Complex Analysis

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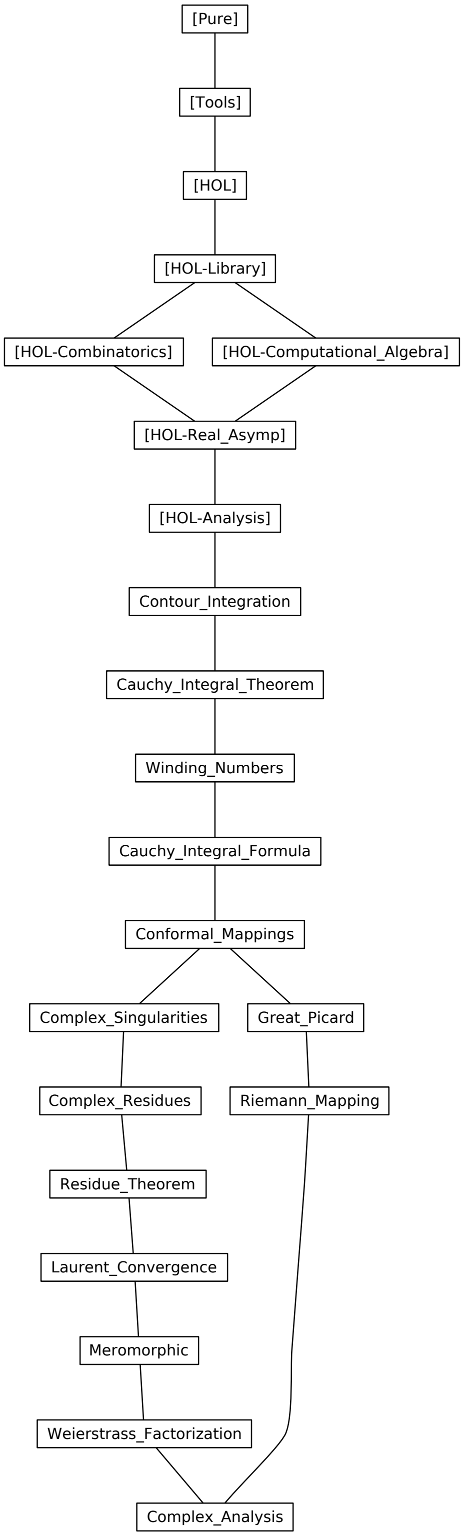
Contents


[Pure]
[Tools]
[HOL]
[HOL-Library]
[HOL-Combinatorics]
[HOL-Computational_Algebra]
[HOL-Real_Asymp]
[HOL-Analysis]
Contour_Integration
Cauchy_Integral_Theorem
Winding_Numbers
Cauchy_Integral_Formula
Conformal_Mappings
Complex_Singularities
Great_Picard
Complex_Residues
Riemann_Mapping
Residue_Theorem
Laurent_Convergence
Meromorphic
Weierstrass_Factorization
Complex_Analysis

1 Contour integration

theory *Contour_Integration*
imports *HOL−Analysis.Analysis*
begin

1.1 Definition

definition *has_contour_integral* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ (*real* ⇒ *complex*) ⇒ *bool*
(**infixr** ‹*has'_contour'_integral*› *50*)
where (*f has_contour_integral i*) *g* ≡
((λ*x*. *f*(*g x*) ∗ *vector_derivative g* (*at x within* {*0*..*1*}))
has_integral i) {*0*..*1*}

definition *contour_integrable_on*
(**infixr** ‹*contour'_integrable'_on*› *50*)
where *f contour_integrable_on g* ≡ ∃ *i*. (*f has_contour_integral i*) *g*

definition *contour_integral*
where *contour_integral g f* ≡ *SOME i*. (*f has_contour_integral i*) *g* ∨ ¬ *f contour_integrable_on g* ∧ *i*=*0*

1.2 Relation to subpath construction

1.3 Cauchy's theorem where there's a primitive

corollary *Cauchy_theorem_primitive*:
assumes ⋀*x*. *x* ∈ *S* ⟹ (*f has_field_derivative f′ x*) (*at x within S*)
and *valid_path g* *path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f′ has_contour_integral 0*) *g*

1.4 Reversing the order in a double path integral

proposition *contour_integral_swap*:
assumes *fcon*: *continuous_on* (*path_image g* × *path_image h*) (λ(*y1*,*y2*). *f y1 y2*)
and *vp*: *valid_path g valid_path h*
and *gvcon*: *continuous_on* {*0*..*1*} (λ*t*. *vector_derivative g* (*at t*))
and *hvcon*: *continuous_on* {*0*..*1*} (λ*t*. *vector_derivative h* (*at t*))
shows *contour_integral g* (λ*w*. *contour_integral h* (*f w*)) =
contour_integral h (λ*z*. *contour_integral g* (λ*w*. *f w z*))

1.5 Partial circle path

definition *part_circlepath* :: [*complex*, *real*, *real*, *real*, *real*] ⇒ *complex*
where *part_circlepath z r s t* ≡ λ*x*. *z* + *of_real r* ∗ *exp* (i ∗ *of_real* (*linepath s t x*))

proposition *path_image_part_circlepath*:
assumes $s \le t$
shows *path_image* (*part_circlepath z r s t*) = {*z* + *r* ∗ *exp*(i ∗ *of_real x*) | *x*. $s \le x \land x \le t$}

corollary *contour_integral_bound_part_circlepath_strong*:
assumes *f contour_integrable_on part_circlepath z r s t*
and *finite k* **and** $0 \le B$ $0 < r$ $s \le t$
and ⋀*x*. *x* ∈ *path_image*(*part_circlepath z r s t*) − *k* ⟹ *norm*(*f x*) ≤ *B*
shows *cmod* (*contour_integral* (*part_circlepath z r s t*) *f*) ≤ *B* ∗ *r* ∗ (*t* − *s*)

1.6 Special case of one complete circle

definition *circlepath* :: [*complex*, *real*, *real*] ⇒ *complex*
where *circlepath z r* ≡ *part_circlepath z r 0* (*2*∗*pi*)

