assumption+*)
apply (*simp add*: *super-def*, *blast*)
done

Lemma 4

lemma *TFin-chain-lemma4*:
  $[\![c \in TFin(S,next);$
    $ch \in (\prod X \in Pow(chain(S)) - \{0\}.\ X);$
    $next \in increasing(S);$
    $\forall\, X \in Pow(S).\ next`X =$
      $if(X \in chain(S) - maxchain(S),\ ch`super(S,X),\ X)]\!]$
  $\Longrightarrow c \in chain(S)$
apply (*erule TFin-induct*)
apply (*simp* (*no-asm-simp*) *add*: *chain-subset-Pow* [*THEN subsetD*, *THEN PowD*]
    choice-super [*THEN super-subset-chain* [*THEN subsetD*]])
  unfolding *chain-def*
apply (*rule CollectI*, *blast*, *safe*)

apply (*rule-tac m1=B* **and** *n1=Ba* **in** *TFin-subset-linear* [*THEN disjE*], *fast*+)

Blast-tac's slow

done

theorem *Hausdorff*: $\exists$ *c*. *c* $\in$ *maxchain*(*S*)
apply (*rule AC-Pi-Pow* [*THEN exE*])
apply (*rule Hausdorff-next-exists* [*THEN bexE*], *assumption*)
apply (*rename-tac ch next*)
apply (*subgoal-tac* $\bigcup$(*TFin* (*S*,*next*)) $\in$ *chain* (*S*))
prefer *2*
 apply (*blast intro*!: *TFin-chain-lemma4 subset-refl* [*THEN TFin-UnionI*])
apply (*rule-tac x* = $\bigcup$(*TFin* (*S*,*next*)) **in** *exI*)
apply (*rule classical*)
apply (*subgoal-tac next* ‘ *Union*(*TFin* (*S*,*next*)) = $\bigcup$(*TFin* (*S*,*next*)))
apply (*rule-tac* [*2*] *equal-next-Union* [*THEN iffD2*, *symmetric*])
apply (*rule-tac* [*2*] *subset-refl* [*THEN TFin-UnionI*])
prefer *2* **apply** *assumption*
apply (*rule-tac* [*2*] *refl*)
apply (*simp add*: *subset-refl* [*THEN TFin-UnionI*,
THEN TFin.dom-subset [*THEN subsetD*, *THEN PowD*]])
apply (*erule choice-not-equals* [*THEN notE*])
apply (*assumption*+)
done

37.5 Zorn's Lemma: If All Chains in S Have Upper Bounds In S, then S contains a Maximal Element

Used in the proof of Zorn's Lemma

lemma *chain-extend*:
[[*c* $\in$ *chain*(*A*); *z* $\in$ *A*; $\forall$ *x* $\in$ *c*. *x*<=*z*]] $\Longrightarrow$ *cons*(*z*,*c*) $\in$ *chain*(*A*)
by (*unfold chain-def*, *blast*)

lemma *Zorn*: $\forall$ *c* $\in$ *chain*(*S*). $\bigcup$(*c*) $\in$ *S* $\Longrightarrow$ $\exists$ *y* $\in$ *S*. $\forall$ *z* $\in$ *S*. *y*<=*z* $\longrightarrow$ *y*=*z*
apply (*rule Hausdorff* [*THEN exE*])
apply (*simp add*: *maxchain-def*)
apply (*rename-tac c*)
apply (*rule-tac x* = $\bigcup$(*c*) **in** *bexI*)
prefer *2* **apply** *blast*
apply *safe*
apply (*rename-tac z*)
apply (*rule classical*)
apply (*subgoal-tac cons* (*z*,*c*) $\in$ *super* (*S*,*c*))
apply (*blast elim*: *equalityE*)
apply (*unfold super-def*, *safe*)
apply (*fast elim*: *chain-extend*)
apply (*fast elim*: *equalityE*)
done

Alternative version of Zorn's Lemma

theorem *Zorn2*:
$\forall\, c \in chain(S).\ \exists\, y \in S.\ \forall\, x \in c.\ x \subseteq y \Longrightarrow \exists\, y \in S.\ \forall\, z \in S.\ y{<=}z \longrightarrow y{=}z$
apply (*cut-tac Hausdorff maxchain-subset-chain*)
apply (*erule exE*)
apply (*drule subsetD*, *assumption*)
apply (*drule bspec*, *assumption*, *erule bexE*)
apply (*rule-tac* $x = y$ **in** *bexI*)
prefer *2* **apply** *assumption*
apply *clarify*
apply *rule* **apply** *assumption*
apply *rule*
apply (*rule ccontr*)
apply (*frule-tac* $z{=}z$ **in** *chain-extend*)
apply (*assumption*, *blast*)
unfolding *maxchain-def super-def*
apply (*blast elim*!: *equalityCE*)
done

