Computational Algebra

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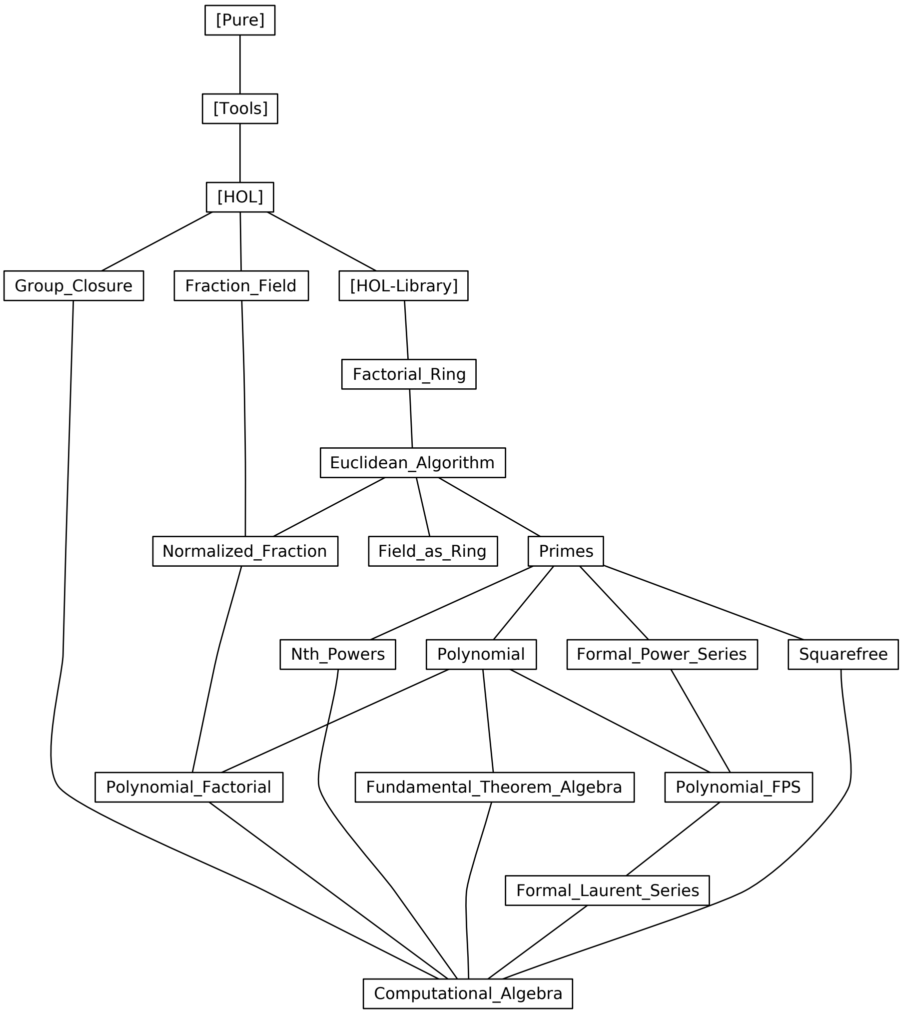
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1 Factorial (semi)rings

theory *Factorial-Ring*
imports
  Main
  HOL−Library.Multiset
begin

unbundle *multiset.lifting*

1.1 Irreducible and prime elements

context *comm-semiring-1*
begin

definition *irreducible* :: $'a \Rightarrow bool$ **where**
  $irreducible\ p \longleftrightarrow p \neq 0 \wedge \neg p\ dvd\ 1 \wedge (\forall a\ b.\ p = a * b \longrightarrow a\ dvd\ 1 \vee b\ dvd\ 1)$

lemma *not-irreducible-zero* [*simp*]: $\neg irreducible\ 0$
  by (*simp add*: *irreducible-def*)

lemma *irreducible-not-unit*: $irreducible\ p \Longrightarrow \neg p\ dvd\ 1$
  by (*simp add*: *irreducible-def*)

lemma *not-irreducible-one* [*simp*]: $\neg irreducible\ 1$
  by (*simp add*: *irreducible-def*)

lemma *irreducibleI*:
  $p \neq 0 \Longrightarrow \neg p\ dvd\ 1 \Longrightarrow (\bigwedge a\ b.\ p = a * b \Longrightarrow a\ dvd\ 1 \vee b\ dvd\ 1) \Longrightarrow irreducible$
p
  by (*simp add*: *irreducible-def*)

lemma *irreducibleD*: $irreducible\ p \Longrightarrow p = a * b \Longrightarrow a\ dvd\ 1 \vee b\ dvd\ 1$
  by (*simp add*: *irreducible-def*)

lemma *irreducible-mono*:
  assumes *irr*: *irreducible b* **and** *a dvd b* $\neg a\ dvd\ 1$
  shows *irreducible a*
proof (*rule irreducibleI*)
  fix *c d* **assume** $a = c * d$
  from *assms* **obtain** *k* **where** [*simp*]: $b = a * k$ **by** *auto*
  from ‹$a = c * d$› **have** $b = c * d * k$
    by *simp*
  hence $c\ dvd\ 1 \vee (d * k)\ dvd\ 1$
    using *irreducibleD*[*OF irr*, *of c d* $*$ *k*] **by** (*auto simp*: *mult.assoc*)
  thus $c\ dvd\ 1 \vee d\ dvd\ 1$
    by *auto*
qed (*use assms* **in** ‹*auto simp*: *irreducible-def*›)

lemma *irreducible-multD*:

```
  assumes l: irreducible (a*b)
  shows a dvd 1 ∧ irreducible b ∨ b dvd 1 ∧ irreducible a
proof−
  have *: irreducible b if l: irreducible (a*b) and a: a dvd 1 for a b :: 'a
  proof (rule irreducibleI)
    show ¬(b dvd 1)
    proof
      assume b dvd 1
      hence a * b dvd 1 * 1
        using ‹a dvd 1› by (intro mult-dvd-mono) auto
      with l show False
        by (auto simp: irreducible-def)
    qed
  next
    fix x y assume b = x * y
    have a * x dvd 1 ∨ y dvd 1
      using l by (rule irreducibleD) (use ‹b = x * y› in ‹auto simp: mult-ac›)
    thus x dvd 1 ∨ y dvd 1
      by auto
  qed (use l a in auto)

  from irreducibleD[OF assms refl] have a dvd 1 ∨ b dvd 1
    by (auto simp: irreducible-def)
  with *[of a b] *[of b a] l show ?thesis
    by (auto simp: mult.commute)
qed

lemma irreducible-power-iff [simp]:
  irreducible (p ^ n) ⟷ irreducible p ∧ n = 1
proof
  assume *: irreducible (p ^ n)
  have irreducible p
    using * by (induction n) (auto dest!: irreducible-multD)
  hence [simp]: ¬p dvd 1
    using * by (auto simp: irreducible-def)

  consider n = 0 | n = 1 | n > 1
    by linarith
  thus irreducible p ∧ n = 1
  proof cases
    assume n > 1
    hence p ^ n = p * p ^ (n − 1)
      by (cases n) auto
    with * ‹¬ p dvd 1› have p ^ (n − 1) dvd 1
      using irreducible-multD[of p p ^ (n − 1)] by auto
    with ‹¬p dvd 1› and ‹n > 1› have False
      by (meson dvd-power dvd-trans zero-less-diff)
    thus ?thesis ..
  qed (use * in auto)
```

qed *auto*

definition *prime-elem* :: $'a \Rightarrow bool$ **where**
prime-elem p $\longleftrightarrow$ $p \neq 0 \wedge \neg p$ *dvd* $1 \wedge (\forall a\ b.\ p$ *dvd* $(a * b) \longrightarrow p$ *dvd* $a \vee p$ *dvd* $b)$

lemma *not-prime-elem-zero* [*simp*]: ¬*prime-elem 0*
by (*simp add*: *prime-elem-def*)

lemma *prime-elem-not-unit*: *prime-elem p* $\Longrightarrow$ ¬*p dvd 1*
by (*simp add*: *prime-elem-def*)

lemma *prime-elemI*:
$p \neq 0 \Longrightarrow \neg p$ *dvd* $1 \Longrightarrow (\bigwedge a\ b.\ p$ *dvd* $(a * b) \Longrightarrow p$ *dvd* $a \vee p$ *dvd* $b) \Longrightarrow$ *prime-elem p*
by (*simp add*: *prime-elem-def*)

lemma *prime-elem-dvd-multD*:
prime-elem p $\Longrightarrow$ *p dvd* $(a * b) \Longrightarrow$ *p dvd a* $\vee$ *p dvd b*
by (*simp add*: *prime-elem-def*)

lemma *prime-elem-dvd-mult-iff*:
prime-elem p $\Longrightarrow$ *p dvd* $(a * b) \longleftrightarrow$ *p dvd a* $\vee$ *p dvd b*
by (*auto simp*: *prime-elem-def*)

lemma *not-prime-elem-one* [*simp*]:
¬ *prime-elem 1*
by (*auto dest*: *prime-elem-not-unit*)

lemma *prime-elem-not-zeroI*:
assumes *prime-elem p*
shows $p \neq 0$
using *assms* **by** (*auto intro*: *ccontr*)

lemma *prime-elem-dvd-power*:
prime-elem p $\Longrightarrow$ *p dvd x* ^ *n* $\Longrightarrow$ *p dvd x*
by (*induction n*) (*auto dest*: *prime-elem-dvd-multD intro*: *dvd-trans*[*of - 1*])

lemma *prime-elem-dvd-power-iff*:
prime-elem p $\Longrightarrow n > 0 \Longrightarrow$ *p dvd x* ^ *n* $\longleftrightarrow$ *p dvd x*
by (*auto dest*: *prime-elem-dvd-power intro*: *dvd-trans*)

lemma *prime-elem-imp-nonzero* [*simp*]:
ASSUMPTION (*prime-elem x*) $\Longrightarrow x \neq 0$
unfolding *ASSUMPTION-def* **by** (*rule prime-elem-not-zeroI*)

lemma *prime-elem-imp-not-one* [*simp*]:
ASSUMPTION (*prime-elem x*) $\Longrightarrow x \neq 1$

unfolding *ASSUMPTION-def* **by** *auto*

end

lemma (**in** *normalization-semidom*) *irreducible-cong*:
assumes *normalize a = normalize b*
shows *irreducible a* ⟷ *irreducible b*
proof (*cases a = 0* ∨ *a dvd 1*)
case *True*
hence ¬*irreducible a* **by** (*auto simp*: *irreducible-def*)
from *True* **have** *normalize a = 0* ∨ *normalize a dvd 1*
by *auto*
also note *assms*
finally have *b = 0* ∨ *b dvd 1* **by** *simp*
hence ¬*irreducible b* **by** (*auto simp*: *irreducible-def*)
with ‹¬*irreducible a*› **show** *?thesis* **by** *simp*
next
case *False*
hence *b*: *b* ≠ *0* ¬*is-unit b* **using** *assms*
by (*auto simp*: *is-unit-normalize*[*of b*])
show *?thesis*
proof
assume *irreducible a*
thus *irreducible b*
by (*rule irreducible-mono*) (*use assms False b* **in** ‹*auto dest*: *associatedD2*›)
next
assume *irreducible b*
thus *irreducible a*
by (*rule irreducible-mono*) (*use assms False b* **in** ‹*auto dest*: *associatedD1*›)
qed
qed

lemma (**in** *normalization-semidom*) *associatedE1*:
assumes *normalize a = normalize b*
obtains *u* **where** *is-unit u a = u* ∗ *b*
proof (*cases a = 0*)
case [*simp*]: *False*
from *assms* **have** [*simp*]: *b* ≠ *0* **by** *auto*
show *?thesis*
proof (*rule that*)
show *is-unit* (*unit-factor a div unit-factor b*)
by *auto*
have *unit-factor a div unit-factor b* ∗ *b = unit-factor a* ∗ (*b div unit-factor b*)
using ‹*b* ≠ *0*› *unit-div-commute unit-div-mult-swap unit-factor-is-unit* **by** *metis*
also have *b div unit-factor b = normalize b* **by** *simp*
finally show *a = unit-factor a div unit-factor b* ∗ *b*
by (*metis assms unit-factor-mult-normalize*)

```
  qed
next
  case [simp]: True
  hence [simp]: b = 0
    using assms[symmetric] by auto
  show ?thesis
    by (intro that[of 1]) auto
qed

lemma (in normalization-semidom) associatedE2:
  assumes normalize a = normalize b
  obtains u where is-unit u b = u * a
proof −
  from assms have normalize b = normalize a
    by simp
  then obtain u where is-unit u b = u * a
    by (elim associatedE1)
  thus ?thesis using that by blast
qed



lemma (in normalization-semidom) normalize-power-normalize:
  normalize (normalize x ^ n) = normalize (x ^ n)
proof (induction n)
  case (Suc n)
  have normalize (normalize x ^ Suc n) = normalize (x * normalize (normalize x
^ n))
    by simp
  also note Suc.IH
  finally show ?case by simp
qed auto

context algebraic-semidom
begin

lemma prime-elem-imp-irreducible:
  assumes prime-elem p
  shows   irreducible p
proof (rule irreducibleI)
  fix a b
  assume p-eq: p = a * b
  with assms have nz: a ≠ 0 b ≠ 0 by auto
  from p-eq have p dvd a * b by simp
  with ‹prime-elem p› have p dvd a ∨ p dvd b by (rule prime-elem-dvd-multD)
  with ‹p = a * b› have a * b dvd 1 * b ∨ a * b dvd a * 1 by auto
  thus a dvd 1 ∨ b dvd 1
   by (simp only: dvd-times-left-cancel-iff[OF nz(1)] dvd-times-right-cancel-iff[OF
nz(2)])
```

qed (*insert assms*, *simp-all add*: *prime-elem-def*)

lemma (**in** *algebraic-semidom*) *unit-imp-no-irreducible-divisors*:
  assumes *is-unit x irreducible p*
  shows ¬*p dvd x*
proof (*rule notI*)
  assume *p dvd x*
  with ‹*is-unit x*› **have** *is-unit p*
    by (*auto intro*: *dvd-trans*)
  with ‹*irreducible p*› **show** *False*
    by (*simp add*: *irreducible-not-unit*)
qed

lemma *unit-imp-no-prime-divisors*:
  assumes *is-unit x prime-elem p*
  shows ¬*p dvd x*
  using *unit-imp-no-irreducible-divisors*[*OF assms*(*1*) *prime-elem-imp-irreducible*[*OF assms*(*2*)]] **.**

lemma *prime-elem-mono*:
  assumes *prime-elem p* ¬*q dvd 1 q dvd p*
  shows *prime-elem q*
proof −
  from ‹*q dvd p*› **obtain** *r* **where** *r*: *p* = *q* ∗ *r* **by** (*elim dvdE*)
  hence *p dvd q* ∗ *r* **by** *simp*
  with ‹*prime-elem p*› **have** *p dvd q* ∨ *p dvd r* **by** (*rule prime-elem-dvd-multD*)
  hence *p dvd q*
  proof
    assume *p dvd r*
    then obtain *s* **where** *s*: *r* = *p* ∗ *s* **by** (*elim dvdE*)
    from *r* **have** *p* ∗ *1* = *p* ∗ (*q* ∗ *s*) **by** (*subst* (*asm*) *s*) (*simp add*: *mult-ac*)
    with ‹*prime-elem p*› **have** *q dvd 1*
      by (*subst* (*asm*) *mult-cancel-left*) *auto*
    with ‹¬*q dvd 1*› **show** *?thesis* **by** *contradiction*
  qed

  show *?thesis*
  proof (*rule prime-elemI*)
    fix *a b* **assume** *q dvd* (*a* ∗ *b*)
    with ‹*p dvd q*› **have** *p dvd* (*a* ∗ *b*) **by** (*rule dvd-trans*)
    with ‹*prime-elem p*› **have** *p dvd a* ∨ *p dvd b* **by** (*rule prime-elem-dvd-multD*)
    with ‹*q dvd p*› **show** *q dvd a* ∨ *q dvd b* **by** (*blast intro*: *dvd-trans*)
  qed (*insert assms*, *auto*)
qed

lemma *irreducibleD'*:
  assumes *irreducible a b dvd a*
  shows *a dvd b* ∨ *is-unit b*
proof −

from *assms* **obtain** *c* **where** *c*: $a = b * c$ **by** (*elim dvdE*)
from *irreducibleD*[*OF assms*(*1*) *this*] **have** *is-unit b* ∨ *is-unit c* **.**
thus *?thesis* **by** (*auto simp*: *c mult-unit-dvd-iff*)
qed

lemma *irreducibleI′*:
assumes $a \neq 0$ ¬*is-unit a* ⋀*b*. *b dvd a* ⟹ *a dvd b* ∨ *is-unit b*
shows *irreducible a*
proof (*rule irreducibleI*)
fix *b c* **assume** *a-eq*: $a = b * c$
hence *a dvd b* ∨ *is-unit b* **by** (*intro assms*) *simp-all*
thus *is-unit b* ∨ *is-unit c*
proof
assume *a dvd b*
hence *b* ∗ *c dvd b* ∗ *1* **by** (*simp add*: *a-eq*)
moreover from ‹$a \neq 0$› *a-eq* **have** $b \neq 0$ **by** *auto*
ultimately show *?thesis* **by** (*subst* (*asm*) *dvd-times-left-cancel-iff*) *auto*
qed *blast*
qed (*simp-all add*: *assms*(*1*,*2*))

lemma *irreducible-altdef*:
irreducible x ⟷ $x \neq 0$ ∧ ¬*is-unit x* ∧ (∀ *b*. *b dvd x* ⟶ *x dvd b* ∨ *is-unit b*)
using *irreducibleI′*[*of x*] *irreducibleD′*[*of x*] *irreducible-not-unit*[*of x*] **by** *auto*

lemma *prime-elem-multD*:
assumes *prime-elem* (*a* ∗ *b*)
shows *is-unit a* ∨ *is-unit b*
proof −
from *assms* **have** $a \neq 0$ $b \neq 0$ **by** (*auto dest*!: *prime-elem-not-zeroI*)
moreover from *assms prime-elem-dvd-multD* [*of a* ∗ *b*] **have** *a* ∗ *b dvd a* ∨ *a* ∗ *b dvd b*
by *auto*
ultimately show *?thesis*
using *dvd-times-left-cancel-iff* [*of a b 1*]
dvd-times-right-cancel-iff [*of b a 1*]
by *auto*
qed

lemma *prime-elemD2*:
assumes *prime-elem p* **and** *a dvd p* **and** ¬ *is-unit a*
shows *p dvd a*
proof −
from ‹*a dvd p*› **obtain** *b* **where** $p = a * b$ **..**
with ‹*prime-elem p*› *prime-elem-multD* ‹¬ *is-unit a*› **have** *is-unit b* **by** *auto*
with ‹$p = a * b$› **show** *?thesis*
by (*auto simp add*: *mult-unit-dvd-iff*)
qed

lemma *prime-elem-dvd-prod-msetE*:

assumes *prime-elem p*
assumes *dvd*: *p dvd prod-mset A*
obtains *a* **where** *a* ∈# *A* **and** *p dvd a*
proof −
from *dvd* **have** ∃ *a*. *a* ∈# *A* ∧ *p dvd a*
proof (*induct A*)
case *empty* **then show** *?case*
using ‹*prime-elem p*› **by** (*simp add*: *prime-elem-not-unit*)
next
case (*add a A*)
then have *p dvd a* ∗ *prod-mset A* **by** *simp*
with ‹*prime-elem p*› **consider** (*A*) *p dvd prod-mset A* | (*B*) *p dvd a*
by (*blast dest*: *prime-elem-dvd-multD*)
then show *?case* **proof** *cases*
case *B* **then show** *?thesis* **by** *auto*
next
case *A*
with *add.hyps* **obtain** *b* **where** *b* ∈# *A p dvd b*
by *auto*
then show *?thesis* **by** *auto*
qed
qed
with *that* **show** *thesis* **by** *blast*

qed

context
begin

lemma *prime-elem-powerD*:
assumes *prime-elem* (*p* ^ *n*)
shows *prime-elem p* ∧ *n* = *1*
proof (*cases n*)
case (*Suc m*)
note *assms*
also from *Suc* **have** *p* ^ *n* = *p* ∗ *p*^*m* **by** *simp*
finally have *is-unit p* ∨ *is-unit* (*p*^*m*) **by** (*rule prime-elem-multD*)
moreover from *assms* **have** ¬*is-unit p* **by** (*simp add*: *prime-elem-def is-unit-power-iff*)
ultimately have *is-unit* (*p* ^ *m*) **by** *simp*
with ‹¬*is-unit p*› **have** *m* = *0* **by** (*simp add*: *is-unit-power-iff*)
with *Suc assms* **show** *?thesis* **by** *simp*
qed (*insert assms*, *simp-all*)

lemma *prime-elem-power-iff*:
prime-elem (*p* ^ *n*) ⟷ *prime-elem p* ∧ *n* = *1*
by (*auto dest*: *prime-elem-powerD*)

end

```
lemma irreducible-mult-unit-left:
  is-unit a ⟹ irreducible (a * p) ⟷ irreducible p
  by (auto simp: irreducible-altdef mult.commute[of a] is-unit-mult-iff
        mult-unit-dvd-iff dvd-mult-unit-iff)

lemma prime-elem-mult-unit-left:
  is-unit a ⟹ prime-elem (a * p) ⟷ prime-elem p
 by (auto simp: prime-elem-def mult.commute[of a] is-unit-mult-iff mult-unit-dvd-iff)

lemma prime-elem-dvd-cases:
  assumes pk: p*k dvd m*n and p: prime-elem p
  shows (∃x. k dvd x*n ∧ m = p*x) ∨ (∃y. k dvd m*y ∧ n = p*y)
proof −
  have p dvd m*n using dvd-mult-left pk by blast
  then consider p dvd m | p dvd n
    using p prime-elem-dvd-mult-iff by blast
  then show ?thesis
  proof cases
    case 1 then obtain a where m = p * a by (metis dvd-mult-div-cancel)
      then have ∃x. k dvd x * n ∧ m = p * x
        using p pk by (auto simp: mult.assoc)
    then show ?thesis ..
  next
    case 2 then obtain b where n = p * b by (metis dvd-mult-div-cancel)
    with p pk have ∃y. k dvd m*y ∧ n = p*y
     by (metis dvd-mult-right dvd-times-left-cancel-iff mult.left-commute mult-zero-left)
    then show ?thesis ..
  qed
qed

lemma prime-elem-power-dvd-prod:
  assumes pc: p^c dvd m*n and p: prime-elem p
  shows ∃a b. a+b = c ∧ p^a dvd m ∧ p^b dvd n
using pc
proof (induct c arbitrary: m n)
  case 0 show ?case by simp
next
  case (Suc c)
  consider x where p^c dvd x*n m = p*x | y where p^c dvd m*y n = p*y
    using prime-elem-dvd-cases [of - p^c, OF - p] Suc.prems by force
  then show ?case
  proof cases
    case (1 x)
    with Suc.hyps[of x n] obtain a b where a + b = c ∧ p ^ a dvd x ∧ p ^ b dvd
n by blast
    with 1 have Suc a + b = Suc c ∧ p ^ Suc a dvd m ∧ p ^ b dvd n
      by (auto intro: mult-dvd-mono)
    thus ?thesis by blast
  next
```

```
    case (2 y)
     with Suc.hyps[of m y] obtain a b where a + b = c ∧ p ^ a dvd m ∧ p ^ b
dvd y by blast
    with 2 have a + Suc b = Suc c ∧ p ^ a dvd m ∧ p ^ Suc b dvd n
      by (auto intro: mult-dvd-mono)
    with Suc.hyps [of m y] show ∃ a b. a + b = Suc c ∧ p ^ a dvd m ∧ p ^ b dvd
n
      by blast
  qed
qed

lemma prime-elem-power-dvd-cases:
  assumes p ^ c dvd m * n and a + b = Suc c and prime-elem p
  shows p ^ a dvd m ∨ p ^ b dvd n
proof −
  from assms obtain r s
    where r + s = c ∧ p ^ r dvd m ∧ p ^ s dvd n
    by (blast dest: prime-elem-power-dvd-prod)
  moreover with assms have
    a ≤ r ∨ b ≤ s by arith
  ultimately show ?thesis by (auto intro: power-le-dvd)
qed

lemma prime-elem-not-unit' [simp]:
  ASSUMPTION (prime-elem x) ⟹ ¬is-unit x
  unfolding ASSUMPTION-def by (rule prime-elem-not-unit)

lemma prime-elem-dvd-power-iff:
  assumes prime-elem p
  shows p dvd a ^ n ⟷ p dvd a ∧ n > 0
 using assms by (induct n) (auto dest: prime-elem-not-unit prime-elem-dvd-multD)

lemma prime-power-dvd-multD:
  assumes prime-elem p
  assumes p ^ n dvd a * b and n > 0 and ¬ p dvd a
  shows p ^ n dvd b
  using ‹p ^ n dvd a * b› and ‹n > 0›
proof (induct n arbitrary: b)
  case 0 then show ?case by simp
next
  case (Suc n) show ?case
  proof (cases n = 0)
    case True with Suc ‹prime-elem p› ‹¬ p dvd a› show ?thesis
      by (simp add: prime-elem-dvd-mult-iff)
  next
    case False then have n > 0 by simp
    from ‹prime-elem p› have p ≠ 0 by auto
    from Suc.prems have *: p * p ^ n dvd a * b
      by simp
```

then have *p dvd a* ∗ *b*
  by (*rule dvd-mult-left*)
with *Suc* ‹*prime-elem p*› ‹¬ *p dvd a*› **have** *p dvd b*
  by (*simp add*: *prime-elem-dvd-mult-iff*)
moreover define *c* **where** *c* = *b div p*
ultimately have *b*: *b* = *p* ∗ *c* **by** *simp*
with ∗ **have** *p* ∗ *p* ^ *n dvd p* ∗ (*a* ∗ *c*)
  by (*simp add*: *ac-simps*)
with ‹*p* ≠ *0*› **have** *p* ^ *n dvd a* ∗ *c*
  by *simp*
with *Suc.hyps* ‹*n* > *0*› **have** *p* ^ *n dvd c*
  by *blast*
with ‹*p* ≠ *0*› **show** *?thesis*
  by (*simp add*: *b*)
qed
qed

end

1.2 Generalized primes: normalized prime elements

context *normalization-semidom*
begin

lemma *irreducible-normalized-divisors*:
  assumes *irreducible x y dvd x normalize y* = *y*
  shows *y* = *1* ∨ *y* = *normalize x*
proof −
  from *assms* **have** *is-unit y* ∨ *x dvd y* **by** (*auto simp*: *irreducible-altdef*)
  thus *?thesis*
  proof (*elim disjE*)
    assume *is-unit y*
    hence *normalize y* = *1* **by** (*simp add*: *is-unit-normalize*)
    with *assms* **show** *?thesis* **by** *simp*
  next
    assume *x dvd y*
    with ‹*y dvd x*› **have** *normalize y* = *normalize x* **by** (*rule associatedI*)
    with *assms* **show** *?thesis* **by** *simp*
  qed
qed

lemma *irreducible-normalize-iff* [*simp*]: *irreducible* (*normalize x*) = *irreducible x*
  using *irreducible-mult-unit-left*[*of 1 div unit-factor x x*]
  by (*cases x* = *0*) (*simp-all add*: *unit-div-commute*)

lemma *prime-elem-normalize-iff* [*simp*]: *prime-elem* (*normalize x*) = *prime-elem x*
  using *prime-elem-mult-unit-left*[*of 1 div unit-factor x x*]
  by (*cases x* = *0*) (*simp-all add*: *unit-div-commute*)

lemma *prime-elem-associated*:
  assumes *prime-elem p* **and** *prime-elem q* **and** *q dvd p*
  shows *normalize q* = *normalize p*
using ‹*q dvd p*› **proof** (*rule associatedI*)
  from ‹*prime-elem q*› **have** ¬ *is-unit q*
    by (*auto simp add*: *prime-elem-not-unit*)
  with ‹*prime-elem p*› ‹*q dvd p*› **show** *p dvd q*
    by (*blast intro*: *prime-elemD2*)
qed

definition *prime* :: *'a* ⇒ *bool* **where**
  prime p ⟷ *prime-elem p* ∧ *normalize p* = *p*

lemma *not-prime-0* [*simp*]: ¬*prime 0* **by** (*simp add*: *prime-def*)

lemma *not-prime-unit*: *is-unit x* ⟹ ¬*prime x*
  using *prime-elem-not-unit*[*of x*] **by** (*auto simp add*: *prime-def*)

lemma *not-prime-1* [*simp*]: ¬*prime 1* **by** (*simp add*: *not-prime-unit*)

lemma *primeI*: *prime-elem x* ⟹ *normalize x* = *x* ⟹ *prime x*
  by (*simp add*: *prime-def*)

lemma *prime-imp-prime-elem* [*dest*]: *prime p* ⟹ *prime-elem p*
  by (*simp add*: *prime-def*)

lemma *normalize-prime*: *prime p* ⟹ *normalize p* = *p*
  by (*simp add*: *prime-def*)

lemma *prime-normalize-iff* [*simp*]: *prime* (*normalize p*) ⟷ *prime-elem p*
  by (*auto simp add*: *prime-def*)

lemma *prime-power-iff*:
  prime (*p* ^ *n*) ⟷ *prime p* ∧ *n* = *1*
  by (*auto simp*: *prime-def prime-elem-power-iff*)

lemma *prime-imp-nonzero* [*simp*]:
  ASSUMPTION (*prime x*) ⟹ *x* ≠ *0*
  unfolding *ASSUMPTION-def prime-def* **by** *auto*

lemma *prime-imp-not-one* [*simp*]:
  ASSUMPTION (*prime x*) ⟹ *x* ≠ *1*
  unfolding *ASSUMPTION-def* **by** *auto*

lemma *prime-not-unit'* [*simp*]:
  ASSUMPTION (*prime x*) ⟹ ¬*is-unit x*
  unfolding *ASSUMPTION-def prime-def* **by** *auto*

lemma *prime-normalize′* [*simp*]: *ASSUMPTION* (*prime x*) $\Longrightarrow$ *normalize x* = *x*
unfolding *ASSUMPTION-def prime-def* **by** *simp*

lemma *unit-factor-prime*: *prime x* $\Longrightarrow$ *unit-factor x* = *1*
using *unit-factor-normalize*[*of x*] **unfolding** *prime-def* **by** *auto*

lemma *unit-factor-prime′* [*simp*]: *ASSUMPTION* (*prime x*) $\Longrightarrow$ *unit-factor x* = *1*
unfolding *ASSUMPTION-def* **by** (*rule unit-factor-prime*)

lemma *prime-imp-prime-elem′* [*simp*]: *ASSUMPTION* (*prime x*) $\Longrightarrow$ *prime-elem x*
by (*simp add*: *prime-def ASSUMPTION-def*)

lemma *prime-dvd-multD*: *prime p* $\Longrightarrow$ *p dvd a* ∗ *b* $\Longrightarrow$ *p dvd a* ∨ *p dvd b*
by (*intro prime-elem-dvd-multD*) *simp-all*

lemma *prime-dvd-mult-iff*: *prime p* $\Longrightarrow$ *p dvd a* ∗ *b* $\longleftrightarrow$ *p dvd a* ∨ *p dvd b*
by (*auto dest*: *prime-dvd-multD*)

lemma *prime-dvd-power*:
prime p $\Longrightarrow$ *p dvd x* ^ *n* $\Longrightarrow$ *p dvd x*
by (*auto dest*!: *prime-elem-dvd-power simp*: *prime-def*)

lemma *prime-dvd-power-iff*:
prime p $\Longrightarrow$ *n* > *0* $\Longrightarrow$ *p dvd x* ^ *n* $\longleftrightarrow$ *p dvd x*
by (*subst prime-elem-dvd-power-iff*) *simp-all*

lemma *prime-dvd-prod-mset-iff*: *prime p* $\Longrightarrow$ *p dvd prod-mset A* $\longleftrightarrow$ (∃ *x*. *x* ∈# *A* ∧ *p dvd x*)
by (*induction A*) (*simp-all add*: *prime-elem-dvd-mult-iff prime-imp-prime-elem*, *blast*+)

lemma *prime-dvd-prod-iff*: *finite A* $\Longrightarrow$ *prime p* $\Longrightarrow$ *p dvd prod f A* $\longleftrightarrow$ (∃ *x*∈*A*. *p dvd f x*)
by (*auto simp*: *prime-dvd-prod-mset-iff prod-unfold-prod-mset*)

lemma *primes-dvd-imp-eq*:
assumes *prime p prime q p dvd q*
shows *p* = *q*
proof −
from *assms* **have** *irreducible q* **by** (*simp add*: *prime-elem-imp-irreducible prime-def*)
from *irreducibleD′*[*OF this* ‹*p dvd q*›] *assms* **have** *q dvd p* **by** *simp*
with ‹*p dvd q*› **have** *normalize p* = *normalize q* **by** (*rule associatedI*)
with *assms* **show** *p* = *q* **by** *simp*
qed

lemma *prime-dvd-prod-mset-primes-iff*:
assumes *prime p* ⋀*q*. *q* ∈# *A* $\Longrightarrow$ *prime q*

shows *p dvd prod-mset A* ⟷ *p* ∈# *A*
proof −
from *assms*(*1*) **have** *p dvd prod-mset A* ⟷ (∃ *x*. *x* ∈# *A* ∧ *p dvd x*) **by** (*rule prime-dvd-prod-mset-iff*)
also from *assms* **have** ... ⟷ *p* ∈# *A* **by** (*auto dest*: *primes-dvd-imp-eq*)
finally show *?thesis* **.**
qed

lemma *prod-mset-primes-dvd-imp-subset*:
assumes *prod-mset A dvd prod-mset B* ⋀*p*. *p* ∈# *A* ⟹ *prime p* ⋀*p*. *p* ∈# *B* ⟹ *prime p*
shows *A* ⊆# *B*
using *assms*
proof (*induction A arbitrary*: *B*)
case *empty*
thus *?case* **by** *simp*
next
case (*add p A B*)
hence *p*: *prime p* **by** *simp*
define *B'* **where** *B'* = *B* − {#*p*#}
from *add.prems* **have** *p dvd prod-mset B* **by** (*simp add*: *dvd-mult-left*)
with *add.prems* **have** *p* ∈# *B*
by (*subst* (*asm*) (*2*) *prime-dvd-prod-mset-primes-iff*) *simp-all*
hence *B*: *B* = *B'* + {#*p*#} **by** (*simp add*: *B'-def*)
from *add.prems p* **have** *A* ⊆# *B'* **by** (*intro add.IH*) (*simp-all add*: *B*)
thus *?case* **by** (*simp add*: *B*)
qed

lemma *prod-mset-dvd-prod-mset-primes-iff*:
assumes ⋀*x*. *x* ∈# *A* ⟹ *prime x* ⋀*x*. *x* ∈# *B* ⟹ *prime x*
shows *prod-mset A dvd prod-mset B* ⟷ *A* ⊆# *B*
using *assms* **by** (*auto intro*: *prod-mset-subset-imp-dvd prod-mset-primes-dvd-imp-subset*)

lemma *is-unit-prod-mset-primes-iff*:
assumes ⋀*x*. *x* ∈# *A* ⟹ *prime x*
shows *is-unit* (*prod-mset A*) ⟷ *A* = {#}
by (*auto simp add*: *is-unit-prod-mset-iff*)
(*meson all-not-in-conv assms not-prime-unit set-mset-eq-empty-iff*)

lemma *prod-mset-primes-irreducible-imp-prime*:
assumes *irred*: *irreducible* (*prod-mset A*)
assumes *A*: ⋀*x*. *x* ∈# *A* ⟹ *prime x*
assumes *B*: ⋀*x*. *x* ∈# *B* ⟹ *prime x*
assumes *C*: ⋀*x*. *x* ∈# *C* ⟹ *prime x*
assumes *dvd*: *prod-mset A dvd prod-mset B* ∗ *prod-mset C*
shows *prod-mset A dvd prod-mset B* ∨ *prod-mset A dvd prod-mset C*
proof −
from *dvd* **have** *prod-mset A dvd prod-mset* (*B* + *C*)
by *simp*

with *A B C* **have** *subset*: *A* ⊆# *B* + *C*
  by (*subst* (*asm*) *prod-mset-dvd-prod-mset-primes-iff*) *auto*
define *A1* **and** *A2* **where** *A1* = *A* ∩# *B* **and** *A2* = *A* − *A1*
have *A* = *A1* + *A2* **unfolding** *A1-def A2-def*
  by (*rule sym*, *intro subset-mset.add-diff-inverse*) *simp-all*
from *subset* **have** *A1* ⊆# *B A2* ⊆# *C*
 by (*auto simp*: *A1-def A2-def Multiset.subset-eq-diff-conv Multiset.union-commute*)
from ‹*A* = *A1* + *A2*› **have** *prod-mset A* = *prod-mset A1* ∗ *prod-mset A2* **by** *simp*
from *irred* **and** *this* **have** *is-unit* (*prod-mset A1*) ∨ *is-unit* (*prod-mset A2*)
  by (*rule irreducibleD*)
with *A* **have** *A1* = {#} ∨ *A2* = {#} **unfolding** *A1-def A2-def*
 by (*subst* (*asm*) (*1 2*) *is-unit-prod-mset-primes-iff*) (*auto dest*: *Multiset.in-diffD*)
with *dvd* ‹*A* = *A1* + *A2*› ‹*A1* ⊆# *B*› ‹*A2* ⊆# *C*› **show** *?thesis*
  by (*auto intro*: *prod-mset-subset-imp-dvd*)
qed

lemma *prod-mset-primes-finite-divisor-powers*:
assumes *A*: ⋀*x*. *x* ∈# *A* ⟹ *prime x*
assumes *B*: ⋀*x*. *x* ∈# *B* ⟹ *prime x*
assumes *A* ≠ {#}
shows *finite* {*n*. *prod-mset A* ^ *n dvd prod-mset B*}
proof −
from ‹*A* ≠ {#}› **obtain** *x* **where** *x*: *x* ∈# *A* **by** *blast*
define *m* **where** *m* = *count B x*
have {*n*. *prod-mset A* ^ *n dvd prod-mset B*} ⊆ {..*m*}
proof *safe*
  fix *n* **assume** *dvd*: *prod-mset A* ^ *n dvd prod-mset B*
  from *x* **have** *x* ^ *n dvd prod-mset A* ^ *n* **by** (*intro dvd-power-same dvd-prod-mset*)
  also note *dvd*
  also have *x* ^ *n* = *prod-mset* (*replicate-mset n x*) **by** *simp*
  finally have *replicate-mset n x* ⊆# *B*
    by (*rule prod-mset-primes-dvd-imp-subset*) (*insert A B x*, *simp-all split*: *if-splits*)
  thus *n* ≤ *m* **by** (*simp add*: *count-le-replicate-mset-subset-eq m-def*)
qed
moreover have *finite* {..*m*} **by** *simp*
ultimately show *?thesis* **by** (*rule finite-subset*)
qed

end

1.3 In a semiring with GCD, each irreducible element is a prime element

context *semiring-gcd*
begin

lemma *irreducible-imp-prime-elem-gcd*:

assumes *irreducible x*
shows *prime-elem x*
proof (*rule prime-elemI*)
fix *a b* **assume** *x dvd a* ∗ *b*
from *dvd-productE*[*OF this*] **obtain** *y z* **where** *yz*: *x* = *y* ∗ *z y dvd a z dvd b* **.**
from ‹*irreducible x*› **and** ‹*x* = *y* ∗ *z*› **have** *is-unit y* ∨ *is-unit z* **by** (*rule irreducibleD*)
with *yz* **show** *x dvd a* ∨ *x dvd b*
by (*auto simp*: *mult-unit-dvd-iff mult-unit-dvd-iff′*)
qed (*insert assms*, *auto simp*: *irreducible-not-unit*)

lemma *prime-elem-imp-coprime*:
assumes *prime-elem p* ¬*p dvd n*
shows *coprime p n*
proof (*rule coprimeI*)
fix *d* **assume** *d dvd p d dvd n*
show *is-unit d*
proof (*rule ccontr*)
assume ¬*is-unit d*
from ‹*prime-elem p*› **and** ‹*d dvd p*› **and** *this* **have** *p dvd d*
by (*rule prime-elemD2*)
from *this* **and** ‹*d dvd n*› **have** *p dvd n* **by** (*rule dvd-trans*)
with ‹¬*p dvd n*› **show** *False* **by** *contradiction*
qed
qed

lemma *prime-imp-coprime*:
assumes *prime p* ¬*p dvd n*
shows *coprime p n*
using *assms* **by** (*simp add*: *prime-elem-imp-coprime*)

lemma *prime-elem-imp-power-coprime*:
prime-elem p ⟹ ¬ *p dvd a* ⟹ *coprime a* (*p* ^ *m*)
by (*cases m* > *0*) (*auto dest*: *prime-elem-imp-coprime simp add*: *ac-simps*)

lemma *prime-imp-power-coprime*:
prime p ⟹ ¬ *p dvd a* ⟹ *coprime a* (*p* ^ *m*)
by (*rule prime-elem-imp-power-coprime*) *simp-all*

lemma *prime-elem-divprod-pow*:
assumes *p*: *prime-elem p* **and** *ab*: *coprime a b* **and** *pab*: *p*^*n dvd a* ∗ *b*
shows *p*^*n dvd a* ∨ *p*^*n dvd b*
using *assms*
proof −
from *p* **have** ¬ *is-unit p*
by *simp*
with *ab p* **have** ¬ *p dvd a* ∨ ¬ *p dvd b*
using *not-coprimeI* **by** *blast*
with *p* **have** *coprime* (*p* ^ *n*) *a* ∨ *coprime* (*p* ^ *n*) *b*

by (*auto dest*: *prime-elem-imp-power-coprime simp add*: *ac-simps*)
with *pab* **show** *?thesis*
by (*auto simp add*: *coprime-dvd-mult-left-iff coprime-dvd-mult-right-iff*)
qed

lemma *primes-coprime*:
prime p $\Longrightarrow$ *prime q* $\Longrightarrow$ $p \neq q$ $\Longrightarrow$ *coprime p q*
using *prime-imp-coprime primes-dvd-imp-eq* **by** *blast*

end

1.4 Factorial semirings: algebraic structures with unique prime factorizations

class *factorial-semiring* = *normalization-semidom* +
assumes *prime-factorization-exists*:
$x \neq 0 \Longrightarrow \exists A.\ (\forall x.\ x \in\#\ A \longrightarrow$ *prime-elem x*) $\wedge$ *normalize* (*prod-mset A*) = *normalize x*

Alternative characterization

lemma (**in** *normalization-semidom*) *factorial-semiring-altI-aux*:
assumes *finite-divisors*: $\bigwedge x.\ x \neq 0 \Longrightarrow$ *finite* {*y*. *y dvd x* $\wedge$ *normalize y* = *y*}
assumes *irreducible-imp-prime-elem*: $\bigwedge x$. *irreducible x* $\Longrightarrow$ *prime-elem x*
assumes $x \neq 0$
shows $\exists A.\ (\forall x.\ x \in\#\ A \longrightarrow$ *prime-elem x*) $\wedge$ *normalize* (*prod-mset A*) = *normalize x*
using ‹$x \neq 0$›
proof (*induction card* {*b*. *b dvd x* $\wedge$ *normalize b* = *b*} *arbitrary*: *x rule*: *less-induct*)
case (*less a*)
let *?fctrs* = λa. {*b*. *b dvd a* $\wedge$ *normalize b* = *b*}
show *?case*
proof (*cases is-unit a*)
case *True*
thus *?thesis* **by** (*intro exI*[*of* - {#}]) (*auto simp*: *is-unit-normalize*)
next
case *False*
show *?thesis*
proof (*cases* $\exists$ *b*. *b dvd a* $\wedge$ ¬*is-unit b* $\wedge$ ¬*a dvd b*)
case *False*
with ‹¬*is-unit a*› *less.prems* **have** *irreducible a* **by** (*auto simp*: *irreducible-altdef*)
hence *prime-elem a* **by** (*rule irreducible-imp-prime-elem*)
thus *?thesis* **by** (*intro exI*[*of* - {#*normalize a*#}]) *auto*
next
case *True*
then obtain *b* **where** *b*: *b dvd a* ¬ *is-unit b* ¬ *a dvd b* **by** *auto*
from *b* **have** *?fctrs b* $\subseteq$ *?fctrs a* **by** (*auto intro*: *dvd-trans*)
moreover from *b* **have** *normalize a* $\notin$ *?fctrs b* *normalize a* $\in$ *?fctrs a* **by** *simp-all*
hence *?fctrs b* $\neq$ *?fctrs a* **by** *blast*

ultimately have *?fctrs b* ⊂ *?fctrs a* **by** (*subst subset-not-subset-eq*) *blast*
with *finite-divisors*[*OF* ‹*a* ≠ *0*›] **have** *card* (*?fctrs b*) < *card* (*?fctrs a*)
by (*rule psubset-card-mono*)
moreover from ‹*a* ≠ *0*› *b* **have** *b* ≠ *0* **by** *auto*
ultimately have ∃ *A*. (∀ *x*. *x* ∈# *A* ⟶ *prime-elem x*) ∧ *normalize* (*prod-mset* *A*) = *normalize b*
by (*intro less*) *auto*
then obtain *A* **where** *A*: (∀ *x*. *x* ∈# *A* ⟶ *prime-elem x*) ∧ *normalize* (∏# *A*) = *normalize b*
by *auto*

define *c* **where** *c* = *a div b*
from *b* **have** *c*: *a* = *b* ∗ *c* **by** (*simp add*: *c-def*)
from *less.prems c* **have** *c* ≠ *0* **by** *auto*
from *b c* **have** *?fctrs c* ⊆ *?fctrs a* **by** (*auto intro*: *dvd-trans*)
moreover have *normalize a* ∉ *?fctrs c*
proof *safe*
assume *normalize a dvd c*
hence *b* ∗ *c dvd 1* ∗ *c* **by** (*simp add*: *c*)
hence *b dvd 1* **by** (*subst* (*asm*) *dvd-times-right-cancel-iff*) *fact+*
with *b* **show** *False* **by** *simp*
qed
with ‹*normalize a* ∈ *?fctrs a*› **have** *?fctrs a* ≠ *?fctrs c* **by** *blast*
ultimately have *?fctrs c* ⊂ *?fctrs a* **by** (*subst subset-not-subset-eq*) *blast*
with *finite-divisors*[*OF* ‹*a* ≠ *0*›] **have** *card* (*?fctrs c*) < *card* (*?fctrs a*)
by (*rule psubset-card-mono*)
with ‹*c* ≠ *0*› **have** ∃ *A*. (∀ *x*. *x* ∈# *A* ⟶ *prime-elem x*) ∧ *normalize* (*prod-mset A*) = *normalize c*
by (*intro less*) *auto*
then obtain *B* **where** *B*: (∀ *x*. *x* ∈# *B* ⟶ *prime-elem x*) ∧ *normalize* (∏# *B*) = *normalize c*
by *auto*

show *?thesis*
proof (*rule exI*[*of - A* + *B*]; *safe*)
have *normalize* (*prod-mset* (*A* + *B*)) =
normalize (*normalize* (*prod-mset A*) ∗ *normalize* (*prod-mset B*))
by *simp*
also have . . . = *normalize* (*b* ∗ *c*)
by (*simp only*: *A B*) *auto*
also have *b* ∗ *c* = *a*
using *c* **by** *simp*
finally show *normalize* (*prod-mset* (*A* + *B*)) = *normalize a* **.**
next
qed (*use A B* **in** *auto*)
qed
qed
qed

lemma *factorial-semiring-altI*:
assumes *finite-divisors*: $\bigwedge x::'a.\ x \neq 0 \Longrightarrow$ *finite* $\{y.\ y$ *dvd* $x \wedge$ *normalize* $y = y\}$
assumes *irreducible-imp-prime*: $\bigwedge x::'a.$ *irreducible* $x \Longrightarrow$ *prime-elem* x
shows *OFCLASS*('a :: *normalization-semidom*, *factorial-semiring-class*)
by *intro-classes* (*rule factorial-semiring-altI-aux*[*OF assms*])

Properties

context *factorial-semiring*
begin

lemma *prime-factorization-exists'*:
assumes $x \neq 0$
obtains A **where** $\bigwedge x.\ x \in\#\ A \Longrightarrow$ *prime* x *normalize* (*prod-mset* A) = *normalize* x
proof −
from *prime-factorization-exists*[*OF assms*] **obtain** A
where A: $\bigwedge x.\ x \in\#\ A \Longrightarrow$ *prime-elem* x *normalize* (*prod-mset* A) = *normalize* x **by** *blast*
define A' **where** A' = *image-mset normalize* A
have *normalize* (*prod-mset* A') = *normalize* (*prod-mset* A)
by (*simp add*: A'*-def normalize-prod-mset-normalize*)
also note $A(2)$
finally have *normalize* (*prod-mset* A') = *normalize* x **by** *simp*
moreover from $A(1)$ **have** $\forall x.\ x \in\#\ A' \longrightarrow$ *prime* x **by** (*auto simp*: *prime-def* A'*-def*)
ultimately show *?thesis* **by** (*intro that*[*of* A']) *blast*
qed

lemma *irreducible-imp-prime-elem*:
assumes *irreducible* x
shows *prime-elem* x
proof (*rule prime-elemI*)
fix a b **assume** *dvd*: x *dvd* $a * b$
from *assms* **have** $x \neq 0$ **by** *auto*
show x *dvd* $a \vee x$ *dvd* b
proof (*cases* $a = 0 \vee b = 0$)
case *False*
hence $a \neq 0$ $b \neq 0$ **by** *blast+*
note *nz* = ‹$x \neq 0$› *this*
from *nz*[*THEN prime-factorization-exists'*] **obtain** A B C
where *ABC*:
$\bigwedge z.\ z \in\#\ A \Longrightarrow$ *prime* z
normalize $(\prod_{\#} A)$ = *normalize* x
$\bigwedge z.\ z \in\#\ B \Longrightarrow$ *prime* z
normalize $(\prod_{\#} B)$ = *normalize* a
$\bigwedge z.\ z \in\#\ C \Longrightarrow$ *prime* z
normalize $(\prod_{\#} C)$ = *normalize* b
by *this blast*

have *irreducible* (*prod-mset A*)
by (*subst irreducible-cong*[*OF ABC*(*2*)]) *fact*
moreover have *normalize* (*prod-mset A*) *dvd*
normalize (*normalize* (*prod-mset B*) ∗ *normalize* (*prod-mset C*))
unfolding *ABC* **using** *dvd* **by** *simp*
hence *prod-mset A dvd prod-mset B* ∗ *prod-mset C*
unfolding *normalize-mult-normalize-left normalize-mult-normalize-right* **by** *simp*
ultimately have *prod-mset A dvd prod-mset B* ∨ *prod-mset A dvd prod-mset C*
by (*intro prod-mset-primes-irreducible-imp-prime*) (*use ABC* **in** *auto*)
hence *normalize* (*prod-mset A*) *dvd normalize* (*prod-mset B*) ∨
normalize (*prod-mset A*) *dvd normalize* (*prod-mset C*) **by** *simp*
thus *?thesis* **unfolding** *ABC* **by** *simp*
qed *auto*
qed (*use assms* **in** ‹*simp-all add*: *irreducible-def*›)

lemma *finite-divisor-powers*:
assumes $y \neq 0$ ¬*is-unit x*
shows *finite* {*n*. *x* ^ *n dvd y*}
proof (*cases* $x = 0$)
case *True*
with *assms* **have** {*n*. *x* ^ *n dvd y*} = {*0*} **by** (*auto simp*: *power-0-left*)
thus *?thesis* **by** *simp*
next
case *False*
note *nz* = *this* ‹$y \neq 0$›
from *nz*[*THEN prime-factorization-exists'*] **obtain** *A B*
where *AB*:
⋀*z*. *z* ∈# *A* ⟹ *prime z*
normalize ($\prod_{\#}$ *A*) = *normalize x*
⋀*z*. *z* ∈# *B* ⟹ *prime z*
normalize ($\prod_{\#}$ *B*) = *normalize y*
by *this blast*

from *AB assms* **have** *A* ≠ {#} **by** (*auto simp*: *normalize-1-iff*)
from *AB*(*2*,*4*) *prod-mset-primes-finite-divisor-powers* [*of A B*, *OF AB*(*1*,*3*) *this*]
have *finite* {*n*. *prod-mset A* ^ *n dvd prod-mset B*} **by** *simp*
also have {*n*. *prod-mset A* ^ *n dvd prod-mset B*} =
{*n*. *normalize* (*normalize* (*prod-mset A*) ^ *n*) *dvd normalize* (*prod-mset B*)}
unfolding *normalize-power-normalize* **by** *simp*
also have ... = {*n*. *x* ^ *n dvd y*}
unfolding *AB* **unfolding** *normalize-power-normalize* **by** *simp*
finally show *?thesis* **.**
qed

lemma *finite-prime-divisors*:

assumes $x \neq 0$
shows *finite* $\{p.\ prime\ p \wedge p\ dvd\ x\}$
proof −
from *prime-factorization-exists'*[*OF assms*] **obtain** *A*
where *A*: $\bigwedge z.\ z \in\#\ A \Longrightarrow prime\ z\ normalize\ (\prod_{\#}\ A) = normalize\ x$ **by** *this blast*
have $\{p.\ prime\ p \wedge p\ dvd\ x\} \subseteq$ *set-mset A*
proof *safe*
fix *p* **assume** *p*: *prime p* **and** *dvd*: *p dvd x*
from *dvd* **have** *p dvd normalize x* **by** *simp*
also from *A* **have** *normalize x* = *normalize* (*prod-mset A*) **by** *simp*
finally have *p dvd prod-mset A*
by *simp*
thus $p \in\#\ A$ **using** *p A*
by (*subst* (*asm*) *prime-dvd-prod-mset-primes-iff*)
qed
moreover have *finite* (*set-mset A*) **by** *simp*
ultimately show *?thesis* **by** (*rule finite-subset*)
qed

lemma *infinite-unit-divisor-powers*:
assumes $y \neq 0$
assumes *is-unit x*
shows *infinite* $\{n.\ x\widehat{\ }n\ dvd\ y\}$
proof −
from ‹*is-unit x*› **have** *is-unit* ($x\widehat{\ }n$) **for** *n*
using *is-unit-power-iff* **by** *auto*
hence $x\widehat{\ }n\ dvd\ y$ **for** *n*
by *auto*
hence $\{n.\ x\widehat{\ }n\ dvd\ y\} = UNIV$
by *auto*
thus *?thesis*
by *auto*
qed

corollary *is-unit-iff-infinite-divisor-powers*:
assumes $y \neq 0$
shows *is-unit x* $\longleftrightarrow$ *infinite* $\{n.\ x\widehat{\ }n\ dvd\ y\}$
using *infinite-unit-divisor-powers finite-divisor-powers assms* **by** *auto*

lemma *prime-elem-iff-irreducible*: *prime-elem x* $\longleftrightarrow$ *irreducible x*
by (*blast intro*: *irreducible-imp-prime-elem prime-elem-imp-irreducible*)

lemma *prime-divisor-exists*:
assumes $a \neq 0$ $\neg$*is-unit a*
shows $\exists\, b.\ b\ dvd\ a \wedge prime\ b$
proof −
from *prime-factorization-exists'*[*OF assms*(*1*)]
obtain *A* **where** *A*: $\bigwedge z.\ z \in\#\ A \Longrightarrow prime\ z\ normalize\ (\prod_{\#}\ A) = normalize\ a$

by *this blast*
with *assms* **have** $A \neq \{\#\}$ **by** *auto*
then obtain x **where** $x \in\# A$ **by** *blast*
with $A(1)$ **have** $*$: *x dvd normalize* (*prod-mset A*) *prime x*
by (*auto simp*: *dvd-prod-mset*)
hence *x dvd a* **by** (*simp add*: $A(2)$)
with $*$ **show** *?thesis* **by** *blast*
qed

lemma *prime-divisors-induct* [*case-names zero unit factor*]:
assumes $P\ 0\ \bigwedge x.$ *is-unit* $x \Longrightarrow P\ x\ \bigwedge p\ x.$ *prime* $p \Longrightarrow P\ x \Longrightarrow P\ (p * x)$
shows $P\ x$
proof (*cases* $x = 0$)
case *False*
from *prime-factorization-exists'*[*OF this*]
obtain A **where** A: $\bigwedge z.\ z \in\# A \Longrightarrow$ *prime z normalize* $(\prod_{\#} A)$ = *normalize x*
by *this blast*
from A **obtain** u **where** u: *is-unit u* $x = u *$ *prod-mset A*
by (*elim associatedE2*)

from $A(1)$ **have** P ($u *$ *prod-mset A*)
proof (*induction A*)
case (*add p A*)
from *add.prems* **have** *prime p* **by** *simp*
moreover from *add.prems* **have** P ($u *$ *prod-mset A*) **by** (*intro add.IH*) *simp-all*
ultimately have P ($p *$ ($u *$ *prod-mset A*)) **by** (*rule assms*(*3*))
thus *?case* **by** (*simp add*: *mult-ac*)
qed (*simp-all add*: *assms False u*)
with *A u* **show** *?thesis* **by** *simp*
qed (*simp-all add*: *assms*(*1*))

lemma *no-prime-divisors-imp-unit*:
assumes $a \neq 0\ \bigwedge b.$ *b dvd a* $\Longrightarrow$ *normalize* $b = b \Longrightarrow \neg$ *prime-elem b*
shows *is-unit a*
proof (*rule ccontr*)
assume $\neg$*is-unit a*
from *prime-divisor-exists*[*OF assms*(*1*) *this*] **obtain** b **where** *b dvd a prime b* **by** *auto*
with *assms*(*2*)[*of b*] **show** *False* **by** (*simp add*: *prime-def*)
qed

lemma *prime-divisorE*:
assumes $a \neq 0$ **and** $\neg$ *is-unit a*
obtains p **where** *prime p* **and** *p dvd a*
using *assms no-prime-divisors-imp-unit* **unfolding** *prime-def* **by** *blast*

definition *multiplicity* :: $'a \Rightarrow 'a \Rightarrow nat$ **where**
multiplicity p x = (*if finite* $\{n.\ p \ \hat{}\ n$ *dvd x*$\}$ *then Max* $\{n.\ p \ \hat{}\ n$ *dvd x*$\}$ *else 0*)