1.7 Uniform convergence of path integral

proposition *contour_integral_uniform_limit*:
assumes *ev_fint*: *eventually* (λ*n*::′*a*. (*f n*) *contour_integrable_on* γ) *F*
and *ul_f*: *uniform_limit* (*path_image* γ) *f l F*
and *noleB*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *norm* (*vector_derivative* γ (*at t*)) ≤ *B*
and γ: *valid_path* γ
and [*simp*]: ¬ *trivial_limit F*
shows *l contour_integrable_on* γ ((λ*n*. *contour_integral* γ (*f n*)) ⟶ *contour_integral* γ *l*) *F*

end

2 Complex Path Integrals and Cauchy's Integral Theorem

theory *Cauchy_Integral_Theorem*
imports
HOL−Analysis.Analysis
Contour_Integration
begin

proposition *Cauchy_theorem_triangle_interior*:
assumes *contf*: *continuous_on* (*convex hull* {*a*,*b*,*c*}) *f*
and *holf*: *f holomorphic_on interior* (*convex hull* {*a*,*b*,*c*})
shows (*f has_contour_integral 0*) (*linepath a b* +++ *linepath b c* +++ *linepath c a*)

2.1 Cauchy's theorem for a convex set

corollary *Cauchy_theorem_convex_simple*:
assumes *holf*: *f holomorphic_on S*
and *convex S valid_path g path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f has_contour_integral 0*) *g*

2.2 Homotopy forms of Cauchy's theorem

proposition *Cauchy_theorem_homotopic_paths*:
assumes *hom*: *homotopic_paths S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

proposition *Cauchy_theorem_homotopic_loops*:
assumes *hom*: *homotopic_loops S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

end

3 Winding numbers

theory *Winding_Numbers*
imports *Cauchy_Integral_Theorem*
begin

3.1 Definition

definition *winding_number_prop* :: [*real* ⇒ *complex*, *complex*, *real*, *real* ⇒ *complex*, *complex*] ⇒ *bool* **where**
winding_number_prop γ *z e p n* ≡
valid_path p ∧ *z* ∉ *path_image p* ∧
pathstart p = *pathstart* γ ∧
pathfinish p = *pathfinish* γ ∧
(∀ *t* ∈ {*0*..*1*}. *norm*(γ *t* − *p t*) < *e*) ∧

$contour\_integral\ p\ (\lambda w.\ 1/(w - z)) = 2 * pi * i * n$

definition *winding_number*:: $[real \Rightarrow complex, complex] \Rightarrow complex$ **where**
$winding\_number\ \gamma\ z \equiv SOME\ n.\ \forall e > 0.\ \exists p.\ winding\_number\_prop\ \gamma\ z\ e\ p\ n$

proposition *winding_number_valid_path*:
assumes $valid\_path\ \gamma\ z \notin path\_image\ \gamma$
shows $winding\_number\ \gamma\ z = 1/(2*pi*i) * contour\_integral\ \gamma\ (\lambda w.\ 1/(w - z))$

proposition *has_contour_integral_winding_number*:
assumes γ: $valid\_path\ \gamma\ z \notin path\_image\ \gamma$
shows $((\lambda w.\ 1/(w - z))\ has\_contour\_integral\ (2*pi*i*winding\_number\ \gamma\ z))\ \gamma$

3.2 The winding number is an integer

theorem *integer_winding_number*:
$[\![path\ \gamma;\ pathfinish\ \gamma = pathstart\ \gamma;\ z \notin path\_image\ \gamma]\!] \Longrightarrow winding\_number\ \gamma\ z \in \mathbb{Z}$

3.3 Continuity of winding number and invariance on connected sets

theorem *continuous_at_winding_number*:
fixes $z::complex$
assumes γ: $path\ \gamma$ **and** z: $z \notin path\_image\ \gamma$
shows $continuous\ (at\ z)\ (winding\_number\ \gamma)$

corollary *continuous_on_winding_number*:
$path\ \gamma \Longrightarrow continuous\_on\ (- path\_image\ \gamma)\ (\lambda w.\ winding\_number\ \gamma\ w)$

3.4 Winding number is zero "outside" a curve

proposition *winding_number_zero_in_outside*:
assumes γ: $path\ \gamma$ **and** *loop*: $pathfinish\ \gamma = pathstart\ \gamma$ **and** z: $z \in outside\ (path\_image\ \gamma)$
shows $winding\_number\ \gamma\ z = 0$

proposition *winding_number_part_circlepath_pos_less*:
assumes $s < t$ **and** *no*: $norm(w - z) < r$
shows $0 < Re\ (winding\_number(part\_circlepath\ z\ r\ s\ t)\ w)$

proposition *winding_number_circlepath*:
assumes $norm(w - z) < r$ **shows** $winding\_number(circlepath\ z\ r)\ w = 1$

3.5 Winding number for a triangle

proposition *winding_number_triangle*:
assumes z: $z \in$ *interior*(*convex hull* $\{a,b,c\}$)
shows *winding_number*(*linepath a b* +++ *linepath b c* +++ *linepath c a*) z =
(*if* $0 < Im((b - a) * cnj\ (b - z))$ *then* 1 *else* -1)

3.6 Winding numbers for simple closed paths

proposition *simple_closed_path_winding_number_inside*:
assumes *simple_path* γ
obtains $\bigwedge z.\ z \in$ *inside*(*path_image* γ) $\Longrightarrow$ *winding_number* $\gamma\ z = 1$
| $\bigwedge z.\ z \in$ *inside*(*path_image* γ) $\Longrightarrow$ *winding_number* $\gamma\ z = -1$

3.7 Winding number for rectangular paths

proposition *winding_number_rectpath*:
assumes $z \in$ *box a1 a3*
shows *winding_number* (*rectpath a1 a3*) $z = 1$

proposition *winding_number_rectpath_outside*:
assumes *Re a1* $\leq$ *Re a3 Im a1* $\leq$ *Im a3*
assumes $z \notin$ *cbox a1 a3*
shows *winding_number* (*rectpath a1 a3*) $z = 0$

end

4 Cauchy's Integral Formula

theory *Cauchy_Integral_Formula*
imports *Winding_Numbers*
begin

4.1 Proof

theorem *Cauchy_integral_formula_convex_simple*:
assumes *convex S* **and** *holf*: *f holomorphic_on S* **and** $z \in$ *interior S valid_path* γ *path_image* $\gamma \subseteq S - \{z\}$
pathfinish γ = *pathstart* γ
shows $((\lambda w.\ f\ w\ /\ (w-z))$ *has_contour_integral* $(2*pi * i *$ *winding_number* $\gamma\ z * f\ z))\ \gamma$
theorem *Cauchy_integral_circlepath*:

assumes *contf*: *continuous_on* (*cball z r*) *f* **and** *holf*: *f holomorphic_on* (*ball z r*) **and** *wz*: *norm*(*w*−*z*) < *r*
shows ((λ*u*. *f u*/(*u*−*w*)) *has_contour_integral* (*2* ∗ *of_real pi* ∗ i ∗ *f w*)) (*circlepath z r*)