37.6 Zermelo's Theorem: Every Set can be Well-Ordered

Lemma 5

lemma *TFin-well-lemma5*:
$[\![n \in TFin(S,next);\ \ Z \subseteq TFin(S,next);\ \ z{:}Z;\ \ \neg \bigcap(Z) \in Z]\!]$
$\Longrightarrow \forall\, m \in Z.\ n \subseteq m$
apply (*erule TFin-induct*)
prefer *2* **apply** *blast*

second induction step is easy

apply (*rule ballI*)
apply (*rule bspec* [*THEN TFin-subsetD*, *THEN disjE*], *auto*)
apply (*subgoal-tac* $m = \bigcap(Z)$)
apply *blast+*
done

Well-ordering of $TFin(S,\ next)$

lemma *well-ord-TFin-lemma*: $[\![Z \subseteq TFin(S,next);\ \ z \in Z]\!] \Longrightarrow \bigcap(Z) \in Z$
apply (*rule classical*)
apply (*subgoal-tac* $Z = \{\bigcup(TFin\ (S,next))\}$)
apply (*simp* (*no-asm-simp*) *add*: *Inter-singleton*)
apply (*erule equal-singleton*)
apply (*rule Union-upper* [*THEN equalityI*])
apply (*rule-tac* [*2*] *subset-refl* [*THEN TFin-UnionI*, *THEN TFin-well-lemma5*, *THEN bspec*], *blast+*)
done

This theorem just packages the previous result

lemma *well-ord-TFin*:

next ∈ *increasing*(*S*)
⟹ *well-ord*(*TFin*(*S*,*next*), *Subset-rel*(*TFin*(*S*,*next*)))
apply (*rule well-ordI*)
unfolding *Subset-rel-def linear-def*

Prove the well-foundedness goal

apply (*rule wf-onI*)
apply (*frule well-ord-TFin-lemma*, *assumption*)
apply (*drule-tac x* = ⋂(*Z*) **in** *bspec*, *assumption*)
apply *blast*

Now prove the linearity goal

apply (*intro ballI*)
apply (*case-tac x*=*y*)
apply *blast*

The $x \neq y$ case remains

apply (*rule-tac n1*=*x* **and** *m1*=*y* **in** *TFin-subset-linear* [*THEN disjE*],
assumption+, *blast*+)
done

* Defining the "next" operation for Zermelo's Theorem *

lemma *choice-Diff*:
⟦*ch* ∈ (∏ *X* ∈ *Pow*(*S*) − {*0*}. *X*); *X* ⊆ *S*; *X*≠*S*⟧ ⟹ *ch* ‘ (*S*−*X*) ∈ *S*−*X*
apply (*erule apply-type*)
apply (*blast elim*!: *equalityE*)
done

This justifies Definition 6.1

lemma *Zermelo-next-exists*:
ch ∈ (∏ *X* ∈ *Pow*(*S*)−{*0*}. *X*) ⟹
∃ *next* ∈ *increasing*(*S*). ∀ *X* ∈ *Pow*(*S*).
next‘*X* = (*if X*=*S then S else cons*(*ch*‘(*S*−*X*), *X*))
apply (*rule-tac x*=λ*X*∈*Pow*(*S*). *if X*=*S then S else cons*(*ch*‘(*S*−*X*), *X*)
in *bexI*)
apply *force*
unfolding *increasing-def*
apply (*rule CollectI*)
apply (*rule lam-type*)

Type checking is surprisingly hard!

apply (*simp* (*no-asm-simp*) *add*: *Pow-iff cons-subset-iff subset-refl*)
apply (*blast intro*!: *choice-Diff* [*THEN DiffD1*])

Verify that it increases

apply (*intro allI impI*)
apply (*simp add*: *Pow-iff subset-consI subset-refl*)
done