lemma *multiplicity-dvd*: *p* ^ *multiplicity p x dvd x*
proof (*cases finite* {*n. p* ^ *n dvd x*})
  case *True*
  hence *multiplicity p x* = *Max* {*n. p* ^ *n dvd x*}
    by (*simp add*: *multiplicity-def*)
  also have ... ∈ {*n. p* ^ *n dvd x*}
    by (*rule Max-in*) (*auto intro*!: *True exI*[*of - 0*::*nat*])
  finally show *?thesis* **by** *simp*
qed (*simp add*: *multiplicity-def*)

lemma *multiplicity-dvd'*: *n* ≤ *multiplicity p x* ⟹ *p* ^ *n dvd x*
  by (*rule dvd-trans*[*OF le-imp-power-dvd multiplicity-dvd*])

context
  fixes *x p* :: *'a*
  assumes *xp*: *x* ≠ *0* ¬*is-unit p*
begin

lemma *multiplicity-eq-Max*: *multiplicity p x* = *Max* {*n. p* ^ *n dvd x*}
  using *finite-divisor-powers*[*OF xp*] **by** (*simp add*: *multiplicity-def*)

lemma *multiplicity-geI*:
  assumes *p* ^ *n dvd x*
  shows *multiplicity p x* ≥ *n*
proof −
  from *assms* **have** *n* ≤ *Max* {*n. p* ^ *n dvd x*}
    by (*intro Max-ge finite-divisor-powers xp*) *simp-all*
  thus *?thesis* **by** (*subst multiplicity-eq-Max*)
qed

lemma *multiplicity-lessI*:
  assumes ¬*p* ^ *n dvd x*
  shows *multiplicity p x* < *n*
proof (*rule ccontr*)
  assume ¬(*n* > *multiplicity p x*)
  hence *p* ^ *n dvd x* **by** (*intro multiplicity-dvd'*) *simp*
  with *assms* **show** *False* **by** *contradiction*
qed

lemma *power-dvd-iff-le-multiplicity*:
  p ^ *n dvd x* ⟷ *n* ≤ *multiplicity p x*
  using *multiplicity-geI*[*of n*] *multiplicity-lessI*[*of n*] **by** (*cases p* ^ *n dvd x*) *auto*

lemma *multiplicity-eq-zero-iff*:
  shows *multiplicity p x* = *0* ⟷ ¬*p dvd x*
  using *power-dvd-iff-le-multiplicity*[*of 1*] **by** *auto*

lemma *multiplicity-gt-zero-iff*:

shows *multiplicity p x* > *0* ⟷ *p dvd x*
using *power-dvd-iff-le-multiplicity*[*of 1*] **by** *auto*

lemma *multiplicity-decompose*:
¬*p dvd* (*x div p* ^ *multiplicity p x*)
proof
assume ∗: *p dvd x div p* ^ *multiplicity p x*
have *x* = *x div p* ^ *multiplicity p x* ∗ (*p* ^ *multiplicity p x*)
using *multiplicity-dvd*[*of p x*] **by** *simp*
also from ∗ **have** *x div p* ^ *multiplicity p x* = (*x div p* ^ *multiplicity p x div p*) ∗ *p* **by** *simp*
also have *x div p* ^ *multiplicity p x div p* ∗ *p* ∗ *p* ^ *multiplicity p x* =
x div p ^ *multiplicity p x div p* ∗ *p* ^ *Suc* (*multiplicity p x*)
by (*simp add*: *mult-assoc*)
also have *p* ^ *Suc* (*multiplicity p x*) *dvd* ... **by** (*rule dvd-triv-right*)
finally show *False* **by** (*subst* (*asm*) *power-dvd-iff-le-multiplicity*) *simp*
qed

lemma *multiplicity-decompose'*:
obtains *y* **where** *x* = *p* ^ *multiplicity p x* ∗ *y* ¬*p dvd y*
using *that*[*of x div p* ^ *multiplicity p x*]
by (*simp add*: *multiplicity-decompose multiplicity-dvd*)

end

lemma *multiplicity-zero* [*simp*]: *multiplicity p 0* = *0*
by (*simp add*: *multiplicity-def*)

lemma *prime-elem-multiplicity-eq-zero-iff*:
prime-elem p ⟹ *x* ≠ *0* ⟹ *multiplicity p x* = *0* ⟷ ¬*p dvd x*
by (*rule multiplicity-eq-zero-iff*) *simp-all*

lemma *prime-multiplicity-other*:
assumes *prime p prime q p* ≠ *q*
shows *multiplicity p q* = *0*
using *assms* **by** (*subst prime-elem-multiplicity-eq-zero-iff*) (*auto dest*: *primes-dvd-imp-eq*)

lemma *prime-multiplicity-gt-zero-iff*:
prime-elem p ⟹ *x* ≠ *0* ⟹ *multiplicity p x* > *0* ⟷ *p dvd x*
by (*rule multiplicity-gt-zero-iff*) *simp-all*

lemma *multiplicity-unit-left*: *is-unit p* ⟹ *multiplicity p x* = *0*
by (*simp add*: *multiplicity-def is-unit-power-iff unit-imp-dvd*)

lemma *multiplicity-unit-right*:
assumes *is-unit x*
shows *multiplicity p x* = *0*
proof (*cases is-unit p* ∨ *x* = *0*)
case *False*

with *multiplicity-lessI*[*of x p 1*] *this assms*
show *?thesis* **by** (*auto dest*: *dvd-unit-imp-unit*)
qed (*auto simp*: *multiplicity-unit-left*)

lemma *multiplicity-one* [*simp*]: *multiplicity p 1* = *0*
by (*rule multiplicity-unit-right*) *simp-all*

lemma *multiplicity-eqI*:
assumes *p* ^ *n dvd x* ¬*p* ^ *Suc n dvd x*
shows *multiplicity p x* = *n*
proof −
consider *x* = *0* | *is-unit p* | *x* ≠ *0* ¬*is-unit p* **by** *blast*
thus *?thesis*
proof *cases*
assume *xp*: *x* ≠ *0* ¬*is-unit p*
from *xp assms*(*1*) **have** *multiplicity p x* ≥ *n* **by** (*intro multiplicity-geI*)
moreover from *assms*(*2*) *xp* **have** *multiplicity p x* < *Suc n* **by** (*intro multiplicity-lessI*)
ultimately show *?thesis* **by** *simp*
next
assume *is-unit p*
hence *is-unit* (*p* ^ *Suc n*) **by** (*simp add*: *is-unit-power-iff del*: *power-Suc*)
hence *p* ^ *Suc n dvd x* **by** (*rule unit-imp-dvd*)
with ‹¬*p* ^ *Suc n dvd x*› **show** *?thesis* **by** *contradiction*
qed (*insert assms*, *simp-all*)
qed

context
fixes *x p* :: *'a*
assumes *xp*: *x* ≠ *0* ¬*is-unit p*
begin

lemma *multiplicity-times-same*:
assumes *p* ≠ *0*
shows *multiplicity p* (*p* ∗ *x*) = *Suc* (*multiplicity p x*)
proof (*rule multiplicity-eqI*)
show *p* ^ *Suc* (*multiplicity p x*) *dvd p* ∗ *x*
by (*auto intro*!: *mult-dvd-mono multiplicity-dvd*)
from *xp assms* **show** ¬ *p* ^ *Suc* (*Suc* (*multiplicity p x*)) *dvd p* ∗ *x*
using *power-dvd-iff-le-multiplicity*[*OF xp*, *of Suc* (*multiplicity p x*)] **by** *simp*
qed

end

lemma *multiplicity-same-power'*: *multiplicity p* (*p* ^ *n*) = (*if p* = *0* ∨ *is-unit p then 0 else n*)
proof −
consider *p* = *0* | *is-unit p* |*p* ≠ *0* ¬*is-unit p* **by** *blast*

thus *?thesis*
proof *cases*
assume $p \neq 0$ ¬*is-unit p*
thus *?thesis* **by** (*induction n*) (*simp-all add*: *multiplicity-times-same*)
qed (*simp-all add*: *power-0-left multiplicity-unit-left*)
qed

lemma *multiplicity-same-power*:
$p \neq 0 \Longrightarrow$ ¬*is-unit p* $\Longrightarrow$ *multiplicity p* $(p$ ^ $n) = n$
by (*simp add*: *multiplicity-same-power′*)

lemma *multiplicity-prime-elem-times-other*:
assumes *prime-elem p* ¬*p dvd q*
shows *multiplicity p* $(q * x) =$ *multiplicity p x*
proof (*cases* $x = 0$)
case *False*
show *?thesis*
proof (*rule multiplicity-eqI*)
have *1* ∗ *p* ^ *multiplicity p x dvd q* ∗ *x*
by (*intro mult-dvd-mono multiplicity-dvd*) *simp-all*
thus *p* ^ *multiplicity p x dvd q* ∗ *x* **by** *simp*
next
define *n* **where** *n* = *multiplicity p x*
from *assms* **have** ¬*is-unit p* **by** *simp*
from *multiplicity-decompose′*[*OF False this*]
obtain *y* **where** *y* [*folded n-def*]: *x* = *p* ^ *multiplicity p x* ∗ *y* ¬ *p dvd y* **.**
from *y* **have** *p* ^ *Suc n dvd q* ∗ *x* ⟷ *p* ^ *n* ∗ *p dvd p* ^ *n* ∗ (*q* ∗ *y*) **by** (*simp add*: *mult-ac*)
also from *assms* **have** ... ⟷ *p dvd q* ∗ *y* **by** *simp*
also have ... ⟷ *p dvd q* ∨ *p dvd y* **by** (*rule prime-elem-dvd-mult-iff*) *fact+*
also from *assms y* **have** ... ⟷ *False* **by** *simp*
finally show ¬(*p* ^ *Suc n dvd q* ∗ *x*) **by** *blast*
qed
qed *simp-all*

lemma *multiplicity-self*:
assumes $p \neq 0$ ¬*is-unit p*
shows *multiplicity p p* = *1*
proof −
from *assms* **have** *multiplicity p p* = *Max* {*n*. *p* ^ *n dvd p*}
by (*simp add*: *multiplicity-eq-Max*)
also from *assms* **have** *p* ^ *n dvd p* ⟷ $n \leq 1$ **for** *n*
using *dvd-power-iff* [*of p n 1*] **by** *auto*
hence {*n*. *p* ^ *n dvd p*} = {..*1*} **by** *auto*
also have ... = {*0*,*1*} **by** *auto*
finally show *?thesis* **by** *simp*
qed

lemma *multiplicity-times-unit-left*:

assumes *is-unit c*
shows *multiplicity* (*c* ∗ *p*) *x* = *multiplicity p x*
proof −
from *assms* **have** {*n*. (*c* ∗ *p*) ^ *n dvd x*} = {*n*. *p* ^ *n dvd x*}
by (*subst mult.commute*) (*simp add*: *mult-unit-dvd-iff power-mult-distrib is-unit-power-iff*)
thus *?thesis* **by** (*simp add*: *multiplicity-def*)
qed

lemma *multiplicity-times-unit-right*:
assumes *is-unit c*
shows *multiplicity p* (*c* ∗ *x*) = *multiplicity p x*
proof −
from *assms* **have** {*n*. *p* ^ *n dvd c* ∗ *x*} = {*n*. *p* ^ *n dvd x*}
by (*subst mult.commute*) (*simp add*: *dvd-mult-unit-iff*)
thus *?thesis* **by** (*simp add*: *multiplicity-def*)
qed

lemma *multiplicity-normalize-left* [*simp*]:
multiplicity (*normalize p*) *x* = *multiplicity p x*
proof (*cases p* = *0*)
case [*simp*]: *False*
have *normalize p* = (*1 div unit-factor p*) ∗ *p*
by (*simp add*: *unit-div-commute is-unit-unit-factor*)
also have *multiplicity* ... *x* = *multiplicity p x*
by (*rule multiplicity-times-unit-left*) (*simp add*: *is-unit-unit-factor*)
finally show *?thesis* **.**
qed *simp-all*

lemma *multiplicity-normalize-right* [*simp*]:
multiplicity p (*normalize x*) = *multiplicity p x*
proof (*cases x* = *0*)
case [*simp*]: *False*
have *normalize x* = (*1 div unit-factor x*) ∗ *x*
by (*simp add*: *unit-div-commute is-unit-unit-factor*)
also have *multiplicity p* ... = *multiplicity p x*
by (*rule multiplicity-times-unit-right*) (*simp add*: *is-unit-unit-factor*)
finally show *?thesis* **.**
qed *simp-all*

lemma *multiplicity-prime* [*simp*]: *prime-elem p* ⟹ *multiplicity p p* = *1*
by (*rule multiplicity-self*) *auto*

lemma *multiplicity-prime-power* [*simp*]: *prime-elem p* ⟹ *multiplicity p* (*p* ^ *n*)
= *n*
by (*subst multiplicity-same-power′*) *auto*

lift-definition *prime-factorization* :: *′a* ⇒ *′a multiset* **is**
λx p. *if prime p then multiplicity p x else 0*
proof −

fix *x* :: *′a*
show *finite* {*p*. *0* < (*if prime p then multiplicity p x else 0*)} (**is** *finite ?A*)
proof (*cases x* = *0*)
case *False*
from *False* **have** *?A* ⊆ {*p*. *prime p* ∧ *p dvd x*}
by (*auto simp*: *multiplicity-gt-zero-iff*)
moreover from *False* **have** *finite* {*p*. *prime p* ∧ *p dvd x*}
by (*rule finite-prime-divisors*)
ultimately show *?thesis* **by** (*rule finite-subset*)
qed *simp-all*
qed

abbreviation *prime-factors* :: *′a* ⇒ *′a set* **where**
prime-factors a ≡ *set-mset* (*prime-factorization a*)

lemma *count-prime-factorization-nonprime*:
¬*prime p* =⇒ *count* (*prime-factorization x*) *p* = *0*
by *transfer simp*

lemma *count-prime-factorization-prime*:
prime p =⇒ *count* (*prime-factorization x*) *p* = *multiplicity p x*
by *transfer simp*

lemma *count-prime-factorization*:
count (*prime-factorization x*) *p* = (*if prime p then multiplicity p x else 0*)
by *transfer simp*

lemma *dvd-imp-multiplicity-le*:
assumes *a dvd b b* ≠ *0*
shows *multiplicity p a* ≤ *multiplicity p b*
proof (*cases is-unit p*)
case *False*
with *assms* **show** *?thesis*
by (*intro multiplicity-geI*) (*auto intro*: *dvd-trans*[*OF multiplicity-dvd′ assms*(*1*)])
qed (*insert assms*, *auto simp*: *multiplicity-unit-left*)

lemma *prime-power-inj*:
assumes *prime a a* ^ *m* = *a* ^ *n*
shows *m* = *n*
proof −
have *multiplicity a* (*a* ^ *m*) = *multiplicity a* (*a* ^ *n*) **by** (*simp only*: *assms*)
thus *?thesis* **using** *assms* **by** (*subst* (*asm*) (*1 2*) *multiplicity-prime-power*) *simp-all*
qed

lemma *prime-power-inj′*:
assumes *prime p prime q*
assumes *p* ^ *m* = *q* ^ *n m* > *0 n* > *0*
shows *p* = *q m* = *n*
proof −

from *assms* **have** *p ^ 1 dvd p ^ m* **by** (*intro le-imp-power-dvd*) *simp*
also have *p ^ m = q ^ n* **by** *fact*
finally have *p dvd q ^ n* **by** *simp*
with *assms* **have** *p dvd q* **using** *prime-dvd-power*[*of p q*] **by** *simp*
with *assms* **show** *p = q* **by** (*simp add*: *primes-dvd-imp-eq*)
with *assms* **show** *m = n* **by** (*simp add*: *prime-power-inj*)
qed

lemma *prime-power-eq-one-iff* [*simp*]: *prime p* ⟹ *p ^ n = 1* ⟷ *n = 0*
using *prime-power-inj*[*of p n 0*] **by** *auto*

lemma *one-eq-prime-power-iff* [*simp*]: *prime p* ⟹ *1 = p ^ n* ⟷ *n = 0*
using *prime-power-inj*[*of p 0 n*] **by** *auto*

lemma *prime-power-inj''*:
assumes *prime p prime q*
shows *p ^ m = q ^ n* ⟷ (*m = 0* ∧ *n = 0*) ∨ (*p = q* ∧ *m = n*)
using *assms*
by (*cases m = 0*; *cases n = 0*)
(*auto dest*: *prime-power-inj'*[*OF assms*])

lemma *prime-factorization-0* [*simp*]: *prime-factorization 0* = {#}
by (*simp add*: *multiset-eq-iff count-prime-factorization*)

lemma *prime-factorization-empty-iff*:
prime-factorization x = {#} ⟷ *x = 0* ∨ *is-unit x*
proof
assume ∗: *prime-factorization x* = {#}
{
assume *x*: *x* ≠ *0* ¬*is-unit x*
{
fix *p* **assume** *p*: *prime p*
have *count* (*prime-factorization x*) *p = 0* **by** (*simp add*: ∗)
also from *p* **have** *count* (*prime-factorization x*) *p = multiplicity p x*
by (*rule count-prime-factorization-prime*)
also from *x p* **have** ... = *0* ⟷ ¬*p dvd x* **by** (*simp add*: *multiplicity-eq-zero-iff*)
finally have ¬*p dvd x* **.**
}
with *prime-divisor-exists*[*OF x*] **have** *False* **by** *blast*
}
thus *x = 0* ∨ *is-unit x* **by** *blast*
next
assume *x = 0* ∨ *is-unit x*
thus *prime-factorization x* = {#}
proof
assume *x*: *is-unit x*
{
fix *p* **assume** *p*: *prime p*

from *p x* **have** *multiplicity p x = 0*
by (*subst multiplicity-eq-zero-iff*)
(*auto simp*: *multiplicity-eq-zero-iff dest*: *unit-imp-no-prime-divisors*)
}
thus *?thesis* **by** (*simp add*: *multiset-eq-iff count-prime-factorization*)
qed *simp-all*
qed

lemma *prime-factorization-unit*:
assumes *is-unit x*
shows *prime-factorization x* = {#}
proof (*rule multiset-eqI*)
fix *p* :: *'a*
show *count* (*prime-factorization x*) *p* = *count* {#} *p*
proof (*cases prime p*)
case *True*
with *assms* **have** *multiplicity p x = 0*
by (*subst multiplicity-eq-zero-iff*)
(*auto simp*: *multiplicity-eq-zero-iff dest*: *unit-imp-no-prime-divisors*)
with *True* **show** *?thesis* **by** (*simp add*: *count-prime-factorization-prime*)
qed (*simp-all add*: *count-prime-factorization-nonprime*)
qed

lemma *prime-factorization-1* [*simp*]: *prime-factorization 1* = {#}
by (*simp add*: *prime-factorization-unit*)

lemma *prime-factorization-times-prime*:
assumes $x \neq 0$ *prime p*
shows *prime-factorization* $(p * x)$ = {#*p*#} + *prime-factorization x*
proof (*rule multiset-eqI*)
fix *q* :: *'a*
consider ¬*prime q* | *p* = *q* | *prime q* $p \neq q$ **by** *blast*
thus *count* (*prime-factorization* $(p * x)$) *q* = *count* ({#*p*#} + *prime-factorization x*) *q*
proof *cases*
assume *q*: *prime q* $p \neq q$
with *assms primes-dvd-imp-eq*[*of q p*] **have** ¬*q dvd p* **by** *auto*
with *q assms* **show** *?thesis*
by (*simp add*: *multiplicity-prime-elem-times-other count-prime-factorization*)
qed (*insert assms*, *auto simp*: *count-prime-factorization multiplicity-times-same*)
qed

lemma *prod-mset-prime-factorization-weak*:
assumes $x \neq 0$
shows *normalize* (*prod-mset* (*prime-factorization x*)) = *normalize x*
using *assms*
proof (*induction x rule*: *prime-divisors-induct*)
case (*factor p x*)
have *normalize* (*prod-mset* (*prime-factorization* $(p * x)$)) =

normalize (*p* ∗ *normalize* (*prod-mset* (*prime-factorization x*)))
using *factor.prems factor.hyps* **by** (*simp add*: *prime-factorization-times-prime*)
also have *normalize* (*prod-mset* (*prime-factorization x*)) = *normalize x*
by (*rule factor.IH*) (*use factor* **in** *auto*)
finally show *?case* **by** *simp*
qed (*auto simp*: *prime-factorization-unit is-unit-normalize*)

lemma *in-prime-factors-iff*:
$p \in$ *prime-factors* $x \longleftrightarrow x \neq 0 \wedge p$ *dvd* $x \wedge$ *prime* p
proof −
have $p \in$ *prime-factors* $x \longleftrightarrow$ *count* (*prime-factorization* x) $p > 0$ **by** *simp*
also have . . . $\longleftrightarrow x \neq 0 \wedge p$ *dvd* $x \wedge$ *prime* p
by (*subst count-prime-factorization*, *cases* $x = 0$)
(*auto simp*: *multiplicity-eq-zero-iff multiplicity-gt-zero-iff*)
finally show *?thesis* **.**
qed

lemma *in-prime-factors-imp-prime* [*intro*]:
$p \in$ *prime-factors* $x \Longrightarrow$ *prime* p
by (*simp add*: *in-prime-factors-iff*)

lemma *in-prime-factors-imp-dvd* [*dest*]:
$p \in$ *prime-factors* $x \Longrightarrow p$ *dvd* x
by (*simp add*: *in-prime-factors-iff*)

lemma *prime-factorsI*:
$x \neq 0 \Longrightarrow$ *prime* $p \Longrightarrow p$ *dvd* $x \Longrightarrow p \in$ *prime-factors* x
by (*auto simp*: *in-prime-factors-iff*)

lemma *prime-factors-dvd*:
$x \neq 0 \Longrightarrow$ *prime-factors* $x = \{p.$ *prime* $p \wedge p$ *dvd* $x\}$
by (*auto intro*: *prime-factorsI*)

lemma *prime-factors-multiplicity*:
prime-factors $n = \{p.$ *prime* $p \wedge$ *multiplicity* $p\ n > 0\}$
by (*cases* $n = 0$) (*auto simp add*: *prime-factors-dvd prime-multiplicity-gt-zero-iff*)

lemma *prime-factorization-prime*:
assumes *prime p*
shows *prime-factorization* $p = \{\#p\#\}$
proof (*rule multiset-eqI*)
fix q :: $'a$
consider ¬*prime* q | $q = p$ | *prime* q $q \neq p$ **by** *blast*
thus *count* (*prime-factorization* p) q = *count* $\{\#p\#\}$ q
by *cases* (*insert assms*, *auto dest*: *primes-dvd-imp-eq*
simp: *count-prime-factorization multiplicity-self multiplicity-eq-zero-iff*)
qed

lemma *prime-factorization-prod-mset-primes*:

assumes $\bigwedge p.\ p \in\#\ A \Longrightarrow prime\ p$
shows *prime-factorization* (*prod-mset A*) = *A*
using *assms*
proof (*induction A*)
case (*add p A*)
from *add.prems*[*of 0*] **have** $0 \notin\#\ A$ **by** *auto*
hence *prod-mset A* $\neq$ *0* **by** *auto*
with *add* **show** *?case*
by (*simp-all add*: *mult-ac prime-factorization-times-prime Multiset.union-commute*)
qed *simp-all*

lemma *prime-factorization-cong*:
normalize x = *normalize y* $\Longrightarrow$ *prime-factorization x* = *prime-factorization y*
by (*simp add*: *multiset-eq-iff count-prime-factorization*
multiplicity-normalize-right [*of - x*, *symmetric*]
multiplicity-normalize-right [*of - y*, *symmetric*]
del: *multiplicity-normalize-right*)

lemma *prime-factorization-unique*:
assumes $x \neq 0$ $y \neq 0$
shows *prime-factorization x* = *prime-factorization y* $\longleftrightarrow$ *normalize x* = *normalize y*
proof
assume *prime-factorization x* = *prime-factorization y*
hence *prod-mset* (*prime-factorization x*) = *prod-mset* (*prime-factorization y*) **by** *simp*
hence *normalize* (*prod-mset* (*prime-factorization x*)) =
normalize (*prod-mset* (*prime-factorization y*))
by (*simp only*:)
with *assms* **show** *normalize x* = *normalize y*
by (*simp add*: *prod-mset-prime-factorization-weak*)
qed (*rule prime-factorization-cong*)

lemma *prime-factorization-normalize* [*simp*]:
prime-factorization (*normalize x*) = *prime-factorization x*
by (*cases x* = *0*, *simp*, *subst prime-factorization-unique*) *auto*

lemma *prime-factorization-eqI-strong*:
assumes $\bigwedge p.\ p \in\#\ P \Longrightarrow prime\ p$ *prod-mset P* = *n*
shows *prime-factorization n* = *P*
using *prime-factorization-prod-mset-primes*[*of P*] *assms* **by** *simp*

lemma *prime-factorization-eqI*:
assumes $\bigwedge p.\ p \in\#\ P \Longrightarrow prime\ p$ *normalize* (*prod-mset P*) = *normalize n*
shows *prime-factorization n* = *P*
proof −
have *P* = *prime-factorization* (*normalize* (*prod-mset P*))
using *prime-factorization-prod-mset-primes*[*of P*] *assms*(*1*) **by** *simp*
with *assms*(*2*) **show** *?thesis* **by** *simp*

qed

lemma *prime-factorization-mult*:
assumes $x \neq 0$ $y \neq 0$
shows *prime-factorization* $(x * y)$ = *prime-factorization* x + *prime-factorization* y
proof −
have *normalize* (*prod-mset* (*prime-factorization* x) $*$ *prod-mset* (*prime-factorization* y)) =
normalize (*normalize* (*prod-mset* (*prime-factorization* x)) $*$
normalize (*prod-mset* (*prime-factorization* y)))
by (*simp only*: *normalize-mult-normalize-left normalize-mult-normalize-right*)
also have ... = *normalize* $(x * y)$
by (*subst* (*1 2*) *prod-mset-prime-factorization-weak*) (*use assms* **in** *auto*)
finally show *?thesis*
by (*intro prime-factorization-eqI*) *auto*
qed

lemma *prime-factorization-prod*:
assumes *finite A* $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0$
shows *prime-factorization* (*prod f A*) = ($\sum n \in A$. *prime-factorization* (*f n*))
using *assms* **by** (*induction A rule*: *finite-induct*)
(*auto simp*: *Sup-multiset-empty prime-factorization-mult*)

lemma *prime-elem-multiplicity-mult-distrib*:
assumes *prime-elem p* $x \neq 0$ $y \neq 0$
shows *multiplicity p* $(x * y)$ = *multiplicity p x* + *multiplicity p y*
proof −
have *multiplicity p* $(x * y)$ = *count* (*prime-factorization* $(x * y)$) (*normalize p*)
by (*subst count-prime-factorization-prime*) (*simp-all add*: *assms*)
also from *assms*
have *prime-factorization* $(x * y)$ = *prime-factorization* x + *prime-factorization* y
by (*intro prime-factorization-mult*)
also have *count* ... (*normalize p*) =
count (*prime-factorization x*) (*normalize p*) + *count* (*prime-factorization y*) (*normalize p*)
by *simp*
also have ... = *multiplicity p x* + *multiplicity p y*
by (*subst* (*1 2*) *count-prime-factorization-prime*) (*simp-all add*: *assms*)
finally show *?thesis* **.**
qed

lemma *prime-elem-multiplicity-prod-mset-distrib*:
assumes *prime-elem p* $0 \notin\# A$
shows *multiplicity p* (*prod-mset A*) = *sum-mset* (*image-mset* (*multiplicity p*) *A*)
using *assms* **by** (*induction A*) (*auto simp*: *prime-elem-multiplicity-mult-distrib*)

lemma *prime-elem-multiplicity-power-distrib*:
  assumes *prime-elem p x* $\neq$ *0*
  shows *multiplicity p* $(x \,\widehat{}\, n) = n * $ *multiplicity p x*
  using *assms prime-elem-multiplicity-prod-mset-distrib* [*of p replicate-mset n x*]
  by *simp*

lemma *prime-elem-multiplicity-prod-distrib*:
  assumes *prime-elem p 0* $\notin$ *f ‘ A finite A*
  shows *multiplicity p* (*prod f A*) = ($\sum$ *x*$\in$*A*. *multiplicity p* (*f x*))
proof −
  have *multiplicity p* (*prod f A*) = ($\sum$ *x*$\in$#*mset-set A*. *multiplicity p* (*f x*))
    using *assms* **by** (*subst prod-unfold-prod-mset*)
        (*simp-all add*: *prime-elem-multiplicity-prod-mset-distrib sum-unfold-sum-mset*
          multiset.map-comp o-def)
  also from ‹*finite A*› **have** ... = ($\sum$ *x*$\in$*A*. *multiplicity p* (*f x*))
    by (*induction A rule*: *finite-induct*) *simp-all*
  finally show *?thesis* **.**
qed

lemma *multiplicity-distinct-prime-power*:
  prime p $\Longrightarrow$ *prime q* $\Longrightarrow$ $p \neq q$ $\Longrightarrow$ *multiplicity p* $(q \,\widehat{}\, n) = 0$
 by (*subst prime-elem-multiplicity-power-distrib*) (*auto simp*: *prime-multiplicity-other*)

lemma *prime-factorization-prime-power*:
  prime p $\Longrightarrow$ *prime-factorization* $(p \,\widehat{}\, n) =$ *replicate-mset n p*
  by (*induction n*)
   (*simp-all add*: *prime-factorization-mult prime-factorization-prime Multiset.union-commute*)

lemma *prime-factorization-subset-iff-dvd*:
  assumes [*simp*]: $x \neq 0$ $y \neq 0$
  shows *prime-factorization x* $\subseteq$# *prime-factorization y* $\longleftrightarrow$ *x dvd y*
proof −
  have *x dvd y* $\longleftrightarrow$
   normalize (*prod-mset* (*prime-factorization x*)) *dvd normalize* (*prod-mset* (*prime-factorization*
y))
    using *assms* **by** (*subst* (*1 2*) *prod-mset-prime-factorization-weak*) *auto*
  also have ... $\longleftrightarrow$ *prime-factorization x* $\subseteq$# *prime-factorization y*
    by (*auto intro*!: *prod-mset-primes-dvd-imp-subset prod-mset-subset-imp-dvd*)
  finally show *?thesis* **..**
qed

lemma *prime-factorization-subset-imp-dvd*:
  $x \neq 0$ $\Longrightarrow$ (*prime-factorization x* $\subseteq$# *prime-factorization y*) $\Longrightarrow$ *x dvd y*
  by (*cases y = 0*) (*simp-all add*: *prime-factorization-subset-iff-dvd*)

lemma *prime-factorization-divide*:
  assumes *b dvd a*
 shows *prime-factorization* (*a div b*) = *prime-factorization a* − *prime-factorization*
b

proof (*cases a* = *0*)
  case [*simp*]: *False*
  from *assms* **have** [*simp*]: *b* ≠ *0* **by** *auto*
   have *prime-factorization* ((*a div b*) ∗ *b*) = *prime-factorization* (*a div b*) + *prime-factorization b*
    by (*intro prime-factorization-mult*) (*insert assms*, *auto elim*!: *dvdE*)
  with *assms* **show** *?thesis* **by** *simp*
qed *simp-all*

lemma *zero-not-in-prime-factors* [*simp*]: *0* ∉ *prime-factors x*
  by (*auto dest*: *in-prime-factors-imp-prime*)

lemma *prime-prime-factors*:
  prime p ⟹ *prime-factors p* = {*p*}
  by (*drule prime-factorization-prime*) *simp*

lemma *prime-factors-product*:
  x ≠ *0* ⟹ *y* ≠ *0* ⟹ *prime-factors* (*x* ∗ *y*) = *prime-factors x* ∪ *prime-factors y*
  by (*simp add*: *prime-factorization-mult*)

lemma *dvd-prime-factors* [*intro*]:
  y ≠ *0* ⟹ *x dvd y* ⟹ *prime-factors x* ⊆ *prime-factors y*
  by (*intro set-mset-mono*, *subst prime-factorization-subset-iff-dvd*) *auto*

lemma *multiplicity-le-imp-dvd*:
  assumes *x* ≠ *0* ⋀*p*. *prime p* ⟹ *multiplicity p x* ≤ *multiplicity p y*
  shows *x dvd y*
proof (*cases y* = *0*)
  case *False*
  from *assms this* **have** *prime-factorization x* ⊆# *prime-factorization y*
    by (*intro mset-subset-eqI*) (*auto simp*: *count-prime-factorization*)
  with *assms False* **show** *?thesis* **by** (*subst* (*asm*) *prime-factorization-subset-iff-dvd*)
qed *auto*

lemma *dvd-multiplicity-eq*:
  x ≠ *0* ⟹ *y* ≠ *0* ⟹ *x dvd y* ⟷ (∀ *p*. *multiplicity p x* ≤ *multiplicity p y*)
  by (*auto intro*: *dvd-imp-multiplicity-le multiplicity-le-imp-dvd*)

lemma *multiplicity-eq-imp-eq*:
  assumes *x* ≠ *0 y* ≠ *0*
  assumes ⋀*p*. *prime p* ⟹ *multiplicity p x* = *multiplicity p y*
  shows *normalize x* = *normalize y*
  using *assms* **by** (*intro associatedI multiplicity-le-imp-dvd*) *simp-all*

lemma *prime-factorization-unique′*:
  assumes ∀ *p* ∈# *M*. *prime p* ∀ *p* ∈# *N*. *prime p* (∏ *i* ∈# *M*. *i*) = (∏ *i* ∈# *N*. *i*)
  shows *M* = *N*

proof −
  have *prime-factorization* ($\prod i \in\# M.\ i$) = *prime-factorization* ($\prod i \in\# N.\ i$)
    by (*simp only*: *assms*)
  also from *assms* **have** *prime-factorization* ($\prod i \in\# M.\ i$) = *M*
    by (*subst prime-factorization-prod-mset-primes*) *simp-all*
  also from *assms* **have** *prime-factorization* ($\prod i \in\# N.\ i$) = *N*
    by (*subst prime-factorization-prod-mset-primes*) *simp-all*
  finally show *?thesis* **.**
qed

lemma *prime-factorization-unique''*:
  assumes $\forall p \in\# M.$ *prime p* $\forall p \in\# N.$ *prime p normalize* ($\prod i \in\# M.\ i$) =
normalize ($\prod i \in\# N.\ i$)
  shows *M* = *N*
proof −
  have *prime-factorization* (*normalize* ($\prod i \in\# M.\ i$)) =
        prime-factorization (*normalize* ($\prod i \in\# N.\ i$))
    by (*simp only*: *assms*)
  also from *assms* **have** *prime-factorization* (*normalize* ($\prod i \in\# M.\ i$)) = *M*
   by (*subst prime-factorization-normalize*, *subst prime-factorization-prod-mset-primes*)
simp-all
  also from *assms* **have** *prime-factorization* (*normalize* ($\prod i \in\# N.\ i$)) = *N*
   by (*subst prime-factorization-normalize*, *subst prime-factorization-prod-mset-primes*)
simp-all
  finally show *?thesis* **.**
qed

lemma *multiplicity-cong*:
  ($\bigwedge r.$ *p* ^ *r dvd a* $\longleftrightarrow$ *p* ^ *r dvd b*) $\Longrightarrow$ *multiplicity p a* = *multiplicity p b*
  by (*simp add*: *multiplicity-def*)

lemma *not-dvd-imp-multiplicity-0*:
  assumes ¬*p dvd x*
  shows *multiplicity p x* = *0*
proof −
  from *assms* **have** *multiplicity p x* < *1*
    by (*intro multiplicity-lessI*) *auto*
  thus *?thesis* **by** *simp*
qed

lemma *multiplicity-zero-left* [*simp*]: *multiplicity 0 x* = *0*
 by (*cases x* = *0*) (*auto intro*: *not-dvd-imp-multiplicity-0*)

lemma *inj-on-Prod-primes*:
  assumes $\bigwedge P\ p.\ P \in A \Longrightarrow p \in P \Longrightarrow$ *prime p*
  assumes $\bigwedge P.\ P \in A \Longrightarrow$ *finite P*
  shows *inj-on Prod A*
proof (*rule inj-onI*)
  fix *P Q* **assume** *PQ*: $P \in A$ $Q \in A$ $\prod P = \prod Q$

with *prime-factorization-unique′*[*of mset-set P mset-set Q*] *assms*[*of P*] *assms*[*of Q*]
  have *mset-set P* = *mset-set Q* **by** (*auto simp*: *prod-unfold-prod-mset*)
  with *assms*[*of P*] *assms*[*of Q*] *PQ* **show** *P* = *Q* **by** *simp*
qed

lemma *divides-primepow-weak*:
 assumes *prime p* **and** *a dvd p ^ n*
 obtains *m* **where** *m* ≤ *n* **and** *normalize a* = *normalize* (*p ^ m*)
proof −
 from *assms* **have** *a* ≠ *0*
  by *auto*
 with *assms*
 have *normalize* (*prod-mset* (*prime-factorization a*)) *dvd*
    normalize (*prod-mset* (*prime-factorization* (*p ^ n*)))
  by (*subst* (*1 2*) *prod-mset-prime-factorization-weak*) *auto*
 then have *prime-factorization a* ⊆# *prime-factorization* (*p ^ n*)
  by (*simp add*: *in-prime-factors-imp-prime prod-mset-dvd-prod-mset-primes-iff*)
 with *assms* **have** *prime-factorization a* ⊆# *replicate-mset n p*
  by (*simp add*: *prime-factorization-prime-power*)
 then obtain *m* **where** *m* ≤ *n* **and** *prime-factorization a* = *replicate-mset m p*
  by (*rule msubseteq-replicate-msetE*)
 then have ∗: *normalize* (*prod-mset* (*prime-factorization a*)) =
     normalize (*prod-mset* (*replicate-mset m p*)) **by** *metis*
 also have *normalize* (*prod-mset* (*prime-factorization a*)) = *normalize a*
  using ‹*a* ≠ *0*› **by** (*simp add*: *prod-mset-prime-factorization-weak*)
 also have *prod-mset* (*replicate-mset m p*) = *p ^ m*
  by *simp*
 finally show *?thesis* **using** ‹*m* ≤ *n*›
  by (*intro that*[*of m*])
qed

lemma *divide-out-primepow-ex*:
 assumes *n* ≠ *0* ∃ *p*∈*prime-factors n*. *P p*
 obtains *p k n′* **where** *P p prime p p dvd n* ¬*p dvd n′ k* > *0 n* = *p ^ k* ∗ *n′*
proof −
 from *assms* **obtain** *p* **where** *p*: *P p prime p p dvd n*
  by *auto*
 define *k* **where** *k* = *multiplicity p n*
 define *n′* **where** *n′* = *n div p ^ k*
 have *n′*: *n* = *p ^ k* ∗ *n′* ¬*p dvd n′*
  using *assms p multiplicity-decompose*[*of n p*]
  by (*auto simp*: *n′-def k-def multiplicity-dvd*)
 from *n′ p* **have** *k* > *0* **by** (*intro Nat.gr0I*) *auto*
 with *n′ p that*[*of p n′ k*] **show** *?thesis* **by** *auto*
qed

lemma *divide-out-primepow*:
 assumes *n* ≠ *0* ¬*is-unit n*

obtains *p k n'* **where** *prime p p dvd n* $\neg$*p dvd n' k* > *0 n* = *p* ^ *k* ∗ *n'*
using *divide-out-primepow-ex*[*OF assms*(*1*), *of* λ-. *True*] *prime-divisor-exists*[*OF assms*] *assms*
prime-factorsI **by** *metis*

1.5 GCD and LCM computation with unique factorizations

definition *gcd-factorial a b* = (*if a* = *0 then normalize b*
else if b = *0 then normalize a*
else normalize (*prod-mset* (*prime-factorization a* ∩# *prime-factorization b*)))

definition *lcm-factorial a b* = (*if a* = *0* ∨ *b* = *0 then 0*
else normalize (*prod-mset* (*prime-factorization a* ∪# *prime-factorization b*)))

definition *Gcd-factorial A* =
(*if A* ⊆ {*0*} *then 0 else normalize* (*prod-mset* (*Inf* (*prime-factorization* ‘ (*A* − {*0*})))))

definition *Lcm-factorial A* =
(*if A* = {} *then 1*
else if 0 ∉ *A* ∧ *subset-mset.bdd-above* (*prime-factorization* ‘ (*A* − {*0*})) *then*
normalize (*prod-mset* (*Sup* (*prime-factorization* ‘ *A*)))
else
0)

lemma *prime-factorization-gcd-factorial*:
assumes [*simp*]: *a* ≠ *0 b* ≠ *0*
shows *prime-factorization* (*gcd-factorial a b*) = *prime-factorization a* ∩# *prime-factorization b*
proof −
have *prime-factorization* (*gcd-factorial a b*) =
prime-factorization (*prod-mset* (*prime-factorization a* ∩# *prime-factorization b*))
by (*simp add*: *gcd-factorial-def*)
also have . . . = *prime-factorization a* ∩# *prime-factorization b*
by (*subst prime-factorization-prod-mset-primes*) *auto*
finally show *?thesis* **.**
qed

lemma *prime-factorization-lcm-factorial*:
assumes [*simp*]: *a* ≠ *0 b* ≠ *0*
shows *prime-factorization* (*lcm-factorial a b*) = *prime-factorization a* ∪# *prime-factorization b*
proof −
have *prime-factorization* (*lcm-factorial a b*) =
prime-factorization (*prod-mset* (*prime-factorization a* ∪# *prime-factorization b*))
by (*simp add*: *lcm-factorial-def*)
also have . . . = *prime-factorization a* ∪# *prime-factorization b*

by (*subst prime-factorization-prod-mset-primes*) *auto*
finally show *?thesis* **.**
qed

lemma *prime-factorization-Gcd-factorial*:
assumes ¬*A* ⊆ {*0*}
shows *prime-factorization* (*Gcd-factorial A*) = *Inf* (*prime-factorization* ‘ (*A* − {*0*}))
proof −
from *assms* **obtain** *x* **where** *x*: *x* ∈ *A* − {*0*} **by** *auto*
hence *Inf* (*prime-factorization* ‘ (*A* − {*0*})) ⊆# *prime-factorization x*
by (*intro subset-mset.cInf-lower*) *simp-all*
hence ∀ *y*. *y* ∈# *Inf* (*prime-factorization* ‘ (*A* − {*0*})) ⟶ *y* ∈ *prime-factors x*
by (*auto dest*: *mset-subset-eqD*)
with *in-prime-factors-imp-prime*[*of - x*]
have ∀ *p*. *p* ∈# *Inf* (*prime-factorization* ‘ (*A* − {*0*})) ⟶ *prime p* **by** *blast*
with *assms* **show** *?thesis*
by (*simp add*: *Gcd-factorial-def prime-factorization-prod-mset-primes*)
qed

lemma *prime-factorization-Lcm-factorial*:
assumes *0* ∉ *A subset-mset.bdd-above* (*prime-factorization* ‘ *A*)
shows *prime-factorization* (*Lcm-factorial A*) = *Sup* (*prime-factorization* ‘ *A*)
proof (*cases A* = {})
case *True*
hence *prime-factorization* ‘ *A* = {} **by** *auto*
also have *Sup* ... = {#} **by** (*simp add*: *Sup-multiset-empty*)
finally show *?thesis* **by** (*simp add*: *Lcm-factorial-def*)
next
case *False*
have ∀ *y*. *y* ∈# *Sup* (*prime-factorization* ‘ *A*) ⟶ *prime y*
by (*auto simp*: *in-Sup-multiset-iff assms*)
with *assms False* **show** *?thesis*
by (*simp add*: *Lcm-factorial-def prime-factorization-prod-mset-primes*)
qed

lemma *gcd-factorial-commute*: *gcd-factorial a b* = *gcd-factorial b a*
by (*simp add*: *gcd-factorial-def multiset-inter-commute*)

lemma *gcd-factorial-dvd1*: *gcd-factorial a b dvd a*
proof (*cases a* = *0* ∨ *b* = *0*)
case *False*
hence *gcd-factorial a b* ≠ *0* **by** (*auto simp*: *gcd-factorial-def*)
with *False* **show** *?thesis*
by (*subst prime-factorization-subset-iff-dvd* [*symmetric*])
(*auto simp*: *prime-factorization-gcd-factorial*)
qed (*auto simp*: *gcd-factorial-def*)

lemma *gcd-factorial-dvd2*: *gcd-factorial a b dvd b*

by (*subst gcd-factorial-commute*) (*rule gcd-factorial-dvd1*)

lemma *normalize-gcd-factorial* [*simp*]: *normalize* (*gcd-factorial a b*) = *gcd-factorial a b*
by (*simp add*: *gcd-factorial-def*)

lemma *normalize-lcm-factorial* [*simp*]: *normalize* (*lcm-factorial a b*) = *lcm-factorial a b*
by (*simp add*: *lcm-factorial-def*)

lemma *gcd-factorial-greatest*: *c dvd gcd-factorial a b* **if** *c dvd a c dvd b* **for** *a b c*
proof (*cases a* = *0* ∨ *b* = *0*)
case *False*
with *that* **have** [*simp*]: *c* ≠ *0* **by** *auto*
let *?p* = *prime-factorization*
from *that False* **have** *?p c* ⊆# *?p a ?p c* ⊆# *?p b*
by (*simp-all add*: *prime-factorization-subset-iff-dvd*)
hence *prime-factorization c* ⊆#
prime-factorization (*prod-mset* (*prime-factorization a* ∩# *prime-factorization b*))
using *False* **by** (*subst prime-factorization-prod-mset-primes*) *auto*
with *False* **show** *?thesis*
by (*auto simp*: *gcd-factorial-def prime-factorization-subset-iff-dvd* [*symmetric*])
qed (*auto simp*: *gcd-factorial-def that*)

lemma *lcm-factorial-gcd-factorial*:
lcm-factorial a b = *normalize* (*a* ∗ *b div gcd-factorial a b*) **for** *a b*
proof (*cases a* = *0* ∨ *b* = *0*)
case *False*
let *?p* = *prime-factorization*
have *1*: *normalize x* ∗ *normalize y dvd z* ⟷ *x* ∗ *y dvd z* **for** *x y z* :: ′*a*
proof −
have *normalize* (*normalize x* ∗ *normalize y*) *dvd z* ⟷ *x* ∗ *y dvd z*
unfolding *normalize-mult-normalize-left normalize-mult-normalize-right* **by** *simp*
thus *?thesis* **unfolding** *normalize-dvd-iff* **by** *simp*
qed

have *?p* (*a* ∗ *b*) = (*?p a* ∪# *?p b*) + (*?p a* ∩# *?p b*)
using *False* **by** (*subst prime-factorization-mult*) (*auto intro*!: *multiset-eqI*)
hence *normalize* (*prod-mset* (*?p* (*a* ∗ *b*))) =
normalize (*prod-mset* ((*?p a* ∪# *?p b*) + (*?p a* ∩# *?p b*)))
by (*simp only*:)
hence ∗: *normalize* (*a* ∗ *b*) = *normalize* (*lcm-factorial a b* ∗ *gcd-factorial a b*) **using** *False*
by (*subst* (*asm*) *prod-mset-prime-factorization-weak*)
(*auto simp*: *lcm-factorial-def gcd-factorial-def*)

have [*simp*]: *gcd-factorial a b dvd a* ∗ *b lcm-factorial a b dvd a* ∗ *b*

using *associatedD2*[*OF* ∗] **by** *auto*
from *False* **have** [*simp*]: *gcd-factorial a b* ≠ *0 lcm-factorial a b* ≠ *0*
by (*auto simp*: *gcd-factorial-def lcm-factorial-def*)

show *?thesis*
by (*rule associated-eqI*)
(*use* ∗ **in** ‹*auto simp*: *dvd-div-iff-mult div-dvd-iff-mult dest*: *associatedD1 associatedD2*›)
qed (*auto simp*: *lcm-factorial-def*)

lemma *normalize-Gcd-factorial*:
normalize (*Gcd-factorial A*) = *Gcd-factorial A*
by (*simp add*: *Gcd-factorial-def*)

lemma *Gcd-factorial-eq-0-iff*:
Gcd-factorial A = *0* ⟷ *A* ⊆ {*0*}
by (*auto simp*: *Gcd-factorial-def in-Inf-multiset-iff split*: *if-splits*)

lemma *Gcd-factorial-dvd*:
assumes *x* ∈ *A*
shows *Gcd-factorial A dvd x*
proof (*cases x* = *0*)
case *False*
with *assms* **have** *prime-factorization* (*Gcd-factorial A*) = *Inf* (*prime-factorization* ‘ (*A* − {*0*}))
by (*intro prime-factorization-Gcd-factorial*) *auto*
also from *False assms* **have** ... ⊆# *prime-factorization x*
by (*intro subset-mset.cInf-lower*) *auto*
finally show *?thesis*
by (*subst* (*asm*) *prime-factorization-subset-iff-dvd*)
(*insert assms False*, *auto simp*: *Gcd-factorial-eq-0-iff*)
qed *simp-all*

lemma *Gcd-factorial-greatest*:
assumes ⋀*y*. *y* ∈ *A* ⟹ *x dvd y*
shows *x dvd Gcd-factorial A*
proof (*cases A* ⊆ {*0*})
case *False*
from *False* **obtain** *y* **where** *y* ∈ *A y* ≠ *0* **by** *auto*
with *assms*[*of y*] **have** *nz*: *x* ≠ *0* **by** *auto*
from *nz assms* **have** *prime-factorization x* ⊆# *prime-factorization y* **if** *y* ∈ *A* − {*0*} **for** *y*
using *that* **by** (*subst prime-factorization-subset-iff-dvd*) *auto*
with *False* **have** *prime-factorization x* ⊆# *Inf* (*prime-factorization* ‘ (*A* − {*0*}))
by (*intro subset-mset.cInf-greatest*) *auto*
also from *False* **have** ... = *prime-factorization* (*Gcd-factorial A*)
by (*rule prime-factorization-Gcd-factorial* [*symmetric*])
finally show *?thesis*
by (*subst* (*asm*) *prime-factorization-subset-iff-dvd*)