4.2 Existence of all higher derivatives

proposition *derivative_is_holomorphic*:
assumes *open S*
and *fder*: ⋀*z*. *z* ∈ *S* ⟹ (*f has_field_derivative f′ z*) (*at z*)
shows *f′ holomorphic_on S*

4.3 Morera's theorem

proposition *Morera_triangle*:
⟦*continuous_on S f*; *open S*;
⋀*a b c*. *convex hull* {*a*,*b*,*c*} ⊆ *S*
⟶ *contour_integral* (*linepath a b*) *f* +
contour_integral (*linepath b c*) *f* +
contour_integral (*linepath c a*) *f* = *0*⟧
⟹ *f analytic_on S*

4.4 Combining theorems for higher derivatives including Leibniz rule

proposition *no_isolated_singularity*:
fixes *z*::*complex*
assumes *f*: *continuous_on S f* **and** *holf*: *f holomorphic_on* (*S*−*K*) **and** *S*: *open S* **and** *K*: *finite K*
shows *f holomorphic_on S*

proposition *Cauchy_integral_formula_convex*:
assumes *S*: *convex S* **and** *K*: *finite K* **and** *contf*: *continuous_on S f*
and *fcd*: (⋀*x*. *x* ∈ *interior S* − *K* ⟹ *f field_differentiable at x*)
and *z*: *z* ∈ *interior S* **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* γ ⊆ *S* − {*z*} **and** *loop*: *pathfinish* γ = *pathstart* γ
shows ((λ*w*. *f w* / (*w*−*z*)) *has_contour_integral* (*2*∗*pi* ∗ i ∗ *winding_number* γ *z* ∗ *f z*)) γ

corollary *Cauchy_contour_integral_circlepath*:
assumes *continuous_on* (*cball z r*) *f f holomorphic_on ball z r w* ∈ *ball z r*
shows *contour_integral*(*circlepath z r*) (λ*u*. *f u*/(*u*−*w*)^(*Suc k*)) = (*2* ∗ *pi* ∗ i) ∗ (*deriv* ^^ *k*) *f w* / (*fact k*)

4.5 A holomorphic function is analytic, i.e. has local power series

theorem *holomorphic_power_series*:
assumes *holf*: *f holomorphic_on ball z r*
and *w*: $w \in$ *ball z r*
shows ((λn. (*deriv* ^^ *n*) *f z* / (*fact n*) $*$ ($w-z$)^*n*) *sums f w*)

4.6 The Liouville theorem and the Fundamental Theorem of Algebra

proposition *Liouville_weak*:
assumes *f holomorphic_on UNIV* **and** ($f \longrightarrow l$) *at_infinity*
shows $f\ z = l$

proposition *Liouville_weak_inverse*:
assumes *f holomorphic_on UNIV* **and** *unbounded*: $\bigwedge B$. *eventually* (λx. *norm* (*f x*) $\geq B$) *at_infinity*
obtains *z* **where** $f\ z = 0$
theorem *fundamental_theorem_of_algebra*:
fixes $a :: nat \Rightarrow complex$
assumes $a\ 0 = 0 \lor (\exists\, i \in \{1..n\}.\ a\ i \neq 0)$
obtains *z* **where** $(\sum i \leq n.\ a\ i * z\hat{\ }i) = 0$

4.7 Weierstrass convergence theorem

proposition *has_complex_derivative_uniform_limit*:
fixes *z*::*complex*
assumes *cont*: *eventually* (λn. *continuous_on* (*cball z r*) (*f n*) $\land$
($\forall\, w \in$ *ball z r*. ((*f n*) *has_field_derivative* ($f'\ n\ w$)) (*at w*))) *F*
and *ulim*: *uniform_limit* (*cball z r*) *f g F*
and *F*: $\neg$ *trivial_limit F* **and** $0 < r$
obtains g' **where**
continuous_on (*cball z r*) *g*
$\bigwedge w$. $w \in$ *ball z r* $\Longrightarrow$ (*g has_field_derivative* ($g'\ w$)) (*at w*) $\land$ ((λn. $f'\ n\ w$) $\longrightarrow g'\ w$) *F*

4.8 On analytic functions defined by a series

corollary *holomorphic_iff_power_series*:
f holomorphic_on ball z r $\longleftrightarrow$
($\forall\, w \in$ *ball z r*. (λn. (*deriv* ^^ *n*) *f z* / (*fact n*) $*$ ($w-z$)^*n*) *sums f w*)

4.9 General, homology form of Cauchy's theorem

theorem *Cauchy_integral_formula_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *z*: $z \in S$ **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* $\gamma \subseteq S - \{z\}$ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow$ *winding_number* γ $w = 0$
shows $((\lambda w.\ f\ w\ /\ (w-z))$ *has_contour_integral* (*2**pi* * i * *winding_number* γ z * $f\ z$)) γ

theorem *Cauchy_theorem_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *vpg*: *valid_path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *pas*: *path_image* $\gamma \subseteq S$
and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow$ *winding_number* γ $w = 0$
shows (*f has_contour_integral 0*) γ

corollary *Cauchy_theorem_global_outside*:
assumes *open S f holomorphic_on S valid_path* γ *pathfinish* γ = *pathstart* γ *path_image* $\gamma \subseteq S$
$\bigwedge w.\ w \notin S \Longrightarrow w \in$ *outside*(*path_image* γ)
shows (*f has_contour_integral 0*) γ

4.10 Cauchy's inequality and more versions of Liouville

theorem *Liouville_theorem*:
assumes *holf*: *f holomorphic_on UNIV*
and *bf*: *bounded* (*range f*)
shows *f constant_on UNIV*