The construction of the injection

lemma *choice-imp-injection*:
  $\llbracket ch \in (\prod X \in Pow(S)-\{0\}.\ X);$
    $next \in increasing(S);$
    $\forall X \in Pow(S).\ next`X = if(X{=}S,\ S,\ cons(ch`(S-X),\ X))\rrbracket$
  $\Longrightarrow (\lambda\ x \in S.\ \bigcup(\{y \in TFin(S,next).\ x \notin y\}))$
    $\in inj(S,\ TFin(S,next) - \{S\})$
apply (*rule-tac d* = $\lambda y.\ ch`\ (S-y)$ **in** *lam-injective*)
apply (*rule DiffI*)
apply (*rule Collect-subset* [*THEN TFin-UnionI*])
apply (*blast intro*!: *Collect-subset* [*THEN TFin-UnionI*] *elim*: *equalityE*)
apply (*subgoal-tac* $x \notin \bigcup(\{y \in TFin\ (S,next)\ .\ x \notin y\})$)
prefer *2* **apply** (*blast elim*: *equalityE*)
apply (*subgoal-tac* $\bigcup(\{y \in TFin\ (S,next)\ .\ x \notin y\}) \neq S$)
prefer *2* **apply** (*blast elim*: *equalityE*)

For proving $x \in next`\bigcup(...)$. Abrial and Laffitte's justification appears to be faulty.

apply (*subgoal-tac* $\neg\ next\ `\ Union(\{y \in TFin\ (S,next)\ .\ x \notin y\})$
    $\subseteq \bigcup(\{y \in TFin\ (S,next)\ .\ x \notin y\})$)
prefer *2*
apply (*simp del*: *Union-iff*
    add: *Collect-subset* [*THEN TFin-UnionI*, *THEN TFin-is-subset*]
    Pow-iff cons-subset-iff subset-refl choice-Diff [*THEN DiffD2*])
apply (*subgoal-tac* $x \in next\ `\ Union(\{y \in TFin\ (S,next)\ .\ x \notin y\})$)
prefer *2*
apply (*blast intro*!: *Collect-subset* [*THEN TFin-UnionI*] *TFin.nextI*)

End of the lemmas!

apply (*simp add*: *Collect-subset* [*THEN TFin-UnionI*, *THEN TFin-is-subset*])
done

The wellordering theorem

theorem *AC-well-ord*: $\exists\, r.\ well\text{-}ord(S,r)$
apply (*rule AC-Pi-Pow* [*THEN exE*])
apply (*rule Zermelo-next-exists* [*THEN bexE*], *assumption*)
apply (*rule exI*)
apply (*rule well-ord-rvimage*)
apply (*erule-tac* [*2*] *well-ord-TFin*)
apply (*rule choice-imp-injection* [*THEN inj-weaken-type*], *blast*+)
done

37.7 Zorn's Lemma for Partial Orders

Reimported from HOL by Clemens Ballarin.

definition *Chain* :: $i \Rightarrow i$ **where**
  $Chain(r) = \{A \in Pow(field(r)).\ \forall a{\in}A.\ \forall b{\in}A.\ \langle a,\ b\rangle \in r \mid \langle b,\ a\rangle \in r\}$

lemma *mono-Chain*:

$r \subseteq s \Longrightarrow Chain(r) \subseteq Chain(s)$
unfolding *Chain-def*
by *blast*

theorem *Zorn-po*:
assumes *po*: $Partial\text{-}order(r)$
and *u*: $\forall C \in Chain(r).\ \exists u \in field(r).\ \forall a \in C.\ \langle a, u\rangle \in r$
shows $\exists m \in field(r).\ \forall a \in field(r).\ \langle m, a\rangle \in r \longrightarrow a = m$
proof –
have $Preorder(r)$ **using** *po* **by** (*simp add*: *partial-order-on-def*)
— Mirror r in the set of subsets below (wrt r) elements of A (?).
let $?B = \lambda x \in field(r).\ r -`` \{x\}$ **let** $?S = ?B `` field(r)$
have $\forall C \in chain(?S).\ \exists U \in ?S.\ \forall A \in C.\ A \subseteq U$
proof (*clarsimp simp*: *chain-def Subset-rel-def bex-image-simp*)
fix *C*
assume *1*: $C \subseteq ?S$ **and** *2*: $\forall A \in C.\ \forall B \in C.\ A \subseteq B \mid B \subseteq A$
let $?A = \{x \in field(r).\ \exists M \in C.\ M = ?B`x\}$
have $C = ?B `` ?A$ **using** *1*
apply (*auto simp*: *image-def*)
apply *rule*
apply *rule*
apply (*drule subsetD*) **apply** *assumption*
apply (*erule CollectE*)
apply *rule* **apply** *assumption*
apply (*erule bexE*)
apply *rule* **prefer** *2* **apply** *assumption*
apply *rule*
apply (*erule lamE*) **apply** *simp*
apply *assumption*