(*insert nz False*, *auto simp*: *Gcd-factorial-eq-0-iff*)
qed (*simp-all add*: *Gcd-factorial-def*)

lemma *normalize-Lcm-factorial*:
normalize (*Lcm-factorial A*) = *Lcm-factorial A*
by (*simp add*: *Lcm-factorial-def*)

lemma *Lcm-factorial-eq-0-iff*:
Lcm-factorial A = *0* ⟷ *0* ∈ *A* ∨ ¬*subset-mset.bdd-above* (*prime-factorization* ‘ *A*)
by (*auto simp*: *Lcm-factorial-def in-Sup-multiset-iff*)

lemma *dvd-Lcm-factorial*:
assumes *x* ∈ *A*
shows *x dvd Lcm-factorial A*
proof (*cases 0* ∉ *A* ∧ *subset-mset.bdd-above* (*prime-factorization* ‘ *A*))
case *True*
with *assms* **have** [*simp*]: *0* ∉ *A x* ≠ *0 A* ≠ {} **by** *auto*
from *assms True* **have** *prime-factorization x* ⊆# *Sup* (*prime-factorization* ‘ *A*)
by (*intro subset-mset.cSup-upper*) *auto*
also have . . . = *prime-factorization* (*Lcm-factorial A*)
by (*rule prime-factorization-Lcm-factorial* [*symmetric*]) (*insert True*, *simp-all*)
finally show *?thesis*
by (*subst* (*asm*) *prime-factorization-subset-iff-dvd*)
(*insert True*, *auto simp*: *Lcm-factorial-eq-0-iff*)
qed (*insert assms*, *auto simp*: *Lcm-factorial-def*)

lemma *Lcm-factorial-least*:
assumes ⋀*y*. *y* ∈ *A* ⟹ *y dvd x*
shows *Lcm-factorial A dvd x*
proof −
consider *A* = {} | *0* ∈ *A* | *x* = *0* | *A* ≠ {} *0* ∉ *A x* ≠ *0* **by** *blast*
thus *?thesis*
proof *cases*
assume ∗: *A* ≠ {} *0* ∉ *A x* ≠ *0*
hence *nz*: *x* ≠ *0* **if** *x* ∈ *A* **for** *x* **using** *that* **by** *auto*
from ∗ **have** *bdd*: *subset-mset.bdd-above* (*prime-factorization* ‘ *A*)
by (*intro subset-mset.bdd-aboveI*[*of - prime-factorization x*])
(*auto simp*: *prime-factorization-subset-iff-dvd nz dest*: *assms*)
have *prime-factorization* (*Lcm-factorial A*) = *Sup* (*prime-factorization* ‘ *A*)
by (*rule prime-factorization-Lcm-factorial*) *fact+*
also from ∗ **have** . . . ⊆# *prime-factorization x*
by (*intro subset-mset.cSup-least*)
(*auto simp*: *prime-factorization-subset-iff-dvd nz dest*: *assms*)
finally show *?thesis*
by (*subst* (*asm*) *prime-factorization-subset-iff-dvd*)
(*insert* ∗ *bdd*, *auto simp*: *Lcm-factorial-eq-0-iff*)
qed (*auto simp*: *Lcm-factorial-def dest*: *assms*)
qed

lemmas *gcd-lcm-factorial* =
gcd-factorial-dvd1 gcd-factorial-dvd2 gcd-factorial-greatest
normalize-gcd-factorial lcm-factorial-gcd-factorial
normalize-Gcd-factorial Gcd-factorial-dvd Gcd-factorial-greatest
normalize-Lcm-factorial dvd-Lcm-factorial Lcm-factorial-least

end

class *factorial-semiring-gcd* = *factorial-semiring* + *gcd* + *Gcd* +
assumes *gcd-eq-gcd-factorial*: *gcd a b* = *gcd-factorial a b*
and *lcm-eq-lcm-factorial*: *lcm a b* = *lcm-factorial a b*
and *Gcd-eq-Gcd-factorial*: *Gcd A* = *Gcd-factorial A*
and *Lcm-eq-Lcm-factorial*: *Lcm A* = *Lcm-factorial A*
begin

lemma *prime-factorization-gcd*:
assumes [*simp*]: $a \neq 0$ $b \neq 0$
shows *prime-factorization* (*gcd a b*) = *prime-factorization a* ∩# *prime-factorization b*
by (*simp add*: *gcd-eq-gcd-factorial prime-factorization-gcd-factorial*)

lemma *prime-factorization-lcm*:
assumes [*simp*]: $a \neq 0$ $b \neq 0$
shows *prime-factorization* (*lcm a b*) = *prime-factorization a* ∪# *prime-factorization b*
by (*simp add*: *lcm-eq-lcm-factorial prime-factorization-lcm-factorial*)

lemma *prime-factorization-Gcd*:
assumes *Gcd A* $\neq 0$
shows *prime-factorization* (*Gcd A*) = *Inf* (*prime-factorization* ‘ ($A - \{0\}$))
using *assms*
by (*simp add*: *prime-factorization-Gcd-factorial Gcd-eq-Gcd-factorial Gcd-factorial-eq-0-iff*)

lemma *prime-factorization-Lcm*:
assumes *Lcm A* $\neq 0$
shows *prime-factorization* (*Lcm A*) = *Sup* (*prime-factorization* ‘ *A*)
using *assms*
by (*simp add*: *prime-factorization-Lcm-factorial Lcm-eq-Lcm-factorial Lcm-factorial-eq-0-iff*)

lemma *prime-factors-gcd* [*simp*]:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *prime-factors* (*gcd a b*) =
prime-factors a ∩ *prime-factors b*
by (*subst prime-factorization-gcd*) *auto*

lemma *prime-factors-lcm* [*simp*]:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *prime-factors* (*lcm a b*) =
prime-factors a ∪ *prime-factors b*
by (*subst prime-factorization-lcm*) *auto*

subclass *semiring-gcd*
  by (*standard*, *unfold gcd-eq-gcd-factorial lcm-eq-lcm-factorial*)
    (*rule gcd-lcm-factorial*; *assumption*)+

subclass *semiring-Gcd*
  by (*standard*, *unfold Gcd-eq-Gcd-factorial Lcm-eq-Lcm-factorial*)
    (*rule gcd-lcm-factorial*; *assumption*)+

lemma
  assumes $x \neq 0$ $y \neq 0$
  shows *gcd-eq-factorial′*:
      gcd x y = *normalize* ($\prod$ *p* ∈ *prime-factors x* ∩ *prime-factors y*.
          p ^ *min* (*multiplicity p x*) (*multiplicity p y*)) (**is** - = *?rhs1*)
   and *lcm-eq-factorial′*:
      lcm x y = *normalize* ($\prod$ *p* ∈ *prime-factors x* ∪ *prime-factors y*.
          p ^ *max* (*multiplicity p x*) (*multiplicity p y*)) (**is** - = *?rhs2*)
proof −
  have *gcd x y* = *gcd-factorial x y* **by** (*rule gcd-eq-gcd-factorial*)
  also have ... = *?rhs1*
   by (*auto simp*: *gcd-factorial-def assms prod-mset-multiplicity*
      count-prime-factorization-prime
      intro!: *arg-cong*[*of* - - *normalize*] *dest*: *in-prime-factors-imp-prime intro*!:
prod.cong)
  finally show *gcd x y* = *?rhs1* **.**
  have *lcm x y* = *lcm-factorial x y* **by** (*rule lcm-eq-lcm-factorial*)
  also have ... = *?rhs2*
   by (*auto simp*: *lcm-factorial-def assms prod-mset-multiplicity*
      count-prime-factorization-prime intro!: *arg-cong*[*of* - - *normalize*]
      dest: *in-prime-factors-imp-prime intro*!: *prod.cong*)
  finally show *lcm x y* = *?rhs2* **.**
qed

lemma
  assumes $x \neq 0$ $y \neq 0$ *prime p*
  shows *multiplicity-gcd*: *multiplicity p* (*gcd x y*) = *min* (*multiplicity p x*)
(*multiplicity p y*)
   and *multiplicity-lcm*: *multiplicity p* (*lcm x y*) = *max* (*multiplicity p x*)
(*multiplicity p y*)
proof −
  have *gcd x y* = *gcd-factorial x y* **by** (*rule gcd-eq-gcd-factorial*)
  also from *assms* **have** *multiplicity p* ... = *min* (*multiplicity p x*) (*multiplicity p*
y)
   by (*simp add*: *count-prime-factorization-prime* [*symmetric*] *prime-factorization-gcd-factorial*)
  finally show *multiplicity p* (*gcd x y*) = *min* (*multiplicity p x*) (*multiplicity p y*) **.**
  have *lcm x y* = *lcm-factorial x y* **by** (*rule lcm-eq-lcm-factorial*)
  also from *assms* **have** *multiplicity p* ... = *max* (*multiplicity p x*) (*multiplicity*
p y)
   by (*simp add*: *count-prime-factorization-prime* [*symmetric*] *prime-factorization-lcm-factorial*)

finally show *multiplicity p* (*lcm x y*) = *max* (*multiplicity p x*) (*multiplicity p y*)
.
qed

lemma *gcd-lcm-distrib*:
gcd x (*lcm y z*) = *lcm* (*gcd x y*) (*gcd x z*)
proof (*cases x* = *0* ∨ *y* = *0* ∨ *z* = *0*)
case *True*
thus *?thesis*
by (*auto simp*: *lcm-proj1-if-dvd lcm-proj2-if-dvd*)
next
case *False*
hence *normalize* (*gcd x* (*lcm y z*)) = *normalize* (*lcm* (*gcd x y*) (*gcd x z*))
by (*intro associatedI prime-factorization-subset-imp-dvd*)
(*auto simp*: *lcm-eq-0-iff prime-factorization-gcd prime-factorization-lcm subset-mset.inf-sup-distrib1*)
thus *?thesis* **by** *simp*
qed

lemma *lcm-gcd-distrib*:
lcm x (*gcd y z*) = *gcd* (*lcm x y*) (*lcm x z*)
proof (*cases x* = *0* ∨ *y* = *0* ∨ *z* = *0*)
case *True*
thus *?thesis*
by (*auto simp*: *lcm-proj1-if-dvd lcm-proj2-if-dvd*)
next
case *False*
hence *normalize* (*lcm x* (*gcd y z*)) = *normalize* (*gcd* (*lcm x y*) (*lcm x z*))
by (*intro associatedI prime-factorization-subset-imp-dvd*)
(*auto simp*: *lcm-eq-0-iff prime-factorization-gcd prime-factorization-lcm subset-mset.sup-inf-distrib1*)
thus *?thesis* **by** *simp*
qed

end

class *factorial-ring-gcd* = *factorial-semiring-gcd* + *idom*
begin

subclass *ring-gcd* **..**

subclass *idom-divide* **..**

end

class *factorial-semiring-multiplicative* =
factorial-semiring + *normalization-semidom-multiplicative*
begin

lemma *normalize-prod-mset-primes*:
  ($\bigwedge p.\ p \in\# A \Longrightarrow prime\ p) \Longrightarrow normalize\ (prod\text{-}mset\ A) = prod\text{-}mset\ A$
proof (*induction A*)
  case (*add p A*)
  hence *prime p* **by** *simp*
  hence *normalize p = p* **by** *simp*
  with *add* **show** *?case* **by** (*simp add*: *normalize-mult*)
qed *simp-all*

lemma *prod-mset-prime-factorization*:
  assumes $x \neq 0$
  shows *prod-mset* (*prime-factorization x*) = *normalize x*
  using *assms*
  by (*induction x rule*: *prime-divisors-induct*)
    (*simp-all add*: *prime-factorization-unit prime-factorization-times-prime*
      is-unit-normalize normalize-mult)

lemma *prime-decomposition*: *unit-factor x* ∗ *prod-mset* (*prime-factorization x*) =
x
  by (*cases x = 0*) (*simp-all add*: *prod-mset-prime-factorization*)

lemma *prod-prime-factors*:
  assumes $x \neq 0$
  shows ($\prod p \in prime\text{-}factors\ x.\ p$ ^ *multiplicity p x*) = *normalize x*
proof −
  have *normalize x* = *prod-mset* (*prime-factorization x*)
    by (*simp add*: *prod-mset-prime-factorization assms*)
  also have ... = ($\prod p \in prime\text{-}factors\ x.\ p$ ^ *count* (*prime-factorization x*) *p*)
    by (*subst prod-mset-multiplicity*) *simp-all*
  also have ... = ($\prod p \in prime\text{-}factors\ x.\ p$ ^ *multiplicity p x*)
    by (*intro prod.cong*)
    (*simp-all add*: *assms count-prime-factorization-prime in-prime-factors-imp-prime*)
  finally show *?thesis* **..**
qed

lemma *prime-factorization-unique''*:
  assumes *S-eq*: $S = \{p.\ 0 < f\ p\}$
    and *finite S*
    and *S*: $\forall p{\in}S.\ prime\ p$ *normalize n* = ($\prod p{\in}S.\ p$ ^ *f p*)
  shows $S = prime\text{-}factors\ n \wedge (\forall p.\ prime\ p \longrightarrow f\ p = multiplicity\ p\ n)$
proof
  define *A* **where** *A* = *Abs-multiset f*
  from ‹*finite S*› *S*(*1*) **have** ($\prod p{\in}S.\ p$ ^ *f p*) $\neq 0$ **by** *auto*
  with *S*(*2*) **have** *nz*: $n \neq 0$ **by** *auto*
  from *S-eq* ‹*finite S*› **have** *count-A*: *count A = f*
    unfolding *A-def* **by** (*subst multiset.Abs-multiset-inverse*) *simp-all*
  from *S-eq count-A* **have** *set-mset-A*: *set-mset A = S*
    by (*simp only*: *set-mset-def*)

from $S(2)$ **have** *normalize* $n = (\prod p{\in}S.\ p\ \widehat{}\ f\ p)$ **.**
also have ... = *prod-mset A* **by** (*simp add*: *prod-mset-multiplicity S-eq set-mset-A count-A*)
also from *nz* **have** *normalize n* = *prod-mset* (*prime-factorization n*)
by (*simp add*: *prod-mset-prime-factorization*)
finally have *prime-factorization* (*prod-mset A*) =
prime-factorization (*prod-mset* (*prime-factorization n*)) **by** *simp*
also from $S(1)$ **have** *prime-factorization* (*prod-mset A*) = *A*
by (*intro prime-factorization-prod-mset-primes*) (*auto simp*: *set-mset-A*)
also have *prime-factorization* (*prod-mset* (*prime-factorization n*)) = *prime-factorization n*
by (*intro prime-factorization-prod-mset-primes*) *auto*
finally show *S* = *prime-factors n* **by** (*simp add*: *set-mset-A* [*symmetric*])

show ($\forall$ *p*. *prime p* $\longrightarrow$ *f p* = *multiplicity p n*)
proof *safe*
fix *p* :: *'a* **assume** *p*: *prime p*
have *multiplicity p n* = *multiplicity p* (*normalize n*) **by** *simp*
also have *normalize n* = *prod-mset A*
by (*simp add*: *prod-mset-multiplicity S-eq set-mset-A count-A S*)
also from *p set-mset-A* $S(1)$
have *multiplicity p* ... = *sum-mset* (*image-mset* (*multiplicity p*) *A*)
by (*intro prime-elem-multiplicity-prod-mset-distrib*) *auto*
also from $S(1)$ *p*
have *image-mset* (*multiplicity p*) *A* = *image-mset* (λq. *if p* = *q then 1 else 0*) *A*
by (*intro image-mset-cong*) (*auto simp*: *set-mset-A multiplicity-self prime-multiplicity-other*)
also have *sum-mset* ... = *f p*
by (*simp add*: *semiring-1-class.sum-mset-delta'* *count-A*)
finally show *f p* = *multiplicity p n* **..**
qed
qed

lemma *divides-primepow*:
assumes *prime p* **and** *a dvd p* ^ *n*
obtains *m* **where** $m \le n$ **and** *normalize a* = *p* ^ *m*
using *divides-primepow-weak*[*OF assms*] *that assms*
by (*auto simp add*: *normalize-power*)

lemma *Ex-other-prime-factor*:
assumes $n \neq 0$ **and** $\neg(\exists k.$ *normalize n* = *p* ^ *k*) *prime p*
shows $\exists q{\in}$*prime-factors n*. $q \neq p$
proof (*rule ccontr*)
assume $*$: $\neg(\exists q{\in}$*prime-factors n*. $q \neq p)$
have *normalize n* = ($\prod p{\in}$*prime-factors n*. *p* ^ *multiplicity p n*)
using *assms*(*1*) **by** (*intro prod-prime-factors* [*symmetric*]) *auto*
also from $*$ **have** ... = ($\prod p{\in}\{p\}$. *p* ^ *multiplicity p n*)
using *assms*(*3*) **by** (*intro prod.mono-neutral-left*) (*auto simp*: *prime-factors-multiplicity*)
finally have *normalize n* = *p* ^ *multiplicity p n* **by** *auto*

  with *assms* **show** *False* **by** *auto*
qed

Now a string of results due to Maya Kdzioka

lemma *multiplicity-dvd-iff-dvd*:
 assumes $x \neq 0$
 shows $p\hat{~}k$ *dvd* $x \longleftrightarrow p\hat{~}k$ *dvd* $p\hat{~}$*multiplicity* p x
proof (*cases is-unit p*)
 case *True*
 then have *is-unit* ($p\hat{~}k$)
  using *is-unit-power-iff* **by** *simp*
 hence $p\hat{~}k$ *dvd* x
  by *auto*
 moreover from ‹*is-unit p*› **have** $p\hat{~}k$ *dvd* $p\hat{~}$*multiplicity* p x
  using *multiplicity-unit-left is-unit-power-iff* **by** *simp*
 ultimately show *?thesis* **by** *simp*
next
 case *False*
 show *?thesis*
 proof (*cases* $p = 0$)
  case *True*
  then have $p\hat{~}$*multiplicity* p $x = 1$
   by *simp*
  moreover have $p\hat{~}k$ *dvd* $x \Longrightarrow k = 0$
  proof (*rule ccontr*)
   assume $p\hat{~}k$ *dvd* x **and** $k \neq 0$
   with ‹$p = 0$› **have** $p\hat{~}k = 0$ **by** *auto*
   with ‹$p\hat{~}k$ *dvd* x› **have** 0 *dvd* x **by** *auto*
   hence $x = 0$ **by** *auto*
   with ‹$x \neq 0$› **show** *False* **by** *auto*
  qed
  ultimately show *?thesis*
   by (*auto simp add*: *is-unit-power-iff* ‹¬ *is-unit p*›)
 next
  case *False*
  with ‹$x \neq 0$› ‹¬ *is-unit p*› **show** *?thesis*
   by (*simp add*: *power-dvd-iff-le-multiplicity dvd-power-iff multiplicity-same-power*)
 qed
qed

lemma *multiplicity-decomposeI*:
 assumes $x = p\hat{~}k * x'$ **and** ¬ p *dvd* x' **and** $p \neq 0$
 shows *multiplicity* p $x = k$
  using *assms local.multiplicity-eqI local.power-Suc2* **by** *force*

lemma *multiplicity-sum-lt*:
 assumes *multiplicity* p $a <$ *multiplicity* p b $a \neq 0$ $b \neq 0$
 shows *multiplicity* p $(a + b) =$ *multiplicity* p a
proof −

```
 let ?vp = multiplicity p
 have unit: ¬ is-unit p
 proof
  assume is-unit p
  then have ?vp a = 0 and ?vp b = 0 using multiplicity-unit-left by auto
  with assms show False by auto
 qed

 from multiplicity-decompose' obtain a' where a': a = p^?vp a * a' ¬ p dvd a'
  using unit assms by metis
 from multiplicity-decompose' obtain b' where b': b = p^?vp b * b'
  using unit assms by metis

 show ?vp (a + b) = ?vp a
 proof (rule multiplicity-decomposeI)
  let ?k = ?vp b − ?vp a
  from assms have k: ?k > 0 by simp
  with b' have b = p^?vp a * p^?k * b'
   by (simp flip: power-add)
  with a' show *: a + b = p^?vp a * (a' + p^?k * b')
   by (simp add: ac-simps distrib-left)
  moreover show ¬ p dvd a' + p^?k * b'
   using a' k dvd-add-left-iff by auto
  show p ≠ 0 using assms by auto
 qed
qed

corollary multiplicity-sum-min:
 assumes multiplicity p a ≠ multiplicity p b a ≠ 0 b ≠ 0
 shows multiplicity p (a + b) = min (multiplicity p a) (multiplicity p b)
proof −
 let ?vp = multiplicity p
 from assms have ?vp a < ?vp b ∨ ?vp a > ?vp b
  by auto
 then show ?thesis
  by (metis assms multiplicity-sum-lt min.commute add-commute min.strict-order-iff)

qed

end

lifting-update multiset.lifting
lifting-forget multiset.lifting

end
```

2 Abstract euclidean algorithm in euclidean (semi)rings

theory *Euclidean-Algorithm*

imports *Factorial-Ring*
begin

2.1 Generic construction of the (simple) euclidean algorithm

class *normalization-euclidean-semiring* = *euclidean-semiring* + *normalization-semidom*
begin

lemma *euclidean-size-normalize* [*simp*]:
euclidean-size (*normalize a*) = *euclidean-size a*
proof (*cases a* = *0*)
case *True*
then show *?thesis*
by *simp*
next
case [*simp*]: *False*
have *euclidean-size* (*normalize a*) ≤ *euclidean-size* (*normalize a* ∗ *unit-factor a*)
by (*rule size-mult-mono*) *simp*
moreover have *euclidean-size a* ≤ *euclidean-size* (*a* ∗ (*1 div unit-factor a*))
by (*rule size-mult-mono*) *simp*
ultimately show *?thesis*
by *simp*
qed

context
begin

qualified function *gcd* :: *'a* ⇒ *'a* ⇒ *'a*
where *gcd a b* = (*if b* = *0 then normalize a else gcd b* (*a mod b*))
by *pat-completeness simp*
termination
by (*relation measure* (*euclidean-size* ∘ *snd*)) (*simp-all add*: *mod-size-less*)

declare *gcd.simps* [*simp del*]

lemma *eucl-induct* [*case-names zero mod*]:
assumes *H1*: ⋀*b*. *P b 0*
and *H2*: ⋀*a b*. *b* ≠ *0* ⟹ *P b* (*a mod b*) ⟹ *P a b*
shows *P a b*
proof (*induct a b rule*: *gcd.induct*)
case (*1 a b*)
show *?case*
proof (*cases b* = *0*)
case *True* **then show** *P a b* **by** *simp* (*rule H1*)
next
case *False*
then have *P b* (*a mod b*)
by (*rule 1.hyps*)
with ‹*b* ≠ *0*› **show** *P a b*

by (*blast intro*: *H2*)
qed
qed

qualified lemma *gcd-0*:
gcd a 0 = *normalize a*
by (*simp add*: *gcd.simps* [*of a 0*])

qualified lemma *gcd-mod*:
a ≠ *0* ⟹ *gcd a* (*b mod a*) = *gcd b a*
by (*simp add*: *gcd.simps* [*of b 0*] *gcd.simps* [*of b a*])

qualified definition *lcm* :: *'a* ⇒ *'a* ⇒ *'a*
where *lcm a b* = *normalize* (*a* ∗ *b div gcd a b*)

qualified definition *Lcm* :: *'a set* ⇒ *'a* — Somewhat complicated definition of Lcm that has the advantage of working for infinite sets as well
where
[*code del*]: *Lcm A* = (*if* ∃ *l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*) *then*
let l = *SOME l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*) ∧ *euclidean-size l* =
(*LEAST n*. ∃ *l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*) ∧ *euclidean-size l* = *n*)
in normalize l
else 0)

qualified definition *Gcd* :: *'a set* ⇒ *'a*
where [*code del*]: *Gcd A* = *Lcm* {*d*. ∀ *a*∈*A*. *d dvd a*}

end

lemma *semiring-gcd*:
class.semiring-gcd one zero times gcd lcm
divide plus minus unit-factor normalize
proof
show *gcd a b dvd a*
and *gcd a b dvd b* **for** *a b*
by (*induct a b rule*: *eucl-induct*)
(*simp-all add*: *local.gcd-0 local.gcd-mod dvd-mod-iff*)
next
show *c dvd a* ⟹ *c dvd b* ⟹ *c dvd gcd a b* **for** *a b c*
proof (*induct a b rule*: *eucl-induct*)
case (*zero a*) **from** ‹*c dvd a*› **show** *?case*
by (*rule dvd-trans*) (*simp add*: *local.gcd-0*)
next
case (*mod a b*)
then show *?case*
by (*simp add*: *local.gcd-mod dvd-mod-iff*)
qed
next
show *normalize* (*gcd a b*) = *gcd a b* **for** *a b*

by (*induct a b rule*: *eucl-induct*)
(*simp-all add*: *local.gcd-0 local.gcd-mod*)
next
show *lcm a b* = *normalize* (*a* ∗ *b div gcd a b*) **for** *a b*
by (*fact local.lcm-def*)
qed

interpretation *semiring-gcd one zero times gcd lcm*
divide plus minus unit-factor normalize
by (*fact semiring-gcd*)

lemma *semiring-Gcd*:
class.semiring-Gcd one zero times gcd lcm Gcd Lcm
divide plus minus unit-factor normalize
proof −
show *?thesis*
proof
have (∀ *a*∈*A*. *a dvd Lcm A*) ∧ (∀ *b*. (∀ *a*∈*A*. *a dvd b*) ⟶ *Lcm A dvd b*) **for** *A*
proof (*cases* ∃ *l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*))
case *False*
then have *Lcm A* = *0*
by (*auto simp add*: *local.Lcm-def*)
with *False* **show** *?thesis*
by *auto*
next
case *True*
then obtain l_0 **where** l_0*-props*: l_0 ≠ *0* ∀ *a*∈*A*. *a dvd* l_0 **by** *blast*
define *n* **where** *n* = (*LEAST n*. ∃ *l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*) ∧ *euclidean-size l* = *n*)
define *l* **where** *l* = (*SOME l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*) ∧ *euclidean-size l* = *n*)
have ∃ *l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*) ∧ *euclidean-size l* = *n*
apply (*subst n-def*)
apply (*rule LeastI* [*of - euclidean-size* l_0])
apply (*rule exI* [*of -* l_0])
apply (*simp add*: l_0*-props*)
done
from *someI-ex* [*OF this*] **have** *l* ≠ *0* **and** ∀ *a*∈*A*. *a dvd l*
and *euclidean-size l* = *n*
unfolding *l-def* **by** *simp-all*
{
fix *l′* **assume** ∀ *a*∈*A*. *a dvd l′*
with ‹∀ *a*∈*A*. *a dvd l*› **have** ∀ *a*∈*A*. *a dvd gcd l l′*
by (*auto intro*: *gcd-greatest*)
moreover from ‹*l* ≠ *0*› **have** *gcd l l′* ≠ *0*
by *simp*
ultimately have ∃ *b*. *b* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd b*) ∧
euclidean-size b = *euclidean-size* (*gcd l l′*)
by (*intro exI* [*of - gcd l l′*], *auto*)

then have *euclidean-size* (*gcd l l'*) ≥ *n*
by (*subst n-def*) (*rule Least-le*)
moreover have *euclidean-size* (*gcd l l'*) ≤ *n*
proof −
have *gcd l l' dvd l*
by *simp*
then obtain *a* where *l* = *gcd l l'* ∗ *a* ..
with ‹*l* ≠ *0*› have *a* ≠ *0*
by *auto*
hence *euclidean-size* (*gcd l l'*) ≤ *euclidean-size* (*gcd l l'* ∗ *a*)
by (*rule size-mult-mono*)
also have *gcd l l'* ∗ *a* = *l* using ‹*l* = *gcd l l'* ∗ *a*› ..
also note ‹*euclidean-size l* = *n*›
finally show *euclidean-size* (*gcd l l'*) ≤ *n* .
qed
ultimately have ∗: *euclidean-size l* = *euclidean-size* (*gcd l l'*)
by (*intro le-antisym*, *simp-all add*: ‹*euclidean-size l* = *n*›)
from ‹*l* ≠ *0*› have *l dvd gcd l l'*
by (*rule dvd-euclidean-size-eq-imp-dvd*) (*auto simp add*: ∗)
hence *l dvd l'* by (*rule dvd-trans* [*OF - gcd-dvd2*])
}
with ‹∀ *a*∈*A*. *a dvd l*› and ‹*l* ≠ *0*›
have (∀ *a*∈*A*. *a dvd normalize l*) ∧
(∀ *l'*. (∀ *a*∈*A*. *a dvd l'*) ⟶ *normalize l dvd l'*)
by *auto*
also from *True* have *normalize l* = *Lcm A*
by (*simp add*: *local.Lcm-def Let-def n-def l-def*)
finally show *?thesis* .
qed
then show *dvd-Lcm*: *a* ∈ *A* ⟹ *a dvd Lcm A*
and *Lcm-least*: (⋀*a*. *a* ∈ *A* ⟹ *a dvd b*) ⟹ *Lcm A dvd b* for *A* and *a b*
by *auto*
show *a* ∈ *A* ⟹ *Gcd A dvd a* for *A* and *a*
by (*auto simp add*: *local.Gcd-def intro*: *Lcm-least*)
show (⋀*a*. *a* ∈ *A* ⟹ *b dvd a*) ⟹ *b dvd Gcd A* for *A* and *b*
by (*auto simp add*: *local.Gcd-def intro*: *dvd-Lcm*)
show [*simp*]: *normalize* (*Lcm A*) = *Lcm A* for *A*
by (*simp add*: *local.Lcm-def*)
show *normalize* (*Gcd A*) = *Gcd A* for *A*
by (*simp add*: *local.Gcd-def*)
qed
qed

interpretation *semiring-Gcd one zero times gcd lcm Gcd Lcm*
divide plus minus unit-factor normalize
by (*fact semiring-Gcd*)

subclass *factorial-semiring*
proof −

show *class.factorial-semiring divide plus minus zero times one unit-factor normalize*
proof (*standard*, *rule factorial-semiring-altI-aux*) — FIXME rule
  fix *x* **assume** $x \neq 0$
  thus *finite* {*p. p dvd x* ∧ *normalize p* = *p*}
  proof (*induction euclidean-size x arbitrary*: *x rule*: *less-induct*)
    case (*less x*)
    show *?case*
    proof (*cases* ∃ *y. y dvd x* ∧ ¬*x dvd y* ∧ ¬*is-unit y*)
      case *False*
      have {*p. p dvd x* ∧ *normalize p* = *p*} ⊆ {*1*, *normalize x*}
      proof
        fix *p* **assume** *p*: *p* ∈ {*p. p dvd x* ∧ *normalize p* = *p*}
        with *False* **have** *is-unit p* ∨ *x dvd p* **by** *blast*
        thus *p* ∈ {*1*, *normalize x*}
        proof (*elim disjE*)
          assume *is-unit p*
          hence *normalize p* = *1* **by** (*simp add*: *is-unit-normalize*)
          with *p* **show** *?thesis* **by** *simp*
        next
          assume *x dvd p*
          with *p* **have** *normalize p* = *normalize x* **by** (*intro associatedI*) *simp-all*
          with *p* **show** *?thesis* **by** *simp*
        qed
      qed
      moreover have *finite* ... **by** *simp*
      ultimately show *?thesis* **by** (*rule finite-subset*)
    next
      case *True*
      then obtain *y* **where** *y*: *y dvd x* ¬*x dvd y* ¬*is-unit y* **by** *blast*
      define *z* **where** *z* = *x div y*
      let *?fctrs* = λ*x*. {*p. p dvd x* ∧ *normalize p* = *p*}
      from *y* **have** *x*: *x* = *y* ∗ *z* **by** (*simp add*: *z-def*)
      with *less.prems* **have** *y* ≠ *0* *z* ≠ *0* **by** *auto*
      have *normalized-factors-product*:
        {*p. p dvd a* ∗ *b* ∧ *normalize p* = *p*} ⊆
          (λ(*x*,*y*). *normalize* (*x* ∗ *y*)) ‘ ({*p. p dvd a* ∧ *normalize p* = *p*} × {*p. p dvd b* ∧ *normalize p* = *p*})
        for *a b*
      proof *safe*
        fix *p* **assume** *p*: *p dvd a* ∗ *b normalize p* = *p*
        from *p*(*1*) **obtain** *x y* **where** *xy*: *p* = *x* ∗ *y x dvd a y dvd b*
          by (*rule dvd-productE*)
        define *x′ y′* **where** *x′* = *normalize x* **and** *y′* = *normalize y*
        have *p* = *normalize* (*x′* ∗ *y′*)
          using *p* **by** (*simp add*: *xy x′-def y′-def*)
        moreover have *x′ dvd a* ∧ *normalize x′* = *x′* **and** *y′ dvd b* ∧ *normalize y′* = *y′*
          using *xy* **by** (*auto simp*: *x′-def y′-def*)

ultimately show *p* ∈ (λ(*x*, *y*). *normalize* (*x* ∗ *y*)) ‘
({*p*. *p dvd a* ∧ *normalize p* = *p*} × {*p*. *p dvd b* ∧ *normalize p* = *p*})
by *fast*
qed
from *x y* **have** ¬*is-unit z* **by** (*auto simp*: *mult-unit-dvd-iff*)
have *?fctrs x* ⊆ (λ(*p*,*p*′). *normalize* (*p* ∗ *p*′)) ‘ (*?fctrs y* × *?fctrs z*)
by (*subst x*) (*rule normalized-factors-product*)
moreover have ¬*y* ∗ *z dvd y* ∗ *1* ¬*y* ∗ *z dvd 1* ∗ *z*
by (*subst dvd-times-left-cancel-iff dvd-times-right-cancel-iff*; *fact*)+
hence *finite* ((λ(*p*,*p*′). *normalize* (*p* ∗ *p*′)) ‘ (*?fctrs y* × *?fctrs z*))
by (*intro finite-imageI finite-cartesian-product less dvd-proper-imp-size-less*)
(*auto simp*: *x*)
ultimately show *?thesis* **by** (*rule finite-subset*)
qed
qed
next
fix *p*
assume *irreducible p*
then show *prime-elem p*
by (*rule irreducible-imp-prime-elem-gcd*)
qed
qed

lemma *Gcd-eucl-set* [*code*]:
Gcd (*set xs*) = *fold gcd xs 0*
by (*fact Gcd-set-eq-fold*)

lemma *Lcm-eucl-set* [*code*]:
Lcm (*set xs*) = *fold lcm xs 1*
by (*fact Lcm-set-eq-fold*)

end

hide-const (**open**) *gcd lcm Gcd Lcm*

lemma *prime-elem-int-abs-iff* [*simp*]:
fixes *p* :: *int*
shows *prime-elem* |*p*| ⟷ *prime-elem p*
using *prime-elem-normalize-iff* [*of p*] **by** *simp*

lemma *prime-elem-int-minus-iff* [*simp*]:
fixes *p* :: *int*
shows *prime-elem* (− *p*) ⟷ *prime-elem p*
using *prime-elem-normalize-iff* [*of* − *p*] **by** *simp*

lemma *prime-int-iff*:
fixes *p* :: *int*
shows *prime p* ⟷ *p* > *0* ∧ *prime-elem p*
by (*auto simp add*: *prime-def dest*: *prime-elem-not-zeroI*)

2.2 The (simple) euclidean algorithm as gcd computation

class *euclidean-semiring-gcd* = *normalization-euclidean-semiring* + *gcd* + *Gcd* +
  assumes *gcd-eucl*: *Euclidean-Algorithm.gcd* = *GCD.gcd*
    and *lcm-eucl*: *Euclidean-Algorithm.lcm* = *GCD.lcm*
  assumes *Gcd-eucl*: *Euclidean-Algorithm.Gcd* = *GCD.Gcd*
    and *Lcm-eucl*: *Euclidean-Algorithm.Lcm* = *GCD.Lcm*
begin

subclass *semiring-gcd*
  unfolding *gcd-eucl* [*symmetric*] *lcm-eucl* [*symmetric*]
  by (*fact semiring-gcd*)

subclass *semiring-Gcd*
  unfolding *gcd-eucl* [*symmetric*] *lcm-eucl* [*symmetric*]
    Gcd-eucl [*symmetric*] *Lcm-eucl* [*symmetric*]
  by (*fact semiring-Gcd*)

subclass *factorial-semiring-gcd*
proof
  show *gcd a b* = *gcd-factorial a b* **for** *a b*
    apply (*rule sym*)
    apply (*rule gcdI*)
      apply (*fact gcd-lcm-factorial*)+
    done
  then show *lcm a b* = *lcm-factorial a b* **for** *a b*
    by (*simp add*: *lcm-factorial-gcd-factorial lcm-gcd*)
  show *Gcd A* = *Gcd-factorial A* **for** *A*
    apply (*rule sym*)
    apply (*rule GcdI*)
      apply (*fact gcd-lcm-factorial*)+
    done
  show *Lcm A* = *Lcm-factorial A* **for** *A*
    apply (*rule sym*)
    apply (*rule LcmI*)
      apply (*fact gcd-lcm-factorial*)+
    done
qed

lemma *gcd-mod-right* [*simp*]:
  $a \neq 0 \Longrightarrow gcd\ a\ (b\ mod\ a) = gcd\ a\ b$
  unfolding *gcd.commute* [*of a b*]
  by (*simp add*: *gcd-eucl* [*symmetric*] *local.gcd-mod*)

lemma *gcd-mod-left* [*simp*]:
  $b \neq 0 \Longrightarrow gcd\ (a\ mod\ b)\ b = gcd\ a\ b$
  by (*drule gcd-mod-right* [*of - a*]) (*simp add*: *gcd.commute*)

lemma *euclidean-size-gcd-le1* [*simp*]:
  assumes $a \neq 0$

shows *euclidean-size* (*gcd a b*) ≤ *euclidean-size a*
proof −
from *gcd-dvd1* **obtain** *c* **where** *A*: *a* = *gcd a b* ∗ *c* **..**
with *assms* **have** *c* ≠ *0*
by *auto*
moreover from *this*
have *euclidean-size* (*gcd a b*) ≤ *euclidean-size* (*gcd a b* ∗ *c*)
by (*rule size-mult-mono*)
with *A* **show** *?thesis*
by *simp*
qed

lemma *euclidean-size-gcd-le2* [*simp*]:
b ≠ *0* =⇒ *euclidean-size* (*gcd a b*) ≤ *euclidean-size b*
by (*subst gcd.commute*, *rule euclidean-size-gcd-le1*)

lemma *euclidean-size-gcd-less1*:
assumes *a* ≠ *0* **and** ¬ *a dvd b*
shows *euclidean-size* (*gcd a b*) < *euclidean-size a*
proof (*rule ccontr*)
assume ¬*euclidean-size* (*gcd a b*) < *euclidean-size a*
with ‹*a* ≠ *0*› **have** *A*: *euclidean-size* (*gcd a b*) = *euclidean-size a*
by (*intro le-antisym*, *simp-all*)
have *a dvd gcd a b*
by (*rule dvd-euclidean-size-eq-imp-dvd*) (*simp-all add*: *assms A*)
hence *a dvd b* **using** *dvd-gcdD2* **by** *blast*
with ‹¬ *a dvd b*› **show** *False* **by** *contradiction*
qed

lemma *euclidean-size-gcd-less2*:
assumes *b* ≠ *0* **and** ¬ *b dvd a*
shows *euclidean-size* (*gcd a b*) < *euclidean-size b*
using *assms* **by** (*subst gcd.commute*, *rule euclidean-size-gcd-less1*)

lemma *euclidean-size-lcm-le1*:
assumes *a* ≠ *0* **and** *b* ≠ *0*
shows *euclidean-size a* ≤ *euclidean-size* (*lcm a b*)
proof −
have *a dvd lcm a b* **by** (*rule dvd-lcm1*)
then obtain *c* **where** *A*: *lcm a b* = *a* ∗ *c* **..**
with ‹*a* ≠ *0*› **and** ‹*b* ≠ *0*› **have** *c* ≠ *0* **by** (*auto simp*: *lcm-eq-0-iff*)
then show *?thesis* **by** (*subst A*, *intro size-mult-mono*)
qed

lemma *euclidean-size-lcm-le2*:
a ≠ *0* =⇒ *b* ≠ *0* =⇒ *euclidean-size b* ≤ *euclidean-size* (*lcm a b*)
using *euclidean-size-lcm-le1* [*of b a*] **by** (*simp add*: *ac-simps*)

lemma *euclidean-size-lcm-less1*:

assumes *b* ≠ *0* **and** ¬ *b dvd a*
shows *euclidean-size a* < *euclidean-size* (*lcm a b*)
proof (*rule ccontr*)
from *assms* **have** *a* ≠ *0* **by** *auto*
assume ¬*euclidean-size a* < *euclidean-size* (*lcm a b*)
with ‹*a* ≠ *0*› **and** ‹*b* ≠ *0*› **have** *euclidean-size* (*lcm a b*) = *euclidean-size a*
by (*intro le-antisym*, *simp*, *intro euclidean-size-lcm-le1*)
with *assms* **have** *lcm a b dvd a*
by (*rule-tac dvd-euclidean-size-eq-imp-dvd*) (*auto simp*: *lcm-eq-0-iff*)
hence *b dvd a* **by** (*rule lcm-dvdD2*)
with ‹¬*b dvd a*› **show** *False* **by** *contradiction*
qed

lemma *euclidean-size-lcm-less2*:
assumes *a* ≠ *0* **and** ¬ *a dvd b*
shows *euclidean-size b* < *euclidean-size* (*lcm a b*)
using *assms euclidean-size-lcm-less1* [*of a b*] **by** (*simp add*: *ac-simps*)

end

lemma *factorial-euclidean-semiring-gcdI*:
OFCLASS(′*a*::{*factorial-semiring-gcd*, *normalization-euclidean-semiring*}, *euclidean-semiring-gcd-class*)
proof
interpret *semiring-Gcd 1 0 times*
Euclidean-Algorithm.gcd Euclidean-Algorithm.lcm
Euclidean-Algorithm.Gcd Euclidean-Algorithm.Lcm
divide plus minus unit-factor normalize
rewrites *dvd.dvd* (∗) = *Rings.dvd*
by (*fact semiring-Gcd*) (*simp add*: *dvd.dvd-def dvd-def fun-eq-iff*)
show [*simp*]: *Euclidean-Algorithm.gcd* = (*gcd* :: ′*a* ⇒ -)
proof (*rule ext*)+
fix *a b* :: ′*a*
show *Euclidean-Algorithm.gcd a b* = *gcd a b*
proof (*induct a b rule*: *eucl-induct*)
case *zero*
then show *?case*
by *simp*
next
case (*mod a b*)
moreover have *gcd b* (*a mod b*) = *gcd b a*
using *GCD.gcd-add-mult* [*of b a div b a mod b*, *symmetric*]
by (*simp add*: *div-mult-mod-eq*)
ultimately show *?case*
by (*simp add*: *Euclidean-Algorithm.gcd-mod ac-simps*)
qed
qed
show [*simp*]: *Euclidean-Algorithm.Lcm* = (*Lcm* :: ′*a set* ⇒ -)
by (*auto intro*!: *Lcm-eqI GCD.dvd-Lcm GCD.Lcm-least*)
show *Euclidean-Algorithm.lcm* = (*lcm* :: ′*a* ⇒ -)

by (*simp add*: *fun-eq-iff Euclidean-Algorithm.lcm-def semiring-gcd-class.lcm-gcd*)
show *Euclidean-Algorithm.Gcd* = (*Gcd* :: $'a$ *set* ⇒ -)
by (*simp add*: *fun-eq-iff Euclidean-Algorithm.Gcd-def semiring-Gcd-class.Gcd-Lcm*)
qed

2.3 The extended euclidean algorithm

class *euclidean-ring-gcd* = *euclidean-semiring-gcd* + *idom*
begin

subclass *euclidean-ring* **..**
subclass *ring-gcd* **..**
subclass *factorial-ring-gcd* **..**

function *euclid-ext-aux* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow ('a \times 'a) \times 'a$
where *euclid-ext-aux* s' s t' t r' r = (
if $r = 0$ *then let* c = *1 div unit-factor* r' *in* (($s' * c$, $t' * c$), *normalize* r')
else let $q = r'$ *div* r
in euclid-ext-aux s ($s' - q * s$) t ($t' - q * t$) r (r' *mod* r))
by *auto*
termination
by (*relation measure* (λ(-, -, -, -, -, b). *euclidean-size* b))
(*simp-all add*: *mod-size-less*)

abbreviation (*input*) *euclid-ext* :: $'a \Rightarrow 'a \Rightarrow ('a \times 'a) \times 'a$
where *euclid-ext* ≡ *euclid-ext-aux 1 0 0 1*

lemma
assumes *gcd* r' r = *gcd* a b
assumes $s' * a + t' * b = r'$
assumes $s * a + t * b = r$
assumes *euclid-ext-aux* s' s t' t r' r = ((x, y), c)
shows *euclid-ext-aux-eq-gcd*: c = *gcd* a b
and *euclid-ext-aux-bezout*: $x * a + y * b$ = *gcd* a b
proof −
have *case euclid-ext-aux* s' s t' t r' r *of* ((x, y), c) ⇒
$x * a + y * b = c \wedge c$ = *gcd* a b (**is** *?P* (*euclid-ext-aux* s' s t' t r' r))
using *assms*(*1*−*3*)
proof (*induction* s' s t' t r' r *rule*: *euclid-ext-aux.induct*)
case (*1* s' s t' t r' r)
show *?case*
proof (*cases* $r = 0$)
case *True*
hence *euclid-ext-aux* s' s t' t r' r =
((s' *div unit-factor* r', t' *div unit-factor* r'), *normalize* r')
by (*subst euclid-ext-aux.simps*) (*simp add*: *Let-def*)
also have *?P* . . .
proof *safe*
have s' *div unit-factor* r' $* a$ + t' *div unit-factor* r' $* b$ =

$(s' * a + t' * b)$ *div unit-factor* r'
by (*cases* $r' = 0$) (*simp-all add*: *unit-div-commute*)
also have $s' * a + t' * b = r'$ **by** *fact*
also have ... *div unit-factor* r' = *normalize* r' **by** *simp*
finally show s' *div unit-factor* $r' * a + t'$ *div unit-factor* $r' * b$ = *normalize* r' **.**
next
from *1.prems True* **show** *normalize* r' = *gcd a b*
by *simp*
qed
finally show *?thesis* **.**
next
case *False*
hence *euclid-ext-aux* s' s t' t r' r =
euclid-ext-aux s $(s' - r'$ *div* $r * s)$ t $(t' - r'$ *div* $r * t)$ r $(r'$ *mod* $r)$
by (*subst euclid-ext-aux.simps*) (*simp add*: *Let-def*)
also from *1.prems False* **have** *?P* ...
proof (*intro 1.IH*)
have $(s' - r'$ *div* $r * s) * a + (t' - r'$ *div* $r * t) * b$ =
$(s' * a + t' * b) - r'$ *div* $r * (s * a + t * b)$ **by** (*simp add*: *algebra-simps*)
also have $s' * a + t' * b = r'$ **by** *fact*
also have $s * a + t * b = r$ **by** *fact*
also have $r' - r'$ *div* $r * r = r'$ *mod* r **using** *div-mult-mod-eq* [*of* r' r]
by (*simp add*: *algebra-simps*)
finally show $(s' - r'$ *div* $r * s) * a + (t' - r'$ *div* $r * t) * b = r'$ *mod* r **.**
qed (*auto simp*: *algebra-simps minus-mod-eq-div-mult* [*symmetric*] *gcd.commute*)
finally show *?thesis* **.**
qed
qed
with *assms*(*4*) **show** c = *gcd a b* $x * a + y * b$ = *gcd a b*
by *simp-all*
qed

declare *euclid-ext-aux.simps* [*simp del*]

definition *bezout-coefficients* :: $'a \Rightarrow 'a \Rightarrow 'a \times 'a$
where [*code*]: *bezout-coefficients a b* = *fst* (*euclid-ext a b*)

lemma *bezout-coefficients-0*:
bezout-coefficients a 0 = (*1 div unit-factor a*, *0*)
by (*simp add*: *bezout-coefficients-def euclid-ext-aux.simps*)

lemma *bezout-coefficients-left-0*:
bezout-coefficients 0 a = (*0*, *1 div unit-factor a*)
by (*simp add*: *bezout-coefficients-def euclid-ext-aux.simps*)

lemma *bezout-coefficients*:
assumes *bezout-coefficients a b* = (x, y)
shows $x * a + y * b$ = *gcd a b*

using *assms* **by** (*simp add*: *bezout-coefficients-def*
euclid-ext-aux-bezout [*of a b a b 1 0 0 1 x y*] *prod-eq-iff*)

lemma *bezout-coefficients-fst-snd*:
fst (*bezout-coefficients a b*) ∗ *a* + *snd* (*bezout-coefficients a b*) ∗ *b* = *gcd a b*
by (*rule bezout-coefficients*) *simp*

lemma *euclid-ext-eq* [*simp*]:
euclid-ext a b = (*bezout-coefficients a b*, *gcd a b*) (**is** *?p* = *?q*)
proof
show *fst ?p* = *fst ?q*
by (*simp add*: *bezout-coefficients-def*)
have *snd* (*euclid-ext-aux 1 0 0 1 a b*) = *gcd a b*
by (*rule euclid-ext-aux-eq-gcd* [*of a b a b 1 0 0 1*])
(*simp-all add*: *prod-eq-iff*)
then show *snd ?p* = *snd ?q*
by *simp*
qed

declare *euclid-ext-eq* [*symmetric*, *code-unfold*]

end

class *normalization-euclidean-semiring-multiplicative* =
normalization-euclidean-semiring + *normalization-semidom-multiplicative*
begin

subclass *factorial-semiring-multiplicative* **..**

end

class *field-gcd* =
field + *unique-euclidean-ring* + *euclidean-ring-gcd* + *normalization-semidom-multiplicative*
begin

subclass *normalization-euclidean-semiring-multiplicative* **..**

subclass *normalization-euclidean-semiring* **..**

subclass *semiring-gcd-mult-normalize* **..**

end

2.4 Typical instances

instance *nat* :: *normalization-euclidean-semiring* **..**

instance *nat* :: *euclidean-semiring-gcd*
proof

interpret *semiring-Gcd 1 0 times*
Euclidean-Algorithm.gcd Euclidean-Algorithm.lcm
Euclidean-Algorithm.Gcd Euclidean-Algorithm.Lcm
divide plus minus unit-factor normalize
rewrites *dvd.dvd* (∗) = *Rings.dvd*
by (*fact semiring-Gcd*) (*simp add*: *dvd.dvd-def dvd-def fun-eq-iff*)
show [*simp*]: (*Euclidean-Algorithm.gcd* :: *nat* ⇒ -) = *gcd*
proof (*rule ext*)+
fix *m n* :: *nat*
show *Euclidean-Algorithm.gcd m n* = *gcd m n*
proof (*induct m n rule*: *eucl-induct*)
case *zero*
then show *?case*
by *simp*
next
case (*mod m n*)
then have *gcd n* (*m mod n*) = *gcd n m*
using *gcd-nat.simps* [*of m n*] **by** (*simp add*: *ac-simps*)
with *mod* **show** *?case*
by (*simp add*: *Euclidean-Algorithm.gcd-mod ac-simps*)
qed
qed
show [*simp*]: (*Euclidean-Algorithm.Lcm* :: *nat set* ⇒ -) = *Lcm*
by (*auto intro*!: *ext Lcm-eqI*)
show (*Euclidean-Algorithm.lcm* :: *nat* ⇒ -) = *lcm*
by (*simp add*: *fun-eq-iff Euclidean-Algorithm.lcm-def semiring-gcd-class.lcm-gcd*)
show (*Euclidean-Algorithm.Gcd* :: *nat set* ⇒ -) = *Gcd*
by (*simp add*: *fun-eq-iff Euclidean-Algorithm.Gcd-def semiring-Gcd-class.Gcd-Lcm*)
qed

instance *nat* :: *normalization-euclidean-semiring-multiplicative* **..**

lemma *prime-factorization-Suc-0* [*simp*]: *prime-factorization* (*Suc 0*) = {#}
unfolding *One-nat-def* [*symmetric*] **using** *prime-factorization-1* **.**

instance *int* :: *normalization-euclidean-semiring* **..**

instance *int* :: *euclidean-ring-gcd*
proof
interpret *semiring-Gcd 1 0 times*
Euclidean-Algorithm.gcd Euclidean-Algorithm.lcm
Euclidean-Algorithm.Gcd Euclidean-Algorithm.Lcm
divide plus minus unit-factor normalize
rewrites *dvd.dvd* (∗) = *Rings.dvd*
by (*fact semiring-Gcd*) (*simp add*: *dvd.dvd-def dvd-def fun-eq-iff*)
show [*simp*]: (*Euclidean-Algorithm.gcd* :: *int* ⇒ -) = *gcd*
proof (*rule ext*)+
fix *k l* :: *int*
show *Euclidean-Algorithm.gcd k l* = *gcd k l*

```
    proof (induct k l rule: eucl-induct)
      case zero
      then show ?case
        by simp
    next
      case (mod k l)
      have gcd l (k mod l) = gcd l k
      proof (cases l 0::int rule: linorder-cases)
        case less
        then show ?thesis
          using gcd-non-0-int [of − l − k] by (simp add: ac-simps)
      next
        case equal
        with mod show ?thesis
          by simp
      next
        case greater
        then show ?thesis
          using gcd-non-0-int [of l k] by (simp add: ac-simps)
      qed
      with mod show ?case
        by (simp add: Euclidean-Algorithm.gcd-mod ac-simps)
    qed
  qed
  show [simp]: (Euclidean-Algorithm.Lcm :: int set ⇒ -) = Lcm
    by (auto intro!: ext Lcm-eqI)
  show (Euclidean-Algorithm.lcm :: int ⇒ -) = lcm
   by (simp add: fun-eq-iff Euclidean-Algorithm.lcm-def semiring-gcd-class.lcm-gcd)
  show (Euclidean-Algorithm.Gcd :: int set ⇒ -) = Gcd
   by (simp add: fun-eq-iff Euclidean-Algorithm.Gcd-def semiring-Gcd-class.Gcd-Lcm)
qed

instance int :: normalization-euclidean-semiring-multiplicative ..

lemma (in idom) prime-CHAR-semidom:
  assumes CHAR('a) > 0
  shows   prime CHAR('a)
proof −
  have False if ab: a ≠ 1 b ≠ 1 CHAR('a) = a ∗ b for a b
  proof −
    from assms ab have a > 0 b > 0
      by (auto intro!: Nat.gr0I)
    have of-nat (a ∗ b) = (0 :: 'a)
      using ab by (metis of-nat-CHAR)
    also have of-nat (a ∗ b) = (of-nat a :: 'a) ∗ of-nat b
      by simp
    finally have of-nat a ∗ of-nat b = (0 :: 'a) .
    moreover have of-nat a ∗ of-nat b ≠ (0 :: 'a)
      using ab ‹a > 0› ‹b > 0›
```

by (*intro no-zero-divisors*) (*auto simp*: *of-nat-eq-0-iff-char-dvd*)
ultimately show *False*
by *contradiction*
qed
moreover have *CHAR('a) > 1*
using *assms CHAR-not-1'* **by** *linarith*
ultimately have *prime-elem CHAR('a)*
by (*intro irreducible-imp-prime-elem*) (*auto simp*: *Factorial-Ring.irreducible-def*)
thus *?thesis*
by (*auto simp*: *prime-def*)
qed

end

3 Primes

theory *Primes*
imports *Euclidean-Algorithm*
begin

3.1 Primes on *nat* and *int*

lemma *Suc-0-not-prime-nat* [*simp*]: ¬ *prime* (*Suc 0*)
using *not-prime-1* [**where** *?'a = nat*] **by** *simp*

lemma *prime-ge-2-nat*:
$p \geq 2$ **if** *prime p* **for** *p* :: *nat*
proof −
from *that* **have** $p \neq 0$ **and** $p \neq 1$
by (*auto dest*: *prime-elem-not-zeroI prime-elem-not-unit*)
then show *?thesis*
by *simp*
qed

lemma *prime-ge-2-int*:
$p \geq 2$ **if** *prime p* **for** *p* :: *int*
proof −
from *that* **have** *prime-elem p* **and** $|p| = p$
by (*auto dest*: *normalize-prime*)
then have $p \neq 0$ **and** $|p| \neq 1$ **and** $p \geq 0$
by (*auto dest*: *prime-elem-not-zeroI prime-elem-not-unit*)
then show *?thesis*
by *simp*
qed

lemma *prime-ge-0-int*: *prime p* ⟹ $p \geq (0::int)$
using *prime-ge-2-int* [*of p*] **by** *simp*

lemma *prime-gt-0-nat*: *prime p* ⟹ $p > (0::nat)$

using *prime-ge-2-nat* [*of p*] **by** *simp*

lemma *prime-gt-0-int*: *prime p* $\Longrightarrow$ *p* > (*0*::*int*)
using *prime-ge-2-int* [*of p*] **by** *simp*

lemma *prime-ge-1-nat*: *prime p* $\Longrightarrow$ *p* ≥ (*1*::*nat*)
using *prime-ge-2-nat* [*of p*] **by** *simp*

lemma *prime-ge-Suc-0-nat*: *prime p* $\Longrightarrow$ *p* ≥ *Suc 0*
using *prime-ge-1-nat* [*of p*] **by** *simp*

lemma *prime-ge-1-int*: *prime p* $\Longrightarrow$ *p* ≥ (*1*::*int*)
using *prime-ge-2-int* [*of p*] **by** *simp*

lemma *prime-gt-1-nat*: *prime p* $\Longrightarrow$ *p* > (*1*::*nat*)
using *prime-ge-2-nat* [*of p*] **by** *simp*

lemma *prime-gt-Suc-0-nat*: *prime p* $\Longrightarrow$ *p* > *Suc 0*
using *prime-gt-1-nat* [*of p*] **by** *simp*

lemma *prime-gt-1-int*: *prime p* $\Longrightarrow$ *p* > (*1*::*int*)
using *prime-ge-2-int* [*of p*] **by** *simp*

lemma *prime-natI*:
prime p **if** *p* ≥ *2* **and** $\bigwedge$*m n*. *p dvd m* ∗ *n* $\Longrightarrow$ *p dvd m* ∨ *p dvd n* **for** *p* :: *nat*
using *that* **by** (*auto intro*!: *primeI prime-elemI*)

lemma *prime-intI*:
prime p **if** *p* ≥ *2* **and** $\bigwedge$*m n*. *p dvd m* ∗ *n* $\Longrightarrow$ *p dvd m* ∨ *p dvd n* **for** *p* :: *int*
using *that* **by** (*auto intro*!: *primeI prime-elemI*)

lemma *prime-elem-nat-iff* [*simp*]:
prime-elem n ⟷ *prime n* **for** *n* :: *nat*
by (*simp add*: *prime-def*)

lemma *prime-elem-iff-prime-abs* [*simp*]:
prime-elem k ⟷ *prime* |*k*| **for** *k* :: *int*
by (*auto intro*: *primeI*)

lemma *prime-nat-int-transfer* [*simp*]:
prime (*int n*) ⟷ *prime n* (**is** *?P* ⟷ *?Q*)
proof
assume *?P*
then have *n* ≥ *2*
by (*auto dest*: *prime-ge-2-int*)
then show *?Q*
proof (*rule prime-natI*)

fix *r s*
assume *n dvd r ∗ s*
with *of-nat-dvd-iff* [*of n r ∗ s*] **have** *int n dvd int r ∗ int s*
by *simp*
with ‹*?P*› **have** *int n dvd int r ∨ int n dvd int s*
using *prime-dvd-mult-iff* [*of int n int r int s*]
by *simp*
then show *n dvd r ∨ n dvd s*
by *simp*
qed
next
assume *?Q*
then have *int n ≥ 2*
by (*auto dest*: *prime-ge-2-nat*)
then show *?P*
proof (*rule prime-intI*)
fix *r s*
assume *int n dvd r ∗ s*
then have *n dvd nat |r ∗ s|*
by *simp*
then have *n dvd nat |r| ∗ nat |s|*
by (*simp add*: *nat-abs-mult-distrib*)
with ‹*?Q*› **have** *n dvd nat |r| ∨ n dvd nat |s|*
using *prime-dvd-mult-iff* [*of n nat |r| nat |s|*]
by *simp*
then show *int n dvd r ∨ int n dvd s*
by *simp*
qed
qed

lemma *prime-nat-iff-prime* [*simp*]:
prime (*nat k*) ⟷ *prime k*
proof (*cases k ≥ 0*)
case *True*
then show *?thesis*
using *prime-nat-int-transfer* [*of nat k*] **by** *simp*
next
case *False*
then show *?thesis*
by (*auto dest*: *prime-ge-2-int*)
qed

lemma *prime-int-nat-transfer*:
prime k ⟷ *k ≥ 0 ∧ prime* (*nat k*)
by (*auto dest*: *prime-ge-2-int*)

lemma *prime-nat-naiveI*:
prime p **if** *p ≥ 2* **and** *dvd*: ⋀*n. n dvd p* ⟹ *n = 1 ∨ n = p* **for** *p* :: *nat*
proof (*rule primeI*, *rule prime-elemI*)

fix *m n* :: *nat*
assume *p dvd m* ∗ *n*
then obtain *r s* **where** *p* = *r* ∗ *s r dvd m s dvd n*
by (*blast dest*: *division-decomp*)
moreover have *r* = *1* ∨ *r* = *p*
using ‹*r dvd m*› ‹*p* = *r* ∗ *s*› *dvd* [*of r*] **by** *simp*
ultimately show *p dvd m* ∨ *p dvd n*
by *auto*
qed (*use* ‹*p* ≥ *2*› **in** *simp-all*)

lemma *prime-int-naiveI*:
prime p **if** *p* ≥ *2* **and** *dvd*: ⋀*k*. *k dvd p* ⟹ |*k*| = *1* ∨ |*k*| = *p* **for** *p* :: *int*
proof −
from ‹*p* ≥ *2*› **have** *nat p* ≥ *2*
by *simp*
then have *prime* (*nat p*)
proof (*rule prime-nat-naiveI*)
fix *n*
assume *n dvd nat p*
with ‹*p* ≥ *2*› **have** *n dvd nat* |*p*|
by *simp*
then have *int n dvd p*
by *simp*
with *dvd* [*of int n*] **show** *n* = *1* ∨ *n* = *nat p*
by *auto*
qed
then show *?thesis*
by *simp*
qed

lemma *prime-nat-iff*:
prime (*n* :: *nat*) ⟷ (*1* < *n* ∧ (∀ *m*. *m dvd n* ⟶ *m* = *1* ∨ *m* = *n*))
proof (*safe intro*!: *prime-gt-1-nat*)
assume *prime n*
then have ∗: *prime-elem n*
by *simp*
fix *m* **assume** *m*: *m dvd n m* ≠ *n*
from ∗ ‹*m dvd n*› **have** *n dvd m* ∨ *is-unit m*
by (*intro irreducibleD′ prime-elem-imp-irreducible*)
with *m* **show** *m* = *1* **by** (*auto dest*: *dvd-antisym*)
next
assume *n* > *1* ∀ *m*. *m dvd n* ⟶ *m* = *1* ∨ *m* = *n*
then show *prime n*
using *prime-nat-naiveI* [*of n*] **by** *auto*
qed

lemma *prime-int-iff*:
prime (*n*::*int*) ⟷ (*1* < *n* ∧ (∀ *m*. *m* ≥ *0* ∧ *m dvd n* ⟶ *m* = *1* ∨ *m* = *n*))
proof (*intro iffI conjI allI impI*; (*elim conjE*)*?*)

assume ∗: *prime n*
hence *irred*: *irreducible n* **by** (*auto intro*: *prime-elem-imp-irreducible*)
from ∗ **have** $n \geq 0$ $n \neq 0$ $n \neq 1$
by (*auto simp add*: *prime-ge-0-int*)
thus $n > 1$ **by** *presburger*
fix *m* **assume** *m dvd n* ‹$m \geq 0$›
with *irred* **have** *m dvd 1* ∨ *n dvd m* **by** (*auto simp*: *irreducible-altdef*)
with ‹*m dvd n*› ‹$m \geq 0$› ‹$n > 1$› **show** $m = 1 \vee m = n$
using *associated-iff-dvd*[*of m n*] **by** *auto*
next
assume *n*: $1 < n$ ∀ *m*. $m \geq 0$ ∧ *m dvd n* ⟶ $m = 1 \vee m = n$
hence *nat* $n > 1$ **by** *simp*
moreover have ∀ *m*. *m dvd nat n* ⟶ $m = 1$ ∨ *m* = *nat n*
proof (*intro allI impI*)
fix *m* **assume** *m dvd nat n*
with ‹$n > 1$› **have** *m dvd nat* |*n*|
by *simp*
then have *int m dvd n*
by *simp*
with *n*(*2*) **have** *int m* = *1* ∨ *int m* = *n*
using *of-nat-0-le-iff* **by** *blast*
thus $m = 1$ ∨ *m* = *nat n* **by** *auto*
qed
ultimately show *prime n*
unfolding *prime-int-nat-transfer prime-nat-iff* **by** *auto*
qed

lemma *prime-nat-not-dvd*:
assumes *prime p* $p > n$ $n \neq (1::nat)$
shows ¬*n dvd p*
proof
assume *n dvd p*
from *assms*(*1*) **have** *irreducible p* **by** (*simp add*: *prime-elem-imp-irreducible*)
from *irreducibleD'*[*OF this* ‹*n dvd p*›] ‹*n dvd p*› ‹$p > n$› *assms* **show** *False*
by (*cases* $n = 0$) (*auto dest*!: *dvd-imp-le*)
qed

lemma *prime-int-not-dvd*:
assumes *prime p* $p > n$ $n > (1::int)$
shows ¬*n dvd p*
proof
assume *n dvd p*
from *assms*(*1*) **have** *irreducible p* **by** (*auto intro*: *prime-elem-imp-irreducible*)
from *irreducibleD'*[*OF this* ‹*n dvd p*›] ‹*n dvd p*› ‹$p > n$› *assms* **show** *False*
by (*auto dest*!: *zdvd-imp-le*)
qed

lemma *prime-odd-nat*: *prime p* ⟹ $p > (2::nat)$ ⟹ *odd p*
by (*intro prime-nat-not-dvd*) *auto*

lemma *prime-odd-int*: *prime p* $\Longrightarrow$ *p* > (*2*::*int*) $\Longrightarrow$ *odd p*
  by (*intro prime-int-not-dvd*) *auto*

lemma *prime-int-altdef*:
  prime p = (*1* < *p* ∧ (∀ *m*::*int*. *m* ≥ *0* ⟶ *m dvd p* ⟶
    m = *1* ∨ *m* = *p*))
  unfolding *prime-int-iff* **by** *blast*

lemma *not-prime-eq-prod-nat*:
  assumes *m* > *1* ¬ *prime* (*m*::*nat*)
  shows ∃ *n k*. *n* = *m* ∗ *k* ∧ *1* < *m* ∧ *m* < *n* ∧ *1* < *k* ∧ *k* < *n*
  using *assms irreducible-altdef*[*of m*]
  by (*auto simp*: *prime-elem-iff-irreducible irreducible-altdef*)