4.11 Complex functions and power series

definition *fps_expansion* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex fps*
where
fps_expansion f z0 = *Abs_fps* (λ*n*. (*deriv* ^^ *n*) *f z0* / *fact n*)

end

5 Conformal Mappings and Consequences of Cauchy's Integral Theorem

theory *Conformal_Mappings*
imports *Cauchy_Integral_Formula*

begin

5.1 Analytic continuation

proposition *isolated_zeros*:
assumes *holf*: f *holomorphic_on* S
and *open* S *connected* S $\xi \in S$ $f\ \xi = 0$ $\beta \in S$ $f\ \beta \neq 0$
obtains r **where** $0 < r$ **and** *ball* $\xi\ r \subseteq S$ **and**
$\bigwedge z.\ z \in ball\ \xi\ r - \{\xi\} \Longrightarrow f\ z \neq 0$

proposition *analytic_continuation*:
assumes *holf*: f *holomorphic_on* S
and *open* S **and** *connected* S
and $U \subseteq S$ **and** $\xi \in S$
and ξ *islimpt* U
and *fU0* [*simp*]: $\bigwedge z.\ z \in U \Longrightarrow f\ z = 0$
and $w \in S$
shows $f\ w = 0$

corollary *analytic_continuation_open*:
assumes *open* s **and** *open* s' **and** $s \neq \{\}$ **and** *connected* s'
and $s \subseteq s'$
assumes f *holomorphic_on* s' **and** g *holomorphic_on* s'
and $\bigwedge z.\ z \in s \Longrightarrow f\ z = g\ z$
assumes $z \in s'$
shows $f\ z = g\ z$

corollary *analytic_continuation'*:
assumes f *holomorphic_on* S *open* S *connected* S
and $U \subseteq S$ $\xi \in S$ ξ *islimpt* U
and f *constant_on* U
shows f *constant_on* S

5.2 Open mapping theorem

theorem *open_mapping_thm*:
assumes *holf*: f *holomorphic_on* S
and S: *open* S **and** *connected* S
and *open* U **and** $U \subseteq S$
and *fne*: $\neg$ f *constant_on* S
shows *open* (f ‘ U)

5.3 Maximum modulus principle

proposition *maximum_modulus_principle*:
assumes *holf*: f *holomorphic_on* S
and S: *open* S **and** *connected* S
and *open* U **and** $U \subseteq S$ **and** $\xi \in U$
and *no*: $\bigwedge z.\ z \in U \Longrightarrow norm(f\ z) \leq norm(f\ \xi)$

shows *f constant_on S*

proposition *maximum_modulus_frontier*:
assumes *holf*: *f holomorphic_on* (*interior S*)
and *contf*: *continuous_on* (*closure S*) *f*
and *bos*: *bounded S*
and *leB*: $\bigwedge z.\ z \in \mathit{frontier}\ S \Longrightarrow \mathit{norm}(f\ z) \leq B$
and $\xi \in S$
shows $\mathit{norm}(f\ \xi) \leq B$

5.4 Relating invertibility and nonvanishing of derivative

proposition *holomorphic_has_inverse*:
assumes *holf*: *f holomorphic_on S*
and *open S* **and** *injf*: *inj_on f S*
obtains *g* **where** *g holomorphic_on* (*f ‘ S*)
$\bigwedge z.\ z \in S \Longrightarrow \mathit{deriv}\ f\ z * \mathit{deriv}\ g\ (f\ z) = 1$
$\bigwedge z.\ z \in S \Longrightarrow g(f\ z) = z$

5.5 The Schwarz Lemma

proposition *Schwarz_Lemma*:
assumes *holf*: *f holomorphic_on* (*ball 0 1*) **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ \mathit{norm}\ z < 1 \Longrightarrow \mathit{norm}\ (f\ z) < 1$
and ξ: $\mathit{norm}\ \xi < 1$
shows $\mathit{norm}\ (f\ \xi) \leq \mathit{norm}\ \xi$ **and** $\mathit{norm}(\mathit{deriv}\ f\ 0) \leq 1$
and $((\exists z.\ \mathit{norm}\ z < 1 \wedge z \neq 0 \wedge \mathit{norm}(f\ z) = \mathit{norm}\ z)$
$\vee\ \mathit{norm}(\mathit{deriv}\ f\ 0) = 1)$
$\Longrightarrow \exists \alpha.\ (\forall z.\ \mathit{norm}\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge \mathit{norm}\ \alpha = 1$
(**is** $?P \Longrightarrow ?Q$)

corollary *Schwarz_Lemma′*:
assumes *holf*: *f holomorphic_on* (*ball 0 1*) **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ \mathit{norm}\ z < 1 \Longrightarrow \mathit{norm}\ (f\ z) < 1$
shows $((\forall \xi.\ \mathit{norm}\ \xi < 1 \longrightarrow \mathit{norm}\ (f\ \xi) \leq \mathit{norm}\ \xi)$
$\wedge\ \mathit{norm}(\mathit{deriv}\ f\ 0) \leq 1)$
$\wedge\ (((\exists z.\ \mathit{norm}\ z < 1 \wedge z \neq 0 \wedge \mathit{norm}(f\ z) = \mathit{norm}\ z)$
$\vee\ \mathit{norm}(\mathit{deriv}\ f\ 0) = 1)$
$\longrightarrow (\exists \alpha.\ (\forall z.\ \mathit{norm}\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge \mathit{norm}\ \alpha = 1))$

5.6 The Schwarz reflection principle

proposition *Schwarz_reflection*:
assumes *open S* **and** *cnjs*: *cnj* ‘ $S \subseteq S$
and *holf*: *f holomorphic_on* $(S \cap \{z.\ 0 < Im\ z\})$
and *contf*: *continuous_on* $(S \cap \{z.\ 0 \le Im\ z\})$ *f*
and *f*: $\bigwedge z.\ [\![z \in S;\ z \in \mathbb{R}]\!] \Longrightarrow (f\ z) \in \mathbb{R}$
shows $(\lambda z.\ \mathit{if}\ 0 \le Im\ z\ \mathit{then}\ f\ z\ \mathit{else}\ cnj(f(cnj\ z)))$ *holomorphic_on S*