apply (*thin-tac* $C \subseteq X$ **for** *X*)
apply (*fast elim*: *lamE*)
done
have $?A \in Chain(r)$
proof (*simp add*: *Chain-def subsetI*, *intro conjI ballI impI*)
fix *a b*
assume $a \in field(r)\ r -`` \{a\} \in C\ b \in field(r)\ r -`` \{b\} \in C$
hence $r -`` \{a\} \subseteq r -`` \{b\} \mid r -`` \{b\} \subseteq r -`` \{a\}$ **using** *2* **by** *auto*
then show $\langle a, b\rangle \in r \mid \langle b, a\rangle \in r$
using ‹$Preorder(r)$› ‹$a \in field(r)$› ‹$b \in field(r)$›
by (*simp add*: *subset-vimage1-vimage1-iff*)
qed
then obtain *u* **where** *uA*: $u \in field(r)\ \forall a \in ?A.\ \langle a, u\rangle \in r$
using *u*
apply *auto*
apply (*drule bspec*) **apply** *assumption*
apply *auto*
done
have $\forall A \in C.\ A \subseteq r -`` \{u\}$

```
    proof (auto intro!: vimageI)
      fix a B
      assume aB: B ∈ C a ∈ B
      with 1 obtain x where x ∈ field(r) B = r -`` {x}
        apply -
        apply (drule subsetD) apply assumption
        apply (erule imageE)
        apply (erule lamE)
        apply simp
        done
      then show ⟨a, u⟩ ∈ r using uA aB ‹Preorder(r)›
        by (auto simp: preorder-on-def refl-def) (blast dest: trans-onD)+
    qed
    then show ∃ U∈field(r). ∀ A∈C. A ⊆ r -`` {U}
      using ‹u ∈ field(r)› ..
  qed
  from Zorn2 [OF this]
  obtain m B where m ∈ field(r) B = r -`` {m}
    ∀ x∈field(r). B ⊆ r -`` {x} ⟶ B = r -`` {x}
    by (auto elim!: lamE simp: ball-image-simp)
  then have ∀ a∈field(r). ⟨m, a⟩ ∈ r ⟶ a = m
    using po ‹Preorder(r)› ‹m ∈ field(r)›
   by (auto simp: subset-vimage1-vimage1-iff Partial-order-eq-vimage1-vimage1-iff)
  then show ?thesis using ‹m ∈ field(r)› by blast
qed

end
```

38 Cardinal Arithmetic Using AC

theory *Cardinal-AC* **imports** *CardinalArith Zorn* **begin**

38.1 Strengthened Forms of Existing Theorems on Cardinals

```
lemma cardinal-eqpoll: |A| ≈ A
apply (rule AC-well-ord [THEN exE])
apply (erule well-ord-cardinal-eqpoll)
done
```

The theorem $||A|| = |A|$

```
lemmas cardinal-idem = cardinal-eqpoll [THEN cardinal-cong, simp]

lemma cardinal-eqE: |X| = |Y| ⟹ X ≈ Y
apply (rule AC-well-ord [THEN exE])
apply (rule AC-well-ord [THEN exE])
apply (rule well-ord-cardinal-eqE, assumption+)
done

lemma cardinal-eqpoll-iff: |X| = |Y| ⟷ X ≈ Y
```

by (*blast intro*: *cardinal-cong cardinal-eqE*)

lemma *cardinal-disjoint-Un*:
$[\![|A|=|B|;\ |C|=|D|;\ A \cap C = 0;\ B \cap D = 0]\!]$
$\Longrightarrow |A \cup C| = |B \cup D|$
by (*simp add*: *cardinal-eqpoll-iff eqpoll-disjoint-Un*)

lemma *lepoll-imp-cardinal-le*: $A \lesssim B \Longrightarrow |A| \le |B|$
apply (*rule AC-well-ord* [*THEN exE*])
apply (*erule well-ord-lepoll-imp-cardinal-le*, *assumption*)
done

lemma *cadd-assoc*: $(i \oplus j) \oplus k = i \oplus (j \oplus k)$
apply (*rule AC-well-ord* [*THEN exE*])
apply (*rule AC-well-ord* [*THEN exE*])
apply (*rule AC-well-ord* [*THEN exE*])
apply (*rule well-ord-cadd-assoc*, *assumption+*)
done

lemma *cmult-assoc*: $(i \otimes j) \otimes k = i \otimes (j \otimes k)$
apply (*rule AC-well-ord* [*THEN exE*])
apply (*rule AC-well-ord* [*THEN exE*])
apply (*rule AC-well-ord* [*THEN exE*])
apply (*rule well-ord-cmult-assoc*, *assumption+*)
done

lemma *cadd-cmult-distrib*: $(i \oplus j) \otimes k = (i \otimes k) \oplus (j \otimes k)$
apply (*rule AC-well-ord* [*THEN exE*])
apply (*rule AC-well-ord* [*THEN exE*])
apply (*rule AC-well-ord* [*THEN exE*])
apply (*rule well-ord-cadd-cmult-distrib*, *assumption+*)
done

lemma *InfCard-square-eq*: $InfCard(|A|) \Longrightarrow A*A \approx A$
apply (*rule AC-well-ord* [*THEN exE*])
apply (*erule well-ord-InfCard-square-eq*, *assumption*)
done