3.2 Largest exponent of a prime factor

Possibly duplicates other material, but avoid the complexities of multisets.

lemma *prime-power-cancel-less*:
  assumes *prime p* **and** *eq*: *m* ∗ (*p* ^ *k*) = *m'* ∗ (*p* ^ *k'*) **and** *less*: *k* < *k'* **and** ¬ *p dvd m*
  shows *False*
proof −
  obtain *l* **where** *l*: *k'* = *k* + *l* **and** *l* > *0*
    using *less less-imp-add-positive* **by** *auto*
  have *m* = *m* ∗ (*p* ^ *k*) *div* (*p* ^ *k*)
    using ‹*prime p*› **by** *simp*
  also have . . . = *m'* ∗ (*p* ^ *k'*) *div* (*p* ^ *k*)
    using *eq* **by** *simp*
  also have . . . = *m'* ∗ (*p* ^ *l*) ∗ (*p* ^ *k*) *div* (*p* ^ *k*)
    by (*simp add*: *l mult.commute mult.left-commute power-add*)
  also have ... = *m'* ∗ (*p* ^ *l*)
    using ‹*prime p*› **by** *simp*
  finally have *p dvd m*
    using ‹*l* > *0*› **by** *simp*
  with *assms* **show** *False*
    by *simp*
qed

lemma *prime-power-cancel*:
  assumes *prime p* **and** *eq*: *m* ∗ (*p* ^ *k*) = *m'* ∗ (*p* ^ *k'*) **and** ¬ *p dvd m* ¬ *p dvd m'*
  shows *k* = *k'*
  using *prime-power-cancel-less* [*OF* ‹*prime p*›] *assms*
  by (*metis linorder-neqE-nat*)

lemma *prime-power-cancel2*:
  assumes *prime p m* ∗ (*p* ^ *k*) = *m'* ∗ (*p* ^ *k'*) ¬ *p dvd m* ¬ *p dvd m'*
  obtains *m* = *m'* *k* = *k'*

```
  using prime-power-cancel [OF assms] assms by auto

lemma prime-power-canonical:
  fixes m :: nat
  assumes prime p m > 0
  shows ∃k n. ¬ p dvd n ∧ m = n ∗ p ^ k
using ‹m > 0›
proof (induction m rule: less-induct)
  case (less m)
  show ?case
  proof (cases p dvd m)
    case True
    then obtain m' where m': m = p ∗ m'
      using dvdE by blast
    with ‹prime p› have 0 < m' m' < m
      using less.prems prime-nat-iff by auto
    with m' less show ?thesis
      by (metis power-Suc mult.left-commute)
  next
    case False
    then show ?thesis
      by (metis mult.right-neutral power-0)
  qed
qed
```

3.2.1 Make prime naively executable

```
lemma prime-nat-iff':
  prime (p :: nat) ⟷ p > 1 ∧ (∀n ∈ {2..<p}. ¬ n dvd p)
proof safe
  assume p > 1 and ∗: ∀n∈{2..<p}. ¬n dvd p
  show prime p unfolding prime-nat-iff
  proof (intro conjI allI impI)
    fix m assume m dvd p
    with ‹p > 1› have m ≠ 0 by (intro notI) auto
    hence m ≥ 1 by simp
    moreover from ‹m dvd p› and ∗ have m ∉ {2..<p} by blast
    with ‹m dvd p› and ‹p > 1› have m ≤ 1 ∨ m = p by (auto dest: dvd-imp-le)
    ultimately show m = 1 ∨ m = p by simp
  qed fact+
qed (auto simp: prime-nat-iff)

lemma prime-int-iff':
  prime (p :: int) ⟷ p > 1 ∧ (∀n ∈ {2..<p}. ¬ n dvd p) (is ?P ⟷ ?Q)
proof (cases p ≥ 0)
  case True
  have ?P ⟷ prime (nat p)
    by simp
  also have ... ⟷ p > 1 ∧ (∀n∈{2..<nat p}. ¬ n dvd nat |p|)
```

using *True* **by** (*simp add*: *prime-nat-iff'*)
also have {*2*..<*nat p*} = *nat* ' {*2*..<*p*}
using *True int-eq-iff* **by** *fastforce*
finally show *?P* ⟷ *?Q* **by** *simp*
next
case *False*
then show *?thesis*
by (*auto simp add*: *prime-ge-0-int*)
qed

lemma *prime-int-numeral-eq* [*simp*]:
prime (*numeral m* :: *int*) ⟷ *prime* (*numeral m* :: *nat*)
by (*simp add*: *prime-int-nat-transfer*)

lemma *two-is-prime-nat* [*simp*]: *prime* (*2*::*nat*)
by (*simp add*: *prime-nat-iff'*)

lemma *prime-nat-numeral-eq* [*simp*]:
prime (*numeral m* :: *nat*) ⟷
(*1*::*nat*) < *numeral m* ∧
(∀ *n*::*nat* ∈ *set* [*2*..<*numeral m*]. ¬ *n dvd numeral m*)
by (*simp only*: *prime-nat-iff' set-upt*) — TODO Sieve Of Erathosthenes might speed this up

A bit of regression testing:

lemma *prime*(*97*::*nat*) **by** *simp*
lemma *prime*(*97*::*int*) **by** *simp*

lemma *prime-factor-nat*:
n ≠ (*1*::*nat*) ⟹ ∃ *p*. *prime p* ∧ *p dvd n*
using *prime-divisor-exists*[*of n*]
by (*cases n* = *0*) (*auto intro*: *exI*[*of* - *2*::*nat*])

lemma *prime-factor-int*:
fixes *k* :: *int*
assumes |*k*| ≠ *1*
obtains *p* **where** *prime p p dvd k*
proof (*cases k* = *0*)
case *True*
then have *prime* (*2*::*int*) **and** *2 dvd k*
by *simp-all*
with *that* **show** *thesis*
by *blast*
next
case *False*
with *assms prime-divisor-exists* [*of k*] **obtain** *p* **where** *prime p p dvd k*
by *auto*
with *that* **show** *thesis*
by *blast*

qed

3.3 Infinitely many primes

```
lemma next-prime-bound: ∃ p::nat. prime p ∧ n < p ∧ p ≤ fact n + 1
proof−
  have f1: fact n + 1 ≠ (1::nat) using fact-ge-1 [of n, where 'a=nat] by arith
  from prime-factor-nat [OF f1]
  obtain p :: nat where prime p and p dvd fact n + 1 by auto
  then have p ≤ fact n + 1 apply (intro dvd-imp-le) apply auto done
  { assume p ≤ n
    from ‹prime p› have p ≥ 1
      by (cases p, simp-all)
    with ‹p <= n› have p dvd fact n
      by (intro dvd-fact)
    with ‹p dvd fact n + 1› have p dvd fact n + 1 − fact n
      by (rule dvd-diff-nat)
    then have p dvd 1 by simp
    then have p <= 1 by auto
    moreover from ‹prime p› have p > 1
      using prime-nat-iff by blast
    ultimately have False by auto}
  then have n < p by presburger
  with ‹prime p› and ‹p <= fact n + 1› show ?thesis by auto
qed

lemma bigger-prime: ∃ p. prime p ∧ p > (n::nat)
  using next-prime-bound by auto

lemma primes-infinite: ¬ (finite {(p::nat). prime p})
proof
  assume finite {(p::nat). prime p}
  with Max-ge have (∃ b. (∀ x ∈ {(p::nat). prime p}. x ≤ b))
    by auto
  then obtain b where ∀ (x::nat). prime x ⟶ x ≤ b
    by auto
  with bigger-prime [of b] show False
    by auto
qed
```

3.4 Powers of Primes

Versions for type nat only

```
lemma prime-product:
  fixes p::nat
  assumes prime (p ∗ q)
  shows p = 1 ∨ q = 1
proof −
  from assms have
```

```
    1 < p ∗ q and P: ⋀m. m dvd p ∗ q ⟹ m = 1 ∨ m = p ∗ q
    unfolding prime-nat-iff by auto
  from ‹1 < p ∗ q› have p ≠ 0 by (cases p) auto
  then have Q: p = p ∗ q ⟷ q = 1 by auto
  have p dvd p ∗ q by simp
  then have p = 1 ∨ p = p ∗ q by (rule P)
  then show ?thesis by (simp add: Q)
qed


lemma prime-power-mult-nat:
  fixes p :: nat
  assumes p: prime p and xy: x ∗ y = p ^ k
  shows ∃ i j. x = p ^ i ∧ y = p^ j
using xy
proof(induct k arbitrary: x y)
  case 0 thus ?case apply simp by (rule exI[where x=0], simp)
next
  case (Suc k x y)
  from Suc.prems have pxy: p dvd x∗y by auto
  from prime-dvd-multD [OF p pxy] have pxyc: p dvd x ∨ p dvd y .
  from p have p0: p ≠ 0 by − (rule ccontr, simp)
  {assume px: p dvd x
    then obtain d where d: x = p∗d unfolding dvd-def by blast
    from Suc.prems d  have p∗d∗y = p^Suc k by simp
    hence th: d∗y = p^k using p0 by simp
    from Suc.hyps[OF th] obtain i j where ij: d = p^i y = p^j by blast
    with d have x = p^Suc i by simp
    with ij(2) have ?case by blast}
  moreover
  {assume px: p dvd y
    then obtain d where d: y = p∗d unfolding dvd-def by blast
    from Suc.prems d  have p∗d∗x = p^Suc k by (simp add: mult.commute)
    hence th: d∗x = p^k using p0 by simp
    from Suc.hyps[OF th] obtain i j where ij: d = p^i x = p^j by blast
    with d have y = p^Suc i by simp
    with ij(2) have ?case by blast}
  ultimately show ?case  using pxyc by blast
qed

lemma prime-power-exp-nat:
  fixes p::nat
  assumes p: prime p and n: n ≠ 0
    and xn: x^n = p^k shows ∃ i. x = p^i
  using n xn
proof(induct n arbitrary: k)
  case 0 thus ?case by simp
next
  case (Suc n k) hence th: x∗x^n = p^k by simp
```

```
  {assume n = 0 with Suc have ?case by simp (rule exI[where x=k], simp)}
  moreover
  {assume n: n ≠ 0
    from prime-power-mult-nat[OF p th]
    obtain i j where ij: x = p^i x^n = p^jby blast
    from Suc.hyps[OF n ij(2)] have ?case .}
  ultimately show ?case by blast
qed

lemma divides-primepow-nat:
  fixes p :: nat
  assumes p: prime p
  shows d dvd p ^ k ⟷ (∃ i≤k. d = p ^ i)
  using assms divides-primepow [of p d k] by (auto intro: le-imp-power-dvd)
```

3.5 Chinese Remainder Theorem Variants

```
lemma bezout-gcd-nat:
  fixes a::nat shows ∃ x y. a * x − b * y = gcd a b ∨ b * x − a * y = gcd a b
  using bezout-nat[of a b]
by (metis bezout-nat diff-add-inverse gcd-add-mult gcd.commute
  gcd-nat.right-neutral mult-0)

lemma gcd-bezout-sum-nat:
  fixes a::nat
  assumes a * x + b * y = d
  shows gcd a b dvd d
proof−
  let ?g = gcd a b
    have dv: ?g dvd a*x ?g dvd b * y
      by simp-all
    from dvd-add[OF dv] assms
    show ?thesis by auto
qed
```

A binary form of the Chinese Remainder Theorem.

```
lemma chinese-remainder:
  fixes a::nat  assumes ab: coprime a b and a: a ≠ 0 and b: b ≠ 0
  shows ∃ x q1 q2. x = u + q1 * a ∧ x = v + q2 * b
proof−
  from bezout-add-strong-nat[OF a, of b] bezout-add-strong-nat[OF b, of a]
  obtain d1 x1 y1 d2 x2 y2 where dxy1: d1 dvd a d1 dvd b a * x1 = b * y1 + d1
    and dxy2: d2 dvd b d2 dvd a b * x2 = a * y2 + d2 by blast
  then have d12: d1 = 1 d2 = 1
    using ab coprime-common-divisor-nat [of a b] by blast+
  let ?x = v * a * x1 + u * b * x2
  let ?q1 = v * x1 + u * y2
  let ?q2 = v * y1 + u * x2
  from dxy2(3)[simplified d12] dxy1(3)[simplified d12]
```

have *?x = u + ?q1 ∗ a ?x = v + ?q2 ∗ b*
by *algebra+*
thus *?thesis* **by** *blast*
qed

Primality

lemma *coprime-bezout-strong*:
fixes *a::nat* **assumes** *coprime a b b ≠ 1*
shows *∃ x y. a ∗ x = b ∗ y + 1*
by (*metis add.commute add.right-neutral assms(1) assms(2) chinese-remainder coprime-1-left coprime-1-right coprime-crossproduct-nat mult.commute mult.right-neutral mult-cancel-left*)

lemma *bezout-prime*:
assumes *p*: *prime p* **and** *pa*: *¬ p dvd a*
shows *∃ x y. a∗x = Suc (p∗y)*
proof −
have *ap*: *coprime a p*
using *coprime-commute p pa prime-imp-coprime* **by** *auto*
moreover from *p* **have** *p ≠ 1* **by** *auto*
ultimately have *∃ x y. a ∗ x = p ∗ y + 1*
by (*rule coprime-bezout-strong*)
then show *?thesis* **by** *simp*
qed

3.6 Multiplicity and primality for natural numbers and integers

lemma *prime-factors-gt-0-nat*:
p ∈ prime-factors x ⟹ p > (0::nat)
by (*simp add*: *in-prime-factors-imp-prime prime-gt-0-nat*)

lemma *prime-factors-gt-0-int*:
p ∈ prime-factors x ⟹ p > (0::int)
by (*simp add*: *in-prime-factors-imp-prime prime-gt-0-int*)

lemma *prime-factors-ge-0-int* [*elim*]:
fixes *n :: int*
shows *p ∈ prime-factors n ⟹ p ≥ 0*
by (*drule prime-factors-gt-0-int*) *simp*

lemma *prod-mset-prime-factorization-int*:
fixes *n :: int*
assumes *n > 0*
shows *prod-mset (prime-factorization n) = n*
using *assms* **by** (*simp add*: *prod-mset-prime-factorization*)

lemma *prime-factorization-exists-nat*:
n > 0 ⟹ (∃ M. (∀ p::nat ∈ set-mset M. prime p) ∧ n = (∏ i ∈# M. i))

using *prime-factorization-exists*[*of n*] **by** *auto*

lemma *prod-mset-prime-factorization-nat* [*simp*]:
$(n::nat) > 0 \Longrightarrow$ *prod-mset* (*prime-factorization n*) = *n*
by (*subst prod-mset-prime-factorization*) *simp-all*

lemma *prime-factorization-nat*:
$n > (0::nat) \Longrightarrow n = (\prod p \in \textit{prime-factors } n.\ p\ \hat{}\ \textit{multiplicity } p\ n)$
by (*simp add*: *prod-prime-factors*)

lemma *prime-factorization-int*:
$n > (0::int) \Longrightarrow n = (\prod p \in \textit{prime-factors } n.\ p\ \hat{}\ \textit{multiplicity } p\ n)$
by (*simp add*: *prod-prime-factors*)

lemma *prime-factorization-unique-nat*:
fixes *f* :: *nat* ⇒ -
assumes *S-eq*: $S = \{p.\ 0 < f\ p\}$
and *finite S*
and *S*: $\forall p \in S.\ \textit{prime } p\ n = (\prod p \in S.\ p\ \hat{}\ f\ p)$
shows $S = \textit{prime-factors } n \wedge (\forall p.\ \textit{prime } p \longrightarrow f\ p = \textit{multiplicity } p\ n)$
using *assms* **by** (*intro prime-factorization-unique''*) *auto*

lemma *prime-factorization-unique-int*:
fixes *f* :: *int* ⇒ -
assumes *S-eq*: $S = \{p.\ 0 < f\ p\}$
and *finite S*
and *S*: $\forall p \in S.\ \textit{prime } p\ \textit{abs } n = (\prod p \in S.\ p\ \hat{}\ f\ p)$
shows $S = \textit{prime-factors } n \wedge (\forall p.\ \textit{prime } p \longrightarrow f\ p = \textit{multiplicity } p\ n)$
using *assms* **by** (*intro prime-factorization-unique''*) *auto*

lemma *prime-factors-characterization-nat*:
$S = \{p.\ 0 < f\ (p::nat)\} \Longrightarrow$
$\textit{finite } S \Longrightarrow \forall p \in S.\ \textit{prime } p \Longrightarrow n = (\prod p \in S.\ p\ \hat{}\ f\ p) \Longrightarrow \textit{prime-factors } n = S$
by (*rule prime-factorization-unique-nat* [*THEN conjunct1*, *symmetric*])

lemma *prime-factors-characterization'-nat*:
$\textit{finite } \{p.\ 0 < f\ (p::nat)\} \Longrightarrow$
$(\forall p.\ 0 < f\ p \longrightarrow \textit{prime } p) \Longrightarrow$
$\textit{prime-factors } (\prod p \mid 0 < f\ p.\ p\ \hat{}\ f\ p) = \{p.\ 0 < f\ p\}$
by (*rule prime-factors-characterization-nat*) *auto*

lemma *prime-factors-characterization-int*:
$S = \{p.\ 0 < f\ (p::int)\} \Longrightarrow \textit{finite } S \Longrightarrow$
$\forall p \in S.\ \textit{prime } p \Longrightarrow \textit{abs } n = (\prod p \in S.\ p\ \hat{}\ f\ p) \Longrightarrow \textit{prime-factors } n = S$
by (*rule prime-factorization-unique-int* [*THEN conjunct1*, *symmetric*])

lemma *abs-prod*: $\textit{abs } (\textit{prod } f\ A :: 'a :: \textit{linordered-idom}) = \textit{prod } (\lambda x.\ \textit{abs } (f\ x))\ A$
by (*cases finite A*, *induction A rule*: *finite-induct*) (*simp-all add*: *abs-mult*)

lemma *primes-characterization′-int* [*rule-format*]:
finite {*p*. *p* ≥ *0* ∧ *0* < *f* (*p*::*int*)} ⟹ ∀ *p*. *0* < *f p* ⟶ *prime p* ⟹
prime-factors (∏ *p* | *p* ≥ *0* ∧ *0* < *f p*. *p* ^ *f p*) = {*p*. *p* ≥ *0* ∧ *0* < *f p*}
by (*rule prime-factors-characterization-int*) (*auto simp*: *abs-prod prime-ge-0-int*)

lemma *multiplicity-characterization-nat*:
S = {*p*. *0* < *f* (*p*::*nat*)} ⟹ *finite S* ⟹ ∀ *p*∈*S*. *prime p* ⟹ *prime p* ⟹
n = (∏ *p*∈*S*. *p* ^ *f p*) ⟹ *multiplicity p n* = *f p*
by (*frule prime-factorization-unique-nat* [*of S f n*, *THEN conjunct2*, *rule-format*, *symmetric*]) *auto*

lemma *multiplicity-characterization′-nat*: *finite* {*p*. *0* < *f* (*p*::*nat*)} ⟶
(∀ *p*. *0* < *f p* ⟶ *prime p*) ⟶ *prime p* ⟶
multiplicity p (∏ *p* | *0* < *f p*. *p* ^ *f p*) = *f p*
by (*intro impI*, *rule multiplicity-characterization-nat*) *auto*

lemma *multiplicity-characterization-int*: *S* = {*p*. *0* < *f* (*p*::*int*)} ⟹
finite S ⟹ ∀ *p*∈*S*. *prime p* ⟹ *prime p* ⟹ *n* = (∏ *p*∈*S*. *p* ^ *f p*) ⟹
multiplicity p n = *f p*
by (*frule prime-factorization-unique-int* [*of S f n*, *THEN conjunct2*, *rule-format*, *symmetric*])
(*auto simp*: *abs-prod power-abs prime-ge-0-int intro*!: *prod.cong*)

lemma *multiplicity-characterization′-int* [*rule-format*]:
finite {*p*. *p* ≥ *0* ∧ *0* < *f* (*p*::*int*)} ⟹
(∀ *p*. *0* < *f p* ⟶ *prime p*) ⟹ *prime p* ⟹
multiplicity p (∏ *p* | *p* ≥ *0* ∧ *0* < *f p*. *p* ^ *f p*) = *f p*
by (*rule multiplicity-characterization-int*) (*auto simp*: *prime-ge-0-int*)

lemma *multiplicity-one-nat* [*simp*]: *multiplicity p* (*Suc 0*) = *0*
unfolding *One-nat-def* [*symmetric*] **by** (*rule multiplicity-one*)

lemma *multiplicity-eq-nat*:
fixes *x* **and** *y*::*nat*
assumes *x* > *0* *y* > *0* ⋀*p*. *prime p* ⟹ *multiplicity p x* = *multiplicity p y*
shows *x* = *y*
using *multiplicity-eq-imp-eq*[*of x y*] *assms* **by** *simp*

lemma *multiplicity-eq-int*:
fixes *x y* :: *int*
assumes *x* > *0* *y* > *0* ⋀*p*. *prime p* ⟹ *multiplicity p x* = *multiplicity p y*
shows *x* = *y*
using *multiplicity-eq-imp-eq*[*of x y*] *assms* **by** *simp*

lemma *multiplicity-prod-prime-powers*:
assumes *finite S* ⋀*x*. *x* ∈ *S* ⟹ *prime x* *prime p*
shows *multiplicity p* (∏ *p* ∈ *S*. *p* ^ *f p*) = (*if p* ∈ *S then f p else 0*)
proof −

define g **where** $g = (\lambda x.\ \textit{if } x \in S \textit{ then } f\ x \textit{ else } 0)$
define A **where** $A = \textit{Abs-multiset } g$
have $\{x.\ g\ x > 0\} \subseteq S$ **by** (*auto simp*: *g-def*)
from *finite-subset*[*OF this assms*(*1*)] **have** [*simp*]: *finite* $\{x.\ 0 < g\ x\}$
by *simp*
from *assms* **have** *count-A*: *count* $A\ x = g\ x$ **for** x **unfolding** *A-def*
by *simp*
have *set-mset-A*: *set-mset* $A = \{x{\in}S.\ f\ x > 0\}$
unfolding *set-mset-def count-A* **by** (*auto simp*: *g-def*)
with *assms* **have** *prime*: *prime* x **if** $x \in\#\ A$ **for** x **using** *that* **by** *auto*
from *set-mset-A assms* **have** $(\prod p \in S.\ p\ \hat{}\ f\ p) = (\prod p \in S.\ p\ \hat{}\ g\ p)$
by (*intro prod.cong*) (*auto simp*: *g-def*)
also from *set-mset-A assms* **have** $\ldots = (\prod p \in \textit{set-mset } A.\ p\ \hat{}\ g\ p)$
by (*intro prod.mono-neutral-right*) (*auto simp*: *g-def set-mset-A*)
also have $\ldots$ = *prod-mset A*
by (*auto simp*: *prod-mset-multiplicity count-A set-mset-A intro*!: *prod.cong*)
also from *assms* **have** *multiplicity p* $\ldots$ = *sum-mset* (*image-mset* (*multiplicity p*) *A*)
by (*subst prime-elem-multiplicity-prod-mset-distrib*) (*auto dest*: *prime*)
also from *assms* **have** *image-mset* (*multiplicity p*) *A* = *image-mset* $(\lambda x.\ \textit{if } x = p \textit{ then } 1 \textit{ else } 0)$ *A*
by (*intro image-mset-cong*) (*auto simp*: *prime-multiplicity-other dest*: *prime*)
also have *sum-mset* $\ldots = (\textit{if } p \in S \textit{ then } f\ p \textit{ else } 0)$ **by** (*simp add*: *sum-mset-delta count-A g-def*)
finally show *?thesis* **.**
qed

lemma *prime-factorization-prod-mset*:
assumes $0 \notin\#\ A$
shows *prime-factorization* (*prod-mset A*) = $\sum_{\#}$(*image-mset prime-factorization A*)
using *assms* **by** (*induct A*) (*auto simp add*: *prime-factorization-mult*)

lemma *prime-factors-prod*:
assumes *finite A* **and** $0 \notin f$ ‘ A
shows *prime-factors* (*prod f A*) = $\bigcup$((*prime-factors* $\circ$ *f*) ‘ *A*)
using *assms* **by** (*simp add*: *prod-unfold-prod-mset prime-factorization-prod-mset*)

lemma *prime-factors-fact*:
prime-factors (*fact n*) = $\{p \in \{2..n\}.\ \textit{prime } p\}$ (**is** *?M* = *?N*)
proof (*rule set-eqI*)
fix p
{ fix m :: *nat*
assume $p \in$ *prime-factors m*
then have *prime p* **and** *p dvd m* **by** *auto*
moreover assume $m > 0$
ultimately have $2 \leq p$ **and** $p \leq m$
by (*auto intro*: *prime-ge-2-nat dest*: *dvd-imp-le*)
moreover assume $m \leq n$

ultimately have *2* ≤ *p* **and** *p* ≤ *n*
by (*auto intro*: *order-trans*)
} **note** ∗ = *this*
show *p* ∈ *?M* ⟷ *p* ∈ *?N*
by (*auto simp add*: *fact-prod prime-factors-prod Suc-le-eq dest*!: *prime-prime-factors intro*: ∗)
qed

lemma *prime-dvd-fact-iff*:
assumes *prime p*
shows *p dvd fact n* ⟷ *p* ≤ *n*
using *assms*
by (*auto simp add*: *prime-factorization-subset-iff-dvd* [*symmetric*]
prime-factorization-prime prime-factors-fact prime-ge-2-nat)

lemma *dvd-choose-prime*:
assumes *kn*: *k* < *n* **and** *k*: *k* ≠ *0* **and** *n*: *n* ≠ *0* **and** *prime-n*: *prime n*
shows *n dvd* (*n choose k*)
proof −
have *n dvd* (*fact n*) **by** (*simp add*: *fact-num-eq-if n*)
moreover have ¬ *n dvd* (*fact k* ∗ *fact* (*n*−*k*))
by (*metis prime-dvd-fact-iff prime-dvd-mult-iff assms neq0-conv diff-less linorder-not-less*)
moreover have (*fact n*::*nat*) = *fact k* ∗ *fact* (*n*−*k*) ∗ (*n choose k*)
using *binomial-fact-lemma kn* **by** *auto*
ultimately show *?thesis* **using** *prime-n*
by (*auto simp add*: *prime-dvd-mult-iff*)
qed

lemma (**in** *ring-1*) *minus-power-prime-CHAR*:
assumes *p* = *CHAR*(′*a*) *prime p*
shows (−*x* :: ′*a*) ^ *p* = −(*x* ^ *p*)
proof (*cases p* = *2*)
case *False*
have *prime p*
using *assms* **by** *blast*
hence *odd p*
using *prime-imp-coprime assms False coprime-right-2-iff-odd gcd-nat.strict-iff-not*
by *blast*
thus *?thesis*
by *simp*
qed (*use assms* **in** ‹*auto simp*: *uminus-CHAR-2*›)

3.7 Rings and fields with prime characteristic

We introduce some type classes for rings and fields with prime characteristic.

class *semiring-prime-char* = *semiring-1* +
assumes *prime-char-aux*: ∃ *n*. *prime n* ∧ *of-nat n* = (*0* :: ′*a*)
begin

```
lemma CHAR-pos [intro, simp]: CHAR('a) > 0
  using local.CHAR-pos-iff local.prime-char-aux prime-gt-0-nat by blast

lemma CHAR-nonzero [simp]: CHAR('a) ≠ 0
  using CHAR-pos by auto

lemma CHAR-prime [intro, simp]: prime CHAR('a)
 by (metis (mono-tags, lifting) gcd-nat.order-iff-strict local.of-nat-1 local.of-nat-eq-0-iff-char-dvd
       local.one-neq-zero local.prime-char-aux prime-nat-iff)

end

lemma semiring-prime-charI [intro?]:
  prime CHAR('a :: semiring-1) ⟹ OFCLASS('a, semiring-prime-char-class)
  by standard auto

lemma idom-prime-charI [intro?]:
  assumes CHAR('a :: idom) > 0
  shows   OFCLASS('a, semiring-prime-char-class)
proof
  show prime CHAR('a)
    using assms prime-CHAR-semidom by blast
qed

class comm-semiring-prime-char = comm-semiring-1 + semiring-prime-char
class comm-ring-prime-char = comm-ring-1 + semiring-prime-char
begin
subclass comm-semiring-prime-char ..
end
class idom-prime-char = idom + semiring-prime-char
begin
subclass comm-ring-prime-char ..
end

class field-prime-char = field +
  assumes pos-char-exists: ∃ n>0. of-nat n = (0 :: 'a)
begin
subclass idom-prime-char
  apply standard
 using pos-char-exists local.CHAR-pos-iff local.of-nat-CHAR local.prime-CHAR-semidom
by blast
end

lemma field-prime-charI [intro?]:
  n > 0 ⟹ of-nat n = (0 :: 'a :: field) ⟹ OFCLASS('a, field-prime-char-class)
  by standard auto

lemma field-prime-charI' [intro?]:
  CHAR('a :: field) > 0 ⟹ OFCLASS('a, field-prime-char-class)
```

by *standard auto*

3.8 Finite fields

class *finite-field* = *field-prime-char* + *finite*

lemma *finite-fieldI* [*intro?*]:
assumes *finite* (*UNIV* :: *'a* :: *field set*)
shows *OFCLASS*(*'a*, *finite-field-class*)
proof *standard*
show ∃ *n*>*0*. *of-nat n* = (*0* :: *'a*)
using *assms prime-CHAR-semidom*[**where** *?'a* = *'a*] *finite-imp-CHAR-pos*[*OF assms*]
by (*intro exI*[*of - CHAR*(*'a*)]) *auto*
qed *fact*+

On a finite field with n elements, taking the n-th power of an element is the identity. This is an obvious consequence of the fact that the multiplicative group of the field is a finite group of order $n - 1$, so $x \hat{} n = 1$ for any non-zero x.

Note that this result is sharp in the sense that the multiplicative group of a finite field is cyclic, i.e. it contains an element of order $n - 1$. (We don't prove this here.)

lemma *finite-field-power-card-eq-same*:
fixes *x* :: *'a* :: *finite-field*
shows *x* ^ *card* (*UNIV* :: *'a set*) = *x*
proof (*cases x* = *0*)
case *False*
have *x* ∗ (∏ *y*∈*UNIV*−{*0*}. *x* ∗ *y*) = *x* ∗ *x* ^ (*card* (*UNIV* :: *'a set*) − *1*) ∗ ∏(*UNIV*−{*0*})
by (*simp add*: *prod.distrib mult-ac*)
also have *x* ∗ *x* ^ (*card* (*UNIV* :: *'a set*) − *1*) = *x* ^ *Suc* (*card* (*UNIV* :: *'a set*) − *1*)
by (*subst power-Suc*) *auto*
also have *Suc* (*card* (*UNIV* :: *'a set*) − *1*) = *card* (*UNIV* :: *'a set*)
using *finite-UNIV-card-ge-0*[**where** *?'a* = *'a*] **by** *simp*
also have (∏ *y*∈*UNIV*−{*0*}. *x* ∗ *y*) = (∏ *y*∈*UNIV*−{*0*}. *y*)
by (*rule prod.reindex-bij-witness*[*of* - λ*y*. *y* / *x* λ*y*. *x* ∗ *y*]) (*use False* **in** *auto*)
finally show *?thesis*
by *simp*
qed (*use finite-UNIV-card-ge-0*[**where** *?'a* = *'a*] **in** *auto*)

lemma *finite-field-power-card-power-eq-same*:
fixes *x* :: *'a* :: *finite-field*
assumes *m* = *card* (*UNIV* :: *'a set*) ^ *n*
shows *x* ^ *m* = *x*
unfolding *assms*
by (*induction n*) (*simp-all add*: *finite-field-power-card-eq-same power-mult*)

```
class enum-finite-field = finite-field +
  fixes enum-finite-field :: nat ⇒ 'a
 assumes enum-finite-field: enum-finite-field ‘ {..<card (UNIV :: 'a set)} = UNIV
begin

lemma inj-on-enum-finite-field: inj-on enum-finite-field {..<card (UNIV :: 'a set)}
  using enum-finite-field by (simp add: eq-card-imp-inj-on)

end
```

To get rid of the pending sort hypotheses, we prove that the field with 2 elements is indeed a finite field.

```
typedef gf2 = {0, 1 :: nat}
  by auto

setup-lifting type-definition-gf2

instantiation gf2 :: field
begin
lift-definition zero-gf2 :: gf2 is 0 by auto
lift-definition one-gf2 :: gf2 is 1 by auto
lift-definition uminus-gf2 :: gf2 ⇒ gf2 is λx. x .
lift-definition plus-gf2 :: gf2 ⇒ gf2 ⇒ gf2 is λx y. if x = y then 0 else 1 by auto
lift-definition minus-gf2 :: gf2 ⇒ gf2 ⇒ gf2 is λx y. if x = y then 0 else 1 by
auto
lift-definition times-gf2 :: gf2 ⇒ gf2 ⇒ gf2 is λx y. x ∗ y by auto
lift-definition inverse-gf2 :: gf2 ⇒ gf2 is λx. x .
lift-definition divide-gf2 :: gf2 ⇒ gf2 ⇒ gf2 is λx y. x ∗ y by auto

instance
  by standard (transfer; fastforce)+

end

instance gf2 :: finite-field
proof
  interpret type-definition Rep-gf2 Abs-gf2 {0, 1 :: nat}
    by (rule type-definition-gf2)
  show finite (UNIV :: gf2 set)
    by (metis Abs-image finite.emptyI finite.insertI finite-imageI)
qed
```

3.9 The Freshman's Dream in rings of prime characteristic

```
lemma (in comm-semiring-1) freshmans-dream:
  fixes x y :: 'a and n :: nat
  assumes prime CHAR('a)
  assumes n-def: n = CHAR('a)
```

shows $(x + y)$ ^ $n = x$ ^ $n + y$ ^ n
proof −
interpret *comm-semiring-prime-char*
by *standard* (*auto intro*!: *exI*[*of* - *CHAR*(′*a*)] *assms*)
have $n > 0$
unfolding *n-def* **by** *simp*
have $(x + y)$ ^ $n = (\sum k \le n.$ *of-nat* (*n choose k*) ∗ x ^ k ∗ y ^ $(n - k))$
by (*rule binomial-ring*)
also have ... = $(\sum k \in \{0,n\}.$ *of-nat* (*n choose k*) ∗ x ^ k ∗ y ^ $(n - k))$
proof (*intro sum.mono-neutral-right ballI*)
fix *k* **assume** $k \in \{..n\} - \{0, n\}$
hence *k*: $k > 0$ $k < n$
by *auto*
have *CHAR*(′*a*) *dvd* (*n choose k*)
unfolding *n-def*
by (*rule dvd-choose-prime*) (*use k* **in** ‹*auto simp*: *n-def*›)
hence *of-nat* (*n choose k*) = (*0* :: ′*a*)
using *of-nat-eq-0-iff-char-dvd* **by** *blast*
thus *of-nat* (*n choose k*) ∗ x ^ k ∗ y ^ $(n - k) = 0$
by *simp*
qed *auto*
finally show *?thesis*
using ‹$n > 0$› **by** (*simp add*: *add-ac*)
qed

lemma (**in** *comm-semiring-1*) *freshmans-dream*′:
assumes [*simp*]: *prime CHAR*(′*a*) **and** $m =$ *CHAR*(′*a*) ^ n
shows $(x + y ::$ ′*a*) ^ $m = x$ ^ $m + y$ ^ m
unfolding *assms*(*2*)
proof (*induction n*)
case (*Suc n*)
have $(x + y)$ ^ (*CHAR*(′*a*) ^ n ∗ *CHAR*(′*a*)) = ((*x* + *y*) ^ (*CHAR*(′*a*) ^ *n*)) ^ *CHAR*(′*a*)
by (*rule power-mult*)
thus *?case*
by (*simp add*: *Suc.IH freshmans-dream Groups.mult-ac flip*: *power-mult*)
qed *auto*

lemma (**in** *comm-semiring-1*) *freshmans-dream-sum*:
fixes *f* :: ′*b* ⇒ ′*a*
assumes *prime CHAR*(′*a*) **and** $n =$ *CHAR*(′*a*)
shows *sum f A* ^ $n =$ *sum* (λ*i*. *f i* ^ *n*) *A*
using *assms*
by (*induct A rule*: *infinite-finite-induct*)
(*auto simp add*: *power-0-left freshmans-dream*)

lemma (**in** *comm-semiring-1*) *freshmans-dream-sum*′:
fixes *f* :: ′*b* ⇒ ′*a*
assumes *prime CHAR*(′*a*) $m =$ *CHAR*(′*a*) ^ n

shows *sum f A ^ m = sum (λi. f i ^ m) A*
using *assms*
by (*induction A rule*: *infinite-finite-induct*)
(*auto simp*: *freshmans-dream′ power-0-left*)

lemmas *prime-imp-coprime-nat = prime-imp-coprime*[**where** *?′a = nat*]
lemmas *prime-imp-coprime-int = prime-imp-coprime*[**where** *?′a = int*]
lemmas *prime-dvd-mult-nat = prime-dvd-mult-iff*[**where** *?′a = nat*]
lemmas *prime-dvd-mult-int = prime-dvd-mult-iff*[**where** *?′a = int*]
lemmas *prime-dvd-mult-eq-nat = prime-dvd-mult-iff*[**where** *?′a = nat*]
lemmas *prime-dvd-mult-eq-int = prime-dvd-mult-iff*[**where** *?′a = int*]
lemmas *prime-dvd-power-nat = prime-dvd-power*[**where** *?′a = nat*]
lemmas *prime-dvd-power-int = prime-dvd-power*[**where** *?′a = int*]
lemmas *prime-dvd-power-nat-iff = prime-dvd-power-iff*[**where** *?′a = nat*]
lemmas *prime-dvd-power-int-iff = prime-dvd-power-iff*[**where** *?′a = int*]
lemmas *prime-imp-power-coprime-nat = prime-imp-power-coprime*[**where** *?′a = nat*]
lemmas *prime-imp-power-coprime-int = prime-imp-power-coprime*[**where** *?′a = int*]
lemmas *primes-coprime-nat = primes-coprime*[**where** *?′a = nat*]
lemmas *primes-coprime-int = primes-coprime*[**where** *?′a = nat*]
lemmas *prime-divprod-pow-nat = prime-elem-divprod-pow*[**where** *?′a = nat*]
lemmas *prime-exp = prime-elem-power-iff*[**where** *?′a = nat*]

Code generation

context
begin

qualified definition *prime-nat* :: *nat ⇒ bool*
where [*simp*, *code-abbrev*]: *prime-nat = prime*

lemma *prime-nat-naive* [*code*]:
prime-nat p ⟷ p > 1 ∧ (∀ n ∈{1<..<p}. ¬ n dvd p)
by (*auto simp add*: *prime-nat-iff′*)

qualified definition *prime-int* :: *int ⇒ bool*
where [*simp*, *code-abbrev*]: *prime-int = prime*

lemma *prime-int-naive* [*code*]:
prime-int p ⟷ p > 1 ∧ (∀ n ∈{1<..<p}. ¬ n dvd p)
by (*auto simp add*: *prime-int-iff′*)

lemma *prime(997::nat)* **by** *eval*

lemma *prime(997::int)* **by** *eval*

end

end

4 Polynomials as type over a ring structure

theory *Polynomial*
imports
  Complex-Main
  HOL−Library.More-List
  HOL−Library.Infinite-Set
  Primes
begin

context *semidom-modulo*
begin

lemma *not-dvd-imp-mod-neq-0*:
  ‹*a mod b* ≠ *0*› **if** ‹¬ *b dvd a*›
  using *that mod-0-imp-dvd* [*of a b*] **by** *blast*

end

4.1 Auxiliary: operations for lists (later) representing coefficients

definition *cCons* :: *′a*::*zero* ⇒ *′a list* ⇒ *′a list* (**infixr** ‹##› *65*)
  where *x* ## *xs* = (*if xs* = [] ∧ *x* = *0 then* [] *else x* # *xs*)

lemma *cCons-0-Nil-eq* [*simp*]: *0* ## [] = []
  by (*simp add*: *cCons-def*)

lemma *cCons-Cons-eq* [*simp*]: *x* ## *y* # *ys* = *x* # *y* # *ys*
  by (*simp add*: *cCons-def*)

lemma *cCons-append-Cons-eq* [*simp*]: *x* ## *xs* @ *y* # *ys* = *x* # *xs* @ *y* # *ys*
  by (*simp add*: *cCons-def*)

lemma *cCons-not-0-eq* [*simp*]: *x* ≠ *0* ⟹ *x* ## *xs* = *x* # *xs*
  by (*simp add*: *cCons-def*)

lemma *strip-while-not-0-Cons-eq* [*simp*]:
  strip-while (λ*x*. *x* = *0*) (*x* # *xs*) = *x* ## *strip-while* (λ*x*. *x* = *0*) *xs*
proof (*cases x* = *0*)
  case *False*
  then show *?thesis* **by** *simp*
next
  case *True*
  show *?thesis*

proof (*induct xs rule*: *rev-induct*)
  case *Nil*
  with *True* **show** *?case* **by** *simp*
next
  case (*snoc y ys*)
  then show *?case*
    by (*cases y* = *0*) (*simp-all add*: *append-Cons* [*symmetric*] *del*: *append-Cons*)
qed
qed

lemma *tl-cCons* [*simp*]: *tl* (*x* ## *xs*) = *xs*
  by (*simp add*: *cCons-def*)

4.2 Definition of type *poly*

typedef (**overloaded**) *'a poly* = {*f* :: *nat* ⇒ *'a*::*zero*. $\forall_\infty$ *n*. *f n* = *0*}
  morphisms *coeff Abs-poly*
  by (*auto intro*!: *ALL-MOST*)

setup-lifting *type-definition-poly*

lemma *poly-eq-iff*: *p* = *q* ⟷ (∀ *n*. *coeff p n* = *coeff q n*)
  by (*simp add*: *coeff-inject* [*symmetric*] *fun-eq-iff*)

lemma *poly-eqI*: (⋀*n*. *coeff p n* = *coeff q n*) ⟹ *p* = *q*
  by (*simp add*: *poly-eq-iff*)

lemma *MOST-coeff-eq-0*: $\forall_\infty$ *n*. *coeff p n* = *0*
  using *coeff* [*of p*] **by** *simp*

lemma *coeff-Abs-poly*:
  assumes ⋀*i*. *i* > *n* ⟹ *f i* = *0*
  shows *coeff* (*Abs-poly f*) = *f*
proof (*rule Abs-poly-inverse*, *clarify*)
  have *eventually* (λ*i*. *i* > *n*) *cofinite*
    by (*auto simp*: *MOST-nat*)
  thus *eventually* (λ*i*. *f i* = *0*) *cofinite*
    by *eventually-elim* (*use assms* **in** *auto*)
qed

4.3 Degree of a polynomial

definition *degree* :: *'a*::*zero poly* ⇒ *nat*
  where *degree p* = (*LEAST n*. ∀ *i*>*n*. *coeff p i* = *0*)

lemma *degree-cong*:
  assumes ⋀*i*. *coeff p i* = *0* ⟷ *coeff q i* = *0*
  shows *degree p* = *degree q*
proof −
  have (λ*n*. ∀ *i*>*n*. *poly.coeff p i* = *0*) = (λ*n*. ∀ *i*>*n*. *poly.coeff q i* = *0*)

using *assms* **by** (*auto simp*: *fun-eq-iff*)
thus *?thesis*
by (*simp only*: *degree-def*)
qed

lemma *coeff-Abs-poly-If-le*:
coeff (*Abs-poly* (λi. *if* $i \leq n$ *then f i else 0*)) = (λi. *if* $i \leq n$ *then f i else 0*)
proof (*rule Abs-poly-inverse*, *clarify*)
have *eventually* (λi. $i > n$) *cofinite*
by (*auto simp*: *MOST-nat*)
thus *eventually* (λi. (*if* $i \leq n$ *then f i else 0*) = *0*) *cofinite*
by *eventually-elim auto*
qed

lemma *coeff-eq-0*:
assumes *degree p* < *n*
shows *coeff p n* = *0*
proof −
have $\exists$ *n*. $\forall$ *i*>*n*. *coeff p i* = *0*
using *MOST-coeff-eq-0* **by** (*simp add*: *MOST-nat*)
then have $\forall$ *i*>*degree p*. *coeff p i* = *0*
unfolding *degree-def* **by** (*rule LeastI-ex*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *le-degree*: *coeff p n* $\neq$ *0* $\Longrightarrow$ *n* $\leq$ *degree p*
using *coeff-eq-0 linorder-le-less-linear* **by** *blast*

lemma *degree-le*: $\forall$ *i*>*n*. *coeff p i* = *0* $\Longrightarrow$ *degree p* $\leq$ *n*
unfolding *degree-def* **by** (*erule Least-le*)

lemma *less-degree-imp*: *n* < *degree p* $\Longrightarrow$ $\exists$ *i*>*n*. *coeff p i* $\neq$ *0*
unfolding *degree-def* **by** (*drule not-less-Least*, *simp*)

lemma *poly-eqI2*:
assumes *degree p* = *degree q* **and** $\bigwedge$*i*. *i* $\leq$ *degree p* $\Longrightarrow$ *coeff p i* = *coeff q i*
shows *p* = *q*
by (*metis assms le-degree poly-eqI*)

4.4 The zero polynomial

instantiation *poly* :: (*zero*) *zero*
begin

lift-definition *zero-poly* :: *'a poly*
is λ-. *0*
by (*rule MOST-I*) *simp*

instance ..

end

lemma *coeff-0* [*simp*]: *coeff 0 n* = *0*
  by *transfer rule*

lemma *degree-0* [*simp*]: *degree 0* = *0*
  by (*rule order-antisym* [*OF degree-le le0*]) *simp*

lemma *leading-coeff-neq-0*:
  assumes *p* ≠ *0*
  shows *coeff p* (*degree p*) ≠ *0*
proof (*cases degree p*)
  case *0*
  from ‹*p* ≠ *0*› **obtain** *n* **where** *coeff p n* ≠ *0*
    by (*auto simp add*: *poly-eq-iff*)
  then have *n* ≤ *degree p*
    by (*rule le-degree*)
  with ‹*coeff p n* ≠ *0*› **and** ‹*degree p* = *0*› **show** *coeff p* (*degree p*) ≠ *0*
    by *simp*
next
  case (*Suc n*)
  from ‹*degree p* = *Suc n*› **have** *n* < *degree p*
    by *simp*
  then have ∃ *i*>*n*. *coeff p i* ≠ *0*
    by (*rule less-degree-imp*)
  then obtain *i* **where** *n* < *i* **and** *coeff p i* ≠ *0*
    by *blast*
  from ‹*degree p* = *Suc n*› **and** ‹*n* < *i*› **have** *degree p* ≤ *i*
    by *simp*
  also from ‹*coeff p i* ≠ *0*› **have** *i* ≤ *degree p*
    by (*rule le-degree*)
  finally have *degree p* = *i* **.**
  with ‹*coeff p i* ≠ *0*› **show** *coeff p* (*degree p*) ≠ *0* **by** *simp*
qed

lemma *leading-coeff-0-iff* [*simp*]: *coeff p* (*degree p*) = *0* ⟷ *p* = *0*
  by (*cases p* = *0*) (*simp-all add*: *leading-coeff-neq-0*)

lemma *degree-lessI*:
  assumes *p* ≠ *0* ∨ *n* > *0* ∀ *k*≥*n*. *coeff p k* = *0*
  shows   *degree p* < *n*
proof (*cases p* = *0*)
  case *False*
  show *?thesis*
  proof (*rule ccontr*)
    assume ∗: ¬(*degree p* < *n*)
    define *d* **where** *d* = *degree p*
    from ‹*p* ≠ *0*› **have** *coeff p d* ≠ *0*

    by (*auto simp*: *d-def*)
  moreover have *coeff p d* = *0*
    using *assms*(*2*) ∗ **by** (*auto simp*: *not-less*)
  ultimately show *False* **by** *contradiction*
 qed
qed (*use assms* **in** *auto*)

lemma *eq-zero-or-degree-less*:
 assumes *degree p* ≤ *n* **and** *coeff p n* = *0*
 shows *p* = *0* ∨ *degree p* < *n*
proof (*cases n*)
 case *0*
 with ‹*degree p* ≤ *n*› **and** ‹*coeff p n* = *0*› **have** *coeff p* (*degree p*) = *0*
  by *simp*
 then have *p* = *0* **by** *simp*
 then show *?thesis* **..**
next
 case (*Suc m*)
 from ‹*degree p* ≤ *n*› **have** ∀ *i*>*n*. *coeff p i* = *0*
  by (*simp add*: *coeff-eq-0*)
 with ‹*coeff p n* = *0*› **have** ∀ *i*≥*n*. *coeff p i* = *0*
  by (*simp add*: *le-less*)
 with ‹*n* = *Suc m*› **have** ∀ *i*>*m*. *coeff p i* = *0*
  by (*simp add*: *less-eq-Suc-le*)
 then have *degree p* ≤ *m*
  by (*rule degree-le*)
 with ‹*n* = *Suc m*› **have** *degree p* < *n*
  by (*simp add*: *less-Suc-eq-le*)
 then show *?thesis* **..**
qed

lemma *coeff-0-degree-minus-1*: *coeff rrr dr* = *0* ⟹ *degree rrr* ≤ *dr* ⟹ *degree rrr* ≤ *dr* − *1*
 using *eq-zero-or-degree-less* **by** *fastforce*

4.5 List-style constructor for polynomials

lift-definition *pCons* :: *'a*::*zero* ⇒ *'a poly* ⇒ *'a poly*
 is λ*a p*. *case-nat a* (*coeff p*)
 by (*rule MOST-SucD*) (*simp add*: *MOST-coeff-eq-0*)

lemmas *coeff-pCons* = *pCons.rep-eq*

lemma *coeff-pCons'*: *poly.coeff* (*pCons c p*) *n* = (*if n* = *0 then c else poly.coeff p* (*n* − *1*))
 by *transfer'*(*auto split*: *nat.splits*)

lemma *coeff-pCons-0* [*simp*]: *coeff* (*pCons a p*) *0* = *a*
 by *transfer simp*

lemma *coeff-pCons-Suc* [*simp*]: *coeff* (*pCons a p*) (*Suc n*) = *coeff p n*
  by (*simp add*: *coeff-pCons*)

lemma *degree-pCons-le*: *degree* (*pCons a p*) ≤ *Suc* (*degree p*)
  by (*rule degree-le*) (*simp add*: *coeff-eq-0 coeff-pCons split*: *nat.split*)

lemma *degree-pCons-eq*: *p* ≠ *0* ⟹ *degree* (*pCons a p*) = *Suc* (*degree p*)
  by (*simp add*: *degree-pCons-le le-antisym le-degree*)

lemma *degree-pCons-0*: *degree* (*pCons a 0*) = *0*
proof −
  have *degree* (*pCons a 0*) ≤ *Suc 0*
    by (*metis* (*no-types*) *degree-0 degree-pCons-le*)
  then show *?thesis*
    by (*metis coeff-0 coeff-pCons-Suc degree-0 eq-zero-or-degree-less less-Suc0*)
qed

lemma *degree-pCons-eq-if* [*simp*]: *degree* (*pCons a p*) = (*if p* = *0 then 0 else Suc* (*degree p*))
  by (*simp add*: *degree-pCons-0 degree-pCons-eq*)

lemma *pCons-0-0* [*simp*]: *pCons 0 0* = *0*
  by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *pCons-eq-iff* [*simp*]: *pCons a p* = *pCons b q* ⟷ *a* = *b* ∧ *p* = *q*
proof *safe*
  assume *pCons a p* = *pCons b q*
  then have *coeff* (*pCons a p*) *0* = *coeff* (*pCons b q*) *0*
    by *simp*
  then show *a* = *b*
    by *simp*
next
  assume *pCons a p* = *pCons b q*
  then have *coeff* (*pCons a p*) (*Suc n*) = *coeff* (*pCons b q*) (*Suc n*) **for** *n*
    by *simp*
  then show *p* = *q*
    by (*simp add*: *poly-eq-iff*)
qed

lemma *pCons-eq-0-iff* [*simp*]: *pCons a p* = *0* ⟷ *a* = *0* ∧ *p* = *0*
  using *pCons-eq-iff* [*of a p 0 0*] **by** *simp*

lemma *pCons-cases* [*cases type*: *poly*]:
  obtains (*pCons*) *a q* **where** *p* = *pCons a q*
proof
  show *p* = *pCons* (*coeff p 0*) (*Abs-poly* (λ*n*. *coeff p* (*Suc n*)))
    by *transfer*
      (*simp-all add*: *MOST-inj*[**where** *f*=*Suc* **and** *P*=λ*n*. *p n* = *0* **for** *p*] *fun-eq-iff*

Abs-poly-inverse
    split: *nat.split*)
qed

lemma *pCons-induct* [*case-names 0 pCons*, *induct type*: *poly*]:
  assumes *zero*: *P 0*
  assumes *pCons*: $\bigwedge a\ p.\ a \neq 0 \vee p \neq 0 \Longrightarrow P\ p \Longrightarrow P\ (pCons\ a\ p)$
  shows *P p*
proof (*induct p rule*: *measure-induct-rule* [**where** *f*=*degree*])
  case (*less p*)
  obtain *a q* **where** *p* = *pCons a q* **by** (*rule pCons-cases*)
  have *P q*
  proof (*cases q* = *0*)
    case *True*
    then show *P q* **by** (*simp add*: *zero*)
  next
    case *False*
    then have *degree* (*pCons a q*) = *Suc* (*degree q*)
      by (*rule degree-pCons-eq*)
    with ‹*p* = *pCons a q*› **have** *degree q* < *degree p*
      by *simp*
    then show *P q*
      by (*rule less.hyps*)
  qed
  have *P* (*pCons a q*)
  proof (*cases* $a \neq 0 \vee q \neq 0$)
    case *True*
    with ‹*P q*› **show** *?thesis* **by** (*auto intro*: *pCons*)
  next
    case *False*
    with *zero* **show** *?thesis* **by** *simp*
  qed
  with ‹*p* = *pCons a q*› **show** *?case*
    by *simp*
qed

lemma *degree-eq-zeroE*:
  fixes *p* :: *'a*::*zero poly*
  assumes *degree p* = *0*
  obtains *a* **where** *p* = *pCons a 0*
proof −
  obtain *a q* **where** *p*: *p* = *pCons a q*
    by (*cases p*)
  with *assms* **have** *q* = *0*
    by (*cases q* = *0*) *simp-all*
  with *p* **have** *p* = *pCons a 0*
    by *simp*
  then show *thesis* **..**
qed

4.6 Quickcheck generator for polynomials

quickcheck-generator *poly constructors*: *0* :: - *poly*, *pCons*

4.7 List-style syntax for polynomials

syntax
-poly :: *args* ⇒ ′*a poly* (‹(‹*indent=2 notation=*‹*mixfix polynomial enumeration*››[:-:])›)
syntax-consts
-poly ⇌ *pCons*
translations
[:*x*, *xs*:] ⇌ *CONST pCons x* [:*xs*:]
[:*x*:] ⇌ *CONST pCons x 0*

lemma *degree-0-id*:
assumes *degree p* = *0*
shows [: *coeff p 0* :] = *p*
by (*metis assms coeff-pCons-0 degree-eq-zeroE*)

lemma *degree0-coeffs*: *degree p* = *0* ⟹ ∃ *a*. *p* = [: *a* :]
by (*meson degree-eq-zeroE*)

lemma *degree1-coeffs*:
fixes *p* :: ′*a*::*zero poly*
assumes *degree p* = *1*
obtains *a b* **where** *p* = [: *b*, *a* :] *a* ≠ *0*
proof −
obtain *b a q* **where** *p* = *pCons b q q* = *pCons a 0*
by (*metis assms degree0-coeffs degree-0 degree-pCons-eq-if lessI less-one pCons-cases*)
then show *thesis*
using *assms that* **by** *force*
qed

lemma *degree2-coeffs*:
fixes *p* :: ′*a*::*zero poly*
assumes *degree p* = *2*
obtains *a b c* **where** *p* = [: *c*, *b*, *a* :] *a* ≠ *0*
proof −
obtain *c q* **where** *p* = *pCons c q degree q* = *1*
by (*metis One-nat-def assms degree-0 degree-pCons-eq-if fact-0 fact-2 nat.inject numeral-2-eq-2 pCons-cases*)
then show *thesis*
by (*metis degree1-coeffs that*)
qed