5.7 Bloch's theorem

proposition *Bloch_unit*:
assumes *holf*: *f holomorphic_on ball a 1* **and** [*simp*]: *deriv f a* = *1*
obtains *b r* **where** $1/12 < r$ **and** *ball b r* $\subseteq$ *f* ‘ (*ball a 1*)

theorem *Bloch*:
assumes *holf*: *f holomorphic_on ball a r* **and** $0 < r$
and r': $r' \le r * norm\ (deriv\ f\ a)\ /\ 12$
obtains *b* **where** *ball b* $r' \subseteq f$ ‘ (*ball a r*)

corollary *Bloch_general*:
assumes *holf*: *f holomorphic_on S* **and** $a \in S$
and *tle*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow t \le dist\ a\ z$
and *rle*: $r \le t * norm(deriv\ f\ a)\ /\ 12$
obtains *b* **where** *ball b r* $\subseteq$ *f* ‘ *S*

end

6 The Great Picard Theorem and its Applications

theory *Great_Picard*
imports *Conformal_Mappings*
begin

6.1 Schottky's theorem

theorem *Schottky*:
assumes *holf*: *f holomorphic_on cball 0 1*
and *nof0*: $norm(f\ 0) \le r$
and *not01*: $\bigwedge z.\ z \in cball\ 0\ 1 \Longrightarrow \neg(f\ z = 0 \lor f\ z = 1)$
and $0 < t$ $t < 1$ $norm\ z \le t$
shows $norm(f\ z) \le exp(pi * exp(pi * (2 + 2 * r + 12 * t\ /\ (1 - t))))$

6.2 The Little Picard Theorem

theorem *Landau_Picard*:
obtains R
where $\bigwedge z.\ 0 < R\ z$
$\bigwedge f.\ [\![f\ holomorphic\_on\ cball\ 0\ (R(f\ 0));$
$\bigwedge z.\ norm\ z \le R(f\ 0) \Longrightarrow f\ z \ne 0 \wedge f\ z \ne 1]\!] \Longrightarrow norm(deriv\ f\ 0) < 1$

theorem *little_Picard*:
assumes *holf*: $f\ holomorphic\_on\ UNIV$
and $a \ne b\ range\ f \cap \{a,b\} = \{\}$
obtains c **where** $f = (\lambda x.\ c)$

6.3 The Arzelà–Ascoli theorem

theorem *Arzela_Ascoli*:
fixes $\mathcal{F} :: [nat,'a::euclidean\_space] \Rightarrow 'b::\{real\_normed\_vector,heine\_borel\}$
assumes $compact\ S$
and M: $\bigwedge n\ x.\ x \in S \Longrightarrow norm(\mathcal{F}\ n\ x) \le M$
and *equicont*:
$\bigwedge x\ e.\ [\![x \in S;\ 0 < e]\!]$
$\Longrightarrow \exists d.\ 0 < d \wedge (\forall n\ y.\ y \in S \wedge norm(x - y) < d \longrightarrow norm(\mathcal{F}\ n\ x - \mathcal{F}\ n\ y) < e)$
obtains $g\ k$ **where** $continuous\_on\ S\ g\ strict\_mono\ (k :: nat \Rightarrow nat)$
$\bigwedge e.\ 0 < e \Longrightarrow \exists N.\ \forall n\ x.\ n \ge N \wedge x \in S \longrightarrow norm(\mathcal{F}(k\ n)\ x - g\ x) < e$

6.3.1 Montel's theorem

theorem *Montel*:
fixes $\mathcal{F} :: [nat,complex] \Rightarrow complex$
assumes $open\ S$
and $\mathcal{H}$: $\bigwedge h.\ h \in \mathcal{H} \Longrightarrow h\ holomorphic\_on\ S$
and *bounded*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow \exists B.\ \forall h \in \mathcal{H}.\ \forall\ z \in K.\ norm(h\ z) \le B$
and *rng_f*: $range\ \mathcal{F} \subseteq \mathcal{H}$
obtains $g\ r$
where $g\ holomorphic\_on\ S\ strict\_mono\ (r :: nat \Rightarrow nat)$
$\bigwedge x.\ x \in S \Longrightarrow ((\lambda n.\ \mathcal{F}\ (r\ n)\ x) \longlongrightarrow g\ x)\ sequentially$
$\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow uniform\_limit\ K\ (\mathcal{F} \circ r)\ g\ sequentially$

6.4 Some simple but useful cases of Hurwitz's theorem

proposition *Hurwitz_no_zeros*:
 assumes *S*: *open S connected S*
  and *holf*: $\bigwedge n::nat.\ \mathcal{F}\ n$ *holomorphic_on S*
  and *holg*: *g holomorphic_on S*
  and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit K* $\mathcal{F}$ *g sequentially*
  and *nonconst*: $\neg$ *g constant_on S*
  and *nz*: $\bigwedge n\ z.\ z \in S \Longrightarrow \mathcal{F}\ n\ z \neq 0$
  and $z0 \in S$
  shows $g\ z0 \neq 0$

corollary *Hurwitz_injective*:
 assumes *S*: *open S connected S*
  and *holf*: $\bigwedge n::nat.\ \mathcal{F}\ n$ *holomorphic_on S*
  and *holg*: *g holomorphic_on S*
  and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit K* $\mathcal{F}$ *g sequentially*
  and *nonconst*: $\neg$ *g constant_on S*
  and *inj*: $\bigwedge n.$ *inj_on* $(\mathcal{F}\ n)\ S$
 shows *inj_on g S*