38.2 The relationship between cardinality and le-pollence

lemma *Card-le-imp-lepoll*:
assumes $|A| \le |B|$ **shows** $A \lesssim B$
proof −
have $A \approx |A|$
by (*rule cardinal-eqpoll* [*THEN eqpoll-sym*])
also have ... $\lesssim |B|$
by (*rule le-imp-subset* [*THEN subset-imp-lepoll*]) (*rule assms*)
also have ... $\approx B$
by (*rule cardinal-eqpoll*)

finally show *?thesis* **.**
qed

lemma *le-Card-iff*: *Card*(K) $\Longrightarrow$ $|A| \leq K \longleftrightarrow A \lesssim K$
apply (*erule Card-cardinal-eq* [*THEN subst*], *rule iffI*,
erule Card-le-imp-lepoll)
apply (*erule lepoll-imp-cardinal-le*)
done

lemma *cardinal-0-iff-0* [*simp*]: $|A| = 0 \longleftrightarrow A = 0$
apply *auto*
apply (*drule cardinal-0* [*THEN ssubst*])
apply (*blast intro*: *eqpoll-0-iff* [*THEN iffD1*] *cardinal-eqpoll-iff* [*THEN iffD1*])
done

lemma *cardinal-lt-iff-lesspoll*:
assumes *i*: *Ord*(i) **shows** $i < |A| \longleftrightarrow i \prec A$
proof
assume $i < |A|$
hence $i \prec |A|$
by (*blast intro*: *lt-Card-imp-lesspoll Card-cardinal*)
also have ... $\approx A$
by (*rule cardinal-eqpoll*)
finally show $i \prec A$ **.**
next
assume $i \prec A$
also have ... $\approx |A|$
by (*blast intro*: *cardinal-eqpoll eqpoll-sym*)
finally have $i \prec |A|$ **.**
thus $i < |A|$ **using** *i*
by (*force intro*: *cardinal-lt-imp-lt lesspoll-cardinal-lt*)
qed

lemma *cardinal-le-imp-lepoll*: $i \leq |A| \Longrightarrow i \lesssim A$
by (*blast intro*: *lt-Ord Card-le-imp-lepoll Ord-cardinal-le le-trans*)

38.3 Other Applications of AC

lemma *surj-implies-inj*:
assumes *f*: $f \in surj(X,Y)$ **shows** $\exists g.\ g \in inj(Y,X)$
proof −
from *f AC-Pi* [*of Y* $\lambda y.\ f-``\{y\}$]
obtain *z* **where** *z*: $z \in (\prod y{\in}Y.\ f -``\ \{y\})$
by (*auto simp add*: *surj-def*) (*fast dest*: *apply-Pair*)
show *?thesis*
proof
show $z \in inj(Y,\ X)$ **using** *z surj-is-fun* [*OF f*]
by (*blast dest*: *apply-type Pi-memberD*
intro: *apply-equality Pi-type f-imp-injective*)

qed
qed

Kunen's Lemma 10.20

lemma *surj-implies-cardinal-le*:
assumes *f*: $f \in surj(X,Y)$ **shows** $|Y| \leq |X|$
proof (*rule lepoll-imp-cardinal-le*)
from *f* [*THEN surj-implies-inj*] **obtain** *g* **where** $g \in inj(Y,X)$ **..**
thus $Y \lesssim X$
by (*auto simp add*: *lepoll-def*)
qed