4.8 Representation of polynomials by lists of coefficients

primrec *Poly* :: ′*a*::*zero list* ⇒ ′*a poly*
where

[*code-post*]: *Poly* [] = *0*
| [*code-post*]: *Poly* (*a* # *as*) = *pCons a* (*Poly as*)

lemma *Poly-replicate-0* [*simp*]: *Poly* (*replicate n 0*) = *0*
by (*induct n*) *simp-all*

lemma *Poly-eq-0*: *Poly as* = *0* ⟷ (∃ *n*. *as* = *replicate n 0*)
by (*induct as*) (*auto simp add*: *Cons-replicate-eq*)

lemma *Poly-append-replicate-zero* [*simp*]: *Poly* (*as* @ *replicate n 0*) = *Poly as*
by (*induct as*) *simp-all*

lemma *Poly-snoc-zero* [*simp*]: *Poly* (*as* @ [*0*]) = *Poly as*
using *Poly-append-replicate-zero* [*of as 1*] **by** *simp*

lemma *Poly-cCons-eq-pCons-Poly* [*simp*]: *Poly* (*a* ## *p*) = *pCons a* (*Poly p*)
by (*simp add*: *cCons-def*)

lemma *Poly-on-rev-starting-with-0* [*simp*]: *hd as* = *0* ⟹ *Poly* (*rev* (*tl as*)) = *Poly* (*rev as*)
by (*cases as*) *simp-all*

lemma *degree-Poly*: *degree* (*Poly xs*) ≤ *length xs*
by (*induct xs*) *simp-all*

lemma *coeff-Poly-eq* [*simp*]: *coeff* (*Poly xs*) = *nth-default 0 xs*
by (*induct xs*) (*simp-all add*: *fun-eq-iff coeff-pCons split*: *nat.splits*)

definition *coeffs* :: *'a poly* ⇒ *'a::zero list*
where *coeffs p* = (*if p* = *0 then* [] *else map* (λ*i*. *coeff p i*) [*0* ..< *Suc* (*degree p*)])

lemma *coeffs-eq-Nil* [*simp*]: *coeffs p* = [] ⟷ *p* = *0*
by (*simp add*: *coeffs-def*)

lemma *not-0-coeffs-not-Nil*: *p* ≠ *0* ⟹ *coeffs p* ≠ []
by *simp*

lemma *coeffs-0-eq-Nil* [*simp*]: *coeffs 0* = []
by *simp*

lemma *coeffs-pCons-eq-cCons* [*simp*]: *coeffs* (*pCons a p*) = *a* ## *coeffs p*
proof −
have ∗: ∀ *m*∈*set ms*. *m* > *0* ⟹ *map* (*case-nat x f*) *ms* = *map f* (*map* (λ*n*. *n* − *1*) *ms*)
for *ms* :: *nat list* **and** *f* :: *nat* ⇒ *'a* **and** *x* :: *'a*
by (*induct ms*) (*auto split*: *nat.split*)
show *?thesis*
by (*simp add*: ∗ *coeffs-def upt-conv-Cons coeff-pCons map-decr-upt del*: *upt-Suc*)
qed

lemma *length-coeffs*: *p* ≠ *0* ⟹ *length* (*coeffs p*) = *degree p* + *1*
  by (*simp add*: *coeffs-def*)

lemma *coeffs-nth*: *p* ≠ *0* ⟹ *n* ≤ *degree p* ⟹ *coeffs p* ! *n* = *coeff p n*
  by (*auto simp*: *coeffs-def simp del*: *upt-Suc*)

lemma *coeff-in-coeffs*: *p* ≠ *0* ⟹ *n* ≤ *degree p* ⟹ *coeff p n* ∈ *set* (*coeffs p*)
  using *coeffs-nth* [*of p n*, *symmetric*] **by** (*simp add*: *length-coeffs*)

lemma *not-0-cCons-eq* [*simp*]: *p* ≠ *0* ⟹ *a* ## *coeffs p* = *a* # *coeffs p*
  by (*simp add*: *cCons-def*)

lemma *Poly-coeffs* [*simp*, *code abstype*]: *Poly* (*coeffs p*) = *p*
  by (*induct p*) *auto*

lemma *coeffs-Poly* [*simp*]: *coeffs* (*Poly as*) = *strip-while* (*HOL.eq 0*) *as*
proof (*induct as*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case (*Cons a as*)
  from *replicate-length-same* [*of as 0*] **have** (∀ *n*. *as* ≠ *replicate n 0*) ⟷ (∃ *a*∈*set as*. *a* ≠ *0*)
    by (*auto dest*: *sym* [*of - as*])
  with *Cons* **show** *?case* **by** *auto*
qed

lemma *no-trailing-coeffs* [*simp*]:
  no-trailing (*HOL.eq 0*) (*coeffs p*)
  by (*induct p*) *auto*

lemma *strip-while-coeffs* [*simp*]:
  strip-while (*HOL.eq 0*) (*coeffs p*) = *coeffs p*
  by *simp*

lemma *coeffs-eq-iff*: *p* = *q* ⟷ *coeffs p* = *coeffs q*
  (**is** *?P* ⟷ *?Q*)
proof
  assume *?P*
  then show *?Q* **by** *simp*
next
  assume *?Q*
  then have *Poly* (*coeffs p*) = *Poly* (*coeffs q*) **by** *simp*
  then show *?P* **by** *simp*
qed

lemma *nth-default-coeffs-eq*: *nth-default 0* (*coeffs p*) = *coeff p*
  by (*metis Poly-coeffs coeff-Poly-eq*)

lemma *range-coeff*: *range* (*coeff p*) = *insert 0* (*set* (*coeffs p*))
  by (*metis nth-default-coeffs-eq range-nth-default*)

lemma [*code*]: *coeff p* = *nth-default 0* (*coeffs p*)
  by (*simp add*: *nth-default-coeffs-eq*)

lemma *coeffs-eqI*:
  assumes *coeff*: $\bigwedge$*n. coeff p n* = *nth-default 0 xs n*
  assumes *zero*: *no-trailing* (*HOL.eq 0*) *xs*
  shows *coeffs p* = *xs*
proof −
  from *coeff* **have** *p* = *Poly xs*
    by (*simp add*: *poly-eq-iff*)
  with *zero* **show** *?thesis* **by** *simp*
qed

lemma *degree-eq-length-coeffs* [*code*]: *degree p* = *length* (*coeffs p*) − *1*
  by (*simp add*: *coeffs-def*)

lemma *length-coeffs-degree*: *p* ≠ *0* ⟹ *length* (*coeffs p*) = *Suc* (*degree p*)
  by (*induct p*) (*auto simp*: *cCons-def*)

lemma [*code abstract*]: *coeffs 0* = []
  by (*fact coeffs-0-eq-Nil*)

lemma [*code abstract*]: *coeffs* (*pCons a p*) = *a* ## *coeffs p*
  by (*fact coeffs-pCons-eq-cCons*)

lemma *set-coeffs-subset-singleton-0-iff* [*simp*]:
  set (*coeffs p*) ⊆ {*0*} ⟷ *p* = *0*
  by (*auto simp add*: *coeffs-def intro*: *classical*)

lemma *set-coeffs-not-only-0* [*simp*]:
  set (*coeffs p*) ≠ {*0*}
  by (*auto simp add*: *set-eq-subset*)

lemma *forall-coeffs-conv*:
  (∀ *n. P* (*coeff p n*)) ⟷ (∀ *c* ∈ *set* (*coeffs p*). *P c*) **if** *P 0*
  using *that* **by** (*auto simp add*: *coeffs-def*)
    (*metis atLeastLessThan-iff coeff-eq-0 not-less-iff-gr-or-eq zero-le*)

instantiation *poly* :: ({*zero*, *equal*}) *equal*
begin

definition [*code*]: *HOL.equal* (*p*::′*a poly*) *q* ⟷ *HOL.equal* (*coeffs p*) (*coeffs q*)

instance
  by *standard* (*simp add*: *equal equal-poly-def coeffs-eq-iff*)

end

lemma [*code nbe*]: *HOL.equal* (*p* :: - *poly*) *p* ⟷ *True*
  by (*fact equal-refl*)

definition *is-zero* :: *'a*::*zero poly* ⇒ *bool*
  where [*code*]: *is-zero p* ⟷ *List.null* (*coeffs p*)

lemma *is-zero-null* [*code-abbrev*]: *is-zero p* ⟷ *p* = *0*
  by (*simp add*: *is-zero-def null-def*)

Reconstructing the polynomial from the list

definition *poly-of-list* :: *'a*::*comm-monoid-add list* ⇒ *'a poly*
  where [*simp*]: *poly-of-list* = *Poly*

lemma *poly-of-list-impl* [*code abstract*]: *coeffs* (*poly-of-list as*) = *strip-while* (*HOL.eq* *0*) *as*
  by *simp*

4.9 Fold combinator for polynomials

definition *fold-coeffs* :: (*'a*::*zero* ⇒ *'b* ⇒ *'b*) ⇒ *'a poly* ⇒ *'b* ⇒ *'b*
  where *fold-coeffs f p* = *foldr f* (*coeffs p*)

lemma *fold-coeffs-0-eq* [*simp*]: *fold-coeffs f 0* = *id*
  by (*simp add*: *fold-coeffs-def*)

lemma *fold-coeffs-pCons-eq* [*simp*]: *f 0* = *id* ⟹ *fold-coeffs f* (*pCons a p*) = *f a* ∘ *fold-coeffs f p*
  by (*simp add*: *fold-coeffs-def cCons-def fun-eq-iff*)

lemma *fold-coeffs-pCons-0-0-eq* [*simp*]: *fold-coeffs f* (*pCons 0 0*) = *id*
  by (*simp add*: *fold-coeffs-def*)

lemma *fold-coeffs-pCons-coeff-not-0-eq* [*simp*]:
  a ≠ *0* ⟹ *fold-coeffs f* (*pCons a p*) = *f a* ∘ *fold-coeffs f p*
  by (*simp add*: *fold-coeffs-def*)

lemma *fold-coeffs-pCons-not-0-0-eq* [*simp*]:
  p ≠ *0* ⟹ *fold-coeffs f* (*pCons a p*) = *f a* ∘ *fold-coeffs f p*
  by (*simp add*: *fold-coeffs-def*)

4.10 Canonical morphism on polynomials – evaluation

definition *poly* :: ‹*'a*::*comm-semiring-0 poly* ⇒ *'a* ⇒ *'a*›
  where ‹*poly p a* = *horner-sum id a* (*coeffs p*)›

lemma *poly-eq-fold-coeffs*:

‹*poly p* = *fold-coeffs* (λ*a f x*. *a* + *x* ∗ *f x*) *p* (λ*x*. *0*)›
by (*induction p*) (*auto simp add*: *fun-eq-iff poly-def*)

lemma *poly-0* [*simp*]: *poly 0 x* = *0*
by (*simp add*: *poly-def*)

lemma *poly-pCons* [*simp*]: *poly* (*pCons a p*) *x* = *a* + *x* ∗ *poly p x*
by (*cases p* = *0* ∧ *a* = *0*) (*auto simp add*: *poly-def*)

lemma *poly-altdef*: *poly p x* = (∑ *i*≤*degree p*. *coeff p i* ∗ *x* ^ *i*)
for *x* :: *'a*::{*comm-semiring-0*,*semiring-1*}
proof (*induction p rule*: *pCons-induct*)
case *0*
then show *?case*
by *simp*
next
case (*pCons a p*)
show *?case*
proof (*cases p* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
let *?p'* = *pCons a p*
note *poly-pCons*[*of a p x*]
also note *pCons.IH*
also have *a* + *x* ∗ (∑ *i*≤*degree p*. *coeff p i* ∗ *x* ^ *i*) =
coeff ?p' 0 ∗ *x*^*0* + (∑ *i*≤*degree p*. *coeff ?p'* (*Suc i*) ∗ *x*^*Suc i*)
by (*simp add*: *field-simps sum-distrib-left coeff-pCons*)
also note *sum.atMost-Suc-shift*[*symmetric*]
also note *degree-pCons-eq*[*OF* ‹*p* ≠ *0*›, *of a*, *symmetric*]
finally show *?thesis* **.**
qed
qed

lemma *poly-0-coeff-0*: *poly p 0* = *coeff p 0*
by (*cases p*) (*auto simp*: *poly-altdef*)

lemma *poly-zero*:
fixes *p* :: *'a* :: *comm-ring-1 poly*
assumes *x*: *poly p x* = *0* **shows** *p* = *0* ⟷ *degree p* = *0*
proof
assume *degp*: *degree p* = *0*
hence *poly p x* = *coeff p* (*degree p*) **by**(*subst degree-0-id*[*OF degp*,*symmetric*], *simp*)
hence *coeff p* (*degree p*) = *0* **using** *x* **by** *auto*
thus *p* = *0* **by** *auto*
qed *auto*

4.11 Monomials

lift-definition *monom* :: *'a* ⇒ *nat* ⇒ *'a::zero poly*
  is $\lambda a\ m\ n.$ *if* $m = n$ *then a else 0*
  by (*simp add*: *MOST-iff-cofinite*)

lemma *coeff-monom* [*simp*]: *coeff* (*monom a m*) *n* = (*if m* = *n then a else 0*)
  by *transfer rule*

lemma *monom-0*: *monom a 0* = [:*a*:]
  by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *monom-Suc*: *monom a* (*Suc n*) = *pCons 0* (*monom a n*)
  by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *monom-eq-0* [*simp*]: *monom 0 n* = *0*
  by (*rule poly-eqI*) *simp*

lemma *monom-eq-0-iff* [*simp*]: *monom a n* = *0* ⟷ *a* = *0*
  by (*simp add*: *poly-eq-iff*)

lemma *monom-eq-iff* [*simp*]: *monom a n* = *monom b n* ⟷ *a* = *b*
  by (*simp add*: *poly-eq-iff*)

lemma *degree-monom-le*: *degree* (*monom a n*) ≤ *n*
  by (*rule degree-le*, *simp*)

lemma *degree-monom-eq*: *a* ≠ *0* ⟹ *degree* (*monom a n*) = *n*
  by (*metis coeff-monom leading-coeff-0-iff*)

lemma *coeffs-monom* [*code abstract*]:
  coeffs (*monom a n*) = (*if a* = *0 then* [] *else replicate n 0* @ [*a*])
  by (*induct n*) (*simp-all add*: *monom-0 monom-Suc*)

lemma *fold-coeffs-monom* [*simp*]: *a* ≠ *0* ⟹ *fold-coeffs f* (*monom a n*) = *f 0* ^^ *n* ∘ *f a*
  by (*simp add*: *fold-coeffs-def coeffs-monom fun-eq-iff*)

lemma *poly-monom*: *poly* (*monom a n*) *x* = *a* ∗ *x* ^ *n*
  for *a x* :: *'a::comm-semiring-1*
  by (*cases a* = *0*, *simp-all*) (*induct n*, *simp-all add*: *mult.left-commute poly-eq-fold-coeffs*)

lemma *monom-eq-iff'*: *monom c n* = *monom d m* ⟷ *c* = *d* ∧ (*c* = *0* ∨ *n* = *m*)
  by (*auto simp*: *poly-eq-iff*)

lemma *monom-eq-const-iff*: *monom c n* = [:*d*:] ⟷ *c* = *d* ∧ (*c* = *0* ∨ *n* = *0*)
  using *monom-eq-iff'*[*of c n d 0*] **by** (*simp add*: *monom-0*)

4.12 Leading coefficient

abbreviation *lead-coeff* :: $'a$::*zero poly* $\Rightarrow$ $'a$
  where *lead-coeff p* $\equiv$ *coeff p* (*degree p*)

lemma *lead-coeff-pCons*[*simp*]:
  $p \neq 0 \Longrightarrow$ *lead-coeff* (*pCons a p*) = *lead-coeff p*
  $p = 0 \Longrightarrow$ *lead-coeff* (*pCons a p*) = *a*
  by *auto*

lemma *lead-coeff-monom* [*simp*]: *lead-coeff* (*monom c n*) = *c*
  by (*cases c* = *0*) (*simp-all add*: *degree-monom-eq*)

lemma *last-coeffs-eq-coeff-degree*:
  last (*coeffs p*) = *lead-coeff p* **if** $p \neq 0$
  using *that* **by** (*simp add*: *coeffs-def*)

4.13 Addition and subtraction

instantiation *poly* :: (*comm-monoid-add*) *comm-monoid-add*
begin

lift-definition *plus-poly* :: $'a$ *poly* $\Rightarrow$ $'a$ *poly* $\Rightarrow$ $'a$ *poly*
  is $\lambda p\ q\ n.$ *coeff p n* + *coeff q n*
proof −
  fix *q p* :: $'a$ *poly*
  show $\forall_\infty n.$ *coeff p n* + *coeff q n* = *0*
    using *MOST-coeff-eq-0*[*of p*] *MOST-coeff-eq-0*[*of q*] **by** *eventually-elim simp*
qed

lemma *coeff-add* [*simp*]: *coeff* (*p* + *q*) *n* = *coeff p n* + *coeff q n*
  by (*simp add*: *plus-poly.rep-eq*)

instance
proof
  fix *p q r* :: $'a$ *poly*
  show (*p* + *q*) + *r* = *p* + (*q* + *r*)
    by (*simp add*: *poly-eq-iff add.assoc*)
  show *p* + *q* = *q* + *p*
    by (*simp add*: *poly-eq-iff add.commute*)
  show *0* + *p* = *p*
    by (*simp add*: *poly-eq-iff*)
qed

end

instantiation *poly* :: (*cancel-comm-monoid-add*) *cancel-comm-monoid-add*
begin

lift-definition *minus-poly* :: $'a$ *poly* $\Rightarrow$ $'a$ *poly* $\Rightarrow$ $'a$ *poly*

is $\lambda p\ q\ n.\ coeff\ p\ n - coeff\ q\ n$
proof −
fix $q\ p :: 'a\ poly$
show $\forall_\infty n.\ coeff\ p\ n - coeff\ q\ n = 0$
using *MOST-coeff-eq-0*[*of p*] *MOST-coeff-eq-0*[*of q*] **by** *eventually-elim simp*
qed

lemma *coeff-diff* [*simp*]: $coeff\ (p - q)\ n = coeff\ p\ n - coeff\ q\ n$
by (*simp add*: *minus-poly.rep-eq*)

instance
proof
fix $p\ q\ r :: 'a\ poly$
show $p + q - p = q$
by (*simp add*: *poly-eq-iff*)
show $p - q - r = p - (q + r)$
by (*simp add*: *poly-eq-iff diff-diff-eq*)
qed

end

instantiation *poly* :: (*ab-group-add*) *ab-group-add*
begin

lift-definition *uminus-poly* :: $'a\ poly \Rightarrow 'a\ poly$
is $\lambda p\ n.\ - coeff\ p\ n$
proof −
fix $p :: 'a\ poly$
show $\forall_\infty n.\ - coeff\ p\ n = 0$
using *MOST-coeff-eq-0* **by** *simp*
qed

lemma *coeff-minus* [*simp*]: $coeff\ (- p)\ n = - coeff\ p\ n$
by (*simp add*: *uminus-poly.rep-eq*)

instance
proof
fix $p\ q :: 'a\ poly$
show $- p + p = 0$
by (*simp add*: *poly-eq-iff*)
show $p - q = p + - q$
by (*simp add*: *poly-eq-iff*)
qed

end

lemma *add-pCons* [*simp*]: $pCons\ a\ p + pCons\ b\ q = pCons\ (a + b)\ (p + q)$
by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *minus-pCons* [*simp*]: − *pCons a p* = *pCons* (− *a*) (− *p*)
  by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *diff-pCons* [*simp*]: *pCons a p* − *pCons b q* = *pCons* (*a* − *b*) (*p* − *q*)
  by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *degree-add-le-max*: *degree* (*p* + *q*) ≤ *max* (*degree p*) (*degree q*)
  by (*rule degree-le*) (*auto simp add*: *coeff-eq-0*)

lemma *degree-add-le*: *degree p* ≤ *n* ⟹ *degree q* ≤ *n* ⟹ *degree* (*p* + *q*) ≤ *n*
  by (*auto intro*: *order-trans degree-add-le-max*)

lemma *degree-add-less*: *degree p* < *n* ⟹ *degree q* < *n* ⟹ *degree* (*p* + *q*) < *n*
  by (*auto intro*: *le-less-trans degree-add-le-max*)

lemma *degree-add-eq-right*: **assumes** *degree p* < *degree q* **shows** *degree* (*p* + *q*) = *degree q*
proof (*cases q* = *0*)
  case *False*
  show *?thesis*
  proof (*rule order-antisym*)
    show *degree* (*p* + *q*) ≤ *degree q*
      by (*simp add*: *assms degree-add-le order.strict-implies-order*)
    show *degree q* ≤ *degree* (*p* + *q*)
      by (*simp add*: *False assms coeff-eq-0 le-degree*)
  qed
qed (*use assms* **in** *auto*)

lemma *degree-add-eq-left*: *degree q* < *degree p* ⟹ *degree* (*p* + *q*) = *degree p*
  using *degree-add-eq-right* [*of q p*] **by** (*simp add*: *add.commute*)

lemma *degree-minus* [*simp*]: *degree* (− *p*) = *degree p*
  by (*simp add*: *degree-def*)

lemma *lead-coeff-add-le*: *degree p* < *degree q* ⟹ *lead-coeff* (*p* + *q*) = *lead-coeff q*
  by (*metis coeff-add coeff-eq-0 monoid-add-class.add.left-neutral degree-add-eq-right*)

lemma *lead-coeff-minus*: *lead-coeff* (− *p*) = − *lead-coeff p*
  by (*metis coeff-minus degree-minus*)

lemma *degree-diff-le-max*: *degree* (*p* − *q*) ≤ *max* (*degree p*) (*degree q*)
  for *p q* :: *'a*::*ab-group-add poly*
  using *degree-add-le* [**where** *p*=*p* **and** *q*=−*q*] **by** *simp*

lemma *degree-diff-le*: *degree p* ≤ *n* ⟹ *degree q* ≤ *n* ⟹ *degree* (*p* − *q*) ≤ *n*
  for *p q* :: *'a*::*ab-group-add poly*
  using *degree-add-le* [*of p n* − *q*] **by** *simp*

lemma *degree-diff-less*: *degree p* < *n* ⟹ *degree q* < *n* ⟹ *degree* (*p* − *q*) < *n*

for *p q* :: *'a::ab-group-add poly*
using *degree-add-less* [*of p n* − *q*] **by** *simp*

lemma *add-monom*: *monom a n* + *monom b n* = *monom* (*a* + *b*) *n*
by (*rule poly-eqI*) *simp*

lemma *diff-monom*: *monom a n* − *monom b n* = *monom* (*a* − *b*) *n*
by (*rule poly-eqI*) *simp*

lemma *minus-monom*: − *monom a n* = *monom* (− *a*) *n*
by (*rule poly-eqI*) *simp*

lemma *coeff-sum*: *coeff* (∑ *x*∈*A*. *p x*) *i* = (∑ *x*∈*A*. *coeff* (*p x*) *i*)
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *monom-sum*: *monom* (∑ *x*∈*A*. *a x*) *n* = (∑ *x*∈*A*. *monom* (*a x*) *n*)
by (*rule poly-eqI*) (*simp add*: *coeff-sum*)

fun *plus-coeffs* :: *'a::comm-monoid-add list* ⇒ *'a list* ⇒ *'a list*
where
plus-coeffs xs [] = *xs*
| *plus-coeffs* [] *ys* = *ys*
| *plus-coeffs* (*x* # *xs*) (*y* # *ys*) = (*x* + *y*) ## *plus-coeffs xs ys*

lemma *coeffs-plus-eq-plus-coeffs* [*code abstract*]:
coeffs (*p* + *q*) = *plus-coeffs* (*coeffs p*) (*coeffs q*)
proof −
have ∗: *nth-default 0* (*plus-coeffs xs ys*) *n* = *nth-default 0 xs n* + *nth-default 0 ys n*
for *xs ys* :: *'a list* **and** *n*
proof (*induct xs ys arbitrary*: *n rule*: *plus-coeffs.induct*)
case (*3 x xs y ys n*)
then show *?case*
by (*cases n*) (*auto simp add*: *cCons-def*)
qed *simp-all*
have ∗∗: *no-trailing* (*HOL.eq 0*) (*plus-coeffs xs ys*)
if *no-trailing* (*HOL.eq 0*) *xs* **and** *no-trailing* (*HOL.eq 0*) *ys*
for *xs ys* :: *'a list*
using *that* **by** (*induct xs ys rule*: *plus-coeffs.induct*) (*simp-all add*: *cCons-def*)
show *?thesis*
by (*rule coeffs-eqI*) (*auto simp add*: ∗ *nth-default-coeffs-eq intro*: ∗∗)
qed

lemma *coeffs-uminus* [*code abstract*]:
coeffs (− *p*) = *map uminus* (*coeffs p*)
proof −
have *eq-0*: *HOL.eq 0* ∘ *uminus* = *HOL.eq* (*0::'a*)
by (*simp add*: *fun-eq-iff*)
show *?thesis*

by (*rule coeffs-eqI*) (*simp-all add*: *nth-default-map-eq nth-default-coeffs-eq no-trailing-map eq-0*)
qed

lemma [*code*]: $p - q = p + - q$
for $p\ q$ :: $'a$::*ab-group-add poly*
by (*fact diff-conv-add-uminus*)

lemma *poly-add* [*simp*]: *poly* $(p + q)$ x = *poly* p x + *poly* q x
proof (*induction p arbitrary*: *q*)
case (*pCons a p*)
then show *?case*
by (*cases q*) (*simp add*: *algebra-simps*)
qed *auto*

lemma *poly-minus* [*simp*]: *poly* $(- p)$ x = $-$ *poly* p x
for x :: $'a$::*comm-ring*
by (*induct p*) *simp-all*

lemma *poly-diff* [*simp*]: *poly* $(p - q)$ x = *poly* p x $-$ *poly* q x
for x :: $'a$::*comm-ring*
using *poly-add* [*of p* $-$ *q x*] **by** *simp*

lemma *poly-sum*: *poly* $(\sum k{\in}A.\ p\ k)$ x = $(\sum k{\in}A.$ *poly* $(p\ k)$ $x)$
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *poly-sum-list*: *poly* $(\sum p{\leftarrow}ps.\ p)$ y = $(\sum p{\leftarrow}ps.$ *poly* p $y)$
by (*induction ps*) *auto*

lemma *poly-sum-mset*: *poly* $(\sum x{\in}\#A.\ p\ x)$ y = $(\sum x{\in}\#A.$ *poly* $(p\ x)$ $y)$
by (*induction A*) *auto*

lemma *degree-sum-le*: *finite* $S \Longrightarrow (\bigwedge p.\ p \in S \Longrightarrow$ *degree* $(f\ p) \le n) \Longrightarrow$ *degree* (*sum f S*) $\le n$
proof (*induct S rule*: *finite-induct*)
case *empty*
then show *?case* **by** *simp*
next
case (*insert p S*)
then have *degree* (*sum f S*) $\le n$ *degree* $(f\ p) \le n$
by *auto*
then show *?case*
unfolding *sum.insert*[*OF insert*(*1*−*2*)] **by** (*metis degree-add-le*)
qed

lemma *degree-sum-less*:
assumes $\bigwedge x.\ x \in A \Longrightarrow$ *degree* $(f\ x) < n$ $n > 0$
shows *degree* (*sum f A*) $< n$
using *assms* **by** (*induction rule*: *infinite-finite-induct*) (*auto intro*!: *degree-add-less*)

lemma *poly-as-sum-of-monoms'*:
  assumes *degree p* $\leq$ *n*
  shows ($\sum$ *i*$\leq$*n. monom* (*coeff p i*) *i*) = *p*
proof −
  have *eq*: $\bigwedge$*i*. {..*n*} $\cap$ {*i*} = (*if i* $\leq$ *n then* {*i*} *else* {})
    by *auto*
  from *assms* **show** *?thesis*
    by (*simp add*: *poly-eq-iff coeff-sum coeff-eq-0 sum.If-cases eq*
      if-distrib[**where** *f*=λ*x. x* $*$ *a* **for** *a*])
qed

lemma *poly-as-sum-of-monoms*: ($\sum$ *i*$\leq$*degree p. monom* (*coeff p i*) *i*) = *p*
  by (*intro poly-as-sum-of-monoms' order-refl*)

lemma *Poly-snoc*: *Poly* (*xs* @ [*x*]) = *Poly xs* + *monom x* (*length xs*)
  by (*induct xs*) (*simp-all add*: *monom-0 monom-Suc*)

4.14 Multiplication by a constant, polynomial multiplication and the unit polynomial

lift-definition *smult* :: *'a*::*comm-semiring-0* ⇒ *'a poly* ⇒ *'a poly*
  is λ*a p n. a* $*$ *coeff p n*
proof −
  fix *a* :: *'a* **and** *p* :: *'a poly*
  show $\forall_\infty$ *i. a* $*$ *coeff p i* = *0*
    using *MOST-coeff-eq-0*[*of p*] **by** *eventually-elim simp*
qed

lemma *coeff-smult* [*simp*]: *coeff* (*smult a p*) *n* = *a* $*$ *coeff p n*
  by (*simp add*: *smult.rep-eq*)

lemma *degree-smult-le*: *degree* (*smult a p*) $\leq$ *degree p*
  by (*rule degree-le*) (*simp add*: *coeff-eq-0*)

lemma *smult-smult* [*simp*]: *smult a* (*smult b p*) = *smult* (*a* $*$ *b*) *p*
  by (*rule poly-eqI*) (*simp add*: *mult.assoc*)

lemma *smult-0-right* [*simp*]: *smult a 0* = *0*
  by (*rule poly-eqI*) *simp*

lemma *smult-0-left* [*simp*]: *smult 0 p* = *0*
  by (*rule poly-eqI*) *simp*

lemma *smult-1-left* [*simp*]: *smult* (*1*::*'a*::*comm-semiring-1*) *p* = *p*
  by (*rule poly-eqI*) *simp*

lemma *smult-add-right*: *smult a* (*p* + *q*) = *smult a p* + *smult a q*
  by (*rule poly-eqI*) (*simp add*: *algebra-simps*)

lemma *smult-add-left*: *smult* (*a* + *b*) *p* = *smult a p* + *smult b p*
by (*rule poly-eqI*) (*simp add*: *algebra-simps*)

lemma *smult-minus-right* [*simp*]: *smult a* (− *p*) = − *smult a p*
for *a* :: ′*a*::*comm-ring*
by (*rule poly-eqI*) *simp*

lemma *smult-minus-left* [*simp*]: *smult* (− *a*) *p* = − *smult a p*
for *a* :: ′*a*::*comm-ring*
by (*rule poly-eqI*) *simp*

lemma *smult-diff-right*: *smult a* (*p* − *q*) = *smult a p* − *smult a q*
for *a* :: ′*a*::*comm-ring*
by (*rule poly-eqI*) (*simp add*: *algebra-simps*)

lemma *smult-diff-left*: *smult* (*a* − *b*) *p* = *smult a p* − *smult b p*
for *a b* :: ′*a*::*comm-ring*
by (*rule poly-eqI*) (*simp add*: *algebra-simps*)

lemmas *smult-distribs* =
smult-add-left smult-add-right
smult-diff-left smult-diff-right

lemma *smult-pCons* [*simp*]: *smult a* (*pCons b p*) = *pCons* (*a* ∗ *b*) (*smult a p*)
by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *smult-monom*: *smult a* (*monom b n*) = *monom* (*a* ∗ *b*) *n*
by (*induct n*) (*simp-all add*: *monom-0 monom-Suc*)

lemma *smult-Poly*: *smult c* (*Poly xs*) = *Poly* (*map* ((∗) *c*) *xs*)
by (*auto simp*: *poly-eq-iff nth-default-def*)

lemma *degree-smult-eq* [*simp*]: *degree* (*smult a p*) = (*if a* = *0 then 0 else degree p*)
for *a* :: ′*a*::{*comm-semiring-0*,*semiring-no-zero-divisors*}
by (*cases a* = *0*) (*simp-all add*: *degree-def*)

lemma *smult-eq-0-iff* [*simp*]: *smult a p* = *0* ⟷ *a* = *0* ∨ *p* = *0*
for *a* :: ′*a*::{*comm-semiring-0*,*semiring-no-zero-divisors*}
by (*simp add*: *poly-eq-iff*)

lemma *coeffs-smult* [*code abstract*]:
coeffs (*smult a p*) = (*if a* = *0 then* [] *else map* (*Groups.times a*) (*coeffs p*))
for *p* :: ′*a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
proof −
have *eq-0*: *HOL.eq 0* ∘ *times a* = *HOL.eq* (*0*::′*a*) **if** *a* ≠ *0*
using *that* **by** (*simp add*: *fun-eq-iff*)
show *?thesis*
by (*rule coeffs-eqI*) (*auto simp add*: *no-trailing-map nth-default-map-eq nth-default-coeffs-eq*

eq-0)
qed

lemma *smult-eq-iff*:
  fixes *b* :: *'a* :: *field*
  assumes $b \neq 0$
  shows *smult a p* = *smult b q* ⟷ *smult* (*a* / *b*) *p* = *q*
    (**is** *?lhs* ⟷ *?rhs*)
proof
  assume *?lhs*
  also from *assms* **have** *smult* (*inverse b*) ... = *q*
    by *simp*
  finally show *?rhs*
    by (*simp add*: *field-simps*)
next
  assume *?rhs*
  with *assms* **show** *?lhs* **by** *auto*
qed

instantiation *poly* :: (*comm-semiring-0*) *comm-semiring-0*
begin

definition *p* ∗ *q* = *fold-coeffs* (λ*a p*. *smult a q* + *pCons 0 p*) *p 0*

lemma *mult-poly-0-left*: (*0*::*'a poly*) ∗ *q* = *0*
  by (*simp add*: *times-poly-def*)

lemma *mult-pCons-left* [*simp*]: *pCons a p* ∗ *q* = *smult a q* + *pCons 0* (*p* ∗ *q*)
  by (*cases p* = *0* ∧ *a* = *0*) (*auto simp add*: *times-poly-def*)

lemma *mult-poly-0-right*: *p* ∗ (*0*::*'a poly*) = *0*
  by (*induct p*) (*simp-all add*: *mult-poly-0-left*)

lemma *mult-pCons-right* [*simp*]: *p* ∗ *pCons a q* = *smult a p* + *pCons 0* (*p* ∗ *q*)
  by (*induct p*) (*simp-all add*: *mult-poly-0-left algebra-simps*)

lemmas *mult-poly-0* = *mult-poly-0-left mult-poly-0-right*

lemma *mult-smult-left* [*simp*]: *smult a p* ∗ *q* = *smult a* (*p* ∗ *q*)
  by (*induct p*) (*simp-all add*: *mult-poly-0 smult-add-right*)

lemma *mult-smult-right* [*simp*]: *p* ∗ *smult a q* = *smult a* (*p* ∗ *q*)
  by (*induct q*) (*simp-all add*: *mult-poly-0 smult-add-right*)

lemma *mult-poly-add-left*: (*p* + *q*) ∗ *r* = *p* ∗ *r* + *q* ∗ *r*
  for *p q r* :: *'a poly*
  by (*induct r*) (*simp-all add*: *mult-poly-0 smult-distribs algebra-simps*)

instance

proof
  fix *p q r* :: *'a poly*
  show *0*: *0* ∗ *p* = *0*
    by (*rule mult-poly-0-left*)
  show *p* ∗ *0* = *0*
    by (*rule mult-poly-0-right*)
  show (*p* + *q*) ∗ *r* = *p* ∗ *r* + *q* ∗ *r*
    by (*rule mult-poly-add-left*)
  show (*p* ∗ *q*) ∗ *r* = *p* ∗ (*q* ∗ *r*)
    by (*induct p*) (*simp-all add*: *mult-poly-0 mult-poly-add-left*)
  show *p* ∗ *q* = *q* ∗ *p*
    by (*induct p*) (*simp-all add*: *mult-poly-0*)
qed

end

lemma *coeff-mult-degree-sum*:
  coeff (*p* ∗ *q*) (*degree p* + *degree q*) = *coeff p* (*degree p*) ∗ *coeff q* (*degree q*)
  by (*induct p*) (*simp-all add*: *coeff-eq-0*)

instance *poly* :: ({*comm-semiring-0*,*semiring-no-zero-divisors*}) *semiring-no-zero-divisors*
proof
  fix *p q* :: *'a poly*
  assume *p* ≠ *0* **and** *q* ≠ *0*
  have *coeff* (*p* ∗ *q*) (*degree p* + *degree q*) = *coeff p* (*degree p*) ∗ *coeff q* (*degree q*)
    by (*rule coeff-mult-degree-sum*)
  also from ‹*p* ≠ *0*› ‹*q* ≠ *0*› **have** *coeff p* (*degree p*) ∗ *coeff q* (*degree q*) ≠ *0*
    by *simp*
  finally have ∃ *n*. *coeff* (*p* ∗ *q*) *n* ≠ *0* **..**
  then show *p* ∗ *q* ≠ *0*
    by (*simp add*: *poly-eq-iff*)
qed

instance *poly* :: (*comm-semiring-0-cancel*) *comm-semiring-0-cancel* **..**

lemma *coeff-mult*: *coeff* (*p* ∗ *q*) *n* = (∑ *i*≤*n*. *coeff p i* ∗ *coeff q* (*n*−*i*))
proof (*induct p arbitrary*: *n*)
  case *0*
  show *?case* **by** *simp*
next
  case (*pCons a p n*)
  then show *?case*
    by (*cases n*) (*simp-all add*: *sum.atMost-Suc-shift del*: *sum.atMost-Suc*)
qed

lemma *coeff-mult-0*: *coeff* (*p* ∗ *q*) *0* = *coeff p 0* ∗ *coeff q 0*
  by (*simp add*: *coeff-mult*)

lemma *degree-mult-le*: *degree* (*p* ∗ *q*) ≤ *degree p* + *degree q*

proof (*rule degree-le*)
  show ∀ *i*>*degree p* + *degree q*. *coeff* (*p* ∗ *q*) *i* = *0*
    by (*induct p*) (*simp-all add*: *coeff-eq-0 coeff-pCons split*: *nat.split*)
qed

lemma *mult-monom*: *monom a m* ∗ *monom b n* = *monom* (*a* ∗ *b*) (*m* + *n*)
  by (*induct m*) (*simp add*: *monom-0 smult-monom*, *simp add*: *monom-Suc*)

instantiation *poly* :: (*comm-semiring-1*) *comm-semiring-1*
begin

lift-definition *one-poly* :: ′*a poly*
  is λ*n*. *of-bool* (*n* = *0*)
  by (*rule MOST-SucD*) *simp*

lemma *coeff-1* [*simp*]:
  coeff 1 n = *of-bool* (*n* = *0*)
  by (*simp add*: *one-poly.rep-eq*)

lemma *one-pCons*:
  1 = [:*1*:]
  by (*simp add*: *poly-eq-iff coeff-pCons split*: *nat.splits*)

lemma *pCons-one*:
  [:*1*:] = *1*
  by (*simp add*: *one-pCons*)

instance
  by *standard* (*simp-all add*: *one-pCons*)

end

lemma *poly-1* [*simp*]:
  poly 1 x = *1*
  by (*simp add*: *one-pCons*)

lemma *one-poly-eq-simps* [*simp*]:
  1 = [:*1*:] ⟷ *True*
  [:*1*:] = *1* ⟷ *True*
  by (*simp-all add*: *one-pCons*)

lemma *degree-1* [*simp*]:
  degree 1 = *0*
  by (*simp add*: *one-pCons*)

lemma *coeffs-1-eq* [*simp*, *code abstract*]:
  coeffs 1 = [*1*]
  by (*simp add*: *one-pCons*)

lemma *smult-one* [*simp*]:
smult c 1 = [:*c*:]
by (*simp add*: *one-pCons*)

lemma *smult-sum*: *smult* ($\sum$ *i* ∈ *S*. *f i*) *p* = ($\sum$ *i* ∈ *S*. *smult* (*f i*) *p*)
by (*induct S rule*: *infinite-finite-induct*, *auto simp*: *smult-add-left*)

lemma *smult-power*: (*smult a p*) ^ *n* = *smult* (*a* ^ *n*) (*p* ^ *n*)
by (*induct n*, *auto simp*: *field-simps*)

lemma *monom-eq-1* [*simp*]:
monom 1 0 = *1*
by (*simp add*: *monom-0 one-pCons*)

lemma *monom-eq-1-iff*:
monom c n = *1* ⟷ *c* = *1* ∧ *n* = *0*
using *monom-eq-const-iff* [*of c n 1*] **by** *auto*

lemma *monom-altdef*:
monom c n = *smult c* ([:*0*, *1*:] ^ *n*)
by (*induct n*) (*simp-all add*: *monom-0 monom-Suc*)

lemma *degree-sum-list-le*: ($\bigwedge$ *p* . *p* ∈ *set ps* ⟹ *degree p* ≤ *n*)
⟹ *degree* (*sum-list ps*) ≤ *n*
proof (*induct ps*)
case (*Cons p ps*)
hence *degree* (*sum-list ps*) ≤ *n degree p* ≤ *n* **by** *auto*
thus *?case* **unfolding** *sum-list.Cons* **by** (*metis degree-add-le*)
qed *simp*

lemma *degree-prod-list-le*: *degree* (*prod-list ps*) ≤ *sum-list* (*map degree ps*)
proof (*induct ps*)
case (*Cons p ps*)
show *?case* **unfolding** *prod-list.Cons*
by (*rule order.trans*[*OF degree-mult-le*], *insert Cons*, *auto*)
qed *simp*

instance *poly* :: ({*comm-semiring-1*,*semiring-1-no-zero-divisors*}) *semiring-1-no-zero-divisors*
..
instance *poly* :: (*comm-ring*) *comm-ring* **..**
instance *poly* :: (*comm-ring-1*) *comm-ring-1* **..**
instance *poly* :: (*comm-ring-1*) *comm-semiring-1-cancel* **..**

lemma *prod-smult*: ($\prod$ *x*∈*A*. *smult* (*c x*) (*p x*)) = *smult* (*prod c A*) (*prod p A*)
by (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *mult-ac*)

lemma *degree-power-le*: *degree* (*p* ^ *n*) ≤ *degree p* ∗ *n*
by (*induct n*) (*auto intro*: *order-trans degree-mult-le*)

lemma *coeff-0-power*: *coeff* $(p \,\hat{}\, n)$ *0* = *coeff p 0* $\hat{}$ *n*
  by (*induct n*) (*simp-all add*: *coeff-mult*)

lemma *poly-smult* [*simp*]: *poly* (*smult a p*) *x* = *a* ∗ *poly p x*
  by (*induct p*) (*simp-all add*: *algebra-simps*)

lemma *poly-mult* [*simp*]: *poly* (*p* ∗ *q*) *x* = *poly p x* ∗ *poly q x*
  by (*induct p*) (*simp-all add*: *algebra-simps*)

lemma *poly-power* [*simp*]: *poly* $(p \,\hat{}\, n)$ *x* = *poly p x* $\hat{}$ *n*
  for *p* :: *'a*::*comm-semiring-1 poly*
  by (*induct n*) *simp-all*

lemma *poly-prod*: *poly* ($\prod$ *k*∈*A*. *p k*) *x* = ($\prod$ *k*∈*A*. *poly* (*p k*) *x*)
  by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *poly-prod-list*: *poly* ($\prod$ *p*←*ps*. *p*) *y* = ($\prod$ *p*←*ps*. *poly p y*)
  by (*induction ps*) *auto*

lemma *poly-prod-mset*: *poly* ($\prod$ *x*∈#*A*. *p x*) *y* = ($\prod$ *x*∈#*A*. *poly* (*p x*) *y*)
  by (*induction A*) *auto*

lemma *poly-const-pow*: [: *c* :] $\hat{}$ *n* = [: *c* $\hat{}$ *n* :]
  by (*induction n*) (*auto simp*: *algebra-simps*)

lemma *monom-power*: *monom c n* $\hat{}$ *k* = *monom* (*c* $\hat{}$ *k*) (*n* ∗ *k*)
  by (*induction k*) (*auto simp*: *mult-monom*)

lemma *degree-prod-sum-le*: *finite S* ⟹ *degree* (*prod f S*) ≤ *sum* (*degree* ∘ *f*) *S*
proof (*induct S rule*: *finite-induct*)
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert a S*)
  show *?case*
    unfolding *prod.insert*[*OF insert*(*1*−*2*)] *sum.insert*[*OF insert*(*1*−*2*)]
    by (*rule le-trans*[*OF degree-mult-le*]) (*use insert* **in** *auto*)
qed

lemma *coeff-0-prod-list*: *coeff* (*prod-list xs*) *0* = *prod-list* (*map* (λ*p*. *coeff p 0*) *xs*)
  by (*induct xs*) (*simp-all add*: *coeff-mult*)

lemma *coeff-monom-mult*: *coeff* (*monom c n* ∗ *p*) *k* = (*if k* < *n then 0 else c* ∗ *coeff p* (*k* − *n*))
proof −
  have *coeff* (*monom c n* ∗ *p*) *k* = ($\sum$ *i*≤*k*. (*if n* = *i then c else 0*) ∗ *coeff p* (*k* − *i*))
    by (*simp add*: *coeff-mult*)
  also have ... = ($\sum$ *i*≤*k*. (*if n* = *i then c* ∗ *coeff p* (*k* − *i*) *else 0*))

by (*intro sum.cong*) *simp-all*
also have ... = (*if k < n then 0 else c ∗ coeff p* (*k* − *n*))
by *simp*
finally show *?thesis* **.**
qed

lemma *coeff-monom-Suc*: *coeff* (*monom a* (*Suc d*) ∗ *p*) (*Suc i*) = *coeff* (*monom a d* ∗ *p*) *i*
by (*simp add*: *monom-Suc*)

lemma *monom-1-dvd-iff* ′: *monom 1 n dvd p* ⟷ (∀ *k*<*n*. *coeff p k* = *0*)
proof
assume *monom 1 n dvd p*
then obtain *r* **where** *p* = *monom 1 n* ∗ *r*
by (*rule dvdE*)
then show ∀ *k*<*n*. *coeff p k* = *0*
by (*simp add*: *coeff-mult*)
next
assume *zero*: (∀ *k*<*n*. *coeff p k* = *0*)
define *r* **where** *r* = *Abs-poly* (λ*k*. *coeff p* (*k* + *n*))
have $\forall_\infty$*k*. *coeff p* (*k* + *n*) = *0*
by (*subst cofinite-eq-sequentially*, *subst eventually-sequentially-seg*, *subst cofinite-eq-sequentially* [*symmetric*]) *transfer*
then have *coeff-r* [*simp*]: *coeff r k* = *coeff p* (*k* + *n*) **for** *k*
unfolding *r-def* **by** (*subst poly.Abs-poly-inverse*) *simp-all*
have *p* = *monom 1 n* ∗ *r*
by (*rule poly-eqI*, *subst coeff-monom-mult*) (*simp-all add*: *zero*)
then show *monom 1 n dvd p* **by** *simp*
qed

lemma *coeff-sum-monom*:
assumes *n*: *n* ≤ *d*
shows *coeff* (∑ *i*≤*d*. *monom* (*f i*) *i*) *n* = *f n* (**is** *?l* = -)
proof −
have *?l* = (∑ *i*≤*d*. *coeff* (*monom* (*f i*) *i*) *n*) (**is** - = *sum ?cmf* -)
using *coeff-sum***.**
also have {..*d*} = *insert n* ({..*d*}−{*n*}) **using** *n* **by** *auto*
hence *sum ?cmf* {..*d*} = *sum ?cmf* ... **by** *auto*
also have ... = *sum ?cmf* ({..*d*}−{*n*}) + *?cmf n* **by** (*subst sum.insert*,*auto*)
also have *sum ?cmf* ({..*d*}−{*n*}) = *0* **by** (*subst sum.neutral*, *auto*)
finally show *?thesis* **by** *simp*
qed

4.15 Mapping polynomials

definition *map-poly* :: (′*a* :: *zero* ⇒ ′*b* :: *zero*) ⇒ ′*a poly* ⇒ ′*b poly*
where *map-poly f p* = *Poly* (*map f* (*coeffs p*))

lemma *map-poly-0* [*simp*]: *map-poly f 0* = *0*

by (*simp add*: *map-poly-def*)

lemma *map-poly-1*: *map-poly f 1* = [:*f 1*:]
by (*simp add*: *map-poly-def*)

lemma *map-poly-1'* [*simp*]: *f 1* = *1* $\Longrightarrow$ *map-poly f 1* = *1*
by (*simp add*: *map-poly-def one-pCons*)

lemma *coeff-map-poly*:
assumes *f 0* = *0*
shows *coeff* (*map-poly f p*) *n* = *f* (*coeff p n*)
by (*auto simp*: *assms map-poly-def nth-default-def coeffs-def not-less Suc-le-eq coeff-eq-0*
simp del: *upt-Suc*)

lemma *coeffs-map-poly* [*code abstract*]:
coeffs (*map-poly f p*) = *strip-while* ((=) *0*) (*map f* (*coeffs p*))
by (*simp add*: *map-poly-def*)

lemma *coeffs-map-poly'*:
assumes $\bigwedge x.\ x \neq 0 \Longrightarrow f\ x \neq 0$
shows *coeffs* (*map-poly f p*) = *map f* (*coeffs p*)
using *assms*
by (*auto simp add*: *coeffs-map-poly strip-while-idem-iff*
last-coeffs-eq-coeff-degree no-trailing-unfold last-map)

lemma *set-coeffs-map-poly*:
($\bigwedge x.\ f\ x = 0 \longleftrightarrow x = 0$) $\Longrightarrow$ *set* (*coeffs* (*map-poly f p*)) = *f* ` *set* (*coeffs p*)
by (*simp add*: *coeffs-map-poly'*)

lemma *degree-map-poly*:
assumes $\bigwedge x.\ x \neq 0 \Longrightarrow f\ x \neq 0$
shows *degree* (*map-poly f p*) = *degree p*
by (*simp add*: *degree-eq-length-coeffs coeffs-map-poly' assms*)

lemma *map-poly-eq-0-iff*:
assumes *f 0* = *0* $\bigwedge x.\ x \in set\ (coeffs\ p) \Longrightarrow x \neq 0 \Longrightarrow f\ x \neq 0$
shows *map-poly f p* = *0* $\longleftrightarrow$ *p* = *0*
proof −
have (*coeff* (*map-poly f p*) *n* = *0*) = (*coeff p n* = *0*) **for** *n*
proof −
have *coeff* (*map-poly f p*) *n* = *f* (*coeff p n*)
by (*simp add*: *coeff-map-poly assms*)
also have ... = *0* $\longleftrightarrow$ *coeff p n* = *0*
proof (*cases n* < *length* (*coeffs p*))
case *True*
then have *coeff p n* $\in$ *set* (*coeffs p*)
by (*auto simp*: *coeffs-def simp del*: *upt-Suc*)
with *assms* **show** *f* (*coeff p n*) = *0* $\longleftrightarrow$ *coeff p n* = *0*

```
        by auto
    next
      case False
      then show ?thesis
      by (auto simp: assms length-coeffs nth-default-coeffs-eq [symmetric] nth-default-def)
    qed
    finally show ?thesis .
  qed
  then show ?thesis by (auto simp: poly-eq-iff)
qed

lemma map-poly-smult:
  assumes f 0 = 0⋀c x. f (c * x) = f c * f x
  shows map-poly f (smult c p) = smult (f c) (map-poly f p)
  by (intro poly-eqI) (simp-all add: assms coeff-map-poly)

lemma map-poly-pCons:
  assumes f 0 = 0
  shows map-poly f (pCons c p) = pCons (f c) (map-poly f p)
 by (intro poly-eqI) (simp-all add: assms coeff-map-poly coeff-pCons split: nat.splits)

lemma map-poly-map-poly:
  assumes f 0 = 0 g 0 = 0
  shows map-poly f (map-poly g p) = map-poly (f ∘ g) p
  by (intro poly-eqI) (simp add: coeff-map-poly assms)

lemma map-poly-id [simp]: map-poly id p = p
  by (simp add: map-poly-def)

lemma map-poly-id' [simp]: map-poly (λx. x) p = p
  by (simp add: map-poly-def)

lemma map-poly-cong:
  assumes (⋀x. x ∈ set (coeffs p) ⟹ f x = g x)
  shows map-poly f p = map-poly g p
proof −
  from assms have map f (coeffs p) = map g (coeffs p)
    by (intro map-cong) simp-all
  then show ?thesis
    by (simp only: coeffs-eq-iff coeffs-map-poly)
qed

lemma map-poly-monom: f 0 = 0 ⟹ map-poly f (monom c n) = monom (f c) n
  by (intro poly-eqI) (simp-all add: coeff-map-poly)

lemma map-poly-idI:
  assumes ⋀x. x ∈ set (coeffs p) ⟹ f x = x
  shows map-poly f p = p
  using map-poly-cong[OF assms, of - id] by simp
```

lemma *map-poly-idI'*:
  assumes $\bigwedge x.\ x \in set\ (coeffs\ p) \Longrightarrow f\ x = x$
  shows $p = map\text{-}poly\ f\ p$
  using *map-poly-cong*[*OF assms*, *of - id*] **by** *simp*

lemma *smult-conv-map-poly*: $smult\ c\ p = map\text{-}poly\ (\lambda x.\ c * x)\ p$
  by (*intro poly-eqI*) (*simp-all add*: *coeff-map-poly*)

lemma *poly-cnj*: $cnj\ (poly\ p\ z) = poly\ (map\text{-}poly\ cnj\ p)\ (cnj\ z)$
  by (*simp add*: *poly-altdef degree-map-poly coeff-map-poly*)

lemma *poly-cnj-real*:
  assumes $\bigwedge n.\ poly.coeff\ p\ n \in \mathbb{R}$
  shows $cnj\ (poly\ p\ z) = poly\ p\ (cnj\ z)$
proof −
  from *assms* **have** $map\text{-}poly\ cnj\ p = p$
    by (*intro poly-eqI*) (*auto simp*: *coeff-map-poly Reals-cnj-iff*)
  with *poly-cnj*[*of p z*] **show** *?thesis* **by** *simp*
qed

lemma *real-poly-cnj-root-iff*:
  assumes $\bigwedge n.\ poly.coeff\ p\ n \in \mathbb{R}$
  shows $poly\ p\ (cnj\ z) = 0 \longleftrightarrow poly\ p\ z = 0$
proof −
  have $poly\ p\ (cnj\ z) = cnj\ (poly\ p\ z)$
    by (*simp add*: *poly-cnj-real assms*)
  also have $\ldots = 0 \longleftrightarrow poly\ p\ z = 0$ **by** *simp*
  finally show *?thesis* **.**
qed

lemma *sum-to-poly*: $(\sum x{\in}A.\ [:f\ x:]) = [:\sum x{\in}A.\ f\ x:]$
  by (*induction A rule*: *infinite-finite-induct*) *auto*

lemma *diff-to-poly*: $[:c:] - [:d:] = [:c - d:]$
  by (*simp add*: *poly-eq-iff mult-ac*)

lemma *mult-to-poly*: $[:c:] * [:d:] = [:c * d:]$
  by (*simp add*: *poly-eq-iff mult-ac*)

lemma *prod-to-poly*: $(\prod x{\in}A.\ [:f\ x:]) = [:\prod x{\in}A.\ f\ x:]$
  by (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *mult-to-poly mult-ac*)

lemma *poly-map-poly-cnj* [*simp*]: $poly\ (map\text{-}poly\ cnj\ p)\ x = cnj\ (poly\ p\ (cnj\ x))$
  by (*induction p*) (*auto simp*: *map-poly-pCons*)

4.16 Conversions

lemma *of-nat-poly*:

of-nat n = [:*of-nat n*:]
by (*induct n*) (*simp-all add*: *one-pCons*)

lemma *of-nat-monom*:
of-nat n = *monom* (*of-nat n*) *0*
by (*simp add*: *of-nat-poly monom-0*)

lemma *degree-of-nat* [*simp*]:
degree (*of-nat n*) = *0*
by (*simp add*: *of-nat-poly*)

lemma *lead-coeff-of-nat* [*simp*]:
lead-coeff (*of-nat n*) = *of-nat n*
by (*simp add*: *of-nat-poly*)

lemma *of-int-poly*:
of-int k = [:*of-int k*:]
by (*simp only*: *of-int-of-nat of-nat-poly*) *simp*

lemma *of-int-monom*:
of-int k = *monom* (*of-int k*) *0*
by (*simp add*: *of-int-poly monom-0*)

lemma *degree-of-int* [*simp*]:
degree (*of-int k*) = *0*
by (*simp add*: *of-int-poly*)

lemma *lead-coeff-of-int* [*simp*]:
lead-coeff (*of-int k*) = *of-int k*
by (*simp add*: *of-int-poly*)

lemma *poly-of-nat* [*simp*]: *poly* (*of-nat n*) *x* = *of-nat n*
by (*simp add*: *of-nat-poly*)

lemma *poly-of-int* [*simp*]: *poly* (*of-int n*) *x* = *of-int n*
by (*simp add*: *of-int-poly*)

lemma *poly-numeral* [*simp*]: *poly* (*numeral n*) *x* = *numeral n*
by (*metis of-nat-numeral poly-of-nat*)

lemma *numeral-poly*: *numeral n* = [:*numeral n*:]
proof −
have *numeral n* = *of-nat* (*numeral n*)
by *simp*
also have ... = [:*of-nat* (*numeral n*):]
by (*simp add*: *of-nat-poly*)
finally show *?thesis*
by *simp*
qed

lemma *numeral-monom*:
numeral n = *monom* (*numeral n*) *0*
by (*simp add*: *numeral-poly monom-0*)

lemma *degree-numeral* [*simp*]:
degree (*numeral n*) = *0*
by (*simp add*: *numeral-poly*)

lemma *lead-coeff-numeral* [*simp*]:
lead-coeff (*numeral n*) = *numeral n*
by (*simp add*: *numeral-poly*)

lemma *coeff-linear-poly-power*:
fixes *c* :: *'a* :: *semiring-1*
assumes $i \leq n$
shows *coeff* ([:*a*, *b*:] ^ *n*) *i* = *of-nat* (*n choose i*) ∗ *b* ^ *i* ∗ *a* ^ (*n* − *i*)
proof −
have [:*a*, *b*:] = *monom b 1* + [:*a*:]
by (*simp add*: *monom-altdef*)
also have *coeff* (... ^ *n*) *i* = ($\sum$ *k*≤*n*. *a*^(*n*−*k*) ∗ *of-nat* (*n choose k*) ∗ (*if k* = *i then b* ^ *k else 0*))
by (*subst binomial-ring*) (*simp add*: *coeff-sum of-nat-poly monom-power poly-const-pow mult-ac*)
also have ... = ($\sum$ *k*∈{*i*}. *a* ^ (*n* − *i*) ∗ *b* ^ *i* ∗ *of-nat* (*n choose k*))
using *assms* **by** (*intro sum.mono-neutral-cong-right*) (*auto simp*: *mult-ac*)
finally show ∗: *?thesis* **by** (*simp add*: *mult-ac*)
qed