6.5 The Great Picard theorem

theorem *great_Picard*:
 assumes *open M* $z \in M$ $a \neq b$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
  and *fab*: $\bigwedge w.\ w \in M - \{z\} \Longrightarrow f\ w \neq a \wedge f\ w \neq b$
 obtains *l* **where** $(f \longrightarrow l)\ (at\ z) \vee ((inverse \circ f) \longrightarrow l)\ (at\ z)$

corollary *great_Picard_alt*:
 assumes *M*: *open M* $z \in M$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
  and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
 obtains *a* **where** $- \{a\} \subseteq f\ `\ (M - \{z\})$

corollary *great_Picard_infinite*:
 assumes *M*: *open M* $z \in M$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
  and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
 obtains *a* **where** $\bigwedge w.\ w \neq a \Longrightarrow$ *infinite* $\{x.\ x \in M - \{z\} \wedge f\ x = w\}$

theorem *Casorati_Weierstrass*:

assumes *open M z* $\in$ *M f holomorphic_on* $(M - \{z\})$
and $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
shows $closure(f\ `\ (M - \{z\})) = UNIV$

end

7 Moebius functions, Equivalents of Simply Connected Sets, Riemann Mapping Theorem

theory *Riemann_Mapping*
imports *Great_Picard*
begin

7.1 Moebius functions are biholomorphisms of the unit disc

definition *Moebius_function* :: [*real,complex,complex*] $\Rightarrow$ *complex* **where**
Moebius_function $\equiv \lambda t\ w\ z.\ exp(i * of\_real\ t) * (z - w) / (1 - cnj\ w * z)$

7.2 A big chain of equivalents of simple connectedness for an open set

proposition
assumes *open S*
shows *simply_connected_eq_winding_number_zero*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall g\ z.\ path\ g \wedge path\_image\ g \subseteq S \wedge$
$pathfinish\ g = pathstart\ g \wedge (z \notin S)$
$\longrightarrow winding\_number\ g\ z = 0)$ (**is** *?wn0*)
and *simply_connected_eq_contour_integral_zero*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall g\ f.\ valid\_path\ g \wedge path\_image\ g \subseteq S \wedge$
$pathfinish\ g = pathstart\ g \wedge f\ holomorphic\_on\ S$
$\longrightarrow (f\ has\_contour\_integral\ 0)\ g)$ (**is** *?ci0*)
and *simply_connected_eq_global_primitive*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall f.\ f\ holomorphic\_on\ S \longrightarrow$
$(\exists h.\ \forall z.\ z \in S \longrightarrow (h\ has\_field\_derivative\ f\ z)\ (at\ z)))$ (**is** *?gp*)
and *simply_connected_eq_holomorphic_log*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall f.\ f\ holomorphic\_on\ S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.\ g\ holomorphic\_on\ S \wedge (\forall z \in S.\ f\ z = exp(g\ z))))$ (**is** *?log*)

and *simply_connected_eq_holomorphic_sqrt*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall f.\ f$ *holomorphic_on* $S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.\ g$ *holomorphic_on* $S \wedge (\forall z \in S.\ \ f\ z = (g\ z)^2)))$ (**is** *?sqrt*)
and *simply_connected_eq_biholomorphic_to_disc*:
simply_connected S $\longleftrightarrow$
$S = \{\} \vee S = UNIV \vee$
$(\exists f\ g.\ f$ *holomorphic_on* $S \wedge g$ *holomorphic_on ball 0 1* $\wedge$
$(\forall z \in S.\ f\ z \in ball\ 0\ 1 \wedge g(f\ z) = z) \wedge$
$(\forall z \in ball\ 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))$ (**is** *?bih*)
and *simply_connected_eq_homeomorphic_to_disc*:
simply_connected S $\longleftrightarrow S = \{\} \vee S$ *homeomorphic ball* $(0{::}complex)\ 1$
(**is** *?disc*)

corollary *contractible_eq_simply_connected_2d*:
fixes S :: *complex set*
assumes *open S*
shows *contractible S* $\longleftrightarrow$ *simply_connected S*

7.3 A further chain of equivalences about components of the complement of a simply connected set

proposition
fixes S :: *complex set*
assumes *open S*
shows *simply_connected_eq_frontier_properties*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
(*if bounded S then connected*(*frontier S*)
else $(\forall C \in$ *components*(*frontier S*). $\neg$*bounded C*)) (**is** *?fp*)
and *simply_connected_eq_unbounded_complement_components*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$ $(\forall C \in$ *components*$(-\ S)$. $\neg$*bounded C*) (**is** *?ucc*)
and *simply_connected_eq_empty_inside*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$ *inside S* $= \{\}$ (**is** *?ei*)

7.4 Further equivalences based on continuous logs and sqrts

proposition
fixes S :: *complex set*
assumes *open S*
shows *simply_connected_eq_continuous_log*:

simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall f{::}complex{\Rightarrow}complex.\ continuous\_on\ S\ f \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.\ continuous\_on\ S\ g \wedge (\forall z \in S.\ f\ z = exp\ (g\ z))))$ (**is** *?log*)
and *simply_connected_eq_continuous_sqrt*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall f{::}complex{\Rightarrow}complex.\ continuous\_on\ S\ f \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.\ continuous\_on\ S\ g \wedge (\forall z \in S.\ f\ z = (g\ z)^2)))$ (**is** *?sqrt*)

7.5 Finally, the Riemann Mapping Theorem

theorem *Riemann_mapping_theorem*:
open S $\wedge$ *simply_connected S* $\longleftrightarrow$
$S = \{\} \vee S = UNIV \vee$
$(\exists f\ g.\ f$ *holomorphic_on* $S \wedge g$ *holomorphic_on* *ball 0 1* $\wedge$
$(\forall z \in S.\ f\ z \in ball\ 0\ 1 \wedge g(f\ z) = z) \wedge$
$(\forall z \in ball\ 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))$
(**is** _ = *?rhs*)

7.6 Applications to Winding Numbers

7.7 Winding number equality is the same as path/loop homotopy in C - 0

proposition *winding_number_homotopic_paths_eq*:
assumes *path p* **and** ζp: $\zeta \notin$ *path_image p*
and *path q* **and** ζq: $\zeta \notin$ *path_image q*
and *qp*: *pathstart q* = *pathstart p pathfinish q* = *pathfinish p*
shows *winding_number p* ζ = *winding_number q* $\zeta \longleftrightarrow$ *homotopic_paths* $(-\{\zeta\})$ *p q*
(**is** *?lhs* = *?rhs*)

end
theory *Complex_Singularities*
imports *Conformal_Mappings*
begin

7.8 Non-essential singular points

definition
is_pole :: ('*a*::*topological_space* $\Rightarrow$ '*b*::*real_normed_vector*) $\Rightarrow$ '*a* $\Rightarrow$ *bool*
where *is_pole f a* = (*LIM x* (*at a*). *f x* :> *at_infinity*)