Kunen's Lemma 10.21

lemma *cardinal-UN-le*:
assumes *K*: *InfCard*(*K*)
shows $(\bigwedge i.\ i{\in}K \Longrightarrow |X(i)| \leq K) \Longrightarrow |\bigcup i{\in}K.\ X(i)| \leq K$
proof (*simp add*: *K InfCard-is-Card le-Card-iff*)
have [*intro*]: *Ord*(*K*) **by** (*blast intro*: *InfCard-is-Card Card-is-Ord K*)
assume $\bigwedge i.\ i{\in}K \Longrightarrow X(i) \lesssim K$
hence $\bigwedge i.\ i{\in}K \Longrightarrow \exists f.\ f \in inj(X(i),\ K)$ **by** (*simp add*: *lepoll-def*)
with *AC-Pi* **obtain** *f* **where** *f*: $f \in (\prod i{\in}K.\ inj(X(i),\ K))$
by *force*
{ fix *z*
assume *z*: $z \in (\bigcup i{\in}K.\ X(i))$
then obtain *i* **where** *i*: $i \in K$ *Ord*(*i*) $z \in X(i)$
by (*blast intro*: *Ord-in-Ord* [*of K*])
hence $(\mu\ i.\ z \in X(i)) \leq i$ **by** (*fast intro*: *Least-le*)
hence $(\mu\ i.\ z \in X(i)) < K$ **by** (*best intro*: *lt-trans1 ltI i*)
hence $(\mu\ i.\ z \in X(i)) \in K$ **and** $z \in X(\mu\ i.\ z \in X(i))$
by (*auto intro*: *LeastI ltD i*)
} note *mems* = *this*
have $(\bigcup i{\in}K.\ X(i)) \lesssim K \times K$
proof (*unfold lepoll-def*)
show $\exists f.\ f \in inj(\bigcup RepFun(K,\ X),\ K \times K)$
apply (*rule exI*)
apply (*rule-tac* $c = \lambda z.\ \langle \mu\ i.\ z \in X(i),\ f\ \text{`}\ (\mu\ i.\ z \in X(i))\ \text{`}\ z\rangle$
and $d = \lambda\langle i,j\rangle.\ converse\ (f\text{`}i)\ \text{`}\ j$ **in** *lam-injective*)
apply (*force intro*: *f inj-is-fun mems apply-type Perm.left-inverse*)+
done
qed
also have ... $\approx K$
by (*simp add*: *K InfCard-square-eq InfCard-is-Card Card-cardinal-eq*)
finally show $(\bigcup i{\in}K.\ X(i)) \lesssim K$ **.**
qed

The same again, using *csucc*

lemma *cardinal-UN-lt-csucc*:
$[\![InfCard(K);\ \bigwedge i.\ i{\in}K \Longrightarrow |X(i)| < csucc(K)]\!]$
$\Longrightarrow |\bigcup i{\in}K.\ X(i)| < csucc(K)$

by (*simp add*: *Card-lt-csucc-iff cardinal-UN-le InfCard-is-Card Card-cardinal*)

The same again, for a union of ordinals. In use, j(i) is a bit like rank(i), the least ordinal j such that i:Vfrom(A,j).

lemma *cardinal-UN-Ord-lt-csucc*:
  $[\![InfCard(K);\ \bigwedge i.\ i{\in}K \Longrightarrow j(i) < csucc(K)]\!]$
  $\Longrightarrow (\bigcup i{\in}K.\ j(i)) < csucc(K)$
apply (*rule cardinal-UN-lt-csucc* [*THEN Card-lt-imp-lt*], *assumption*)
apply (*blast intro*: *Ord-cardinal-le* [*THEN lt-trans1*] *elim*: *ltE*)
apply (*blast intro*!: *Ord-UN elim*: *ltE*)
apply (*erule InfCard-is-Card* [*THEN Card-is-Ord*, *THEN Card-csucc*])
done

38.4 The Main Result for Infinite-Branching Datatypes

As above, but the index set need not be a cardinal. Work backwards along the injection from W into K, given that $W \neq 0$.

lemma *inj-UN-subset*:
 assumes f: $f \in inj(A,B)$ **and** a: $a \in A$
 shows $(\bigcup x{\in}A.\ C(x)) \subseteq (\bigcup y{\in}B.\ C(if\ y \in range(f)\ then\ converse(f)`y\ else\ a))$
proof (*rule UN-least*)
 fix x
 assume x: $x \in A$
 hence fx: $f ` x \in B$ **by** (*blast intro*: *f inj-is-fun* [*THEN apply-type*])
 have $C(x) \subseteq C(if\ f ` x \in range(f)\ then\ converse(f) ` (f ` x)\ else\ a)$
  using *f x* **by** (*simp add*: *inj-is-fun* [*THEN apply-rangeI*])
 also have ... $\subseteq (\bigcup y{\in}B.\ C(if\ y \in range(f)\ then\ converse(f)\ `\ y\ else\ a))$
  by (*rule UN-upper* [*OF fx*])
 finally show $C(x) \subseteq (\bigcup y{\in}B.\ C(if\ y \in range(f)\ then\ converse(f)`y\ else\ a))$ **.**
qed