4.17 Lemmas about divisibility

lemma *dvd-smult*:
assumes *p dvd q*
shows *p dvd smult a q*
proof −
from *assms* **obtain** *k* **where** *q* = *p* ∗ *k* **..**
then have *smult a q* = *p* ∗ *smult a k* **by** *simp*
then show *p dvd smult a q* **..**
qed

lemma *dvd-smult-cancel*: *p dvd smult a q* ⟹ *a* ≠ *0* ⟹ *p dvd q*
for *a* :: *'a*::*field*
by (*drule dvd-smult* [**where** *a*=*inverse a*]) *simp*

lemma *dvd-smult-iff*: *a* ≠ *0* ⟹ *p dvd smult a q* ⟷ *p dvd q*
for *a* :: *'a*::*field*
by (*safe elim*!: *dvd-smult dvd-smult-cancel*)

lemma *smult-dvd-cancel*:

assumes *smult a p dvd q*
shows *p dvd q*
proof −
from *assms* **obtain** *k* **where** $q = smult\ a\ p * k$ **..**
then have $q = p * smult\ a\ k$ **by** *simp*
then show *p dvd q* **..**
qed

lemma *smult-dvd*: $p\ dvd\ q \Longrightarrow a \neq 0 \Longrightarrow smult\ a\ p\ dvd\ q$
for *a* :: *'a::field*
by (*rule smult-dvd-cancel* [**where** *a=inverse a*]) *simp*

lemma *smult-dvd-iff*: $smult\ a\ p\ dvd\ q \longleftrightarrow (if\ a = 0\ then\ q = 0\ else\ p\ dvd\ q)$
for *a* :: *'a::field*
by (*auto elim*: *smult-dvd smult-dvd-cancel*)

lemma *is-unit-smult-iff*: $smult\ c\ p\ dvd\ 1 \longleftrightarrow c\ dvd\ 1 \wedge p\ dvd\ 1$
proof −
have $smult\ c\ p = [:c:] * p$ **by** *simp*
also have $\ldots\ dvd\ 1 \longleftrightarrow c\ dvd\ 1 \wedge p\ dvd\ 1$
proof *safe*
assume ∗: $[:c:] * p\ dvd\ 1$
then show *p dvd 1*
by (*rule dvd-mult-right*)
from ∗ **obtain** *q* **where** *q*: $1 = [:c:] * p * q$
by (*rule dvdE*)
have $c\ dvd\ c * (coeff\ p\ 0 * coeff\ q\ 0)$
by *simp*
also have $\ldots = coeff\ ([:c:] * p * q)\ 0$
by (*simp add*: *mult.assoc coeff-mult*)
also note *q* [*symmetric*]
finally have *c dvd coeff 1 0* **.**
then show *c dvd 1* **by** *simp*
next
assume *c dvd 1 p dvd 1*
from *this*(*1*) **obtain** *d* **where** $1 = c * d$
by (*rule dvdE*)
then have $1 = [:c:] * [:d:]$
by (*simp add*: *one-pCons ac-simps*)
then have *[:c:] dvd 1*
by (*rule dvdI*)
from *mult-dvd-mono*[*OF this* ‹*p dvd 1*›] **show** $[:c:] * p\ dvd\ 1$
by *simp*
qed
finally show *?thesis* **.**
qed

4.18 Polynomials form an integral domain

instance *poly* :: (*idom*) *idom* **..**

instance *poly* :: ({*ring-char-0*, *comm-ring-1*}) *ring-char-0*
  by *standard* (*auto simp add*: *of-nat-poly intro*: *injI*)

lemma *semiring-char-poly* [*simp*]: $CHAR('a :: comm\text{-}semiring\text{-}1\ poly) = CHAR('a)$
  by (*rule CHAR-eqI*) (*auto simp*: *of-nat-poly of-nat-eq-0-iff-char-dvd*)

instance *poly* :: ({*semiring-prime-char*,*comm-semiring-1*}) *semiring-prime-char*
  by (*rule semiring-prime-charI*) *auto*
instance *poly* :: ({*comm-semiring-prime-char*,*comm-semiring-1*}) *comm-semiring-prime-char*
  by *standard*
instance *poly* :: ({*comm-ring-prime-char*,*comm-semiring-1*}) *comm-ring-prime-char*
  by *standard*
instance *poly* :: ({*idom-prime-char*,*comm-semiring-1*}) *idom-prime-char*
  by *standard*

lemma *degree-mult-eq*: $p \neq 0 \Longrightarrow q \neq 0 \Longrightarrow degree\ (p * q) = degree\ p + degree\ q$
  for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
 by (*rule order-antisym* [*OF degree-mult-le le-degree*]) (*simp add*: *coeff-mult-degree-sum*)

lemma *degree-prod-sum-eq*:
  $(\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0) \Longrightarrow$
    $degree\ (prod\ f\ A :: 'a :: idom\ poly) = (\sum x{\in}A.\ degree\ (f\ x))$
  by (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *degree-mult-eq*)

lemma *dvd-imp-degree*:
  ‹$degree\ x \leq degree\ y$› **if** ‹*x dvd y*› ‹$x \neq 0$› ‹$y \neq 0$›
    for *x y* :: ‹*'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*›
proof −
  from ‹*x dvd y*› **obtain** *z* **where** ‹$y = x * z$› **..**
  with ‹$x \neq 0$› ‹$y \neq 0$› **show** *?thesis*
    by (*simp add*: *degree-mult-eq*)
qed

lemma *degree-prod-eq-sum-degree*:
  fixes *A* :: *'a set*
  and $f :: 'a \Rightarrow 'b::idom\ poly$
  assumes *f0*: $\forall i{\in}A.\ f\ i \neq 0$
  shows $degree\ (\prod i{\in}A.\ (f\ i)) = (\sum i{\in}A.\ degree\ (f\ i))$
  using *assms*
  by (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *degree-mult-eq*)

lemma *degree-mult-eq-0*:
  $degree\ (p * q) = 0 \longleftrightarrow p = 0 \lor q = 0 \lor (p \neq 0 \land q \neq 0 \land degree\ p = 0 \land degree\ q = 0)$
  for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
  by (*auto simp*: *degree-mult-eq*)

lemma *degree-power-eq*: $p \neq 0 \Longrightarrow$ *degree* $((p :: 'a :: idom\ poly)$ ^ $n) = n * degree$ p
by (*induction n*) (*simp-all add*: *degree-mult-eq*)

lemma *degree-mult-right-le*:
fixes $p\ q :: 'a::\{comm\text{-}semiring\text{-}0, semiring\text{-}no\text{-}zero\text{-}divisors\}\ poly$
assumes $q \neq 0$
shows *degree* $p \leq$ *degree* $(p * q)$
using *assms* **by** (*cases* $p = 0$) (*simp-all add*: *degree-mult-eq*)

lemma *coeff-degree-mult*: *coeff* $(p * q)$ (*degree* $(p * q)) =$ *coeff* q (*degree* $q) *$ *coeff* p (*degree* p)
for $p\ q :: 'a::\{comm\text{-}semiring\text{-}0, semiring\text{-}no\text{-}zero\text{-}divisors\}\ poly$
by (*cases* $p = 0 \vee q = 0$) (*auto simp*: *degree-mult-eq coeff-mult-degree-sum mult-ac*)

lemma *dvd-imp-degree-le*: p *dvd* $q \Longrightarrow q \neq 0 \Longrightarrow$ *degree* $p \leq$ *degree* q
for $p\ q :: 'a::\{comm\text{-}semiring\text{-}1, semiring\text{-}no\text{-}zero\text{-}divisors\}\ poly$
by (*erule dvdE*, *hypsubst*, *subst degree-mult-eq*) *auto*

lemma *divides-degree*:
fixes $p\ q :: 'a ::\{comm\text{-}semiring\text{-}1, semiring\text{-}no\text{-}zero\text{-}divisors\}\ poly$
assumes p *dvd* q
shows *degree* $p \leq$ *degree* $q \vee q = 0$
by (*metis dvd-imp-degree-le assms*)

lemma *const-poly-dvd-iff*:
fixes $c :: 'a::\{comm\text{-}semiring\text{-}1, semiring\text{-}no\text{-}zero\text{-}divisors\}$
shows $[:c:]$ *dvd* $p \longleftrightarrow (\forall n.\ c$ *dvd coeff* $p\ n)$
proof (*cases* $c = 0 \vee p = 0$)
case *True*
then show *?thesis*
by (*auto intro*!: *poly-eqI*)
next
case *False*
show *?thesis*
proof
assume $[:c:]$ *dvd* p
then show $\forall n.\ c$ *dvd coeff* $p\ n$
by (*auto simp*: *coeffs-def*)
next
assume $*$: $\forall n.\ c$ *dvd coeff* $p\ n$
define *mydiv* **where** *mydiv* $x\ y = (SOME\ z.\ x = y * z)$ **for** $x\ y :: 'a$
have *mydiv*: $x = y *$ *mydiv* $x\ y$ **if** y *dvd* x **for** $x\ y$
using *that* **unfolding** *mydiv-def dvd-def* **by** (*rule someI-ex*)
define q **where** $q = Poly$ (*map* ($\lambda a.$ *mydiv* $a\ c$) (*coeffs* p))
from *False* $*$ **have** $p = q * [:c:]$
by (*intro poly-eqI*)

(*auto simp*: *q-def nth-default-def not-less length-coeffs-degree coeffs-nth*
intro!: *coeff-eq-0 mydiv*)
then show [:*c*:] *dvd p*
by (*simp only*: *dvd-triv-right*)
qed
qed

lemma *const-poly-dvd-const-poly-iff* [*simp*]: [:*a*:] *dvd* [:*b*:] ⟷ *a dvd b*
for *a b* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*}
by (*subst const-poly-dvd-iff*) (*auto simp*: *coeff-pCons split*: *nat.splits*)

lemma *lead-coeff-mult*: *lead-coeff* (*p* ∗ *q*) = *lead-coeff p* ∗ *lead-coeff q*
for *p q* :: *'a*::{*comm-semiring-0*, *semiring-no-zero-divisors*} *poly*
by (*cases p* = *0* ∨ *q* = *0*) (*auto simp*: *coeff-mult-degree-sum degree-mult-eq*)

lemma *lead-coeff-prod*: *lead-coeff* (*prod f A*) = (∏ *x*∈*A*. *lead-coeff* (*f x*))
for *f* :: *'a* ⇒ *'b*::{*comm-semiring-1*, *semiring-no-zero-divisors*} *poly*
by (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *lead-coeff-mult*)

lemma *lead-coeff-smult*: *lead-coeff* (*smult c p*) = *c* ∗ *lead-coeff p*
for *p* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
proof −
have *smult c p* = [:*c*:] ∗ *p* **by** *simp*
also have *lead-coeff* . . . = *c* ∗ *lead-coeff p*
by (*subst lead-coeff-mult*) *simp-all*
finally show *?thesis* **.**
qed

lemma *lead-coeff-1* [*simp*]: *lead-coeff 1* = *1*
by *simp*

lemma *lead-coeff-power*: *lead-coeff* (*p* ^ *n*) = *lead-coeff p* ^ *n*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
by (*induct n*) (*simp-all add*: *lead-coeff-mult*)

4.19 Polynomials form an ordered integral domain

definition *pos-poly* :: *'a*::*linordered-semidom poly* ⇒ *bool*
where *pos-poly p* ⟷ *0* < *coeff p* (*degree p*)

lemma *pos-poly-pCons*: *pos-poly* (*pCons a p*) ⟷ *pos-poly p* ∨ (*p* = *0* ∧ *0* < *a*)
by (*simp add*: *pos-poly-def*)

lemma *not-pos-poly-0* [*simp*]: ¬ *pos-poly 0*
by (*simp add*: *pos-poly-def*)

lemma *pos-poly-add*: *pos-poly p* ⟹ *pos-poly q* ⟹ *pos-poly* (*p* + *q*)
proof (*induction p arbitrary*: *q*)
case (*pCons a p*)

then show *?case*
by (*cases q*; *force simp add*: *pos-poly-pCons add-pos-pos*)
qed *auto*

lemma *pos-poly-mult*: *pos-poly p* ⟹ *pos-poly q* ⟹ *pos-poly* (*p* ∗ *q*)
by (*simp add*: *pos-poly-def coeff-degree-mult*)

lemma *pos-poly-total*: *p* = *0* ∨ *pos-poly p* ∨ *pos-poly* (− *p*)
for *p* :: ′*a*::*linordered-idom poly*
by (*induct p*) (*auto simp*: *pos-poly-pCons*)

lemma *pos-poly-coeffs* [*code*]: *pos-poly p* ⟷ (*let as* = *coeffs p in as* ≠ [] ∧ *last as* > *0*)
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *?rhs*
then show *?lhs*
by (*auto simp add*: *pos-poly-def last-coeffs-eq-coeff-degree*)
next
assume *?lhs*
then have ∗: *0* < *coeff p* (*degree p*)
by (*simp add*: *pos-poly-def*)
then have *p* ≠ *0*
by *auto*
with ∗ **show** *?rhs*
by (*simp add*: *last-coeffs-eq-coeff-degree*)
qed

instantiation *poly* :: (*linordered-idom*) *linordered-idom*
begin

definition *x* < *y* ⟷ *pos-poly* (*y* − *x*)

definition *x* ≤ *y* ⟷ *x* = *y* ∨ *pos-poly* (*y* − *x*)

definition |*x*::′*a poly*| = (*if x* < *0 then* − *x else x*)

definition *sgn* (*x*::′*a poly*) = (*if x* = *0 then 0 else if 0* < *x then 1 else* − *1*)

instance
proof
fix *x y z* :: ′*a poly*
show *x* < *y* ⟷ *x* ≤ *y* ∧ ¬ *y* ≤ *x*
unfolding *less-eq-poly-def less-poly-def*
using *pos-poly-add* **by** *force*
then show *x* ≤ *y* ⟹ *y* ≤ *x* ⟹ *x* = *y*
using *less-eq-poly-def less-poly-def* **by** *force*
show *x* ≤ *x*
by (*simp add*: *less-eq-poly-def*)

show $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
using *less-eq-poly-def pos-poly-add* **by** *fastforce*
show $x \leq y \Longrightarrow z + x \leq z + y$
by (*simp add*: *less-eq-poly-def*)
show $x \leq y \lor y \leq x$
unfolding *less-eq-poly-def*
using *pos-poly-total* [*of* $x - y$]
by *auto*
show $x < y \Longrightarrow 0 < z \Longrightarrow z * x < z * y$
by (*simp add*: *less-poly-def right-diff-distrib* [*symmetric*] *pos-poly-mult*)
show $|x| = (\textit{if } x < 0 \textit{ then } - x \textit{ else } x)$
by (*rule abs-poly-def*)
show $sgn\ x = (\textit{if } x = 0 \textit{ then } 0 \textit{ else if } 0 < x \textit{ then } 1 \textit{ else } - 1)$
by (*rule sgn-poly-def*)
qed

end

TODO: Simplification rules for comparisons

4.20 Synthetic division and polynomial roots

4.20.1 Synthetic division

Synthetic division is simply division by the linear polynomial $x - c$.

definition *synthetic-divmod* :: *'a::comm-semiring-0 poly* ⇒ *'a* ⇒ *'a poly* × *'a*
where *synthetic-divmod p c* = *fold-coeffs* (λ*a* (*q*, *r*). (*pCons r q*, *a* + *c* ∗ *r*)) *p* (*0*, *0*)

definition *synthetic-div* :: *'a::comm-semiring-0 poly* ⇒ *'a* ⇒ *'a poly*
where *synthetic-div p c* = *fst* (*synthetic-divmod p c*)

lemma *synthetic-divmod-0* [*simp*]: *synthetic-divmod 0 c* = (*0*, *0*)
by (*simp add*: *synthetic-divmod-def*)

lemma *synthetic-divmod-pCons* [*simp*]:
synthetic-divmod (*pCons a p*) *c* = (λ(*q*, *r*). (*pCons r q*, *a* + *c* ∗ *r*)) (*synthetic-divmod p c*)
by (*cases p* = *0* ∧ *a* = *0*) (*auto simp add*: *synthetic-divmod-def*)

lemma *synthetic-div-0* [*simp*]: *synthetic-div 0 c* = *0*
by (*simp add*: *synthetic-div-def*)

lemma *synthetic-div-unique-lemma*: *smult c p* = *pCons a p* ⟹ *p* = *0*
by (*induct p arbitrary*: *a*) *simp-all*

lemma *snd-synthetic-divmod*: *snd* (*synthetic-divmod p c*) = *poly p c*
by (*induct p*) (*simp-all add*: *split-def*)

lemma *synthetic-div-pCons* [*simp*]:
  synthetic-div (*pCons a p*) *c* = *pCons* (*poly p c*) (*synthetic-div p c*)
  by (*simp add*: *synthetic-div-def split-def snd-synthetic-divmod*)

lemma *synthetic-div-eq-0-iff*: *synthetic-div p c* = *0* ⟷ *degree p* = *0*
proof (*induct p*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*pCons a p*)
  then show *?case* **by** (*cases p*) *simp*
qed

lemma *degree-synthetic-div*: *degree* (*synthetic-div p c*) = *degree p* − *1*
  by (*induct p*) (*simp-all add*: *synthetic-div-eq-0-iff*)

lemma *synthetic-div-correct*:
  p + *smult c* (*synthetic-div p c*) = *pCons* (*poly p c*) (*synthetic-div p c*)
  by (*induct p*) *simp-all*

lemma *synthetic-div-unique*: *p* + *smult c q* = *pCons r q* ⟹ *r* = *poly p c* ∧ *q* = *synthetic-div p c*
proof (*induction p arbitrary*: *q r*)
  case *0*
  then show *?case*
    using *synthetic-div-unique-lemma* **by** *fastforce*
next
  case (*pCons a p*)
  then show *?case*
    by (*cases q*; *force*)
qed

lemma *synthetic-div-correct'*: [:−*c*, *1*:] ∗ *synthetic-div p c* + [:*poly p c*:] = *p*
  for *c* :: *'a*::*comm-ring-1*
  using *synthetic-div-correct* [*of p c*] **by** (*simp add*: *algebra-simps*)

4.20.2 Polynomial roots

lemma *poly-eq-0-iff-dvd*: *poly p c* = *0* ⟷ [:− *c*, *1*:] *dvd p*
  (**is** *?lhs* ⟷ *?rhs*)
  for *c* :: *'a*::*comm-ring-1*
proof
  assume *?lhs*
  with *synthetic-div-correct'* [*of c p*] **have** *p* = [:−*c*, *1*:] ∗ *synthetic-div p c* **by** *simp*
  then show *?rhs* **..**
next
  assume *?rhs*
  then obtain *k* **where** *p* = [:−*c*, *1*:] ∗ *k* **by** (*rule dvdE*)
  then show *?lhs* **by** *simp*

```
qed

lemma dvd-iff-poly-eq-0: [:c, 1:] dvd p ⟷ poly p (− c) = 0
  for c :: 'a::comm-ring-1
  by (simp add: poly-eq-0-iff-dvd)

lemma poly-roots-finite: p ≠ 0 ⟹ finite {x. poly p x = 0}
  for p :: 'a::{comm-ring-1,ring-no-zero-divisors} poly
proof (induct n ≡ degree p arbitrary: p)
  case 0
  then obtain a where a ≠ 0 and p = [:a:]
    by (cases p) (simp split: if-splits)
  then show finite {x. poly p x = 0}
    by simp
next
  case (Suc n)
  show finite {x. poly p x = 0}
  proof (cases ∃x. poly p x = 0)
    case False
    then show finite {x. poly p x = 0} by simp
  next
    case True
    then obtain a where poly p a = 0 ..
    then have [:−a, 1:] dvd p
      by (simp only: poly-eq-0-iff-dvd)
    then obtain k where k: p = [:−a, 1:] ∗ k ..
    with ‹p ≠ 0› have k ≠ 0
      by auto
    with k have degree p = Suc (degree k)
      by (simp add: degree-mult-eq del: mult-pCons-left)
    with ‹Suc n = degree p› have n = degree k
      by simp
    from this ‹k ≠ 0› have finite {x. poly k x = 0}
      by (rule Suc.hyps)
    then have finite (insert a {x. poly k x = 0})
      by simp
    then show finite {x. poly p x = 0}
      by (simp add: k Collect-disj-eq del: mult-pCons-left)
  qed
qed

lemma poly-eq-poly-eq-iff: poly p = poly q ⟷ p = q
  (is ?lhs ⟷ ?rhs)
  for p q :: 'a::{comm-ring-1,ring-no-zero-divisors,ring-char-0} poly
proof
  assume ?rhs
  then show ?lhs by simp
next
  assume ?lhs
```

have *poly p* = *poly 0* ⟷ *p* = *0* **for** *p* :: *'a poly*
proof (*cases p* = *0*)
case *False*
then show *?thesis*
by (*auto simp add*: *infinite-UNIV-char-0 dest*: *poly-roots-finite*)
qed *auto*
from ‹*?lhs*› **and** *this* [*of p* − *q*] **show** *?rhs*
by *auto*
qed

A nice extension rule for polynomials.

lemma *poly-ext*:
fixes *p q* :: *'a* :: {*ring-char-0*, *idom*} *poly*
assumes ⋀*x*. *poly p x* = *poly q x* **shows** *p* = *q*
unfolding *poly-eq-poly-eq-iff* [*symmetric*]
using *assms* **by** (*rule ext*)

Copied from non-negative variants.

lemma *coeff-linear-power-neg*[*simp*]:
fixes *a* :: *'a*::*comm-ring-1*
shows *coeff* ([:*a*, −*1*:] ^ *n*) *n* = (−*1*)^*n*
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
then have *degree* ([:*a*, − *1*:] ^ *n*) < *Suc n*
by (*auto intro*: *le-less-trans degree-power-le*)
with *Suc* **show** *?case*
by (*simp add*: *coeff-eq-0*)
qed

lemma *degree-linear-power-neg*[*simp*]:
fixes *a* :: *'a*::{*idom*,*comm-ring-1*}
shows *degree* ([:*a*, −*1*:] ^ *n*) = *n*
by (*simp add*: *degree-power-eq*)

lemma *poly-all-0-iff-0*: (∀ *x*. *poly p x* = *0*) ⟷ *p* = *0*
for *p* :: *'a*::{*ring-char-0*,*comm-ring-1*,*ring-no-zero-divisors*} *poly*
by (*auto simp add*: *poly-eq-poly-eq-iff* [*symmetric*])

lemma *card-poly-roots-bound*:
fixes *p* :: *'a*::{*comm-ring-1*,*ring-no-zero-divisors*} *poly*
assumes *p* ≠ *0*
shows *card* {*x*. *poly p x* = *0*} ≤ *degree p*
using *assms*
proof (*induction degree p arbitrary*: *p rule*: *less-induct*)
case (*less p*)
show *?case*

proof (*cases* ∃ *x*. *poly p x* = *0*)
  case *False*
  hence {*x*. *poly p x* = *0*} = {} **by** *blast*
  thus *?thesis* **by** *simp*
next
  case *True*
  then obtain *x* **where** *x*: *poly p x* = *0* **by** *blast*
  hence [:−*x*, *1*:] *dvd p* **by** (*subst* (*asm*) *poly-eq-0-iff-dvd*)
  then obtain *q* **where** *q*: *p* = [:−*x*, *1*:] ∗ *q* **by** (*auto simp*: *dvd-def*)
  with ‹*p* ≠ *0*› **have** [*simp*]: *q* ≠ *0* **by** *auto*
  have *deg*: *degree p* = *Suc* (*degree q*)
    by (*subst q*, *subst degree-mult-eq*) *auto*
  have *card* {*x*. *poly p x* = *0*} ≤ *card* (*insert x* {*x*. *poly q x* = *0*})
    by (*intro card-mono*) (*auto intro*: *poly-roots-finite simp*: *q*)
  also have . . . ≤ *Suc* (*card* {*x*. *poly q x* = *0*})
    by (*rule card-insert-le-m1*) *auto*
  also from *deg* **have** *card* {*x*. *poly q x* = *0*} ≤ *degree q*
    using ‹*p* ≠ *0*› **and** *q* **by** (*intro less*) *auto*
  also have *Suc* . . . = *degree p* **by** (*simp add*: *deg*)
  finally show *?thesis* **by** − *simp-all*
qed
qed

lemma *poly-eqI-degree*:
  fixes *p q* :: ′*a* :: {*comm-ring-1*, *ring-no-zero-divisors*} *poly*
  assumes ⋀*x*. *x* ∈ *A* ⟹ *poly p x* = *poly q x*
  assumes *card A* > *degree p card A* > *degree q*
  shows *p* = *q*
proof (*rule ccontr*)
  assume *neq*: *p* ≠ *q*
  have *degree* (*p* − *q*) ≤ *max* (*degree p*) (*degree q*)
    by (*rule degree-diff-le-max*)
  also from *assms* **have** . . . < *card A* **by** *linarith*
  also have . . . ≤ *card* {*x*. *poly* (*p* − *q*) *x* = *0*}
    using *neq* **and** *assms* **by** (*intro card-mono poly-roots-finite*) *auto*
  finally have *degree* (*p* − *q*) < *card* {*x*. *poly* (*p* − *q*) *x* = *0*} **.**
  moreover have *degree* (*p* − *q*) ≥ *card* {*x*. *poly* (*p* − *q*) *x* = *0*}
    using *neq* **by** (*intro card-poly-roots-bound*) *auto*
  ultimately show *False* **by** *linarith*
qed

4.20.3 Order of polynomial roots

definition *order* :: ′*a*::*idom* ⇒ ′*a poly* ⇒ *nat*
  where *order a p* = (*LEAST n*. ¬ [:−*a*, *1*:] ^ *Suc n dvd p*)

lemma *coeff-linear-power*: *coeff* ([:*a*, *1*:] ^ *n*) *n* = *1*
  for *a* :: ′*a*::*comm-semiring-1*
proof (*induct n*)

case (*Suc n*)
have *degree* ([:*a*, *1*:] ^ *n*) ≤ *1* ∗ *n*
by (*metis One-nat-def degree-pCons-eq-if degree-power-le one-neq-zero one-pCons*)
then have *coeff* ([:*a*, *1*:] ^ *n*) (*Suc n*) = *0*
by (*simp add*: *coeff-eq-0*)
then show *?case*
using *Suc.hyps* **by** *fastforce*
qed *auto*

lemma *degree-linear-power*: *degree* ([:*a*, *1*:] ^ *n*) = *n*
for *a* :: ′*a*::*comm-semiring-1*
proof (*rule order-antisym*)
show *degree* ([:*a*, *1*:] ^ *n*) ≤ *n*
by (*metis One-nat-def degree-pCons-eq-if degree-power-le mult.left-neutral one-neq-zero one-pCons*)
qed (*simp add*: *coeff-linear-power le-degree*)

lemma *order-1*: [:−*a*, *1*:] ^ *order a p dvd p*
proof (*cases p* = *0*)
case *False*
show *?thesis*
proof (*cases order a p*)
case (*Suc n*)
then show *?thesis*
by (*metis lessI not-less-Least order-def*)
qed *auto*
qed *auto*

lemma *order-2*:
assumes *p* ≠ *0*
shows ¬ [:−*a*, *1*:] ^ *Suc* (*order a p*) *dvd p*
proof −
have *False* **if** [:− *a*, *1*:] ^ *Suc* (*degree p*) *dvd p*
using *dvd-imp-degree-le* [*OF that*]
by (*metis Suc-n-not-le-n assms degree-linear-power*)
then show *?thesis*
unfolding *order-def*
by (*metis* (*no-types*, *lifting*) *LeastI*)
qed

lemma *order*: *p* ≠ *0* ⟹ [:−*a*, *1*:] ^ *order a p dvd p* ∧ ¬ [:−*a*, *1*:] ^ *Suc* (*order a p*) *dvd p*
by (*rule conjI* [*OF order-1 order-2*])

lemma *order-degree*:
assumes *p*: *p* ≠ *0*
shows *order a p* ≤ *degree p*
proof −
have *order a p* = *degree* ([:−*a*, *1*:] ^ *order a p*)

by (*simp only*: *degree-linear-power*)
also from *order-1 p* **have** ... ≤ *degree p*
by (*rule dvd-imp-degree-le*)
finally show *?thesis* **.**
qed

lemma *order-root*: *poly p a = 0* ⟷ *p = 0* ∨ *order a p* ≠ *0* (**is** *?lhs = ?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*metis One-nat-def order-2 poly-eq-0-iff-dvd power-one-right*)
show *?rhs* ⟹ *?lhs*
by (*meson dvd-power dvd-trans neq0-conv order-1 poly-0 poly-eq-0-iff-dvd*)
qed

lemma *order-0I*: *poly p a* ≠ *0* ⟹ *order a p = 0*
by (*subst* (*asm*) *order-root*) *auto*

lemma *order-unique-lemma*:
fixes *p* :: *'a*::*idom poly*
assumes [:−*a*, *1*:] ^ *n dvd p* ¬ [:−*a*, *1*:] ^ *Suc n dvd p*
shows *order a p = n*
unfolding *Polynomial.order-def*
by (*metis* (*mono-tags*, *lifting*) *Least-equality assms not-less-eq-eq power-le-dvd*)

lemma *order-mult*:
assumes *p* ∗ *q* ≠ *0* **shows** *order a* (*p* ∗ *q*) = *order a p* + *order a q*
proof −
define *i* **where** *i* ≡ *order a p*
define *j* **where** *j* ≡ *order a q*
define *t* **where** *t* ≡ [:−*a*, *1*:]
have *t-dvd-iff*: ⋀*u*. *t dvd u* ⟷ *poly u a = 0*
by (*simp add*: *t-def dvd-iff-poly-eq-0*)
have *dvd*: *t* ^ *i dvd p t* ^ *j dvd q* **and** ¬ *t* ^ *Suc i dvd p* ¬ *t* ^ *Suc j dvd q*
using *assms i-def j-def order-1 order-2 t-def* **by** *auto*
then have ¬ *t* ^ *Suc*(*i* + *j*) *dvd p* ∗ *q*
by (*elim dvdE*) (*simp add*: *power-add t-dvd-iff*)
moreover have *t* ^ (*i* + *j*) *dvd p* ∗ *q*
using *dvd* **by** (*simp add*: *mult-dvd-mono power-add*)
ultimately show *order a* (*p* ∗ *q*) = *i* + *j*
using *order-unique-lemma t-def* **by** *blast*
qed

lemma *order-smult*:
assumes *c* ≠ *0*
shows *order x* (*smult c p*) = *order x p*
proof (*cases p = 0*)
case *True*
then show *?thesis*

```
    by simp
next
  case False
  have smult c p = [:c:] * p by simp
  also from assms False have order x ... = order x [:c:] + order x p
    by (subst order-mult) simp-all
  also have order x [:c:] = 0
    by (rule order-0I) (use assms in auto)
  finally show ?thesis
    by simp
qed

lemma order-gt-0-iff: p ≠ 0 ⟹ order x p > 0 ⟷ poly p x = 0
  by (subst order-root) auto

lemma order-eq-0-iff: p ≠ 0 ⟹ order x p = 0 ⟷ poly p x ≠ 0
  by (subst order-root) auto
```

Next three lemmas contributed by Wenda Li

```
lemma order-1-eq-0 [simp]:order x 1 = 0
  by (metis order-root poly-1 zero-neq-one)

lemma order-uminus[simp]: order x (−p) = order x p
  by (metis neg-equal-0-iff-equal order-smult smult-1-left smult-minus-left)

lemma order-power-n-n: order a ([:−a,1:]^n)=n
proof (induct n)
  case 0
  then show ?case
    by (metis order-root poly-1 power-0 zero-neq-one)
next
  case (Suc n)
  have order a ([:− a, 1:] ^ Suc n) = order a ([:− a, 1:] ^ n) + order a [:−a,1:]
   by (metis (no-types, opaque-lifting) One-nat-def add-Suc-right monoid-add-class.add.right-neutral
     one-neq-zero order-mult pCons-eq-0-iff power-add power-eq-0-iff power-one-right)
  moreover have order a [:−a,1:] = 1
    unfolding order-def
  proof (rule Least-equality, rule notI)
    assume [:− a, 1:] ^ Suc 1 dvd [:− a, 1:]
    then have degree ([:− a, 1:] ^ Suc 1) ≤ degree ([:− a, 1:])
      by (rule dvd-imp-degree-le) auto
    then show False
      by auto
  next
    fix y
    assume *: ¬ [:− a, 1:] ^ Suc y dvd [:− a, 1:]
    show 1 ≤ y
    proof (rule ccontr)
      assume ¬ 1 ≤ y
```

then have *y* = *0* **by** *auto*
then have [:− *a*, *1*:] ^ *Suc y dvd* [:− *a*, *1*:] **by** *auto*
with ∗ **show** *False* **by** *auto*
qed
qed
ultimately show *?case*
using *Suc* **by** *auto*
qed

lemma *order-0-monom* [*simp*]: *c* ≠ *0* ⟹ *order 0* (*monom c n*) = *n*
using *order-power-n-n*[*of 0 n*] **by** (*simp add*: *monom-altdef order-smult*)

lemma *dvd-imp-order-le*: *q* ≠ *0* ⟹ *p dvd q* ⟹ *Polynomial.order a p* ≤ *Polynomial.order a q*
by (*auto simp*: *order-mult*)

Now justify the standard squarefree decomposition, i.e. *f* / *gcd f f′*.

lemma *order-divides*: [:−*a*, *1*:] ^ *n dvd p* ⟷ *p* = *0* ∨ *n* ≤ *order a p*
by (*meson dvd-0-right not-less-eq-eq order-1 order-2 power-le-dvd*)

lemma *order-decomp*:
assumes *p* ≠ *0*
shows ∃ *q*. *p* = [:− *a*, *1*:] ^ *order a p* ∗ *q* ∧ ¬ [:− *a*, *1*:] *dvd q*
proof −
from *assms* **have** ∗: [:− *a*, *1*:] ^ *order a p dvd p*
and ∗∗: ¬ [:− *a*, *1*:] ^ *Suc* (*order a p*) *dvd p*
by (*auto dest*: *order*)
from ∗ **obtain** *q* **where** *q*: *p* = [:− *a*, *1*:] ^ *order a p* ∗ *q* **..**
with ∗∗ **have** ¬ [:− *a*, *1*:] ^ *Suc* (*order a p*) *dvd* [:− *a*, *1*:] ^ *order a p* ∗ *q*
by *simp*
then have ¬ [:− *a*, *1*:] ^ *order a p* ∗ [:− *a*, *1*:] *dvd* [:− *a*, *1*:] ^ *order a p* ∗ *q*
by *simp*
with *idom-class.dvd-mult-cancel-left* [*of* [:− *a*, *1*:] ^ *order a p* [:− *a*, *1*:] *q*]
have ¬ [:− *a*, *1*:] *dvd q* **by** *auto*
with *q* **show** *?thesis* **by** *blast*
qed

lemma *monom-1-dvd-iff*: *p* ≠ *0* ⟹ *monom 1 n dvd p* ⟷ *n* ≤ *order 0 p*
using *order-divides*[*of 0 n p*] **by** (*simp add*: *monom-altdef*)

lemma *poly-root-order-induct* [*case-names 0 no-roots root*]:
fixes *p* :: ′*a* :: *idom poly*
assumes *P 0* ⋀*p*. (⋀*x*. *poly p x* ≠ *0*) ⟹ *P p*
⋀*p x n*. *n* > *0* ⟹ *poly p x* ≠ *0* ⟹ *P p* ⟹ *P* ([:−*x*, *1*:] ^ *n* ∗ *p*)
shows *P p*
proof (*induction degree p arbitrary*: *p rule*: *less-induct*)
case (*less p*)
consider *p* = *0* | *p* ≠ *0* ∃ *x*. *poly p x* = *0* | ⋀*x*. *poly p x* ≠ *0* **by** *blast*
thus *?case*

proof *cases*
case *3*
with *assms*(*2*)[*of p*] **show** *?thesis* **by** *simp*
next
case *2*
then obtain *x* **where** *x*: *poly p x* = *0* **by** *auto*
have [:−*x*, *1*:] ^ *order x p dvd p* **by** (*intro order-1*)
then obtain *q* **where** *q*: *p* = [:−*x*, *1*:] ^ *order x p* ∗ *q* **by** (*auto simp*: *dvd-def*)
with *2* **have** [*simp*]: *q* ≠ *0* **by** *auto*
have *order-pos*: *order x p* > *0*
using ‹*p* ≠ *0*› **and** *x* **by** (*auto simp*: *order-root*)
have *order x p* = *order x p* + *order x q*
by (*subst q*, *subst order-mult*) (*auto simp*: *order-power-n-n*)
hence [*simp*]: *order x q* = *0* **by** *simp*
have *deg*: *degree p* = *order x p* + *degree q*
by (*subst q*, *subst degree-mult-eq*) (*auto simp*: *degree-power-eq*)
with *order-pos* **have** *degree q* < *degree p* **by** *simp*
hence *P q* **by** (*rule less*)
with *order-pos* **have** *P* ([:−*x*, *1*:] ^ *order x p* ∗ *q*)
by (*intro assms*(*3*)) (*auto simp*: *order-root*)
with *q* **show** *?thesis* **by** *simp*
qed (*simp-all add*: *assms*(*1*))
qed

context
includes *multiset.lifting*
begin

lift-definition *proots* :: (′*a* :: *idom*) *poly* ⇒ ′*a multiset* **is**
λ(*p* :: ′*a poly*) (*x* :: ′*a*). *if p* = *0 then 0 else order x p*
proof −
fix *p* :: ′*a poly*
show *finite* {*x*. *0* < (*if p* = *0 then 0 else order x p*)}
by (*cases p* = *0*)
(*auto simp*: *order-gt-0-iff intro*: *finite-subset*[*OF* - *poly-roots-finite*[*of p*]])
qed

lemma *proots-0* [*simp*]: *proots* (*0* :: ′*a* :: *idom poly*) = {#}
by *transfer*′ *auto*

lemma *proots-1* [*simp*]: *proots* (*1* :: ′*a* :: *idom poly*) = {#}
by *transfer*′ *auto*

lemma *proots-const* [*simp*]: *proots* [: *x* :] = *0*
by *transfer*′ (*auto split*: *if-splits simp*: *fun-eq-iff order-eq-0-iff*)

lemma *proots-numeral* [*simp*]: *proots* (*numeral n*) = *0*
by (*simp add*: *numeral-poly*)

lemma *count-proots* [*simp*]:
  $p \neq 0 \Longrightarrow$ *count* (*proots p*) $a =$ *order a p*
  by *transfer′ auto*

lemma *set-count-proots* [*simp*]:
  $p \neq 0 \Longrightarrow$ *set-mset* (*proots p*) $= \{x.$ *poly p x* $= 0\}$
  by (*auto simp*: *set-mset-def order-gt-0-iff*)

lemma *proots-uminus* [*simp*]: *proots* $(-p) =$ *proots p*
  by (*cases* $p = 0$; *rule multiset-eqI*) *auto*

lemma *proots-smult* [*simp*]: $c \neq 0 \Longrightarrow$ *proots* (*smult c p*) $=$ *proots p*
  by (*cases* $p = 0$; *rule multiset-eqI*) (*auto simp*: *order-smult*)

lemma *proots-mult*:
  assumes $p \neq 0$ $q \neq 0$
  shows *proots* $(p * q) =$ *proots p* $+$ *proots q*
  using *assms* **by** (*intro multiset-eqI*) (*auto simp*: *order-mult*)

lemma *proots-prod*:
  assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0$
  shows *proots* $(\prod x{\in}A.\ f\ x) = (\sum x{\in}A.$ *proots* $(f\ x))$
  using *assms* **by** (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *proots-mult*)

lemma *proots-prod-mset*:
  assumes $0 \notin\# A$
  shows *proots* $(\prod p{\in}\#A.\ p) = (\sum p{\in}\#A.$ *proots* $p)$
  using *assms* **by** (*induction A*) (*auto simp*: *proots-mult*)

lemma *proots-prod-list*:
  assumes $0 \notin$ *set ps*
  shows *proots* $(\prod p{\leftarrow}ps.\ p) = (\sum p{\leftarrow}ps.$ *proots* $p)$
  using *assms* **by** (*induction ps*) (*auto simp*: *proots-mult prod-list-zero-iff*)

lemma *proots-power*: *proots* $(p\ \hat{}\ n) =$ *repeat-mset n* (*proots p*)
proof (*cases* $p = 0$)
  case *False*
  thus *?thesis*
    by (*induction n*) (*auto simp*: *proots-mult*)
qed (*auto simp*: *power-0-left*)

lemma *proots-linear-factor* [*simp*]: *proots* $[:x,\ 1:] = \{\#-x\#\}$
proof −
  have *order* $(-x)\ [:x,\ 1:] > 0$
    by (*subst order-gt-0-iff*) *auto*
  moreover have *order* $(-x)\ [:x,\ 1:] \leq$ *degree* $[:x,\ 1:]$
    by (*rule order-degree*) *auto*
  moreover have *order* $y\ [:x,\ 1:] = 0$ **if** $y \neq -x$ **for** y

by (*rule order-0I*) (*use that* **in** ‹*auto simp*: *add-eq-0-iff*›)
ultimately show *?thesis*
by (*intro multiset-eqI*) *auto*
qed

lemma *size-proots-le*: *size* (*proots p*) $\leq$ *degree p*
proof (*induction p rule*: *poly-root-order-induct*)
case (*no-roots p*)
hence *proots p* = *0*
by (*simp add*: *multiset-eqI order-root*)
thus *?case* **by** *simp*
next
case (*root p x n*)
have [*simp*]: $p \neq 0$
using *root.hyps* **by** *auto*
from *root.IH* **show** *?case*
by (*auto simp*: *proots-mult proots-power degree-mult-eq degree-power-eq*)
qed *auto*

end

4.21 Additional induction rules on polynomials

An induction rule for induction over the roots of a polynomial with a certain property. (e.g. all positive roots)

lemma *poly-root-induct* [*case-names 0 no-roots root*]:
fixes *p* :: *'a* :: *idom poly*
assumes *Q 0*
and $\bigwedge p. (\bigwedge a.\ P\ a \Longrightarrow poly\ p\ a \neq 0) \Longrightarrow Q\ p$
and $\bigwedge a\ p.\ P\ a \Longrightarrow Q\ p \Longrightarrow Q\ ([:a, -1:] * p)$
shows *Q p*
proof (*induction degree p arbitrary*: *p rule*: *less-induct*)
case (*less p*)
show *?case*
proof (*cases p* = *0*)
case *True*
with *assms*(*1*) **show** *?thesis* **by** *simp*
next
case *False*
show *?thesis*
proof (*cases* $\exists a.\ P\ a \wedge poly\ p\ a = 0$)
case *False*
then show *?thesis* **by** (*intro assms*(*2*)) *blast*
next
case *True*
then obtain *a* **where** *a*: *P a poly p a* = *0*
by *blast*
then have $-[:-a, 1:]$ *dvd p*
by (*subst minus-dvd-iff*) (*simp add*: *poly-eq-0-iff-dvd*)

```
      then obtain q where q: p = [:a, −1:] ∗ q by (elim dvdE) simp
      with False have q ≠ 0 by auto
      have degree p = Suc (degree q)
        by (subst q, subst degree-mult-eq) (simp-all add: ‹q ≠ 0›)
      then have Q q by (intro less) simp
      with a(1) have Q ([:a, −1:] ∗ q)
        by (rule assms(3))
      with q show ?thesis by simp
    qed
  qed
qed

lemma dropWhile-replicate-append:
  dropWhile ((=) a) (replicate n a @ ys) = dropWhile ((=) a) ys
  by (induct n) simp-all

lemma Poly-append-replicate-0: Poly (xs @ replicate n 0) = Poly xs
  by (subst coeffs-eq-iff) (simp-all add: strip-while-def dropWhile-replicate-append)
```

An induction rule for simultaneous induction over two polynomials, prepending one coefficient in each step.

```
lemma poly-induct2 [case-names 0 pCons]:
  assumes P 0 0 ⋀a p b q. P p q ⟹ P (pCons a p) (pCons b q)
  shows P p q
proof −
  define n where n = max (length (coeffs p)) (length (coeffs q))
  define xs where xs = coeffs p @ (replicate (n − length (coeffs p)) 0)
  define ys where ys = coeffs q @ (replicate (n − length (coeffs q)) 0)
  have length xs = length ys
    by (simp add: xs-def ys-def n-def)
  then have P (Poly xs) (Poly ys)
    by (induct rule: list-induct2) (simp-all add: assms)
  also have Poly xs = p
    by (simp add: xs-def Poly-append-replicate-0)
  also have Poly ys = q
    by (simp add: ys-def Poly-append-replicate-0)
  finally show ?thesis .
qed
```

4.22 Composition of polynomials

```
definition pcompose :: 'a::comm-semiring-0 poly ⇒ 'a poly ⇒ 'a poly
  where pcompose p q = fold-coeffs (λa c. [:a:] + q ∗ c) p 0

notation pcompose (infixl ‹∘p› 71)

lemma pcompose-0 [simp]: pcompose 0 q = 0
  by (simp add: pcompose-def)
```

lemma *pcompose-pCons*: *pcompose* (*pCons a p*) *q* = [:*a*:] + *q* ∗ *pcompose p q*
  by (*cases p* = *0* ∧ *a* = *0*) (*auto simp add*: *pcompose-def*)

lemma *pcompose-altdef*: *pcompose p q* = *poly* (*map-poly* (λ*x*. [:*x*:]) *p*) *q*
  by (*induction p*) (*simp-all add*: *map-poly-pCons pcompose-pCons*)

lemma *coeff-pcompose-0* [*simp*]:
  coeff (*pcompose p q*) *0* = *poly p* (*coeff q 0*)
  by (*induction p*) (*simp-all add*: *coeff-mult-0 pcompose-pCons*)

lemma *pcompose-1*: *pcompose 1 p* = *1*
  for *p* :: ′*a*::*comm-semiring-1 poly*
  by (*auto simp*: *one-pCons pcompose-pCons*)

lemma *poly-pcompose*: *poly* (*pcompose p q*) *x* = *poly p* (*poly q x*)
  by (*induct p*) (*simp-all add*: *pcompose-pCons*)

lemma *degree-pcompose-le*: *degree* (*pcompose p q*) ≤ *degree p* ∗ *degree q*
proof (*induction p*)
  case (*pCons a p*)
  then show *?case*
  proof (*clarsimp simp add*: *pcompose-pCons*)
    assume *degree* ($p \circ_p q$) ≤ *degree p* ∗ *degree q p* ≠ *0*
    then have *degree* ($q * p \circ_p q$) ≤ *degree q* + *degree p* ∗ *degree q*
      by (*meson add-le-cancel-left degree-mult-le dual-order.trans pCons.IH*)
    then show *degree* ([:*a*:] + $q * p \circ_p q$) ≤ *degree q* + *degree p* ∗ *degree q*
      by (*simp add*: *degree-add-le*)
  qed
qed *auto*

lemma *pcompose-add*: *pcompose* (*p* + *q*) *r* = *pcompose p r* + *pcompose q r*
  for *p q r* :: ′*a*::{*comm-semiring-0*, *ab-semigroup-add*} *poly*
proof (*induction p q rule*: *poly-induct2*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*pCons a p b q*)
  have *pcompose* (*pCons a p* + *pCons b q*) *r* = [:*a* + *b*:] + *r* ∗ *pcompose p r* + *r*
∗ *pcompose q r*
    by (*simp-all add*: *pcompose-pCons pCons.IH algebra-simps*)
  also have [:*a* + *b*:] = [:*a*:] + [:*b*:] **by** *simp*
  also have . . . + *r* ∗ *pcompose p r* + *r* ∗ *pcompose q r* = *pcompose* (*pCons a p*)
r + *pcompose* (*pCons b q*) *r*
    by (*simp only*: *pcompose-pCons add-ac*)
  finally show *?case* **.**
qed

lemma *pcompose-uminus*: *pcompose* (−*p*) *r* = −*pcompose p r*
  for *p r* :: ′*a*::*comm-ring poly*

by (*induct p*) (*simp-all add*: *pcompose-pCons*)

lemma *pcompose-diff*: *pcompose* (*p* − *q*) *r* = *pcompose p r* − *pcompose q r*
for *p q r* :: ′*a*::*comm-ring poly*
using *pcompose-add*[*of p* −*q*] **by** (*simp add*: *pcompose-uminus*)

lemma *pcompose-smult*: *pcompose* (*smult a p*) *r* = *smult a* (*pcompose p r*)
for *p r* :: ′*a*::*comm-semiring-0 poly*
by (*induct p*) (*simp-all add*: *pcompose-pCons pcompose-add smult-add-right*)

lemma *pcompose-mult*: *pcompose* (*p* ∗ *q*) *r* = *pcompose p r* ∗ *pcompose q r*
for *p q r* :: ′*a*::*comm-semiring-0 poly*
by (*induct p arbitrary*: *q*) (*simp-all add*: *pcompose-add pcompose-smult pcompose-pCons algebra-simps*)

lemma *pcompose-assoc*: *pcompose p* (*pcompose q r*) = *pcompose* (*pcompose p q*) *r*
for *p q r* :: ′*a*::*comm-semiring-0 poly*
by (*induct p arbitrary*: *q*) (*simp-all add*: *pcompose-pCons pcompose-add pcompose-mult*)

lemma *pcompose-idR*[*simp*]: *pcompose p* [: *0*, *1* :] = *p*
for *p* :: ′*a*::*comm-semiring-1 poly*
by (*induct p*) (*simp-all add*: *pcompose-pCons*)

lemma *pcompose-sum*: *pcompose* (*sum f A*) *p* = *sum* (λ*i*. *pcompose* (*f i*) *p*) *A*
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *pcompose-1 pcompose-add*)

lemma *pcompose-prod*: *pcompose* (*prod f A*) *p* = *prod* (λ*i*. *pcompose* (*f i*) *p*) *A*
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *pcompose-1 pcompose-mult*)

lemma *pcompose-const* [*simp*]: *pcompose* [:*a*:] *q* = [:*a*:]
by (*subst pcompose-pCons*) *simp*

lemma *pcompose-0′*: *pcompose p 0* = [:*coeff p 0*:]
by (*induct p*) (*auto simp add*: *pcompose-pCons*)

lemma *degree-pcompose*: *degree* (*pcompose p q*) = *degree p* ∗ *degree q*
for *p q* :: ′*a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
proof (*induct p*)
case *0*
then show *?case* **by** *auto*
next
case (*pCons a p*)
consider *degree* (*q* ∗ *pcompose p q*) = *0* | *degree* (*q* ∗ *pcompose p q*) > *0*
by *blast*
then show *?case*
proof *cases*
case *prems*: *1*
show *?thesis*

```
    proof (cases p = 0)
      case True
      then show ?thesis by auto
    next
      case False
      from prems have degree q = 0 ∨ pcompose p q = 0
        by (auto simp add: degree-mult-eq-0)
      moreover have False if pcompose p q = 0 degree q ≠ 0
      proof −
        from pCons.hyps(2) that have degree p = 0
          by auto
        then obtain a1 where p = [:a1:]
          by (metis degree-pCons-eq-if old.nat.distinct(2) pCons-cases)
        with ‹pcompose p q = 0› ‹p ≠ 0› show False
          by auto
      qed
      ultimately have degree (pCons a p) ∗ degree q = 0
        by auto
      moreover have degree (pcompose (pCons a p) q) = 0
      proof −
        from prems have 0 = max (degree [:a:]) (degree (q ∗ pcompose p q))
          by simp
        also have . . . ≥ degree ([:a:] + q ∗ pcompose p q)
          by (rule degree-add-le-max)
        finally show ?thesis
          by (auto simp add: pcompose-pCons)
      qed
      ultimately show ?thesis by simp
    qed
  next
    case prems: 2
    then have p ≠ 0 q ≠ 0 pcompose p q ≠ 0
      by auto
    from prems degree-add-eq-right [of [:a:]]
    have degree (pcompose (pCons a p) q) = degree (q ∗ pcompose p q)
      by (auto simp: pcompose-pCons)
   with pCons.hyps(2) degree-mult-eq[OF ‹q≠0› ‹pcompose p q≠0›] show ?thesis
      by auto
  qed
qed

lemma pcompose-eq-0:
  fixes p q :: 'a::{comm-semiring-0,semiring-no-zero-divisors} poly
  assumes pcompose p q = 0 degree q > 0
  shows p = 0
proof −
  from assms degree-pcompose [of p q] have degree p = 0
    by auto
  then obtain a where p = [:a:]
```

by (*metis degree-pCons-eq-if gr0-conv-Suc neq0-conv pCons-cases*)
with *assms*(*1*) **have** $a = 0$
by *auto*
with ‹$p = [:a:]$› **show** *?thesis*
by *simp*
qed

lemma *pcompose-eq-0-iff*:
fixes p q :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
assumes *degree* $q > 0$
shows *pcompose* p $q = 0 \longleftrightarrow p = 0$
using *pcompose-eq-0*[*OF* - *assms*] **by** *auto*

lemma *coeff-pcompose-linear*:
coeff (*pcompose* p [:*0*, a :: *'a* :: *comm-semiring-1*:]) $i = a$ ^ i ∗ *coeff* p i
by (*induction p arbitrary*: *i*) (*auto simp*: *pcompose-pCons coeff-pCons mult-ac*
split: *nat.splits*)

lemma *lead-coeff-comp*:
fixes p q :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
assumes *degree* $q > 0$
shows *lead-coeff* (*pcompose* p q) = *lead-coeff* p ∗ *lead-coeff* q ^ (*degree* p)
proof (*induct p*)
case *0*
then show *?case* **by** *auto*
next
case (*pCons a p*)
consider *degree* (q ∗ *pcompose* p q) = *0* | *degree* (q ∗ *pcompose* p q) > *0*
by *blast*
then show *?case*
proof *cases*
case *prems*: *1*
then have *pcompose* p $q = 0$
by (*metis assms degree-0 degree-mult-eq-0 neq0-conv*)
with *pcompose-eq-0*[*OF* - ‹*degree* $q > 0$›] **have** $p = 0$
by *simp*
then show *?thesis*
by *auto*
next
case *prems*: *2*
then have *degree* [:a:] < *degree* (q ∗ *pcompose* p q)
by *simp*
then have *lead-coeff* ([:a:] + q ∗ $p \circ_p q$) = *lead-coeff* (q ∗ $p \circ_p q$)
by (*rule lead-coeff-add-le*)
then have *lead-coeff* (*pcompose* (*pCons a p*) q) = *lead-coeff* (q ∗ *pcompose* p
q)
by (*simp add*: *pcompose-pCons*)
also have ... = *lead-coeff* q ∗ (*lead-coeff* p ∗ *lead-coeff* q ^ *degree* p)
using *pCons.hyps*(*2*) *lead-coeff-mult*[*of q pcompose p q*] **by** *simp*

also have ... = *lead-coeff p* ∗ *lead-coeff q* ^ (*degree p* + *1*)
  by (*auto simp*: *mult-ac*)
finally show *?thesis* **by** *auto*
qed
qed

lemma *coeff-pcompose-monom-linear* [*simp*]:
  fixes *p* :: *'a* :: *comm-ring-1 poly*
  shows *coeff* (*pcompose p* (*monom c* (*Suc 0*))) *k* = *c* ^ *k* ∗ *coeff p k*
  by (*induction p arbitrary*: *k*)
    (*auto simp*: *coeff-pCons coeff-monom-mult pcompose-pCons split*: *nat.splits*)

lemma *of-nat-mult-conv-smult*: *of-nat n* ∗ *P* = *smult* (*of-nat n*) *P*
  by (*simp add*: *monom-0 of-nat-monom*)

lemma *numeral-mult-conv-smult*: *numeral n* ∗ *P* = *smult* (*numeral n*) *P*
  by (*simp add*: *numeral-poly*)

lemma *sum-order-le-degree*:
  assumes *p* ≠ *0*
  shows (∑ *x* | *poly p x* = *0*. *order x p*) ≤ *degree p*
  using *assms*
proof (*induction degree p arbitrary*: *p rule*: *less-induct*)
  case (*less p*)
  show *?case*
  proof (*cases* ∃ *x*. *poly p x* = *0*)
    case *False*
    thus *?thesis*
      by *auto*
  next
    case *True*
    then obtain *x* **where** *x*: *poly p x* = *0*
      by *auto*
    have [:−*x*, *1*:] ^ *order x p dvd p*
      by (*simp add*: *order-1*)
    then obtain *q* **where** *q*: *p* = [:−*x*, *1*:] ^ *order x p* ∗ *q*
      by (*elim dvdE*)
    have [*simp*]: *q* ≠ *0*
      using *q less.prems* **by** *auto*
    have *order x p* = *order x p* + *order x q*
      by (*subst q*, *subst order-mult*) (*auto simp*: *order-power-n-n*)
    hence *order x q* = *0*
      by *auto*
    hence [*simp*]: *poly q x* ≠ *0*
      by (*simp add*: *order-root*)
    have *deg-p*: *degree p* = *degree q* + *order x p*
      by (*subst q*, *subst degree-mult-eq*) (*auto simp*: *degree-power-eq*)
    moreover have *order x p* > *0*
      using *x less.prems* **by** (*simp add*: *order-root*)

ultimately have *degree q* $<$ *degree p*
by *linarith*
hence $(\sum x \mid$ *poly q x* $= 0.$ *order x q*$) \leq$ *degree q*
by (*intro less.hyps*) *auto*
hence *order x p* $+ (\sum x \mid$ *poly q x* $= 0.$ *order x q*$) \leq$ *degree p*
by (*simp add*: *deg-p*)
also have $\{y.$ *poly q y* $= 0\} = \{y.$ *poly p y* $= 0\} - \{x\}$
by (*subst q*) *auto*
also have $(\sum y \in \{y.$ *poly p y* $= 0\} - \{x\}.$ *order y q*$) =$
$(\sum y \in \{y.$ *poly p y* $= 0\} - \{x\}.$ *order y p*$)$
by (*intro sum.cong refl*, *subst q*)
(*auto simp*: *order-mult order-power-n-n intro*!: *order-0I*)
also have *order x p* $+ \ldots = (\sum y \in$ *insert x* $(\{y.$ *poly p y* $= 0\} - \{x\}).$ *order y p*$)$
using ‹$p \neq 0$› **by** (*subst sum.insert*) (*auto simp*: *poly-roots-finite*)
also have *insert x* $(\{y.$ *poly p y* $= 0\} - \{x\}) = \{y.$ *poly p y* $= 0\}$
using ‹*poly p x* $= 0$› **by** *auto*
finally show *?thesis* **.**
qed
qed