7.9 The order of non-essential singularities (i.e. removable singularities or poles)

```
definition zorder :: (complex ⇒ complex) ⇒ complex ⇒ int where
  zorder f z = (THE n. (∃h r. r>0 ∧ h holomorphic_on cball z r ∧ h z≠0
                  ∧ (∀w∈cball z r − {z}. f w =  h w ∗ (w−z) powi n
                  ∧ h w ≠0)))
```

```
definition zor_poly
    :: [complex ⇒ complex, complex] ⇒ complex ⇒ complex where
  zor_poly f z = (SOME h. ∃r. r > 0 ∧ h holomorphic_on cball z r ∧ h z ≠ 0
                  ∧ (∀w∈cball z r − {z}. f w =  h w ∗ (w−z) powi (zorder f z)
                  ∧ h w ≠0))
```

7.10 Isolated zeroes

7.11 Isolated points

```
end
theory Complex_Residues
  imports Complex_Singularities
begin
```

7.12 Definition of residues

```
definition residue :: (complex ⇒ complex) ⇒ complex ⇒ complex where
  residue f z = (SOME int. ∃e>0. ∀ε>0. ε<e
    ⟶ (f has_contour_integral 2∗pi∗ i ∗int) (circlepath z ε))
```

```
theorem residue_fps_expansion_over_power_at_0:
  assumes f has_fps_expansion F
  shows   residue (λz. f z / z ^ Suc n) 0 = fps_nth F n
```

7.13 Poles and residues of some well-known functions

```
end
```

8 The Residue Theorem, the Argument Principle and Rouché's Theorem

```
theory Residue_Theorem
  imports Complex_Residues HOL−Library.Landau_Symbols
```

begin

8.1 Cauchy's residue theorem

theorem *Residue_theorem*:
  fixes *S pts*::*complex set* **and** *f*::*complex* ⇒ *complex*
    and *g*::*real* ⇒ *complex*
  assumes *open S connected S finite pts* **and**
    holo:*f holomorphic_on S−pts* **and**
    valid_path g **and**
    loop:*pathfinish g* = *pathstart g* **and**
    path_image g ⊆ *S−pts* **and**
    homo:∀ *z*. (*z* ∉ *S*) ⟶ *winding_number g z* = *0*
  shows *contour_integral g f* = *2* ∗ *pi* ∗ i ∗(∑ *p*∈*pts*. *winding_number g p* ∗ *residue f p*)

8.2 The argument principle

theorem *argument_principle*:
  fixes *f*::*complex* ⇒ *complex* **and** *poles S*:: *complex set*
  defines *pz* ≡ {*w*∈*S*. *f w* = *0* ∨ *w* ∈ *poles*} — *pz* is the set of poles and zeros
  assumes *open S connected S* **and**
    f_holo:*f holomorphic_on S−poles* **and**
    h_holo:*h holomorphic_on S* **and**
    valid_path g **and**
    loop:*pathfinish g* = *pathstart g* **and**
    path_img:*path_image g* ⊆ *S* − *pz* **and**
    homo:∀ *z*. (*z* ∉ *S*) ⟶ *winding_number g z* = *0* **and**
    finite:*finite pz* **and**
    poles:∀ *p*∈*S*∩*poles*. *is_pole f p*
  shows *contour_integral g* (λ*x*. *deriv f x* ∗ *h x* / *f x*) = *2* ∗ *pi* ∗ i ∗
    (∑ *p*∈*pz*. *winding_number g p* ∗ *h p* ∗ *zorder f p*)
    (**is** *?L*=*?R*)

8.3 Coefficient asymptotics for generating functions

theorem
  fixes *f* :: *complex* ⇒ *complex* **and** *n* :: *nat* **and** *r* :: *real*
  defines *g* ≡ (λ*w*. *f w* / *w* ^ *Suc n*) **and** γ ≡ *circlepath 0 r*
  assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
  assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*
  shows *fps_coeff_conv_residues*:
    (*deriv* ^^ *n*) *f 0* / *fact n* =
     contour_integral γ *g* / (*2* ∗ *pi* ∗ i) − (∑ *z*∈*S*. *residue g z*) (**is** *?thesis1*)
    and *fps_coeff_residues_bound*:
    (⋀*z*. *norm z* = *r* ⟹ *z* ∉ *k* ⟹ *norm* (*f z*) ≤ *C*) ⟹ *C* ≥ *0* ⟹ *finite k* ⟹

```
        norm ((deriv ^^ n) f 0 / fact n + (∑ z∈S. residue g z)) ≤ C / r ^ n
corollary fps_coeff_residues_bigo:
  fixes f :: complex ⇒ complex and r :: real
  assumes open A connected A cball 0 r ⊆ A r > 0
  assumes f holomorphic_on A − S S ⊆ ball 0 r finite S 0 ∉ S
  assumes g: eventually (λn. g n = −(∑ z∈S. residue (λz. f z / z ^ Suc n) z))
sequentially
          (is eventually (λn. _ = −?g′ n) _)
  shows  (λn. (deriv ^^ n) f 0 / fact n − g n) ∈ O(λn. 1 / r ^ n) (is (λn. ?c n
− _) ∈ O(_))

corollary fps_coeff_residues_bigo′:
  fixes f :: complex ⇒ complex and r :: real
  assumes exp: f has_fps_expansion F
  assumes open A connected A cball 0 r ⊆ A r > 0
  assumes f holomorphic_on A − S S ⊆ ball 0 r finite S 0 ∉ S
  assumes eventually (λn. g n = −(∑ z∈S. residue (λz. f z / z ^ Suc n) z))
sequentially
        (is eventually (λn. _ = −?g′ n) _)
  shows  (λn. fps_nth F n − g n) ∈ O(λn. 1 / r ^ n) (is (λn. ?c n − _) ∈
O(_))
```