theorem *le-UN-Ord-lt-csucc*:
 assumes *IK*: *InfCard*(*K*) **and** *WK*: $|W| \leq K$ **and** *j*: $\bigwedge w.\ w{\in}W \Longrightarrow j(w) < csucc(K)$
 shows $(\bigcup w{\in}W.\ j(w)) < csucc(K)$
proof −
 have *CK*: *Card*(*K*)
  by (*simp add*: *InfCard-is-Card IK*)
 then obtain *f* **where** f: $f \in inj(W, K)$ **using** *WK*
  by(*auto simp add*: *le-Card-iff lepoll-def*)
 have *OU*: $Ord(\bigcup w{\in}W.\ j(w))$ **using** *j*
  by (*blast elim*: *ltE*)
 note *lt-subset-trans* [*OF - - OU*, *trans*]
 show *?thesis*
  proof (*cases W=0*)
   case *True* — solve the easy 0 case
   thus *?thesis* **by** (*simp add*: *CK Card-is-Ord Card-csucc Ord-0-lt-csucc*)
  next

case *False*
then obtain *x* **where** *x*: *x* ∈ *W* **by** *blast*
have $(\bigcup x\in W.\ j(x)) \subseteq (\bigcup k\in K.\ j($*if* $k \in$ *range*(f) *then converse*(f) ‘ *k else* *x*))
by (*rule inj-UN-subset* [*OF f x*])
also have ... < *csucc*(*K*) **using** *IK*
proof (*rule cardinal-UN-Ord-lt-csucc*)
fix *k*
assume *k* ∈ *K*
thus *j*(*if k* ∈ *range*(*f*) *then converse*(*f*) ‘ *k else x*) < *csucc*(*K*) **using** *f x j*
by (*simp add*: *inj-converse-fun* [*THEN apply-type*])
qed
finally show *?thesis* **.**
qed
qed

end

39 Infinite-Branching Datatype Definitions

theory *InfDatatype* **imports** *Datatype Univ Finite Cardinal-AC* **begin**

lemmas *fun-Limit-VfromE* =
Limit-VfromE [*OF apply-funtype InfCard-csucc* [*THEN InfCard-is-Limit*]]

lemma *fun-Vcsucc-lemma*:
assumes *f*: *f* ∈ *D* −> *Vfrom*(*A*,*csucc*(*K*)) **and** *DK*: |*D*| ≤ *K* **and** *ICK*: *InfCard*(*K*)
shows ∃ *j*. *f* ∈ *D* −> *Vfrom*(*A*,*j*) ∧ *j* < *csucc*(*K*)
proof (*rule exI*, *rule conjI*)
show *f* ∈ *D* → *Vfrom*(*A*, ⋃ *z*∈*D*. μ *i*. *f*‘*z* ∈ *Vfrom* (*A*,*i*))
proof (*rule Pi-type* [*OF f*])
fix *d*
assume *d*: *d* ∈ *D*
show *f* ‘ *d* ∈ *Vfrom*(*A*, ⋃ *z*∈*D*. μ *i*. *f* ‘ *z* ∈ *Vfrom*(*A*, *i*))
proof (*rule fun-Limit-VfromE* [*OF f d ICK*])
fix *x*
assume *x* < *csucc*(*K*) *f* ‘ *d* ∈ *Vfrom*(*A*, *x*)
hence *f*‘*d* ∈ *Vfrom*(*A*, μ *i*. *f*‘*d* ∈ *Vfrom* (*A*,*i*)) **using** *d*
by (*fast elim*: *LeastI ltE*)
also have ... ⊆ *Vfrom*(*A*, ⋃ *z*∈*D*. μ *i*. *f* ‘ *z* ∈ *Vfrom*(*A*, *i*))
by (*rule Vfrom-mono*) (*auto intro*: *d*)
finally show *f*‘*d* ∈ *Vfrom*(*A*, ⋃ *z*∈*D*. μ *i*. *f* ‘ *z* ∈ *Vfrom*(*A*, *i*)) **.**
qed
qed
next
show (⋃ *d*∈*D*. μ *i*. *f* ‘ *d* ∈ *Vfrom*(*A*, *i*)) < *csucc*(*K*)
proof (*rule le-UN-Ord-lt-csucc* [*OF ICK DK*])
fix *d*

assume *d*: $d \in D$
show $(\mu\ i.\ f\ `\ d \in Vfrom(A,\ i)) < csucc(K)$
proof (*rule fun-Limit-VfromE* [*OF f d ICK*])
fix *x*
assume $x < csucc(K)\ \ f\ `\ d \in Vfrom(A,\ x)$
thus $(\mu\ i.\ f\ `\ d \in Vfrom(A,\ i)) < csucc(K)$
by (*blast intro*: *Least-le lt-trans1 lt-Ord*)
qed
qed
qed