4.23 Closure properties of coefficients

context
fixes R :: *'a* :: *comm-semiring-1 set*
assumes *R-0*: $0 \in R$
assumes *R-plus*: $\bigwedge x\ y.\ x \in R \Longrightarrow y \in R \Longrightarrow x + y \in R$
assumes *R-mult*: $\bigwedge x\ y.\ x \in R \Longrightarrow y \in R \Longrightarrow x * y \in R$
begin

lemma *coeff-mult-semiring-closed*:
assumes $\bigwedge i.$ *coeff p i* $\in R$ $\bigwedge i.$ *coeff q i* $\in R$
shows *coeff* $(p * q)$ $i \in R$
proof −
have *R-sum*: *sum f A* $\in R$ **if** $\bigwedge x.\ x \in A \Longrightarrow f\ x \in R$ **for** A **and** f :: *nat* ⇒ *'a*
using *that* **by** (*induction A rule*: *infinite-finite-induct*) (*auto intro*: *R-0 R-plus*)
show *?thesis*
unfolding *coeff-mult* **by** (*auto intro*!: *R-sum R-mult assms*)
qed

lemma *coeff-pcompose-semiring-closed*:
assumes $\bigwedge i.$ *coeff p i* $\in R$ $\bigwedge i.$ *coeff q i* $\in R$
shows *coeff* (*pcompose p q*) $i \in R$
using *assms*(*1*)
proof (*induction p arbitrary*: *i*)
case (*pCons a p i*)
have [*simp*]: $a \in R$
using *pCons.prems*[*of 0*] **by** *auto*
have *coeff p i* $\in R$ **for** i

using *pCons.prems*[*of Suc i*] **by** *auto*
hence *coeff* (*p* $\circ_p$ *q*) *i* ∈ *R* **for** *i*
using *pCons.prems* **by** (*intro pCons.IH*)
thus *?case*
by (*auto simp*: *pcompose-pCons coeff-pCons split*: *nat.splits*
intro!: *assms R-plus coeff-mult-semiring-closed*)
qed *auto*

end

4.24 Shifting polynomials

definition *poly-shift* :: *nat* ⇒ *'a*::*zero poly* ⇒ *'a poly*
where *poly-shift n p* = *Abs-poly* (λ*i*. *coeff p* (*i* + *n*))

lemma *nth-default-drop*: *nth-default x* (*drop n xs*) *m* = *nth-default x xs* (*m* + *n*)
by (*auto simp add*: *nth-default-def add-ac*)

lemma *nth-default-take*: *nth-default x* (*take n xs*) *m* = (*if m* < *n then nth-default x xs m else x*)
by (*auto simp add*: *nth-default-def add-ac*)

lemma *coeff-poly-shift*: *coeff* (*poly-shift n p*) *i* = *coeff p* (*i* + *n*)
proof −
from *MOST-coeff-eq-0*[*of p*] **obtain** *m* **where** ∀ *k*>*m*. *coeff p k* = *0*
by (*auto simp*: *MOST-nat*)
then have ∀ *k*>*m*. *coeff p* (*k* + *n*) = *0*
by *auto*
then have $\forall_\infty$*k*. *coeff p* (*k* + *n*) = *0*
by (*auto simp*: *MOST-nat*)
then show *?thesis*
by (*simp add*: *poly-shift-def poly.Abs-poly-inverse*)
qed

lemma *poly-shift-id* [*simp*]: *poly-shift 0* = (λ*x*. *x*)
by (*simp add*: *poly-eq-iff fun-eq-iff coeff-poly-shift*)

lemma *poly-shift-0* [*simp*]: *poly-shift n 0* = *0*
by (*simp add*: *poly-eq-iff coeff-poly-shift*)

lemma *poly-shift-1*: *poly-shift n 1* = (*if n* = *0 then 1 else 0*)
by (*simp add*: *poly-eq-iff coeff-poly-shift*)

lemma *poly-shift-monom*: *poly-shift n* (*monom c m*) = (*if m* ≥ *n then monom c* (*m* − *n*) *else 0*)
by (*auto simp add*: *poly-eq-iff coeff-poly-shift*)

lemma *coeffs-shift-poly* [*code abstract*]:
coeffs (*poly-shift n p*) = *drop n* (*coeffs p*)

proof (*cases p = 0*)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  then show *?thesis*
    by (*intro coeffs-eqI*)
      (*simp-all add*: *coeff-poly-shift nth-default-drop nth-default-coeffs-eq*)
qed

4.25 Truncating polynomials

definition *poly-cutoff*
  where *poly-cutoff n p* = *Abs-poly* (*λk. if k < n then coeff p k else 0*)

lemma *coeff-poly-cutoff*: *coeff* (*poly-cutoff n p*) *k* = (*if k < n then coeff p k else 0*)
  unfolding *poly-cutoff-def*
  by (*subst poly.Abs-poly-inverse*) (*auto simp*: *MOST-nat intro*: *exI*[*of - n*])

lemma *poly-cutoff-0* [*simp*]: *poly-cutoff n 0* = *0*
  by (*simp add*: *poly-eq-iff coeff-poly-cutoff*)

lemma *poly-cutoff-1* [*simp*]: *poly-cutoff n 1* = (*if n = 0 then 0 else 1*)
  by (*simp add*: *poly-eq-iff coeff-poly-cutoff*)

lemma *coeffs-poly-cutoff* [*code abstract*]:
  coeffs (*poly-cutoff n p*) = *strip-while* ((=) *0*) (*take n* (*coeffs p*))
proof (*cases strip-while* ((=) *0*) (*take n* (*coeffs p*)) = [])
  case *True*
  then have *coeff* (*poly-cutoff n p*) *k* = *0* **for** *k*
    unfolding *coeff-poly-cutoff*
    by (*auto simp*: *nth-default-coeffs-eq* [*symmetric*] *nth-default-def set-conv-nth*)
  then have *poly-cutoff n p* = *0*
    by (*simp add*: *poly-eq-iff*)
  then show *?thesis*
    by (*subst True*) *simp-all*
next
  case *False*
  have *no-trailing* ((=) *0*) (*strip-while* ((=) *0*) (*take n* (*coeffs p*)))
    by *simp*
  with *False* **have** *last* (*strip-while* ((=) *0*) (*take n* (*coeffs p*))) ≠ *0*
    unfolding *no-trailing-unfold* **by** *auto*
  then show *?thesis*
    by (*intro coeffs-eqI*)
      (*simp-all add*: *coeff-poly-cutoff nth-default-take nth-default-coeffs-eq*)
qed

4.26 Reflecting polynomials

definition *reflect-poly* :: *'a::zero poly* ⇒ *'a poly*
  where *reflect-poly p* = *Poly* (*rev* (*coeffs p*))

lemma *coeffs-reflect-poly* [*code abstract*]:
  coeffs (*reflect-poly p*) = *rev* (*dropWhile* ((=) *0*) (*coeffs p*))
  by (*simp add*: *reflect-poly-def*)

lemma *reflect-poly-0* [*simp*]: *reflect-poly 0* = *0*
  by (*simp add*: *reflect-poly-def*)

lemma *reflect-poly-1* [*simp*]: *reflect-poly 1* = *1*
  by (*simp add*: *reflect-poly-def one-pCons*)

lemma *coeff-reflect-poly*:
  coeff (*reflect-poly p*) *n* = (*if n* > *degree p then 0 else coeff p* (*degree p* − *n*))
  by (*cases p* = *0*)
    (*auto simp add*: *reflect-poly-def nth-default-def*
      rev-nth degree-eq-length-coeffs coeffs-nth not-less
      dest: *le-imp-less-Suc*)

lemma *coeff-0-reflect-poly-0-iff* [*simp*]: *coeff* (*reflect-poly p*) *0* = *0* ⟷ *p* = *0*
  by (*simp add*: *coeff-reflect-poly*)

lemma *reflect-poly-at-0-eq-0-iff* [*simp*]: *poly* (*reflect-poly p*) *0* = *0* ⟷ *p* = *0*
  by (*simp add*: *coeff-reflect-poly poly-0-coeff-0*)

lemma *reflect-poly-pCons'*:
  p ≠ *0* ⟹ *reflect-poly* (*pCons c p*) = *reflect-poly p* + *monom c* (*Suc* (*degree p*))
  by (*intro poly-eqI*)
    (*auto simp*: *coeff-reflect-poly coeff-pCons not-less Suc-diff-le split*: *nat.split*)

lemma *reflect-poly-const* [*simp*]: *reflect-poly* [:*a*:] = [:*a*:]
  by (*cases a* = *0*) (*simp-all add*: *reflect-poly-def*)

lemma *poly-reflect-poly-nz*:
  x ≠ *0* ⟹ *poly* (*reflect-poly p*) *x* = *x* ^ *degree p* ∗ *poly p* (*inverse x*)
  for *x* :: *'a::field*
  by (*induct rule*: *pCons-induct*) (*simp-all add*: *field-simps reflect-poly-pCons' poly-monom*)

lemma *coeff-0-reflect-poly* [*simp*]: *coeff* (*reflect-poly p*) *0* = *lead-coeff p*
  by (*simp add*: *coeff-reflect-poly*)

lemma *poly-reflect-poly-0* [*simp*]: *poly* (*reflect-poly p*) *0* = *lead-coeff p*
  by (*simp add*: *poly-0-coeff-0*)

lemma *reflect-poly-reflect-poly* [*simp*]: *coeff p 0* ≠ *0* ⟹ *reflect-poly* (*reflect-poly p*) = *p*
  by (*cases p rule*: *pCons-cases*) (*simp add*: *reflect-poly-def*)

lemma *degree-reflect-poly-le*: *degree* (*reflect-poly p*) ≤ *degree p*
by (*simp add*: *degree-eq-length-coeffs coeffs-reflect-poly length-dropWhile-le diff-le-mono*)

lemma *reflect-poly-pCons*: *a* ≠ *0* ⟹ *reflect-poly* (*pCons a p*) = *Poly* (*rev* (*a* # *coeffs p*))
by (*subst coeffs-eq-iff*) (*simp add*: *coeffs-reflect-poly*)

lemma *degree-reflect-poly-eq* [*simp*]: *coeff p 0* ≠ *0* ⟹ *degree* (*reflect-poly p*) = *degree p*
by (*cases p rule*: *pCons-cases*) (*simp add*: *reflect-poly-pCons degree-eq-length-coeffs*)

lemma *reflect-poly-eq-0-iff* [*simp*]: *reflect-poly p* = *0* ⟷ *p* = *0*
using *coeff-0-reflect-poly-0-iff* **by** *fastforce*

lemma *reflect-poly-mult*: *reflect-poly* (*p* ∗ *q*) = *reflect-poly p* ∗ *reflect-poly q*
for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
proof (*cases p* = *0* ∨ *q* = *0*)
case *False*
then have [*simp*]: *p* ≠ *0 q* ≠ *0* **by** *auto*
show *?thesis*
proof (*rule poly-eqI*)
show *coeff* (*reflect-poly* (*p* ∗ *q*)) *i* = *coeff* (*reflect-poly p* ∗ *reflect-poly q*) *i* **for** *i*
proof (*cases i* ≤ *degree* (*p* ∗ *q*))
case *True*
define *A* **where** *A* = {..*i*} ∩ {*i* − *degree q*..*degree p*}
define *B* **where** *B* = {..*degree p*} ∩ {*degree p* − *i*..*degree* (*p*∗*q*) − *i*}
let *?f* = λ*j*. *degree p* − *j*

from *True* **have** *coeff* (*reflect-poly* (*p* ∗ *q*)) *i* = *coeff* (*p* ∗ *q*) (*degree* (*p* ∗ *q*) − *i*)
by (*simp add*: *coeff-reflect-poly*)
also have . . . = (∑*j*≤*degree* (*p* ∗ *q*) − *i*. *coeff p j* ∗ *coeff q* (*degree* (*p* ∗ *q*) − *i* − *j*))
by (*simp add*: *coeff-mult*)
also have . . . = (∑*j*∈*B*. *coeff p j* ∗ *coeff q* (*degree* (*p* ∗ *q*) − *i* − *j*))
by (*intro sum.mono-neutral-right*) (*auto simp*: *B-def degree-mult-eq not-le coeff-eq-0*)
also from *True* **have** . . . = (∑*j*∈*A*. *coeff p* (*degree p* − *j*) ∗ *coeff q* (*degree q* − (*i* − *j*)))
by (*intro sum.reindex-bij-witness*[*of - ?f ?f*])
(*auto simp*: *A-def B-def degree-mult-eq add-ac*)
also have . . . =
(∑*j*≤*i*.
if j ∈ {*i* − *degree q*..*degree p*}
then coeff p (*degree p* − *j*) ∗ *coeff q* (*degree q* − (*i* − *j*))
else 0)
by (*subst sum.inter-restrict* [*symmetric*]) (*simp-all add*: *A-def*)

also have . . . = *coeff* (*reflect-poly p* ∗ *reflect-poly q*) *i*
by (*fastforce simp*: *coeff-mult coeff-reflect-poly intro*!: *sum.cong*)
finally show *?thesis* **.**
qed (*auto simp*: *coeff-mult coeff-reflect-poly coeff-eq-0 degree-mult-eq intro*!: *sum.neutral*)
qed
qed *auto*

lemma *reflect-poly-smult*: *reflect-poly* (*smult c p*) = *smult c* (*reflect-poly p*)
for *p* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
using *reflect-poly-mult*[*of* [:*c*:] *p*] **by** *simp*

lemma *reflect-poly-power*: *reflect-poly* (*p* ^ *n*) = *reflect-poly p* ^ *n*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
by (*induct n*) (*simp-all add*: *reflect-poly-mult*)

lemma *reflect-poly-prod*: *reflect-poly* (*prod f A*) = *prod* (λ*x*. *reflect-poly* (*f x*)) *A*
for *f* :: - ⇒ -::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *reflect-poly-mult*)

lemma *reflect-poly-prod-list*: *reflect-poly* (*prod-list xs*) = *prod-list* (*map reflect-poly xs*)
for *xs* :: -::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly list*
by (*induct xs*) (*simp-all add*: *reflect-poly-mult*)

lemma *reflect-poly-Poly-nz*:
no-trailing (*HOL.eq 0*) *xs* ⟹ *reflect-poly* (*Poly xs*) = *Poly* (*rev xs*)
by (*simp add*: *reflect-poly-def*)

lemmas *reflect-poly-simps* =
reflect-poly-0 reflect-poly-1 reflect-poly-const reflect-poly-smult reflect-poly-mult reflect-poly-power reflect-poly-prod reflect-poly-prod-list

4.27 Derivatives

function *pderiv* :: (*'a* :: {*comm-semiring-1*,*semiring-no-zero-divisors*}) *poly* ⇒ *'a poly*
where *pderiv* (*pCons a p*) = (*if p* = *0 then 0 else p* + *pCons 0* (*pderiv p*))
by (*auto intro*: *pCons-cases*)

termination *pderiv*
by (*relation measure degree*) *simp-all*

declare *pderiv.simps*[*simp del*]

lemma *pderiv-0* [*simp*]: *pderiv 0* = *0*
using *pderiv.simps* [*of 0 0*] **by** *simp*

lemma *pderiv-pCons*: *pderiv* (*pCons a p*) = *p* + *pCons 0* (*pderiv p*)

```
  by (simp add: pderiv.simps)

lemma pderiv-1 [simp]: pderiv 1 = 0
  by (simp add: one-pCons pderiv-pCons)

lemma pderiv-of-nat [simp]: pderiv (of-nat n) = 0
  and pderiv-numeral [simp]: pderiv (numeral m) = 0
  by (simp-all add: of-nat-poly numeral-poly pderiv-pCons)

lemma coeff-pderiv: coeff (pderiv p) n = of-nat (Suc n) * coeff p (Suc n)
  by (induct p arbitrary: n)
    (auto simp add: pderiv-pCons coeff-pCons algebra-simps split: nat.split)

fun pderiv-coeffs-code :: 'a::{comm-semiring-1,semiring-no-zero-divisors} ⇒ 'a list
⇒ 'a list
  where
    pderiv-coeffs-code f (x # xs) = cCons (f * x) (pderiv-coeffs-code (f+1) xs)
  | pderiv-coeffs-code f [] = []

definition pderiv-coeffs :: 'a::{comm-semiring-1,semiring-no-zero-divisors} list ⇒
'a list
  where pderiv-coeffs xs = pderiv-coeffs-code 1 (tl xs)


lemma pderiv-coeffs-code:
  nth-default 0 (pderiv-coeffs-code f xs) n = (f + of-nat n) * nth-default 0 xs n
proof (induct xs arbitrary: f n)
  case Nil
  then show ?case by simp
next
  case (Cons x xs)
  show ?case
  proof (cases n)
    case 0
    then show ?thesis
     by (cases pderiv-coeffs-code (f + 1) xs = [] ∧ f * x = 0) (auto simp: cCons-def)
  next
    case n: (Suc m)
    show ?thesis
    proof (cases pderiv-coeffs-code (f + 1) xs = [] ∧ f * x = 0)
      case False
      then have nth-default 0 (pderiv-coeffs-code f (x # xs)) n =
          nth-default 0 (pderiv-coeffs-code (f + 1) xs) m
        by (auto simp: cCons-def n)
      also have ... = (f + of-nat n) * nth-default 0 xs m
        by (simp add: Cons n add-ac)
      finally show ?thesis
        by (simp add: n)
    next
```

case *True*
have *empty*: *pderiv-coeffs-code g xs* = [] $\Longrightarrow$ *g* + *of-nat m* = *0* ∨ *nth-default 0 xs m* = *0* **for** *g*
proof (*induct xs arbitrary*: *g m*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons x xs*)
from *Cons*(*2*) **have** *empty*: *pderiv-coeffs-code* (*g* + *1*) *xs* = [] **and** *g*: *g* = *0* ∨ *x* = *0*
by (*auto simp*: *cCons-def split*: *if-splits*)
note *IH* = *Cons*(*1*)[*OF empty*]
from *IH*[*of m*] *IH*[*of m* − *1*] *g* **show** *?case*
by (*cases m*) (*auto simp*: *field-simps*)
qed
from *True* **have** *nth-default 0* (*pderiv-coeffs-code f* (*x* # *xs*)) *n* = *0*
by (*auto simp*: *cCons-def n*)
moreover from *True* **have** (*f* + *of-nat n*) ∗ *nth-default 0* (*x* # *xs*) *n* = *0*
by (*simp add*: *n*) (*use empty*[*of f+1*] **in** ‹*auto simp*: *field-simps*›)
ultimately show *?thesis* **by** *simp*
qed
qed
qed

lemma *coeffs-pderiv-code* [*code abstract*]: *coeffs* (*pderiv p*) = *pderiv-coeffs* (*coeffs p*)
unfolding *pderiv-coeffs-def*
proof (*rule coeffs-eqI*, *unfold pderiv-coeffs-code coeff-pderiv*, *goal-cases*)
case (*1 n*)
have *id*: *coeff p* (*Suc n*) = *nth-default 0* (*map* (λ*i*. *coeff p* (*Suc i*)) [*0*..<*degree p*]) *n*
by (*cases n* < *degree p*) (*auto simp*: *nth-default-def coeff-eq-0*)
show *?case*
unfolding *coeffs-def map-upt-Suc* **by** (*auto simp*: *id*)
next
case *2*
obtain *n* :: ′*a* **and** *xs* **where** *defs*: *tl* (*coeffs p*) = *xs 1* = *n*
by *simp*
from *2* **show** *?case*
unfolding *defs* **by** (*induct xs arbitrary*: *n*) (*auto simp*: *cCons-def*)
qed

lemma *pderiv-eq-0-iff*: *pderiv p* = *0* ⟷ *degree p* = *0*
for *p* :: ′*a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
proof (*cases degree p*)
case *0*
then show *?thesis*
by (*metis degree-eq-zeroE pderiv.simps*)
next

case (*Suc n*)
then show *?thesis*
using *coeff-0 coeff-pderiv degree-0 leading-coeff-0-iff mult-eq-0-iff nat.distinct*(*1*) *of-nat-eq-0-iff*
by (*metis coeff-0 coeff-pderiv degree-0 leading-coeff-0-iff mult-eq-0-iff nat.distinct*(*1*) *of-nat-eq-0-iff*)
qed

lemma *degree-pderiv*: *degree* (*pderiv p*) = *degree p* − *1*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
proof −
have *degree p* − *1* ≤ *degree* (*pderiv p*)
proof (*cases degree p*)
case (*Suc n*)
then show *?thesis*
by (*metis coeff-pderiv degree-0 diff-Suc-1 le-degree leading-coeff-0-iff mult-eq-0-iff nat.distinct*(*1*) *of-nat-eq-0-iff*)
qed *auto*
moreover have ∀ *i*>*degree p* − *1*. *coeff* (*pderiv p*) *i* = *0*
by (*simp add*: *coeff-eq-0 coeff-pderiv*)
ultimately show *?thesis*
using *order-antisym* [*OF degree-le*] **by** *blast*
qed

lemma *not-dvd-pderiv*:
fixes *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
assumes *degree p* ≠ *0*
shows ¬ *p dvd pderiv p*
proof
assume *dvd*: *p dvd pderiv p*
then obtain *q* **where** *p*: *pderiv p* = *p* ∗ *q*
unfolding *dvd-def* **by** *auto*
from *dvd* **have** *le*: *degree p* ≤ *degree* (*pderiv p*)
by (*simp add*: *assms dvd-imp-degree-le pderiv-eq-0-iff*)
from *assms* **and** *this* [*unfolded degree-pderiv*]
show *False* **by** *auto*
qed

lemma *dvd-pderiv-iff* [*simp*]: *p dvd pderiv p* ⟷ *degree p* = *0*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
using *not-dvd-pderiv*[*of p*] **by** (*auto simp*: *pderiv-eq-0-iff* [*symmetric*])

lemma *pderiv-singleton* [*simp*]: *pderiv* [:*a*:] = *0*
by (*simp add*: *pderiv-pCons*)

lemma *pderiv-add*: *pderiv* (*p* + *q*) = *pderiv p* + *pderiv q*
by (*rule poly-eqI*) (*simp add*: *coeff-pderiv algebra-simps*)

lemma *pderiv-minus*: *pderiv* (− *p* :: *'a* :: *idom poly*) = − *pderiv p*

by (*rule poly-eqI*) (*simp add*: *coeff-pderiv algebra-simps*)

lemma *pderiv-diff*: *pderiv* ((*p* :: - :: *idom poly*) − *q*) = *pderiv p* − *pderiv q*
by (*rule poly-eqI*) (*simp add*: *coeff-pderiv algebra-simps*)

lemma *pderiv-smult*: *pderiv* (*smult a p*) = *smult a* (*pderiv p*)
by (*rule poly-eqI*) (*simp add*: *coeff-pderiv algebra-simps*)

lemma *pderiv-mult*: *pderiv* (*p* ∗ *q*) = *p* ∗ *pderiv q* + *q* ∗ *pderiv p*
by (*induct p*) (*auto simp*: *pderiv-add pderiv-smult pderiv-pCons algebra-simps*)

lemma *pderiv-power-Suc*: *pderiv* (*p* ^ *Suc n*) = *smult* (*of-nat* (*Suc n*)) (*p* ^ *n*) ∗ *pderiv p*
proof (*induction n*)
case (*Suc n*)
then show *?case*
by (*simp add*: *pderiv-mult smult-add-left algebra-simps*)
qed *auto*

lemma *pderiv-power*:
pderiv (*p* ^ *n*) = *smult* (*of-nat n*) (*p* ^ (*n* − *1*) ∗ *pderiv p*)
by (*cases n*) (*simp-all add*: *pderiv-power-Suc del*: *power-Suc*)

lemma *pderiv-monom*:
pderiv (*monom c n*) = *monom* (*of-nat n* ∗ *c*) (*n* − *1*)
by (*cases n*)
(*simp-all add*: *monom-altdef pderiv-power-Suc pderiv-smult pderiv-pCons mult-ac del*: *power-Suc*)

lemma *pderiv-pcompose*: *pderiv* (*pcompose p q*) = *pcompose* (*pderiv p*) *q* ∗ *pderiv q*
by (*induction p rule*: *pCons-induct*)
(*auto simp*: *pcompose-pCons pderiv-add pderiv-mult pderiv-pCons pcompose-add algebra-simps*)

lemma *pderiv-prod*: *pderiv* (*prod f* (*as*)) = (∑ *a*∈*as*. *prod f* (*as* − {*a*}) ∗ *pderiv* (*f a*))
proof (*induct as rule*: *infinite-finite-induct*)
case (*insert a as*)
then have *id*: *prod f* (*insert a as*) = *f a* ∗ *prod f as*
⋀*g*. *sum g* (*insert a as*) = *g a* + *sum g as*
insert a as − {*a*} = *as*
by *auto*
have *prod f* (*insert a as* − {*b*}) = *f a* ∗ *prod f* (*as* − {*b*}) **if** *b* ∈ *as* **for** *b*
proof −
from ‹*a* ∉ *as*› *that* **have** ∗: *insert a as* − {*b*} = *insert a* (*as* − {*b*})
by *auto*
show *?thesis*
unfolding ∗ **by** (*subst prod.insert*) (*use insert* **in** *auto*)

qed
then show *?case*
unfolding *id pderiv-mult insert*(*3*) *sum-distrib-left*
by (*auto simp add*: *ac-simps intro*!: *sum.cong*)
qed *auto*

lemma *coeff-higher-pderiv*:
coeff ((*pderiv* ^^ *m*) *f*) *n* = *pochhammer* (*of-nat* (*Suc n*)) *m* ∗ *coeff f* (*n* + *m*)
by (*induction m arbitrary*: *n*) (*simp-all add*: *coeff-pderiv pochhammer-rec algebra-simps*)

lemma *higher-pderiv-0* [*simp*]: (*pderiv* ^^ *n*) *0* = *0*
by (*induction n*) *simp-all*

lemma *higher-pderiv-add*: (*pderiv* ^^ *n*) (*p* + *q*) = (*pderiv* ^^ *n*) *p* + (*pderiv* ^^ *n*) *q*
by (*induction n arbitrary*: *p q*) (*simp-all del*: *funpow.simps add*: *funpow-Suc-right pderiv-add*)

lemma *higher-pderiv-smult*: (*pderiv* ^^ *n*) (*smult c p*) = *smult c* ((*pderiv* ^^ *n*) *p*)
by (*induction n arbitrary*: *p*) (*simp-all del*: *funpow.simps add*: *funpow-Suc-right pderiv-smult*)

lemma *higher-pderiv-monom*:
m ≤ *n* + *1* ⟹ (*pderiv* ^^ *m*) (*monom c n*) = *monom* (*pochhammer* (*int n* − *int m* + *1*) *m* ∗ *c*) (*n* − *m*)
proof (*induction m arbitrary*: *c n*)
case (*Suc m*)
thus *?case*
by (*cases n*)
(*simp-all del*: *funpow.simps add*: *funpow-Suc-right pderiv-monom pochhammer-rec′ Suc.IH*)
qed *simp-all*

lemma *higher-pderiv-monom-eq-zero*:
m > *n* + *1* ⟹ (*pderiv* ^^ *m*) (*monom c n*) = *0*
proof (*induction m arbitrary*: *c n*)
case (*Suc m*)
thus *?case*
by (*cases n*)
(*simp-all del*: *funpow.simps add*: *funpow-Suc-right pderiv-monom pochhammer-rec′ Suc.IH*)
qed *simp-all*

lemma *higher-pderiv-sum*: (*pderiv* ^^ *n*) (*sum f A*) = (∑ *x*∈*A*. (*pderiv* ^^ *n*) (*f x*))
by (*induction A rule*: *infinite-finite-induct*) (*simp-all add*: *higher-pderiv-add*)

lemma *higher-pderiv-sum-mset*: (*pderiv* ^^ *n*) (*sum-mset A*) = (∑ *p*∈#*A*. (*pderiv*

$\frown\frown$ *n*) *p*)
  by (*induction A*) (*simp-all add*: *higher-pderiv-add*)

lemma *higher-pderiv-sum-list*: (*pderiv* $\frown\frown$ *n*) (*sum-list ps*) = ($\sum$ *p*←*ps*. (*pderiv* $\frown\frown$ *n*) *p*)
  by (*induction ps*) (*simp-all add*: *higher-pderiv-add*)

lemma *degree-higher-pderiv*: *Polynomial.degree* ((*pderiv* $\frown\frown$ *n*) *p*) = *Polynomial.degree* *p* − *n*
  for *p* :: ′*a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
  by (*induction n*) (*auto simp*: *degree-pderiv*)

lemma *DERIV-pow2*: *DERIV* (λ*x*. *x* ^ *Suc n*) *x* :> *real* (*Suc n*) ∗ (*x* ^ *n*)
  by (*rule DERIV-cong*, *rule DERIV-pow*) *simp*
declare *DERIV-pow2* [*simp*] *DERIV-pow* [*simp*]

lemma *DERIV-add-const*: *DERIV f x* :> *D* ⟹ *DERIV* (λ*x*. *a* + *f x* :: ′*a*::*real-normed-field*) *x* :> *D*
  by (*rule DERIV-cong*, *rule DERIV-add*) *auto*

lemma *poly-DERIV* [*simp*]: *DERIV* (λ*x*. *poly p x*) *x* :> *poly* (*pderiv p*) *x*
  by (*induct p*) (*auto intro*!: *derivative-eq-intros simp add*: *pderiv-pCons*)

lemma *poly-isCont*[*simp*]:
  fixes *x*::′*a*::*real-normed-field*
  shows *isCont* (λ*x*. *poly p x*) *x*
by (*rule poly-DERIV* [*THEN DERIV-isCont*])

lemma *tendsto-poly* [*tendsto-intros*]: (*f* ⟶ *a*) *F* ⟹ ((λ*x*. *poly p* (*f x*)) ⟶ *poly p a*) *F*
  for *f* :: - ⇒ ′*a*::*real-normed-field*
  by (*rule isCont-tendsto-compose* [*OF poly-isCont*])

lemma *continuous-within-poly*: *continuous* (*at z within s*) (*poly p*)
  for *z* :: ′*a*::{*real-normed-field*}
  by (*simp add*: *continuous-within tendsto-poly*)

lemma *continuous-poly* [*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *poly p* (*f x*))
  for *f* :: - ⇒ ′*a*::*real-normed-field*
  unfolding *continuous-def* **by** (*rule tendsto-poly*)

lemma *continuous-on-poly* [*continuous-intros*]:
  fixes *p* :: ′*a* :: {*real-normed-field*} *poly*
  assumes *continuous-on A f*
  shows *continuous-on A* (λ*x*. *poly p* (*f x*))
  by (*metis DERIV-continuous-on assms continuous-on-compose2 poly-DERIV subset-UNIV*)

Consequences of the derivative theorem above.

lemma *poly-differentiable*[*simp*]: (*λx. poly p x*) *differentiable* (*at x*)
  for *x* :: *real*
  by (*simp add*: *real-differentiable-def*) (*blast intro*: *poly-DERIV*)

lemma *poly-IVT-pos*: $a < b \Longrightarrow poly\ p\ a < 0 \Longrightarrow 0 < poly\ p\ b \Longrightarrow \exists x.\ a < x \wedge x < b \wedge poly\ p\ x = 0$
  for *a b* :: *real*
  using *IVT* [*of poly p a 0 b*] **by** (*auto simp add*: *order-le-less*)

lemma *poly-IVT-neg*: $a < b \Longrightarrow 0 < poly\ p\ a \Longrightarrow poly\ p\ b < 0 \Longrightarrow \exists x.\ a < x \wedge x < b \wedge poly\ p\ x = 0$
  for *a b* :: *real*
  using *poly-IVT-pos* [**where** $p = - p$] **by** *simp*

lemma *poly-IVT*: $a < b \Longrightarrow poly\ p\ a * poly\ p\ b < 0 \Longrightarrow \exists x>a.\ x < b \wedge poly\ p\ x = 0$
  for *p* :: *real poly*
  by (*metis less-not-sym mult-less-0-iff poly-IVT-neg poly-IVT-pos*)

lemma *poly-MVT*: $a < b \Longrightarrow \exists x.\ a < x \wedge x < b \wedge poly\ p\ b - poly\ p\ a = (b - a) * poly\ (pderiv\ p)\ x$
  for *a b* :: *real*
  by (*simp add*: *MVT2*)

lemma *poly-MVT'*:
  fixes *a b* :: *real*
  assumes {*min a b*..*max a b*} ⊆ *A*
  shows $\exists x\in A.\ poly\ p\ b - poly\ p\ a = (b - a) * poly\ (pderiv\ p)\ x$
proof (*cases a b rule*: *linorder-cases*)
  case *less*
  from *poly-MVT*[*OF less*, *of p*] **obtain** *x*
    where $a < x$ $x < b$ $poly\ p\ b - poly\ p\ a = (b - a) * poly\ (pderiv\ p)\ x$
    by *auto*
  then show *?thesis* **by** (*intro bexI*[*of - x*]) (*auto intro*!: *subsetD*[*OF assms*])
next
  case *greater*
  from *poly-MVT*[*OF greater*, *of p*] **obtain** *x*
    where $b < x$ $x < a$ $poly\ p\ a - poly\ p\ b = (a - b) * poly\ (pderiv\ p)\ x$ **by** *auto*
  then show *?thesis* **by** (*intro bexI*[*of - x*]) (*auto simp*: *algebra-simps intro*!: *subsetD*[*OF assms*])
qed (*use assms* **in** *auto*)

lemma *poly-pinfty-gt-lc*:
  fixes *p* :: *real poly*
  assumes $lead\text{-}coeff\ p > 0$
  shows $\exists n.\ \forall\ x \geq n.\ poly\ p\ x \geq lead\text{-}coeff\ p$
  using *assms*
proof (*induct p*)

  case *0*
  then show *?case* **by** *auto*
next
  case (*pCons a p*)
  from *this*(*1*) **consider** $a \neq 0$ $p = 0$ | $p \neq 0$ **by** *auto*
  then show *?case*
  proof *cases*
    case *1*
    then show *?thesis* **by** *auto*
  next
    case *2*
    with *pCons* **obtain** *n1* **where** *gte-lcoeff*: $\forall x \geq n1$. *lead-coeff p* $\leq$ *poly p x*
      by *auto*
    from *pCons*(*3*) ‹$p \neq 0$› **have** *gt-0*: *lead-coeff p* > 0 **by** *auto*
    define *n* **where** *n* = *max n1* (*1* + |*a*| / *lead-coeff p*)
    have *lead-coeff* (*pCons a p*) $\leq$ *poly* (*pCons a p*) *x* **if** $n \leq x$ **for** *x*
    proof −
      from *gte-lcoeff that* **have** *lead-coeff p* $\leq$ *poly p x*
        by (*auto simp*: *n-def*)
      with *gt-0* **have** |*a*| / *lead-coeff p* $\geq$ |*a*| / *poly p x* **and** *poly p x* > 0
        by (*auto intro*: *frac-le*)
      with ‹$n \leq x$›[*unfolded n-def*] **have** $x \geq 1$ + |*a*| / *poly p x*
        by *auto*
      with ‹*lead-coeff p* $\leq$ *poly p x*› ‹*poly p x* > 0› ‹$p \neq 0$›
      show *lead-coeff* (*pCons a p*) $\leq$ *poly* (*pCons a p*) *x*
        by (*auto simp*: *field-simps*)
    qed
    then show *?thesis* **by** *blast*
  qed
qed

lemma *lemma-order-pderiv1*:
  pderiv ([:− *a*, *1*:] ^ *Suc n* ∗ *q*) = [:− *a*, *1*:] ^ *Suc n* ∗ *pderiv q* +
    smult (*of-nat* (*Suc n*)) (*q* ∗ [:− *a*, *1*:] ^ *n*)
   by (*simp only*: *pderiv-mult pderiv-power-Suc*) (*simp del*: *power-Suc of-nat-Suc add*: *pderiv-pCons*)

lemma *lemma-order-pderiv*:
  fixes *p* :: *'a* :: *field-char-0 poly*
  assumes *n*: $0 < n$
    and *pd*: *pderiv p* $\neq 0$
    and *pe*: *p* = [:− *a*, *1*:] ^ *n* ∗ *q*
    and *nd*: ¬ [:− *a*, *1*:] *dvd q*
  shows *n* = *Suc* (*order a* (*pderiv p*))
proof −
  from *assms* **have** *pderiv* ([:− *a*, *1*:] ^ *n* ∗ *q*) $\neq 0$
    by *auto*
  from *assms* **obtain** *n'* **where** *n* = *Suc n'* $0 <$ *Suc n'* *pderiv* ([:− *a*, *1*:] ^ *Suc n'* ∗ *q*) $\neq 0$

by (*cases n*) *auto*
have *order a* (*pderiv* ([:− *a*, *1*:] ^ *Suc n′* ∗ *q*)) = *n′*
proof (*rule order-unique-lemma*)
show [:− *a*, *1*:] ^ *n′ dvd pderiv* ([:− *a*, *1*:] ^ *Suc n′* ∗ *q*)
unfolding *lemma-order-pderiv1*
proof (*rule dvd-add*)
show [:− *a*, *1*:] ^ *n′ dvd* [:− *a*, *1*:] ^ *Suc n′* ∗ *pderiv q*
by (*metis dvdI dvd-mult2 power-Suc2*)
show [:− *a*, *1*:] ^ *n′ dvd smult* (*of-nat* (*Suc n′*)) (*q* ∗ [:− *a*, *1*:] ^ *n′*)
by (*metis dvd-smult dvd-triv-right*)
qed
have *k dvd k* ∗ *pderiv q* + *smult* (*of-nat* (*Suc n′*)) *l* ⟹ *k dvd l* **for** *k l*
by (*auto simp del*: *of-nat-Suc simp*: *dvd-add-right-iff dvd-smult-iff*)
then show ¬ [:− *a*, *1*:] ^ *Suc n′ dvd pderiv* ([:− *a*, *1*:] ^ *Suc n′* ∗ *q*)
unfolding *lemma-order-pderiv1*
by (*metis nd dvd-mult-cancel-right power-not-zero pCons-eq-0-iff power-Suc zero-neq-one*)
qed
then show *?thesis*
by (*metis* ‹*n* = *Suc n′*› *pe*)
qed

lemma *order-pderiv*: *order a p* = *Suc* (*order a* (*pderiv p*))
if *pderiv p* ≠ *0 order a p* ≠ *0*
for *p* :: *′a*::*field-char-0 poly*
proof (*cases p* = *0*)
case *False*
obtain *q* **where** *p* = [:− *a*, *1*:] ^ *order a p* ∗ *q* ∧ ¬ [:− *a*, *1*:] *dvd q*
using *False order-decomp* **by** *blast*
then show *?thesis*
using *lemma-order-pderiv that* **by** *blast*
qed (*use that* **in** *auto*)

lemma *poly-squarefree-decomp-order*:
fixes *p* :: *′a*::*field-char-0 poly*
assumes *pderiv p* ≠ *0*
and *p*: *p* = *q* ∗ *d*
and *p′*: *pderiv p* = *e* ∗ *d*
and *d*: *d* = *r* ∗ *p* + *s* ∗ *pderiv p*
shows *order a q* = (*if order a p* = *0 then 0 else 1*)
proof (*rule classical*)
assume *1*: ¬ *?thesis*
from ‹*pderiv p* ≠ *0*› **have** *p* ≠ *0* **by** *auto*
with *p* **have** *order a p* = *order a q* + *order a d*
by (*simp add*: *order-mult*)
with *1* **have** *order a p* ≠ *0*
by (*auto split*: *if-splits*)
from ‹*pderiv p* ≠ *0*› ‹*pderiv p* = *e* ∗ *d*› **have** *oapp*: *order a* (*pderiv p*) = *order a e* + *order a d*

by (*simp add*: *order-mult*)
from ‹*pderiv p* ≠ *0*› ‹*order a p* ≠ *0*› **have** *oap*: *order a p* = *Suc* (*order a* (*pderiv p*))
by (*rule order-pderiv*)
from ‹*p* ≠ *0*› ‹*p* = *q* ∗ *d*› **have** *d* ≠ *0*
by *simp*
have [:− *a*, *1*:] ^ *order a* (*pderiv p*) *dvd r* ∗ *p*
by (*metis dvd-trans dvd-triv-right oap order-1 power-Suc*)
then have ([:−*a*, *1*:] ^ (*order a* (*pderiv p*))) *dvd d*
by (*simp add*: *d order-1*)
with ‹*d* ≠ *0*› **have** *order a* (*pderiv p*) ≤ *order a d*
by (*simp add*: *order-divides*)
show *?thesis*
using ‹*order a p* = *order a q* + *order a d*›
and *oapp oap*
and ‹*order a* (*pderiv p*) ≤ *order a d*›
by *auto*
qed

lemma *poly-squarefree-decomp-order2*:
pderiv p ≠ *0* ⟹ *p* = *q* ∗ *d* ⟹ *pderiv p* = *e* ∗ *d* ⟹
d = *r* ∗ *p* + *s* ∗ *pderiv p* ⟹ ∀ *a*. *order a q* = (*if order a p* = *0 then 0 else 1*)
for *p* :: ′*a*::*field-char-0 poly*
by (*blast intro*: *poly-squarefree-decomp-order*)

lemma *order-pderiv2*:
pderiv p ≠ *0* ⟹ *order a p* ≠ *0* ⟹ *order a* (*pderiv p*) = *n* ⟷ *order a p* = *Suc n*
for *p* :: ′*a*::*field-char-0 poly*
by (*auto dest*: *order-pderiv*)

definition *rsquarefree* :: ′*a*::*idom poly* ⇒ *bool*
where *rsquarefree p* ⟷ *p* ≠ *0* ∧ (∀ *a*. *order a p* = *0* ∨ *order a p* = *1*)

lemma *pderiv-iszero*: *pderiv p* = *0* ⟹ ∃ *h*. *p* = [:*h*:]
for *p* :: ′*a*::{*semidom*,*semiring-char-0*} *poly*
by (*cases p*) (*auto simp*: *pderiv-eq-0-iff split*: *if-splits*)

lemma *rsquarefree-roots*: *rsquarefree p* ⟷ (∀ *a*. ¬ (*poly p a* = *0* ∧ *poly* (*pderiv p*) *a* = *0*))
for *p* :: ′*a*::*field-char-0 poly*
proof (*cases p* = *0*)
case *False*
show *?thesis*
proof (*cases pderiv p* = *0*)
case *True*
with ‹*p* ≠ *0*› *pderiv-iszero* **show** *?thesis*
by (*force simp add*: *order-0I rsquarefree-def*)
next

case *False*
with ‹$p \neq 0$› *order-pderiv2* **show** *?thesis*
by (*force simp add*: *rsquarefree-def order-root*)
qed
qed (*simp add*: *rsquarefree-def*)

lemma *rsquarefree-root-order*:
assumes *rsquarefree p poly p z* = *0 p* ≠ *0*
shows *order z p* = *1*
proof −
from *assms* **have** *order z p* ∈ {*0*, *1*} **by** (*auto simp*: *rsquarefree-def*)
moreover from *assms* **have** *order z p* > *0* **by** (*auto simp*: *order-root*)
ultimately show *order z p* = *1* **by** *auto*
qed

lemma *poly-squarefree-decomp*:
fixes *p* :: *'a*::*field-char-0 poly*
assumes *pderiv p* ≠ *0*
and *p* = *q* ∗ *d*
and *pderiv p* = *e* ∗ *d*
and *d* = *r* ∗ *p* + *s* ∗ *pderiv p*
shows *rsquarefree q* ∧ (∀ *a*. *poly q a* = *0* ⟷ *poly p a* = *0*)
proof −
from ‹*pderiv p* ≠ *0*› **have** *p* ≠ *0* **by** *auto*
with ‹*p* = *q* ∗ *d*› **have** *q* ≠ *0* **by** *simp*
from *assms* **have** ∀ *a*. *order a q* = (*if order a p* = *0 then 0 else 1*)
by (*rule poly-squarefree-decomp-order2*)
with ‹*p* ≠ *0*› ‹*q* ≠ *0*› **show** *?thesis*
by (*simp add*: *rsquarefree-def order-root*)
qed

lemma *has-field-derivative-poly* [*derivative-intros*]:
assumes (*f has-field-derivative f'*) (*at x within A*)
shows ((λ*x*. *poly p* (*f x*)) *has-field-derivative*
(*f'* ∗ *poly* (*pderiv p*) (*f x*))) (*at x within A*)
using *DERIV-chain*[*OF poly-DERIV assms*, *of p*] **by** (*simp add*: *o-def mult-ac*)

4.28 Algebraic numbers

lemma *intpolyE*:
assumes ⋀*i*. *poly.coeff p i* ∈ ℤ
obtains *q* **where** *p* = *map-poly of-int q*
proof −
have ∀ *i*∈{..*Polynomial.degree p*}. ∃ *x*. *poly.coeff p i* = *of-int x*
using *assms* **by** (*auto simp*: *Ints-def*)
from *bchoice*[*OF this*] **obtain** *f*
where *f*: ⋀*i*. *i* ≤ *Polynomial.degree p* ⟹ *poly.coeff p i* = *of-int* (*f i*) **by** *blast*
define *q* **where** *q* = *Poly* (*map f* [*0*..<*Suc* (*Polynomial.degree p*)])
have *p* = *map-poly of-int q*

by (*intro poly-eqI*)
(*auto simp*: *coeff-map-poly q-def nth-default-def f coeff-eq-0 simp del*: *upt-Suc*)
with *that* **show** *?thesis* **by** *blast*
qed

lemma *ratpolyE*:
assumes $\bigwedge i.$ *poly.coeff p i* $\in \mathbb{Q}$
obtains *q* **where** *p* = *map-poly of-rat q*
proof −
have $\forall i \in \{..$*Polynomial.degree p*$\}$. $\exists x$. *poly.coeff p i* = *of-rat x*
using *assms* **by** (*auto simp*: *Rats-def*)
from *bchoice*[*OF this*] **obtain** *f*
where *f*: $\bigwedge i.$ *i* ≤ *Polynomial.degree p* ⟹ *poly.coeff p i* = *of-rat* (*f i*) **by** *blast*
define *q* **where** *q* = *Poly* (*map f* [*0..<Suc* (*Polynomial.degree p*)])
have *p* = *map-poly of-rat q*
by (*intro poly-eqI*)
(*auto simp*: *coeff-map-poly q-def nth-default-def f coeff-eq-0 simp del*: *upt-Suc*)
with *that* **show** *?thesis* **by** *blast*
qed

Algebraic numbers can be defined in two equivalent ways: all real numbers that are roots of rational polynomials or of integer polynomials. The Algebraic-Numbers AFP entry uses the rational definition, but we need the integer definition.

The equivalence is obvious since any rational polynomial can be multiplied with the LCM of its coefficients, yielding an integer polynomial with the same roots.

definition *algebraic* :: *'a* :: *field-char-0* ⇒ *bool*
where *algebraic x* ⟷ (∃*p*. (∀*i*. *coeff p i* ∈ ℤ) ∧ *p* ≠ *0* ∧ *poly p x* = *0*)

lemma *algebraicI*: (⋀*i*. *coeff p i* ∈ ℤ) ⟹ *p* ≠ *0* ⟹ *poly p x* = *0* ⟹ *algebraic x*
unfolding *algebraic-def* **by** *blast*

lemma *algebraicE*:
assumes *algebraic x*
obtains *p* **where** ⋀*i*. *coeff p i* ∈ ℤ *p* ≠ *0* *poly p x* = *0*
using *assms* **unfolding** *algebraic-def* **by** *blast*

lemma *algebraic-altdef*: *algebraic x* ⟷ (∃*p*. (∀*i*. *coeff p i* ∈ ℚ) ∧ *p* ≠ *0* ∧ *poly p x* = *0*)
for *p* :: *'a::field-char-0 poly*
proof *safe*
fix *p*
assume *rat*: ∀*i*. *coeff p i* ∈ ℚ **and** *root*: *poly p x* = *0* **and** *nz*: *p* ≠ *0*
define *cs* **where** *cs* = *coeffs p*
from *rat* **have** ∀*c*∈*range* (*coeff p*). ∃*c'*. *c* = *of-rat c'*
unfolding *Rats-def* **by** *blast*

```
  then obtain f where f: coeff p i = of-rat (f (coeff p i)) for i
    by (subst (asm) bchoice-iff) blast
  define cs' where cs' = map (quotient-of ∘ f) (coeffs p)
  define d where d = Lcm (set (map snd cs'))
  define p' where p' = smult (of-int d) p

  have coeff p' n ∈ ℤ for n
  proof (cases n ≤ degree p)
    case True
    define c where c = coeff p n
    define a where a = fst (quotient-of (f (coeff p n)))
    define b where b = snd (quotient-of (f (coeff p n)))
    have b-pos: b > 0
      unfolding b-def using quotient-of-denom-pos' by simp
    have coeff p' n = of-int d * coeff p n
      by (simp add: p'-def)
    also have coeff p n = of-rat (of-int a / of-int b)
      unfolding a-def b-def
        by (subst quotient-of-div [of f (coeff p n), symmetric]) (simp-all add: f
[symmetric])
    also have of-int d * ... = of-rat (of-int (a*d) / of-int b)
      by (simp add: of-rat-mult of-rat-divide)
    also from nz True have b ∈ snd ‘ set cs'
      by (force simp: cs'-def o-def b-def coeffs-def simp del: upt-Suc)
    then have b dvd (a * d)
      by (simp add: d-def)
    then have of-int (a * d) / of-int b ∈ (ℤ :: rat set)
      by (rule of-int-divide-in-Ints)
    then have of-rat (of-int (a * d) / of-int b) ∈ ℤ by (elim Ints-cases) auto
    finally show ?thesis .
  next
    case False
    then show ?thesis
      by (auto simp: p'-def not-le coeff-eq-0)
  qed
  moreover have set (map snd cs') ⊆ {0<..}
   unfolding cs'-def using quotient-of-denom-pos' by (auto simp: coeffs-def simp
del: upt-Suc)
  then have d ≠ 0
    unfolding d-def by (induct cs') simp-all
  with nz have p' ≠ 0 by (simp add: p'-def)
  moreover from root have poly p' x = 0
    by (simp add: p'-def)
  ultimately show algebraic x
    unfolding algebraic-def by blast
next
  assume algebraic x
  then obtain p where p: coeff p i ∈ ℤ poly p x = 0 p ≠ 0 for i
    by (force simp: algebraic-def)
```

moreover have *coeff p i* ∈ ℤ ⟹ *coeff p i* ∈ ℚ **for** *i*
  by (*elim Ints-cases*) *simp*
ultimately show ∃ *p*. (∀ *i*. *coeff p i* ∈ ℚ) ∧ *p* ≠ *0* ∧ *poly p x* = *0* **by** *auto*
qed

lemma *algebraicI'*: (⋀*i*. *coeff p i* ∈ ℚ) ⟹ *p* ≠ *0* ⟹ *poly p x* = *0* ⟹ *algebraic x*
  unfolding *algebraic-altdef* **by** *blast*

lemma *algebraicE'*:
  assumes *algebraic* (*x* :: *'a* :: *field-char-0*)
  obtains *p* **where** *p* ≠ *0 poly* (*map-poly of-int p*) *x* = *0*
proof −
  from *assms* **obtain** *q* **where** *q*: ⋀*i*. *coeff q i* ∈ ℤ *q* ≠ *0 poly q x* = *0*
    by (*erule algebraicE*)
  moreover from *this*(*1*) **obtain** *q'* **where** *q'*: *q* = *map-poly of-int q'* **by** (*erule intpolyE*)
  moreover have *q'* ≠ *0*
    using *q' q* **by** *auto*
  ultimately show *?thesis* **by** (*intro that*[*of q'*]) *simp-all*
qed

lemma *algebraicE'-nonzero*:
  assumes *algebraic* (*x* :: *'a* :: *field-char-0*) *x* ≠ *0*
  obtains *p* **where** *p* ≠ *0 coeff p 0* ≠ *0 poly* (*map-poly of-int p*) *x* = *0*
proof −
  from *assms*(*1*) **obtain** *p* **where** *p*: *p* ≠ *0 poly* (*map-poly of-int p*) *x* = *0*
    by (*erule algebraicE'*)
  define *n* :: *nat* **where** *n* = *order 0 p*
  have *monom 1 n dvd p* **by** (*simp add*: *monom-1-dvd-iff p n-def*)
  then obtain *q* **where** *q*: *p* = *monom 1 n* ∗ *q* **by** (*erule dvdE*)
  have [*simp*]: *map-poly of-int* (*monom 1 n* ∗ *q*) = *monom* (*1* :: *'a*) *n* ∗ *map-poly of-int q*
    by (*induction n*) (*auto simp*: *monom-0 monom-Suc map-poly-pCons*)
  from *p* **have** *q* ≠ *0 poly* (*map-poly of-int q*) *x* = *0* **by** (*auto simp*: *q poly-monom assms*(*2*))
  moreover from *this* **have** *order 0 p* = *n* + *order 0 q* **by** (*simp add*: *q order-mult*)
  hence *order 0 q* = *0* **by** (*simp add*: *n-def*)
  with ‹*q* ≠ *0*› **have** *poly q 0* ≠ *0* **by** (*simp add*: *order-root*)
  ultimately show *?thesis* **using** *that*[*of q*] **by** (*auto simp*: *poly-0-coeff-0*)
qed

lemma *rat-imp-algebraic*: *x* ∈ ℚ ⟹ *algebraic x*
proof (*rule algebraicI'*)
  show *poly* [:−*x*, *1*:] *x* = *0*
    by *simp*
qed (*auto simp*: *coeff-pCons split*: *nat.splits*)

lemma *algebraic-0* [*simp*, *intro*]: *algebraic 0*

and *algebraic-1* [*simp*, *intro*]: *algebraic 1*
and *algebraic-numeral* [*simp*, *intro*]: *algebraic* (*numeral n*)
and *algebraic-of-nat* [*simp*, *intro*]: *algebraic* (*of-nat k*)
and *algebraic-of-int* [*simp*, *intro*]: *algebraic* (*of-int m*)
by (*simp-all add*: *rat-imp-algebraic*)

lemma *algebraic-ii* [*simp*, *intro*]: *algebraic* i
proof (*rule algebraicI*)
show *poly* [:*1*, *0*, *1*:] i = *0*
by *simp*
qed (*auto simp*: *coeff-pCons split*: *nat.splits*)

lemma *algebraic-minus* [*intro*]:
assumes *algebraic x*
shows *algebraic* ($-x$)
proof −
from *assms* **obtain** *p* **where** *p*: $\forall$ *i*. *coeff p i* $\in \mathbb{Z}$ *poly p x* = *0* *p* $\neq$ *0*
by (*elim algebraicE*) *auto*
define *s* **where** *s* = (*if even* (*degree p*) *then 1 else* −*1* :: *'a*)

define *q* **where** *q* = *Polynomial.smult s* (*pcompose p* [:*0*, −*1*:])
have *poly q* ($-x$) = *0*
using *p* **by** (*auto simp*: *q-def poly-pcompose s-def*)
moreover have *q* $\neq$ *0*
using *p* **by** (*auto simp*: *q-def s-def pcompose-eq-0-iff*)
find-theorems *pcompose* - - = *0*
moreover have *coeff q i* $\in \mathbb{Z}$ **for** *i*
proof −
have *coeff* (*pcompose p* [:*0*, −*1*:]) *i* $\in \mathbb{Z}$
using *p* **by** (*intro coeff-pcompose-semiring-closed*) (*auto simp*: *coeff-pCons*
split: *nat.splits*)
thus *?thesis* **by** (*simp add*: *q-def s-def*)
qed
ultimately show *?thesis*
by (*auto simp*: *algebraic-def*)
qed

lemma *algebraic-minus-iff* [*simp*]:
algebraic ($-x$) $\longleftrightarrow$ *algebraic* (*x* :: *'a* :: *field-char-0*)
using *algebraic-minus*[*of x*] *algebraic-minus*[*of* −*x*] **by** *auto*

lemma *algebraic-inverse* [*intro*]:
assumes *algebraic x*
shows *algebraic* (*inverse x*)
proof (*cases x* = *0*)
case [*simp*]: *False*
from *assms* **obtain** *p* **where** *p*: $\forall$ *i*. *coeff p i* $\in \mathbb{Z}$ *poly p x* = *0* *p* $\neq$ *0*
by (*elim algebraicE*) *auto*
show *?thesis*

proof (*rule algebraicI*)
show *poly* (*reflect-poly p*) (*inverse x*) = *0*
using *assms p* **by** (*simp add*: *poly-reflect-poly-nz*)
qed (*use assms p* **in** ‹*auto simp*: *coeff-reflect-poly*›)
qed *auto*

lemma *algebraic-root*:
assumes *algebraic y*
and *poly p x* = *y* **and** ∀ *i*. *coeff p i* ∈ ℤ **and** *lead-coeff p* = *1* **and** *degree p* > *0*
shows *algebraic x*
proof −
from *assms* **obtain** *q* **where** *q*: *poly q y* = *0* ∀ *i*. *coeff q i* ∈ ℤ *q* ≠ *0*
by (*elim algebraicE*) *auto*
show *?thesis*
proof (*rule algebraicI*)
from *assms q* **show** *pcompose q p* ≠ *0*
by (*auto simp*: *pcompose-eq-0-iff*)
from *assms q* **show** *coeff* (*pcompose q p*) *i* ∈ ℤ **for** *i*
by (*intro allI coeff-pcompose-semiring-closed*) *auto*
show *poly* (*pcompose q p*) *x* = *0*
using *assms q* **by** (*simp add*: *poly-pcompose*)
qed
qed

lemma *algebraic-abs-real* [*simp*]:
algebraic |*x* :: *real*| ⟷ *algebraic x*
by (*auto simp*: *abs-if*)

lemma *algebraic-nth-root-real* [*intro*]:
assumes *algebraic x*
shows *algebraic* (*root n x*)
proof (*cases n* = *0*)
case *False*
show *?thesis*
proof (*rule algebraic-root*)
show *poly* (*monom 1 n*) (*root n x*) = (*if even n then* |*x*| *else x*)
using *sgn-power-root*[*of n x*] *False*
by (*auto simp add*: *poly-monom sgn-if split*: *if-splits*)
qed (*use False assms* **in** ‹*auto simp*: *degree-monom-eq*›)
qed *auto*

lemma *algebraic-sqrt* [*intro*]: *algebraic x* ⟹ *algebraic* (*sqrt x*)
by (*auto simp*: *sqrt-def*)

lemma *algebraic-csqrt* [*intro*]: *algebraic x* ⟹ *algebraic* (*csqrt x*)
by (*rule algebraic-root*[**where** *p* = *monom 1 2*])
(*auto simp*: *poly-monom degree-monom-eq*)

lemma *algebraic-cnj* [*intro*]:
  assumes *algebraic x*
  shows *algebraic* (*cnj x*)
proof −
  from *assms* **obtain** *p* **where** *p*: *poly p x* = *0* ∀ *i*. *coeff p i* ∈ ℤ *p* ≠ *0*
    by (*elim algebraicE*) *auto*
  show *?thesis*
  proof (*rule algebraicI*)
    show *poly* (*map-poly cnj p*) (*cnj x*) = *0*
      using *p* **by** *simp*
    show *map-poly cnj p* ≠ *0*
      using *p* **by** (*auto simp*: *map-poly-eq-0-iff*)
    show *coeff* (*map-poly cnj p*) *i* ∈ ℤ **for** *i*
      using *p* **by** (*auto simp*: *coeff-map-poly*)
  qed
qed

lemma *algebraic-cnj-iff* [*simp*]: *algebraic* (*cnj x*) ⟷ *algebraic x*
  using *algebraic-cnj*[*of x*] *algebraic-cnj*[*of cnj x*] **by** *auto*

lemma *algebraic-of-real* [*intro*]:
  assumes *algebraic x*
  shows *algebraic* (*of-real x*)
proof −
  from *assms* **obtain** *p* **where** *p*: *p* ≠ *0 poly* (*map-poly of-int p*) *x* = *0* **by** (*erule algebraicE′*)
  have *1*: *map-poly of-int p* ≠ (*0* :: ′*a poly*)
    using *p* **by** (*metis coeff-0 coeff-map-poly leading-coeff-0-iff of-int-eq-0-iff*)

  have *poly* (*map-poly of-int p*) (*of-real x* :: ′*a*) = *of-real* (*poly* (*map-poly of-int p*) *x*)
    by (*simp add*: *poly-altdef degree-map-poly coeff-map-poly*)
  also note *p*(*2*)
  finally have *2*: *poly* (*map-poly of-int p*) (*of-real x* :: ′*a*) = *0*
    by *simp*

  from *1 2* **show** *algebraic* (*of-real x* :: ′*a*)
    by (*intro algebraicI*[*of map-poly of-int p*]) (*auto simp*: *coeff-map-poly*)
qed

lemma *algebraic-of-real-iff* [*simp*]:
   algebraic (*of-real x* :: ′*a* :: {*real-algebra-1*,*field-char-0*}) ⟷ *algebraic x*
proof
  assume *algebraic* (*of-real x* :: ′*a*)
  then obtain *p* **where** *p*: *p* ≠ *0 poly* (*map-poly of-int p*) (*of-real x* :: ′*a*) = *0*
    by (*erule algebraicE′*)
  have *1*: (*map-poly of-int p* :: *real poly*) ≠ *0*
   using *p* **by** (*metis coeff-0 coeff-map-poly leading-coeff-0-iff of-int-0 of-int-eq-iff*)

note *p*(*2*)
also have *poly* (*map-poly of-int p*) (*of-real x* :: *'a*) = *of-real* (*poly* (*map-poly of-int p*) *x*)
by (*simp add*: *poly-altdef degree-map-poly coeff-map-poly*)
also have ... = *0* ⟷ *poly* (*map-poly of-int p*) *x* = *0*
using *of-real-eq-0-iff* **by** *blast*
finally have *2*: *poly* (*map-poly real-of-int p*) *x* = *0* **.**

from *1* **and** *2* **show** *algebraic x*
by (*intro algebraicI*[*of map-poly of-int p*]) (*auto simp*: *coeff-map-poly*)
qed *auto*