8.4 Rouche's theorem

```
theorem Rouche_theorem:
  fixes f g::complex ⇒ complex and s:: complex set
  defines fg≡(λp. f p + g p)
  defines zeros_fg≡{p∈s. fg p = 0} and zeros_f≡{p∈s. f p = 0}
  assumes
    open s and connected s and
    finite zeros_fg and
    finite zeros_f and
    f_holo:f holomorphic_on s and
    g_holo:g holomorphic_on s and
    valid_path γ and
    loop:pathfinish γ = pathstart γ and
    path_img:path_image γ ⊆ s and
    path_less:∀ z∈path_image γ. cmod(f z) > cmod(g z) and
    homo:∀ z. (z ∉ s) ⟶ winding_number γ z = 0
  shows (∑ p∈zeros_fg. winding_number γ p ∗ zorder fg p)
        = (∑ p∈zeros_f. winding_number γ p ∗ zorder f p)

end
theory Laurent_Convergence
 imports HOL−Computational_Algebra.Formal_Laurent_Series HOL−Library.Landau_Symbols
        Residue_Theorem

begin
```

definition *fls_conv_radius* :: *complex fls* ⇒ *ereal* **where**
fls_conv_radius f = *fps_conv_radius* (*fls_regpart f*)

definition *eval_fls* :: *complex fls* ⇒ *complex* ⇒ *complex* **where**
eval_fls F z = *eval_fps* (*fls_base_factor_to_fps F*) *z* ∗ *z powi fls_subdegree F*

definition
has_laurent_expansion :: (*complex* ⇒ *complex*) ⇒ *complex fls* ⇒ *bool*
(**infixl** ‹*has'_laurent'_expansion*› *60*)
where (*f has_laurent_expansion F*) ⟷
fls_conv_radius F > *0* ∧ *eventually* (λ*z*. *eval_fls F z* = *f z*) (*at 0*)

theorem *sums_eval_fls*:
fixes *f*
defines *n* ≡ *fls_subdegree f*
assumes *norm z* < *fls_conv_radius f* **and** *z* ≠ *0* ∨ *n* ≥ *0*
shows (λ*k*. *fls_nth f* (*int k* + *n*) ∗ *z powi* (*int k* + *n*)) *sums eval_fls f z*

theorem *not_essential_has_laurent_expansion_0*:
assumes *isolated_singularity_at f 0 not_essential f 0*
shows *f has_laurent_expansion laurent_expansion f 0*

8.5 More Laurent expansions

end

theory *Meromorphic* **imports**
Laurent_Convergence
Cauchy_Integral_Formula
HOL−Analysis.Sparse_In
begin

8.6 Remove singular points

definition *remove_sings* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex* **where**
remove_sings f z = (*if* ∃ *c*. *f* −*z*→ *c then Lim* (*at z*) *f else 0*)

8.7 Meromorphicity

definition *meromorphic_on* :: (*complex* ⇒ *complex*) ⇒ *complex set* ⇒ *bool*
(**infixl** ‹(*meromorphic′_on*)› *50*) **where**
f meromorphic_on A ⟷ (∀ *z*∈*A*. ∃ *F*. (λ*w*. *f* (*z* + *w*)) *has_laurent_expansion* *F*)

8.8 Nice meromorphicity

8.9 Closure properties and proofs for individual functions

8.10 Meromorphic functions and zorder

end

9 The Weierstraß Factorisation Theorem

theory *Weierstrass_Factorization*
imports *Meromorphic*
begin

9.1 The elementary factors

9.2 Infinite products of elementary factors

9.3 Writing a quotient as an exponential

9.4 Constructing the sequence of zeros

9.5 The factorisation theorem for holomorphic functions

theorem *weierstrass_factorization*:
assumes *g holomorphic_on A open A connected A*
assumes ⋀*z*. *z* ∈ *frontier A* ⟹ ¬*z islimpt* {*w*∈*A*. *g w* = *0*}
obtains *h f* **where**
h holomorphic_on A f holomorphic_on UNIV
∀ *z*. *f z* = *0* ⟷ (∀ *z*∈*A*. *g z* = *0*) ∨ (*z* ∈ *A* ∧ *g z* = *0*)
∀ *z*∈*A*. *zorder f z* = *zorder g z*
∀ *z*∈*A*. *h z* ≠ *0*
∀ *z*∈*A*. *g z* = *h z* ∗ *f z*
theorem *weierstrass_factorization_UNIV*:
assumes *g holomorphic_on UNIV*
obtains *h f* **where**
h holomorphic_on UNIV f holomorphic_on UNIV

∀ z. f z = 0 ⟷ g z = 0
∀ z. zorder f z = zorder g z
∀ z. h z ≠ 0
∀ z. g z = h z ∗ f z

9.6 The factorisation theorem for meromorphic functions

theorem *weierstrass_factorization_meromorphic*:
assumes *mero*: *g nicely_meromorphic_on A* **and** *A*: *open A connected A*
assumes *no_limpt*: ⋀*z. z* ∈ *frontier A* ⟹ ¬*z islimpt* {*w*∈*A. g w* = *0* ∨ *is_pole g w*}
obtains *h f1 f2* **where**
h holomorphic_on A f1 holomorphic_on UNIV f2 holomorphic_on UNIV
∀ *z*∈*A. f1 z* = *0* ⟷ ¬*is_pole g z* ∧ *g z* = *0*
∀ *z*∈*A. f2 z* = *0* ⟷ *is_pole g z*
∀ *z*∈*A.* ¬*is_pole g z* ⟶ *zorder f1 z* = *zorder g z*
∀ *z*∈*A. is_pole g z* ⟶ *zorder f2 z* = −*zorder g z*
∀ *z*∈*A. h z* ≠ *0*
∀ *z*∈*A. g z* = *h z* ∗ *f1 z* / *f2 z*
theorem *weierstrass_factorization_meromorphic_UNIV*:
assumes *g nicely_meromorphic_on UNIV*
obtains *h f1 f2* **where**
h holomorphic_on UNIV f1 holomorphic_on UNIV f2 holomorphic_on UNIV
∀ *z. f1 z* = *0* ⟷ ¬*is_pole g z* ∧ *g z* = *0*
∀ *z. f2 z* = *0* ⟷ *is_pole g z*
∀ *z.* ¬*is_pole g z* ⟶ *zorder f1 z* = *zorder g z*
∀ *z. is_pole g z* ⟶ *zorder f2 z* = −*zorder g z*
∀ *z. h z* ≠ *0*
∀ *z. g z* = *h z* ∗ *f1 z* / *f2 z*

end
theory *Complex_Analysis*
imports
Riemann_Mapping
Residue_Theorem
Weierstrass_Factorization
begin

end

References

[1]