lemma *subset-Vcsucc*:
$[\![D \subseteq Vfrom(A,csucc(K));\ \ |D| \le K;\ \ InfCard(K)]\!]$
$\Longrightarrow \exists j.\ D \subseteq Vfrom(A,j) \wedge j < csucc(K)$
by (*simp add*: *subset-iff-id fun-Vcsucc-lemma*)

lemma *fun-Vcsucc*:
$[\![|D| \le K;\ \ InfCard(K);\ \ D \subseteq Vfrom(A,csucc(K))]\!] \Longrightarrow$
$D \ {-}{>}\ Vfrom(A,csucc(K)) \subseteq Vfrom(A,csucc(K))$
apply (*safe dest*!: *fun-Vcsucc-lemma subset-Vcsucc*)
apply (*rule Vfrom* [*THEN ssubst*])
apply (*drule fun-is-rel*)

apply (*rule-tac a1 = succ* (*succ* $(j \cup ja)$) **in** *UN-I* [*THEN UnI2*])
apply (*blast intro*: *ltD InfCard-csucc InfCard-is-Limit Limit-has-succ*
Un-least-lt)
apply (*erule subset-trans* [*THEN PowI*])
apply (*fast intro*: *Pair-in-Vfrom Vfrom-UnI1 Vfrom-UnI2*)
done

lemma *fun-in-Vcsucc*:
$[\![f\colon D \ {-}{>}\ Vfrom(A,\ csucc(K));\ \ |D| \le K;\ \ InfCard(K);$
$D \subseteq Vfrom(A,csucc(K))]\!]$
$\Longrightarrow f\colon Vfrom(A,csucc(K))$
by (*blast intro*: *fun-Vcsucc* [*THEN subsetD*])

Remove $\subseteq$ from the rule above

lemmas $\textit{fun-in-Vcsucc}' = \textit{fun-in-Vcsucc}$ [*OF - - - subsetI*]

lemma *Card-fun-Vcsucc*:
$InfCard(K) \Longrightarrow K \ {-}{>}\ Vfrom(A,csucc(K)) \subseteq Vfrom(A,csucc(K))$
apply (*frule InfCard-is-Card* [*THEN Card-is-Ord*])
apply (*blast del*: *subsetI*
intro: *fun-Vcsucc Ord-cardinal-le i-subset-Vfrom*
lt-csucc [*THEN leI*, *THEN le-imp-subset*, *THEN subset-trans*])
done

lemma *Card-fun-in-Vcsucc*:
  $[\![f\colon K \mathbin{-\!\!>} Vfrom(A,\ csucc(K));\ \ InfCard(K)]\!] \Longrightarrow f\colon Vfrom(A,csucc(K))$
by (*blast intro*: *Card-fun-Vcsucc* [*THEN subsetD*])

lemma *Limit-csucc*: $InfCard(K) \Longrightarrow Limit(csucc(K))$
by (*erule InfCard-csucc* [*THEN InfCard-is-Limit*])

lemmas *Pair-in-Vcsucc* = *Pair-in-VLimit* [*OF - - Limit-csucc*]
lemmas *Inl-in-Vcsucc* = *Inl-in-VLimit* [*OF - Limit-csucc*]
lemmas *Inr-in-Vcsucc* = *Inr-in-VLimit* [*OF - Limit-csucc*]
lemmas *zero-in-Vcsucc* = *Limit-csucc* [*THEN zero-in-VLimit*]
lemmas *nat-into-Vcsucc* = *nat-into-VLimit* [*OF - Limit-csucc*]

lemmas *InfCard-nat-Un-cardinal* = *InfCard-Un* [*OF InfCard-nat Card-cardinal*]

lemmas *le-nat-Un-cardinal* =
  Un-upper2-le [*OF Ord-nat Card-cardinal* [*THEN Card-is-Ord*]]

lemmas *UN-upper-cardinal* = *UN-upper* [*THEN subset-imp-lepoll*, *THEN lepoll-imp-cardinal-le*]

lemmas *Data-Arg-intros* =
  SigmaI InlI InrI
  Pair-in-univ Inl-in-univ Inr-in-univ
  zero-in-univ A-into-univ nat-into-univ UnCI

lemmas *inf-datatype-intros* =
  InfCard-nat InfCard-nat-Un-cardinal
  Pair-in-Vcsucc Inl-in-Vcsucc Inr-in-Vcsucc
  zero-in-Vcsucc A-into-Vfrom nat-into-Vcsucc
  Card-fun-in-Vcsucc fun-in-Vcsucc′ UN-I

end
theory *ZFC* **imports** *ZF InfDatatype*
begin

end