4.29 Algebraic integers

inductive *algebraic-int* :: *'a* :: *field* ⇒ *bool* **where**
⟦*lead-coeff p* = *1*; ∀ *i*. *coeff p i* ∈ ℤ; *poly p x* = *0*⟧ ⟹ *algebraic-int x*

lemma *algebraic-int-altdef-ipoly*:
fixes *x* :: *'a* :: *field-char-0*
shows *algebraic-int x* ⟷ (∃ *p*. *poly* (*map-poly of-int p*) *x* = *0* ∧ *lead-coeff p* = *1*)
proof
assume *algebraic-int x*
then obtain *p* **where** *p*: *lead-coeff p* = *1* ∀ *i*. *coeff p i* ∈ ℤ *poly p x* = *0*
by (*auto elim*: *algebraic-int.cases*)
define *the-int* **where** *the-int* = (λ*x*::*'a*. *THE r*. *x* = *of-int r*)
define *p'* **where** *p'* = *map-poly the-int p*
have *of-int-the-int*: *of-int* (*the-int x*) = *x* **if** *x* ∈ ℤ **for** *x*
unfolding *the-int-def* **by** (*rule sym*, *rule theI'*) (*insert that*, *auto simp*: *Ints-def*)
have *the-int-0-iff*: *the-int x* = *0* ⟷ *x* = *0* **if** *x* ∈ ℤ **for** *x*
using *of-int-the-int*[*OF that*] **by** *auto*
have [*simp*]: *the-int 0* = *0*
by (*subst the-int-0-iff*) *auto*
have *map-poly of-int p'* = *map-poly* (*of-int* ∘ *the-int*) *p*
by (*simp add*: *p'-def map-poly-map-poly*)
also from *p of-int-the-int* **have** ... = *p*
by (*subst poly-eq-iff*) (*auto simp*: *coeff-map-poly*)
finally have *p-p'*: *map-poly of-int p'* = *p* **.**

show (∃ *p*. *poly* (*map-poly of-int p*) *x* = *0* ∧ *lead-coeff p* = *1*)
proof (*intro exI conjI notI*)
from *p* **show** *poly* (*map-poly of-int p'*) *x* = *0* **by** (*simp add*: *p-p'*)
next
show *lead-coeff p'* = *1*
using *p* **by** (*simp flip*: *p-p' add*: *degree-map-poly coeff-map-poly*)
qed
next
assume ∃ *p*. *poly* (*map-poly of-int p*) *x* = *0* ∧ *lead-coeff p* = *1*
then obtain *p* **where** *p*: *poly* (*map-poly of-int p*) *x* = *0* *lead-coeff p* = *1*

by *auto*
define *p′* **where** *p′* = (*map-poly of-int p* :: *′a poly*)
from *p* **have** *lead-coeff p′* = *1 poly p′ x* = *0* ∀ *i*. *coeff p′ i* ∈ ℤ
by (*auto simp*: *p′-def coeff-map-poly degree-map-poly*)
thus *algebraic-int x*
by (*intro algebraic-int.intros*)
qed

theorem *rational-algebraic-int-is-int*:
assumes *algebraic-int x* **and** *x* ∈ ℚ
shows *x* ∈ ℤ
proof −
from *assms*(*2*) **obtain** *a b* **where** *ab*: *b* > *0 Rings.coprime a b* **and** *x-eq*: *x* = *of-int a* / *of-int b*
by (*auto elim*: *Rats-cases′*)
from ‹*b* > *0*› **have** [*simp*]: *b* ≠ *0*
by *auto*
from *assms*(*1*) **obtain** *p*
where *p*: *lead-coeff p* = *1* ∀ *i*. *coeff p i* ∈ ℤ *poly p x* = *0*
by (*auto simp*: *algebraic-int.simps*)

define *q* :: *′a poly* **where** *q* = [:−*of-int a*, *of-int b*:]
have *poly q x* = *0 q* ≠ *0* ∀ *i*. *coeff q i* ∈ ℤ
by (*auto simp*: *x-eq q-def coeff-pCons split*: *nat.splits*)
define *n* **where** *n* = *degree p*
have *n* > *0*
using *p* **by** (*intro Nat.gr0I*) (*auto simp*: *n-def elim*!: *degree-eq-zeroE*)
have (∑ *i*<*n*. *coeff p i* ∗ *of-int* (*a* ^ *i* ∗ *b* ^ (*n* − *i* − *1*))) ∈ ℤ
using *p* **by** *auto*
then obtain *R* **where** *R*: *of-int R* = (∑ *i*<*n*. *coeff p i* ∗ *of-int* (*a* ^ *i* ∗ *b* ^ (*n* − *i* − *1*)))
by (*auto simp*: *Ints-def*)
have [*simp*]: *coeff p n* = *1*
using *p* **by** (*auto simp*: *n-def*)

have *0* = *poly p x* ∗ *of-int b* ^ *n*
using *p* **by** *simp*
also have ... = (∑ *i*≤*n*. *coeff p i* ∗ *x* ^ *i* ∗ *of-int b* ^ *n*)
by (*simp add*: *poly-altdef n-def sum-distrib-right*)
also have ... = (∑ *i*≤*n*. *coeff p i* ∗ *of-int* (*a* ^ *i* ∗ *b* ^ (*n* − *i*)))
by (*intro sum.cong*) (*auto simp*: *x-eq field-simps simp flip*: *power-add*)
also have {..*n*} = *insert n* {..<*n*}
using ‹*n* > *0*› **by** *auto*
also have (∑ *i*∈. . . . *coeff p i* ∗ *of-int* (*a* ^ *i* ∗ *b* ^ (*n* − *i*))) =
coeff p n ∗ *of-int* (*a* ^ *n*) + (∑ *i*<*n*. *coeff p i* ∗ *of-int* (*a* ^ *i* ∗ *b* ^ (*n* − *i*)))
by (*subst sum.insert*) *auto*
also have (∑ *i*<*n*. *coeff p i* ∗ *of-int* (*a* ^ *i* ∗ *b* ^ (*n* − *i*))) =
(∑ *i*<*n*. *coeff p i* ∗ *of-int* (*a* ^ *i* ∗ *b* ∗ *b* ^ (*n* − *i* − *1*)))

by (*intro sum.cong*) (*auto simp flip*: *power-add power-Suc simp*: *Suc-diff-Suc*)
also have ... = *of-int* (*b* ∗ *R*)
by (*simp add*: *R sum-distrib-left sum-distrib-right mult-ac*)
finally have *of-int* (*a* ^ *n*) = (−*of-int* (*b* ∗ *R*) :: ′*a*)
by (*auto simp*: *add-eq-0-iff*)
hence *a* ^ *n* = −*b* ∗ *R*
by (*simp flip*: *of-int-mult of-int-power of-int-minus*)
hence *b dvd a* ^ *n*
by *simp*
with ‹*Rings.coprime a b*› **have** *b dvd 1*
by (*meson coprime-power-left-iff dvd-refl not-coprimeI*)
with *x-eq* **and** ‹*b* > *0*› **show** *?thesis*
by *auto*
qed

lemma *algebraic-int-imp-algebraic* [*dest*]: *algebraic-int x* ⟹ *algebraic x*
by (*auto simp*: *algebraic-int.simps algebraic-def*)

lemma *int-imp-algebraic-int*:
assumes $x \in \mathbb{Z}$
shows *algebraic-int x*
proof
show ∀ *i*. *coeff* [:−*x*, *1*:] *i* ∈ ℤ
using *assms* **by** (*auto simp*: *coeff-pCons split*: *nat.splits*)
qed *auto*

lemma *algebraic-int-0* [*simp*, *intro*]: *algebraic-int 0*
and *algebraic-int-1* [*simp*, *intro*]: *algebraic-int 1*
and *algebraic-int-numeral* [*simp*, *intro*]: *algebraic-int* (*numeral n*)
and *algebraic-int-of-nat* [*simp*, *intro*]: *algebraic-int* (*of-nat k*)
and *algebraic-int-of-int* [*simp*, *intro*]: *algebraic-int* (*of-int m*)
by (*simp-all add*: *int-imp-algebraic-int*)

lemma *algebraic-int-ii* [*simp*, *intro*]: *algebraic-int* i
proof
show *poly* [:*1*, *0*, *1*:] i = *0*
by *simp*
qed (*auto simp*: *coeff-pCons split*: *nat.splits*)

lemma *algebraic-int-minus* [*intro*]:
assumes *algebraic-int x*
shows *algebraic-int* (−*x*)
proof −
from *assms* **obtain** *p* **where** *p*: *lead-coeff p* = *1* ∀ *i*. *coeff p i* ∈ ℤ *poly p x* = *0*
by (*auto simp*: *algebraic-int.simps*)
define *s* **where** *s* = (*if even* (*degree p*) *then 1 else* −*1* :: ′*a*)

define *q* **where** *q* = *Polynomial.smult s* (*pcompose p* [:*0*, −*1*:])
have *lead-coeff q* = *s* ∗ *lead-coeff* (*pcompose p* [:*0*, −*1*:])

by (*simp add*: *q-def*)
also have *lead-coeff* (*pcompose p* [:*0*, −*1*:]) = *lead-coeff p* ∗ (− *1*) ^ *degree p*
by (*subst lead-coeff-comp*) *auto*
finally have *poly q* (−*x*) = *0* **and** *lead-coeff q* = *1*
using *p* **by** (*auto simp*: *q-def poly-pcompose s-def*)
moreover have *coeff q i* ∈ ℤ **for** *i*
proof −
have *coeff* (*pcompose p* [:*0*, −*1*:]) *i* ∈ ℤ
using *p* **by** (*intro coeff-pcompose-semiring-closed*) (*auto simp*: *coeff-pCons split*: *nat.splits*)
thus *?thesis* **by** (*simp add*: *q-def s-def*)
qed
ultimately show *?thesis*
by (*auto simp*: *algebraic-int.simps*)
qed

lemma *algebraic-int-minus-iff* [*simp*]:
algebraic-int (−*x*) ⟷ *algebraic-int* (*x* :: ′*a* :: *field-char-0*)
using *algebraic-int-minus*[*of x*] *algebraic-int-minus*[*of* −*x*] **by** *auto*

lemma *algebraic-int-inverse* [*intro*]:
assumes *poly p x* = *0* **and** ∀ *i*. *coeff p i* ∈ ℤ **and** *coeff p 0* = *1*
shows *algebraic-int* (*inverse x*)
proof
from *assms* **have** [*simp*]: *x* ≠ *0*
by (*auto simp*: *poly-0-coeff-0*)
show *poly* (*reflect-poly p*) (*inverse x*) = *0*
using *assms* **by** (*simp add*: *poly-reflect-poly-nz*)
qed (*use assms* **in** ‹*auto simp*: *coeff-reflect-poly*›)

lemma *algebraic-int-root*:
assumes *algebraic-int y*
and *poly p x* = *y* **and** ∀ *i*. *coeff p i* ∈ ℤ **and** *lead-coeff p* = *1* **and** *degree p* > *0*
shows *algebraic-int x*
proof −
from *assms* **obtain** *q* **where** *q*: *poly q y* = *0* ∀ *i*. *coeff q i* ∈ ℤ *lead-coeff q* = *1*
by (*auto simp*: *algebraic-int.simps*)
show *?thesis*
proof
from *assms q* **show** *lead-coeff* (*pcompose q p*) = *1*
by (*subst lead-coeff-comp*) *auto*
from *assms q* **show** ∀ *i*. *coeff* (*pcompose q p*) *i* ∈ ℤ
by (*intro allI coeff-pcompose-semiring-closed*) *auto*
show *poly* (*pcompose q p*) *x* = *0*
using *assms q* **by** (*simp add*: *poly-pcompose*)
qed
qed

lemma *algebraic-int-abs-real* [*simp*]:
  algebraic-int |*x* :: *real*| ⟷ *algebraic-int x*
  by (*auto simp*: *abs-if*)

lemma *algebraic-int-nth-root-real* [*intro*]:
  assumes *algebraic-int x*
  shows *algebraic-int* (*root n x*)
proof (*cases n* = *0*)
  case *False*
  show *?thesis*
  proof (*rule algebraic-int-root*)
    show *poly* (*monom 1 n*) (*root n x*) = (*if even n then* |*x*| *else x*)
      using *sgn-power-root*[*of n x*] *False*
      by (*auto simp add*: *poly-monom sgn-if split*: *if-splits*)
  qed (*use False assms* **in** ‹*auto simp*: *degree-monom-eq*›)
qed *auto*

lemma *algebraic-int-sqrt* [*intro*]: *algebraic-int x* ⟹ *algebraic-int* (*sqrt x*)
  by (*auto simp*: *sqrt-def*)

lemma *algebraic-int-csqrt* [*intro*]: *algebraic-int x* ⟹ *algebraic-int* (*csqrt x*)
  by (*rule algebraic-int-root*[**where** *p* = *monom 1 2*])
    (*auto simp*: *poly-monom degree-monom-eq*)

lemma *algebraic-int-cnj* [*intro*]:
  assumes *algebraic-int x*
  shows *algebraic-int* (*cnj x*)
proof −
  from *assms* **obtain** *p* **where** *p*: *lead-coeff p* = *1* ∀ *i*. *coeff p i* ∈ ℤ *poly p x* = *0*
    by (*auto simp*: *algebraic-int.simps*)
  show *?thesis*
  proof
    show *poly* (*map-poly cnj p*) (*cnj x*) = *0*
      using *p* **by** *simp*
    show *lead-coeff* (*map-poly cnj p*) = *1*
      using *p* **by** (*simp add*: *coeff-map-poly degree-map-poly*)
    show ∀ *i*. *coeff* (*map-poly cnj p*) *i* ∈ ℤ
      using *p* **by** (*auto simp*: *coeff-map-poly*)
  qed
qed

lemma *algebraic-int-cnj-iff* [*simp*]: *algebraic-int* (*cnj x*) ⟷ *algebraic-int x*
  using *algebraic-int-cnj*[*of x*] *algebraic-int-cnj*[*of cnj x*] **by** *auto*

lemma *algebraic-int-of-real* [*intro*]:
  assumes *algebraic-int x*
  shows *algebraic-int* (*of-real x*)
proof −
  from *assms* **obtain** *p* **where** *p*: *poly p x* = *0* ∀ *i*. *coeff p i* ∈ ℤ *lead-coeff p* = *1*

by (*auto simp*: *algebraic-int.simps*)
show *algebraic-int* (*of-real x* :: *'a*)
proof
have *poly* (*map-poly of-real p*) (*of-real x*) = (*of-real* (*poly p x*) :: *'a*)
by (*induction p*) (*auto simp*: *map-poly-pCons*)
thus *poly* (*map-poly of-real p*) (*of-real x*) = (*0* :: *'a*)
using *p* **by** *simp*
qed (*use p* **in** ‹*auto simp*: *coeff-map-poly degree-map-poly*›)
qed

lemma *algebraic-int-of-real-iff* [*simp*]:
algebraic-int (*of-real x* :: *'a* :: {*field-char-0*, *real-algebra-1*}) ⟷ *algebraic-int x*
proof
assume *algebraic-int* (*of-real x* :: *'a*)
then obtain *p*
where *p*: *poly* (*map-poly of-int p*) (*of-real x* :: *'a*) = *0 lead-coeff p* = *1*
by (*auto simp*: *algebraic-int-altdef-ipoly*)
show *algebraic-int x*
unfolding *algebraic-int-altdef-ipoly*
proof (*intro exI*[*of - p*] *conjI*)
have *of-real* (*poly* (*map-poly real-of-int p*) *x*) = *poly* (*map-poly of-int p*) (*of-real x* :: *'a*)
by (*induction p*) (*auto simp*: *map-poly-pCons*)
also note *p*(*1*)
finally show *poly* (*map-poly real-of-int p*) *x* = *0* **by** *simp*
qed (*use p* **in** *auto*)
qed *auto*

4.30 Division of polynomials

4.30.1 Division in general

instantiation *poly* :: (*idom-divide*) *idom-divide*
begin

fun *divide-poly-main* :: *'a* ⇒ *'a poly* ⇒ *'a poly* ⇒ *'a poly* ⇒ *nat* ⇒ *nat* ⇒ *'a poly*
where
divide-poly-main lc q r d dr (*Suc n*) =
(*let cr* = *coeff r dr*; *a* = *cr div lc*; *mon* = *monom a n in*
if False ∨ *a* ∗ *lc* = *cr then* — *False* ∨ is only because of problem in function-package
divide-poly-main
lc
(*q* + *mon*)
(*r* − *mon* ∗ *d*)
d (*dr* − *1*) *n else 0*)
| *divide-poly-main lc q r d dr 0* = *q*

definition *divide-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly*
where *divide-poly f g* =

(*if* $g = 0$ *then* 0
else
divide-poly-main (*coeff* g (*degree* g)) 0 f g (*degree* f)
(1 + *length* (*coeffs* f) − *length* (*coeffs* g)))

lemma *divide-poly-main*:
assumes d: $d \neq 0$ $lc = coeff\ d\ (degree\ d)$
and *degree* ($d * r$) ≤ *dr* *divide-poly-main* *lc* q ($d * r$) d *dr* $n = q'$
and $n = 1 + dr - degree\ d \vee dr = 0 \wedge n = 0 \wedge d * r = 0$
shows $q' = q + r$
using *assms*(*3*−)
proof (*induct* n *arbitrary*: q r *dr*)
case (*Suc* n)
let *?rr* = $d * r$
let *?a* = *coeff* *?rr* *dr*
let *?qq* = *?a* *div* *lc*
define b **where** [*simp*]: b = *monom* *?qq* n
let *?rrr* = $d * (r - b)$
let *?qqq* = $q + b$
note *res* = *Suc*(*3*)
from *Suc*(*4*) **have** *dr*: *dr* = n + *degree* d **by** *auto*
from d **have** *lc*: $lc \neq 0$ **by** *auto*
have *coeff* ($b * d$) *dr* = *coeff* b n ∗ *coeff* d (*degree* d)
proof (*cases* *?qq* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have n: n = *degree* b
by (*simp add*: *degree-monom-eq*)
show *?thesis*
unfolding n *dr* **by** (*simp add*: *coeff-mult-degree-sum*)
qed
also have . . . = *lc* ∗ *coeff* b n
by (*simp add*: d)
finally have *c2*: *coeff* ($b * d$) *dr* = *lc* ∗ *coeff* b n **.**
have *rrr*: *?rrr* = *?rr* − $b * d$
by (*simp add*: *field-simps*)
have *c1*: *coeff* ($d * r$) *dr* = *lc* ∗ *coeff* r n
proof (*cases* *degree* $r = n$)
case *True*
with *Suc*(*2*) **show** *?thesis*
unfolding *dr* **using** *coeff-mult-degree-sum*[*of* d r] d **by** (*auto simp*: *ac-simps*)
next
case *False*
from *dr* *Suc*(*2*) **have** *degree* $r \leq n$
by *auto*
(*metis add.commute add-le-cancel-left d*(*1*) *degree-0 degree-mult-eq*
diff-is-0-eq diff-zero le-cases)

with *False* **have** *r-n*: *degree r* < *n*
by *auto*
then have *right*: *lc* ∗ *coeff r n* = *0*
by (*simp add*: *coeff-eq-0*)
have *coeff* (*d* ∗ *r*) *dr* = *coeff* (*d* ∗ *r*) (*degree d* + *n*)
by (*simp add*: *dr ac-simps*)
also from *r-n* **have** ... = *0*
by (*metis False Suc.prems*(*1*) *add.commute add-left-imp-eq coeff-degree-mult coeff-eq-0*
coeff-mult-degree-sum degree-mult-le dr le-eq-less-or-eq)
finally show *?thesis*
by (*simp only*: *right*)
qed
have *c0*: *coeff ?rrr dr* = *0*
and *id*: *lc* ∗ (*coeff* (*d* ∗ *r*) *dr div lc*) = *coeff* (*d* ∗ *r*) *dr*
unfolding *rrr coeff-diff c2*
unfolding *b-def coeff-monom coeff-smult c1* **using** *lc* **by** *auto*
from *res*[*unfolded divide-poly-main.simps*[*of lc q*] *Let-def*] *id*
have *res*: *divide-poly-main lc ?qqq ?rrr d* (*dr* − *1*) *n* = *q'*
by (*simp del*: *divide-poly-main.simps add*: *field-simps*)
note *IH* = *Suc*(*1*)[*OF* - *res*]
from *Suc*(*4*) **have** *dr*: *dr* = *n* + *degree d* **by** *auto*
from *Suc*(*2*) **have** *deg-rr*: *degree ?rr* ≤ *dr* **by** *auto*
have *deg-bd*: *degree* (*b* ∗ *d*) ≤ *dr*
unfolding *dr b-def* **by** (*rule order.trans*[*OF degree-mult-le*]) (*auto simp*: *degree-monom-le*)
have *degree ?rrr* ≤ *dr*
unfolding *rrr* **by** (*rule degree-diff-le*[*OF deg-rr deg-bd*])
with *c0* **have** *deg-rrr*: *degree ?rrr* ≤ (*dr* − *1*)
by (*rule coeff-0-degree-minus-1*)
have *n* = *1* + (*dr* − *1*) − *degree d* ∨ *dr* − *1* = *0* ∧ *n* = *0* ∧ *?rrr* = *0*
proof (*cases dr*)
case *0*
with *Suc*(*4*) **have** *0*: *dr* = *0 n* = *0 degree d* = *0*
by *auto*
with *deg-rrr* **have** *degree ?rrr* = *0*
by *simp*
from *degree-eq-zeroE*[*OF this*] **obtain** *a* **where** *rrr*: *?rrr* = [:*a*:]
by *metis*
show *?thesis*
unfolding *0* **using** *c0* **unfolding** *rrr 0* **by** *simp*
next
case -: *Suc*
with *Suc*(*4*) **show** *?thesis* **by** *auto*
qed
from *IH*[*OF deg-rrr this*] **show** *?case*
by *simp*
next
case *0*

```
  show ?case
  proof (cases r = 0)
    case True
    with 0 show ?thesis by auto
  next
    case False
    from d False have degree (d * r) = degree d + degree r
      by (subst degree-mult-eq) auto
    with 0 d show ?thesis by auto
  qed
qed

lemma divide-poly-main-0: divide-poly-main 0 0 r d dr n = 0
proof (induct n arbitrary: r d dr)
  case 0
  then show ?case by simp
next
  case Suc
  show ?case
    unfolding divide-poly-main.simps[of - - r] Let-def
    by (simp add: Suc del: divide-poly-main.simps)
qed

lemma divide-poly:
  assumes g: g ≠ 0
  shows (f * g) div g = (f :: 'a poly)
proof -
  have len: length (coeffs f) = Suc (degree f) if f ≠ 0 for f :: 'a poly
    using that unfolding degree-eq-length-coeffs by auto
  have divide-poly-main (coeff g (degree g)) 0 (g * f) g (degree (g * f))
    (1 + length (coeffs (g * f)) − length (coeffs  g)) = (f * g) div g
    by (simp add: divide-poly-def Let-def ac-simps)
  note main = divide-poly-main[OF g refl le-refl this]
  have (f * g) div g = 0 + f
  proof (rule main, goal-cases)
    case 1
    show ?case
    proof (cases f = 0)
      case True
      with g show ?thesis
        by (auto simp: degree-eq-length-coeffs)
    next
      case False
      with g have fg: g * f ≠ 0 by auto
      show ?thesis
        unfolding len[OF fg] len[OF g] by auto
    qed
  qed
  then show ?thesis by simp
```

qed

lemma *divide-poly-0*: *f div 0* = *0*
  for *f* :: *'a poly*
  by (*simp add*: *divide-poly-def Let-def divide-poly-main-0*)

instance
  by *standard* (*auto simp*: *divide-poly divide-poly-0*)

end

instance *poly* :: (*idom-divide*) *algebraic-semidom* **..**

lemma *div-const-poly-conv-map-poly*:
  assumes [:*c*:] *dvd p*
  shows *p div* [:*c*:] = *map-poly* (λ*x*. *x div c*) *p*
proof (*cases c* = *0*)
  case *True*
  then show *?thesis*
    by (*auto intro*!: *poly-eqI simp*: *coeff-map-poly*)
next
  case *False*
  from *assms* **obtain** *q* **where** *p*: *p* = [:*c*:] ∗ *q* **by** (*rule dvdE*)
  moreover {
    have *smult c q* = [:*c*:] ∗ *q*
      by *simp*
    also have ... *div* [:*c*:] = *q*
      by (*rule nonzero-mult-div-cancel-left*) (*use False* **in** *auto*)
    finally have *smult c q div* [:*c*:] = *q* **.**
  }
  ultimately show *?thesis* **by** (*intro poly-eqI*) (*auto simp*: *coeff-map-poly False*)
qed

lemma *is-unit-monom-0*:
  fixes *a* :: *'a*::*field*
  assumes *a* ≠ *0*
  shows *is-unit* (*monom a 0*)
proof
  from *assms* **show** *1* = *monom a 0* ∗ *monom* (*inverse a*) *0*
    by (*simp add*: *mult-monom*)
qed

lemma *is-unit-triv*: *a* ≠ *0* ⟹ *is-unit* [:*a*:]
  for *a* :: *'a*::*field*
  by (*simp add*: *is-unit-monom-0 monom-0* [*symmetric*])

lemma *is-unit-iff-degree*:
  fixes *p* :: *'a*::*field poly*
  assumes *p* ≠ *0*

shows *is-unit p* ⟷ *degree p* = *0*
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *?rhs*
then obtain *a* **where** *p* = [:*a*:]
by (*rule degree-eq-zeroE*)
with *assms* **show** *?lhs*
by (*simp add*: *is-unit-triv*)
next
assume *?lhs*
then obtain *q* **where** *q* ≠ *0 p* ∗ *q* = *1* **..**
then have *degree* (*p* ∗ *q*) = *degree 1*
by *simp*
with ‹*p* ≠ *0*› ‹*q* ≠ *0*› **have** *degree p* + *degree q* = *0*
by (*simp add*: *degree-mult-eq*)
then show *?rhs* **by** *simp*
qed

lemma *is-unit-pCons-iff*: *is-unit* (*pCons a p*) ⟷ *p* = *0* ∧ *a* ≠ *0*
for *p* :: *'a*::*field poly*
by (*cases p* = *0*) (*auto simp*: *is-unit-triv is-unit-iff-degree*)

lemma *is-unit-monom-trivial*: *is-unit p* ⟹ *monom* (*coeff p* (*degree p*)) *0* = *p*
for *p* :: *'a*::*field poly*
by (*cases p*) (*simp-all add*: *monom-0 is-unit-pCons-iff*)

lemma *is-unit-const-poly-iff*: [:*c*:] *dvd 1* ⟷ *c dvd 1*
for *c* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*}
by (*auto simp*: *one-pCons*)

lemma *is-unit-polyE*:
fixes *p* :: *'a* :: {*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
assumes *p dvd 1*
obtains *c* **where** *p* = [:*c*:] *c dvd 1*
proof −
from *assms* **obtain** *q* **where** *1* = *p* ∗ *q*
by (*rule dvdE*)
then have *p* ≠ *0* **and** *q* ≠ *0*
by *auto*
from ‹*1* = *p* ∗ *q*› **have** *degree 1* = *degree* (*p* ∗ *q*)
by *simp*
also from ‹*p* ≠ *0*› **and** ‹*q* ≠ *0*› **have** ... = *degree p* + *degree q*
by (*simp add*: *degree-mult-eq*)
finally have *degree p* = *0* **by** *simp*
with *degree-eq-zeroE* **obtain** *c* **where** *c*: *p* = [:*c*:] **.**
with ‹*p dvd 1*› **have** *c dvd 1*
by (*simp add*: *is-unit-const-poly-iff*)
with *c* **show** *thesis* **..**
qed

```
lemma is-unit-polyE':
  fixes p :: 'a::field poly
  assumes is-unit p
  obtains a where p = monom a 0 and a ≠ 0
proof −
  obtain a q where p = pCons a q
    by (cases p)
  with assms have p = [:a:] and a ≠ 0
    by (simp-all add: is-unit-pCons-iff)
  with that show thesis by (simp add: monom-0)
qed

lemma is-unit-poly-iff: p dvd 1 ⟷ (∃ c. p = [:c:] ∧ c dvd 1)
  for p :: 'a::{comm-semiring-1,semiring-no-zero-divisors} poly
  by (auto elim: is-unit-polyE simp add: is-unit-const-poly-iff)

lemma root-imp-reducible-poly:
  fixes x :: 'a :: field
  assumes poly p x = 0 and degree p > 1
  shows  ¬irreducible p
proof −
  from assms have p ≠ 0
    by auto
  define q where q = [:−x, 1:]
  have q dvd p
    using assms by (simp add: poly-eq-0-iff-dvd q-def)
  then obtain r where p-eq: p = q ∗ r
    by (elim dvdE)
  have [simp]: q ≠ 0 r ≠ 0
    using ‹p ≠ 0› by (auto simp: p-eq)
  have degree p = Suc (degree r)
    unfolding p-eq by (subst degree-mult-eq) (auto simp: q-def)
  with assms(2) have degree r > 0
    by auto
  hence ¬r dvd 1
    by (auto simp: is-unit-poly-iff)
  moreover have ¬q dvd 1
    by (auto simp: is-unit-poly-iff q-def)
  ultimately show ?thesis using p-eq
    by (auto simp: irreducible-def)
qed

lemma reducible-polyI:
  fixes p :: 'a :: field poly
  assumes p = q ∗ r degree q > 0 degree r > 0
  shows  ¬irreducible p
  using assms unfolding irreducible-def
  by (metis (no-types, opaque-lifting) is-unitE is-unit-iff-degree not-gr0)
```

4.30.2 Pseudo-Division

This part is by René Thiemann and Akihisa Yamada.

fun *pseudo-divmod-main* ::
 $'a$ :: *comm-ring-1* $\Rightarrow$ $'a$ *poly* $\Rightarrow$ $'a$ *poly* $\Rightarrow$ $'a$ *poly* $\Rightarrow$ *nat* $\Rightarrow$ *nat* $\Rightarrow$ $'a$ *poly* $\times$ $'a$ *poly*
 where
  pseudo-divmod-main lc q r d dr (*Suc n*) =
   (*let*
    rr = *smult lc r*;
    qq = *coeff r dr*;
    rrr = *rr* − *monom qq n* ∗ *d*;
    qqq = *smult lc q* + *monom qq n*
   in pseudo-divmod-main lc qqq rrr d (*dr* − *1*) *n*)
 | *pseudo-divmod-main lc q r d dr 0* = (*q*,*r*)

definition *pseudo-divmod* :: $'a$ :: *comm-ring-1 poly* $\Rightarrow$ $'a$ *poly* $\Rightarrow$ $'a$ *poly* $\times$ $'a$ *poly*
 where *pseudo-divmod p q* ≡
  if q = *0 then* (*0*, *p*)
  else
   pseudo-divmod-main (*coeff q* (*degree q*)) *0 p q* (*degree p*)
    (*1* + *length* (*coeffs p*) − *length* (*coeffs q*))

lemma *pseudo-divmod-main*:
 assumes *d*: *d* ≠ *0 lc* = *coeff d* (*degree d*)
  and *degree r* ≤ *dr pseudo-divmod-main lc q r d dr n* = (q',r')
  and *n* = *1* + *dr* − *degree d* ∨ *dr* = *0* ∧ *n* = *0* ∧ *r* = *0*
 shows (r' = *0* ∨ *degree* r' < *degree d*) ∧ *smult* (*lc*^*n*) (*d* ∗ *q* + *r*) = *d* ∗ q' + r'
 using *assms*(*3*−)
proof (*induct n arbitrary*: *q r dr*)
 case *0*
 then show *?case* **by** *auto*
next
 case (*Suc n*)
 let *?rr* = *smult lc r*
 let *?qq* = *coeff r dr*
 define *b* **where** [*simp*]: *b* = *monom ?qq n*
 let *?rrr* = *?rr* − *b* ∗ *d*
 let *?qqq* = *smult lc q* + *b*
 note *res* = *Suc*(*3*)
 from *res*[*unfolded pseudo-divmod-main.simps*[*of lc q*] *Let-def*]
 have *res*: *pseudo-divmod-main lc ?qqq ?rrr d* (*dr* − *1*) *n* = (q',r')
  by (*simp del*: *pseudo-divmod-main.simps*)
 from *Suc*(*4*) **have** *dr*: *dr* = *n* + *degree d* **by** *auto*
 have *coeff* (*b* ∗ *d*) *dr* = *coeff b n* ∗ *coeff d* (*degree d*)
 proof (*cases ?qq* = *0*)
  case *True*
  then show *?thesis* **by** *auto*
 next

case *False*
then have *n*: *n* = *degree b*
by (*simp add*: *degree-monom-eq*)
show *?thesis*
unfolding *n dr* **by** (*simp add*: *coeff-mult-degree-sum*)
qed
also have ... = *lc* ∗ *coeff b n* **by** (*simp add*: *d*)
finally have *coeff* (*b* ∗ *d*) *dr* = *lc* ∗ *coeff b n* **.**
moreover have *coeff ?rr dr* = *lc* ∗ *coeff r dr*
by *simp*
ultimately have *c0*: *coeff ?rrr dr* = *0*
by *auto*
from *Suc*(*4*) **have** *dr*: *dr* = *n* + *degree d* **by** *auto*
have *deg-rr*: *degree ?rr* ≤ *dr*
using *Suc*(*2*) *degree-smult-le dual-order.trans* **by** *blast*
have *deg-bd*: *degree* (*b* ∗ *d*) ≤ *dr*
unfolding *dr* **by** (*rule order.trans*[*OF degree-mult-le*]) (*auto simp*: *degree-monom-le*)
have *degree ?rrr* ≤ *dr*
using *degree-diff-le*[*OF deg-rr deg-bd*] **by** *auto*
with *c0* **have** *deg-rrr*: *degree ?rrr* ≤ (*dr* − *1*)
by (*rule coeff-0-degree-minus-1*)
have *n* = *1* + (*dr* − *1*) − *degree d* ∨ *dr* − *1* = *0* ∧ *n* = *0* ∧ *?rrr* = *0*
proof (*cases dr*)
case *0*
with *Suc*(*4*) **have** *0*: *dr* = *0 n* = *0 degree d* = *0* **by** *auto*
with *deg-rrr* **have** *degree ?rrr* = *0* **by** *simp*
then have ∃ *a*. *?rrr* = [:*a*:]
by (*metis degree-pCons-eq-if old.nat.distinct*(*2*) *pCons-cases*)
from *this* **obtain** *a* **where** *rrr*: *?rrr* = [:*a*:]
by *auto*
show *?thesis*
unfolding *0* **using** *c0* **unfolding** *rrr 0* **by** *simp*
next
case -: *Suc*
with *Suc*(*4*) **show** *?thesis* **by** *auto*
qed
note *IH* = *Suc*(*1*)[*OF deg-rrr res this*]
show *?case*
proof (*intro conjI*)
from *IH* **show** *r′* = *0* ∨ *degree r′* < *degree d*
by *blast*
show *smult* (*lc* ^ *Suc n*) (*d* ∗ *q* + *r*) = *d* ∗ *q′* + *r′*
unfolding *IH*[*THEN conjunct2*,*symmetric*]
by (*simp add*: *field-simps smult-add-right*)
qed
qed

lemma *pseudo-divmod*:
assumes *g*: *g* ≠ *0*

and ∗: *pseudo-divmod f g* = (*q*,*r*)
shows *smult* (*coeff g* (*degree g*) ^ (*Suc* (*degree f*) − *degree g*)) *f* = *g* ∗ *q* + *r* (**is** *?A*)
and *r* = *0* ∨ *degree r* < *degree g* (**is** *?B*)
proof −
from ∗[*unfolded pseudo-divmod-def Let-def*]
have *pseudo-divmod-main* (*coeff g* (*degree g*)) *0 f g* (*degree f*)
(*1* + *length* (*coeffs f*) − *length* (*coeffs g*)) = (*q*, *r*)
by (*auto simp*: *g*)
note *main* = *pseudo-divmod-main*[*OF - - - this*, *OF g refl le-refl*]
from *g* **have** *1* + *length* (*coeffs f*) − *length* (*coeffs g*) = *1* + *degree f* − *degree g* ∨
degree f = *0* ∧ *1* + *length* (*coeffs f*) − *length* (*coeffs g*) = *0* ∧ *f* = *0*
by (*cases f* = *0*; *cases coeffs g*) (*auto simp*: *degree-eq-length-coeffs*)
note *main′* = *main*[*OF this*]
then show *r* = *0* ∨ *degree r* < *degree g* **by** *auto*
show *smult* (*coeff g* (*degree g*) ^ (*Suc* (*degree f*) − *degree g*)) *f* = *g* ∗ *q* + *r*
by (*subst main′*[*THEN conjunct2*, *symmetric*], *simp add*: *degree-eq-length-coeffs*,
cases f = *0*; *cases coeffs g*, *use g* **in** *auto*)
qed

definition *pseudo-mod-main lc r d dr n* = *snd* (*pseudo-divmod-main lc 0 r d dr n*)

lemma *snd-pseudo-divmod-main*:
snd (*pseudo-divmod-main lc q r d dr n*) = *snd* (*pseudo-divmod-main lc q′ r d dr n*)
by (*induct n arbitrary*: *q q′ lc r d dr*) (*simp-all add*: *Let-def*)

definition *pseudo-mod* :: *′a*::{*comm-ring-1*,*semiring-1-no-zero-divisors*} *poly* ⇒ *′a poly* ⇒ *′a poly*
where *pseudo-mod f g* = *snd* (*pseudo-divmod f g*)

lemma *pseudo-mod*:
fixes *f g* :: *′a*::{*comm-ring-1*,*semiring-1-no-zero-divisors*} *poly*
defines *r* ≡ *pseudo-mod f g*
assumes *g*: *g* ≠ *0*
shows ∃ *a q*. *a* ≠ *0* ∧ *smult a f* = *g* ∗ *q* + *r r* = *0* ∨ *degree r* < *degree g*
proof −
let *?cg* = *coeff g* (*degree g*)
let *?cge* = *?cg* ^ (*Suc* (*degree f*) − *degree g*)
define *a* **where** *a* = *?cge*
from *r-def*[*unfolded pseudo-mod-def*] **obtain** *q* **where** *pdm*: *pseudo-divmod f g* = (*q*, *r*)
by (*cases pseudo-divmod f g*) *auto*
from *pseudo-divmod*[*OF g pdm*] **have** *id*: *smult a f* = *g* ∗ *q* + *r* **and** *r* = *0* ∨ *degree r* < *degree g*
by (*auto simp*: *a-def*)
show *r* = *0* ∨ *degree r* < *degree g* **by** *fact*

from *g* **have** $a \neq 0$
  by (*auto simp*: *a-def*)
with *id* **show** $\exists\, a\ q.\ a \neq 0 \wedge smult\ a\ f = g * q + r$
  by *auto*
qed

lemma *fst-pseudo-divmod-main-as-divide-poly-main*:
assumes *d*: $d \neq 0$
defines *lc*: $lc \equiv coeff\ d\ (degree\ d)$
shows *fst* (*pseudo-divmod-main lc q r d dr n*) =
  divide-poly-main lc (*smult* (*lc^n*) *q*) (*smult* (*lc^n*) *r*) *d dr n*
proof (*induct n arbitrary*: *q r dr*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
note *lc0* = *leading-coeff-neq-0*[*OF d*, *folded lc*]
then have *pseudo-divmod-main lc q r d dr* (*Suc n*) =
  pseudo-divmod-main lc (*smult lc q* + *monom* (*coeff r dr*) *n*)
    (*smult lc r* − *monom* (*coeff r dr*) *n* ∗ *d*) *d* (*dr* − *1*) *n*
  by (*simp add*: *Let-def ac-simps*)
also have *fst* . . . = *divide-poly-main lc*
   (*smult* (*lc^n*) (*smult lc q* + *monom* (*coeff r dr*) *n*))
   (*smult* (*lc^n*) (*smult lc r* − *monom* (*coeff r dr*) *n* ∗ *d*))
   d (*dr* − *1*) *n*
  by (*simp only*: *Suc*[*unfolded divide-poly-main.simps Let-def*])
also have . . . = *divide-poly-main lc* (*smult* (*lc ^ Suc n*) *q*) (*smult* (*lc ^ Suc n*) *r*) *d dr* (*Suc n*)
  unfolding *smult-monom smult-distribs mult-smult-left*[*symmetric*]
  using *lc0* **by** (*simp add*: *Let-def ac-simps*)
finally show *?case* **.**
qed

4.30.3 Division in polynomials over fields

lemma *pseudo-divmod-field*:
fixes *g* :: *'a::field poly*
assumes *g*: $g \neq 0$
  and ∗: *pseudo-divmod f g* = (*q*,*r*)
defines $c \equiv coeff\ g\ (degree\ g)$ ^(*Suc* (*degree f*) − *degree g*)
shows *f* = *g* ∗ *smult* (*1*/*c*) *q* + *smult* (*1*/*c*) *r*
proof −
from *leading-coeff-neq-0*[*OF g*] **have** *c0*: $c \neq 0$
  by (*auto simp*: *c-def*)
from *pseudo-divmod*(*1*)[*OF g* ∗, *folded c-def*] **have** *smult c f* = *g* ∗ *q* + *r*
  by *auto*
also have *smult* (*1* / *c*) . . . = *g* ∗ *smult* (*1* / *c*) *q* + *smult* (*1* / *c*) *r*
  by (*simp add*: *smult-add-right*)
finally show *?thesis*

using *c0* **by** *auto*
qed

lemma *divide-poly-main-field*:
fixes *d* :: *'a::field poly*
assumes *d*: *d* ≠ *0*
defines *lc*: *lc* ≡ *coeff d* (*degree d*)
shows *divide-poly-main lc q r d dr n* =
fst (*pseudo-divmod-main lc* (*smult* ((*1* / *lc*)^*n*) *q*) (*smult* ((*1* / *lc*)^*n*) *r*) *d dr n*)
unfolding *lc* **by** (*subst fst-pseudo-divmod-main-as-divide-poly-main*) (*auto simp*: *d power-one-over*)

lemma *divide-poly-field*:
fixes *f g* :: *'a::field poly*
defines *f'* ≡ *smult* ((*1* / *coeff g* (*degree g*)) ^(*Suc* (*degree f*) − *degree g*)) *f*
shows *f div g* = *fst* (*pseudo-divmod f' g*)
proof (*cases g* = *0*)
case *True*
show *?thesis*
unfolding *divide-poly-def pseudo-divmod-def Let-def f'-def True*
by (*simp add*: *divide-poly-main-0*)
next
case *False*
from *leading-coeff-neq-0*[*OF False*] **have** *degree f'* = *degree f*
by (*auto simp*: *f'-def*)
then show *?thesis*
using *length-coeffs-degree*[*of f'*] *length-coeffs-degree*[*of f*]
unfolding *divide-poly-def pseudo-divmod-def Let-def*
divide-poly-main-field[*OF False*]
length-coeffs-degree[*OF False*]
f'-def
by *force*
qed

instantiation *poly* :: ({*semidom-divide-unit-factor*,*idom-divide*}) *normalization-semidom*
begin

definition *unit-factor-poly* :: *'a poly* ⇒ *'a poly*
where *unit-factor-poly p* = [:*unit-factor* (*lead-coeff p*):]

definition *normalize-poly* :: *'a poly* ⇒ *'a poly*
where *normalize p* = *p div* [:*unit-factor* (*lead-coeff p*):]

instance
proof
fix *p* :: *'a poly*
show *unit-factor p* ∗ *normalize p* = *p*
proof (*cases p* = *0*)

case *True*
then show *?thesis*
by (*simp add*: *unit-factor-poly-def normalize-poly-def*)
next
case *False*
then have *lead-coeff p* ≠ *0*
by *simp*
then have ∗: *unit-factor* (*lead-coeff p*) ≠ *0*
using *unit-factor-is-unit* [*of lead-coeff p*] **by** *auto*
then have *unit-factor* (*lead-coeff p*) *dvd 1*
by (*auto intro*: *unit-factor-is-unit*)
then have ∗∗: *unit-factor* (*lead-coeff p*) *dvd c* **for** *c*
by (*rule dvd-trans*) *simp*
have ∗∗∗: *unit-factor* (*lead-coeff p*) ∗ (*c div unit-factor* (*lead-coeff p*)) = *c* **for**
c
proof −
from ∗∗ **obtain** *b* **where** *c* = *unit-factor* (*lead-coeff p*) ∗ *b* **..**
with *False* ∗ **show** *?thesis* **by** *simp*
qed
have *p div* [:*unit-factor* (*lead-coeff p*):] =
map-poly (λ*c. c div unit-factor* (*lead-coeff p*)) *p*
by (*simp add*: *const-poly-dvd-iff div-const-poly-conv-map-poly* ∗∗)
then show *?thesis*
by (*simp add*: *normalize-poly-def unit-factor-poly-def*
smult-conv-map-poly map-poly-map-poly o-def ∗∗∗)
qed
next
fix *p* :: ′*a poly*
assume *is-unit p*
then obtain *c* **where** *p*: *p* = [:*c*:] *c dvd 1*
by (*auto simp*: *is-unit-poly-iff*)
then show *unit-factor p* = *p*
by (*simp add*: *unit-factor-poly-def monom-0 is-unit-unit-factor*)
next
fix *p* :: ′*a poly*
assume *p* ≠ *0*
then show *is-unit* (*unit-factor p*)
by (*simp add*: *unit-factor-poly-def monom-0 is-unit-poly-iff unit-factor-is-unit*)
next
fix *a b* :: ′*a poly* **assume** *is-unit a*
thus *unit-factor* (*a* ∗ *b*) = *a* ∗ *unit-factor b*
by (*auto simp*: *unit-factor-poly-def lead-coeff-mult unit-factor-mult elim*!: *is-unit-polyE*)
qed (*simp-all add*: *normalize-poly-def unit-factor-poly-def monom-0 lead-coeff-mult*
unit-factor-mult)

end

instance *poly* :: ({*semidom-divide-unit-factor*,*idom-divide*,*normalization-semidom-multiplicative*})
normalization-semidom-multiplicative

by *intro-classes* (*auto simp*: *unit-factor-poly-def lead-coeff-mult unit-factor-mult*)

lemma *normalize-poly-eq-map-poly*: *normalize p* = *map-poly* (*λx. x div unit-factor* (*lead-coeff p*)) *p*
proof −
  have [:*unit-factor* (*lead-coeff p*):] *dvd p*
    by (*metis unit-factor-poly-def unit-factor-self*)
  then show *?thesis*
    by (*simp add*: *normalize-poly-def div-const-poly-conv-map-poly*)
qed

lemma *coeff-normalize* [*simp*]:
  coeff (*normalize p*) *n* = *coeff p n div unit-factor* (*lead-coeff p*)
  by (*simp add*: *normalize-poly-eq-map-poly coeff-map-poly*)

class *field-unit-factor* = *field* + *unit-factor* +
  assumes *unit-factor-field* [*simp*]: *unit-factor* = *id*
begin

subclass *semidom-divide-unit-factor*
proof
  fix *a*
  assume *a* ≠ *0*
  then have *1* = *a* ∗ *inverse a* **by** *simp*
  then have *a dvd 1* **..**
  then show *unit-factor a dvd 1* **by** *simp*
qed *simp-all*

end

lemma *unit-factor-pCons*:
  unit-factor (*pCons a p*) = (*if p* = *0 then* [:*unit-factor a*:] *else unit-factor p*)
  by (*simp add*: *unit-factor-poly-def*)

lemma *normalize-monom* [*simp*]: *normalize* (*monom a n*) = *monom* (*normalize a*) *n*
  by (*cases a* = *0*) (*simp-all add*: *map-poly-monom normalize-poly-eq-map-poly degree-monom-eq*)

lemma *unit-factor-monom* [*simp*]: *unit-factor* (*monom a n*) = [:*unit-factor a*:]
  by (*cases a* = *0*) (*simp-all add*: *unit-factor-poly-def degree-monom-eq*)

lemma *normalize-const-poly*: *normalize* [:*c*:] = [:*normalize c*:]
  by (*simp add*: *normalize-poly-eq-map-poly map-poly-pCons*)

lemma *normalize-smult*:
  fixes *c* :: *'a* :: {*normalization-semidom-multiplicative*, *idom-divide*}
  shows *normalize* (*smult c p*) = *smult* (*normalize c*) (*normalize p*)
proof −

have *smult c p* = [:*c*:] ∗ *p* **by** *simp*
also have *normalize* ... = *smult* (*normalize c*) (*normalize p*)
by (*subst normalize-mult*) (*simp add*: *normalize-const-poly*)
finally show *?thesis* **.**
qed

instantiation *poly* :: (*field*) *idom-modulo*
begin

definition *modulo-poly* :: ′*a poly* ⇒ ′*a poly* ⇒ ′*a poly*
where *mod-poly-def*: *f mod g* =
(*if g* = *0 then f else pseudo-mod* (*smult* ((*1* / *lead-coeff g*) ^ (*Suc* (*degree f*) − *degree g*)) *f*) *g*)

instance
proof
fix *x y* :: ′*a poly*
show *x div y* ∗ *y* + *x mod y* = *x*
proof (*cases y* = *0*)
case *True*
then show *?thesis*
by (*simp add*: *divide-poly-0 mod-poly-def*)
next
case *False*
then have *pseudo-divmod* (*smult* ((*1* / *lead-coeff y*) ^ (*Suc* (*degree x*) − *degree y*)) *x*) *y* =
(*x div y*, *x mod y*)
by (*simp add*: *divide-poly-field mod-poly-def pseudo-mod-def*)
with *False pseudo-divmod* [*OF False this*] **show** *?thesis*
by (*simp add*: *power-mult-distrib* [*symmetric*] *ac-simps*)
qed
qed

end

lemma *pseudo-divmod-eq-div-mod*:
‹*pseudo-divmod f g* = (*f div g*, *f mod g*)› **if** ‹*lead-coeff g* = *1*›
using *that* **by** (*auto simp add*: *divide-poly-field mod-poly-def pseudo-mod-def*)

lemma *degree-mod-less-degree*:
‹*degree* (*x mod y*) < *degree y*› **if** ‹*y* ≠ *0*› ‹¬ *y dvd x*›
proof −
from *pseudo-mod*(*2*) [*of y*] ‹*y* ≠ *0*›
have ∗: ‹*pseudo-mod f y* ≠ *0* ⟹ *degree* (*pseudo-mod f y*) < *degree y*› **for** *f*
by *blast*
from ‹¬ *y dvd x*› **have** ‹*x mod y* ≠ *0*›
by *blast*
with ‹*y* ≠ *0*› **show** *?thesis*
by (*auto simp add*: *mod-poly-def intro*: ∗)

qed

instantiation *poly* :: (*field*) *unique-euclidean-ring*
begin

definition *euclidean-size-poly* :: *'a poly* ⇒ *nat*
  where *euclidean-size-poly p* = (*if p* = *0 then 0 else 2* ^ *degree p*)

definition *division-segment-poly* :: *'a poly* ⇒ *'a poly*
  where [*simp*]: *division-segment-poly p* = *1*

instance proof
  show ‹(*q* ∗ *p* + *r*) *div p* = *q*› **if** ‹*p* ≠ *0*›
    and ‹*euclidean-size r* < *euclidean-size p*› **for** *q p r* :: ‹*'a poly*›
  proof (*cases* ‹*r* = *0*›)
    case *True*
    with *that* **show** *?thesis*
      by *simp*
  next
    case *False*
    with ‹*p* ≠ *0*› ‹*euclidean-size r* < *euclidean-size p*›
    have ‹*degree r* < *degree p*›
      by (*simp add*: *euclidean-size-poly-def*)
    with ‹*r* ≠ *0*› **have** ‹¬ *p dvd r*›
      by (*auto dest*: *dvd-imp-degree*)
    have ‹(*q* ∗ *p* + *r*) *div p* = *q* ∧ (*q* ∗ *p* + *r*) *mod p* = *r*›
    proof (*rule ccontr*)
      assume ‹¬ *?thesis*›
      moreover have ∗: ‹((*q* ∗ *p* + *r*) *div p* − *q*) ∗ *p* = *r* − (*q* ∗ *p* + *r*) *mod p*›
        by (*simp add*: *algebra-simps*)
      ultimately have ‹(*q* ∗ *p* + *r*) *div p* ≠ *q*› **and** ‹(*q* ∗ *p* + *r*) *mod p* ≠ *r*›
        using ‹*p* ≠ *0*› **by** *auto*
      from ‹¬ *p dvd r*› **have** ‹¬ *p dvd* (*q* ∗ *p* + *r*)›
        by *simp*
      with ‹*p* ≠ *0*› **have** ‹*degree* ((*q* ∗ *p* + *r*) *mod p*) < *degree p*›
        by (*rule degree-mod-less-degree*)
      with ‹*degree r* < *degree p*› ‹(*q* ∗ *p* + *r*) *mod p* ≠ *r*›
      have ‹*degree* (*r* − (*q* ∗ *p* + *r*) *mod p*) < *degree p*›
        by (*auto intro*: *degree-diff-less*)
      also have ‹*degree p* ≤ *degree* ((*q* ∗ *p* + *r*) *div p* − *q*) + *degree p*›
        by *simp*
      also from ‹(*q* ∗ *p* + *r*) *div p* ≠ *q*› ‹*p* ≠ *0*›
      have ‹. . . = *degree* (((*q* ∗ *p* + *r*) *div p* − *q*) ∗ *p*)›
        by (*simp add*: *degree-mult-eq*)
      also from ∗ **have** ‹. . . = *degree* (*r* − (*q* ∗ *p* + *r*) *mod p*)›
        by *simp*
      finally have ‹*degree* (*r* − (*q* ∗ *p* + *r*) *mod p*) < *degree* (*r* − (*q* ∗ *p* + *r*) *mod p*)› **.**
      then show *False*

```
      by simp
    qed
    then show ‹(q * p + r) div p = q› ..
  qed
qed (auto simp: euclidean-size-poly-def degree-mult-eq power-add intro: degree-mod-less-degree)

end

lemma euclidean-relation-polyI [case-names by0 divides euclidean-relation]:
  ‹(x div y, x mod y) = (q, r)›
    if by0: ‹y = 0 ⟹ q = 0 ∧ r = x›
    and divides: ‹y ≠ 0 ⟹ y dvd x ⟹ r = 0 ∧ x = q * y›
    and euclidean-relation: ‹y ≠ 0 ⟹ ¬ y dvd x ⟹ degree r < degree y ∧ x = q
* y + r›
  by (rule euclidean-relationI)
    (use that in ‹simp-all add: euclidean-size-poly-def›)

lemma div-poly-eq-0-iff:
  ‹x div y = 0 ⟷ x = 0 ∨ y = 0 ∨ degree x < degree y› for x y :: ‹'a::field poly›
  by (simp add: unique-euclidean-semiring-class.div-eq-0-iff euclidean-size-poly-def)

lemma div-poly-less:
  ‹x div y = 0› if ‹degree x < degree y› for x y :: ‹'a::field poly›
  using that by (simp add: div-poly-eq-0-iff)

lemma mod-poly-less:
  ‹x mod y = x› if ‹degree x < degree y›
  using that by (simp add: mod-eq-self-iff-div-eq-0 div-poly-eq-0-iff)

lemma degree-div-less:
  ‹degree (x div y) < degree x›
    if ‹degree x > 0› ‹degree y > 0›
    for x y :: ‹'a::field poly›
proof (cases ‹x div y = 0›)
  case True
  with ‹degree x > 0› show ?thesis
    by simp
next
  case False
  from that have ‹x ≠ 0› ‹y ≠ 0›
    and *: ‹degree (x div y * y + x mod y) > 0›
    by auto
  show ?thesis
  proof (cases ‹y dvd x›)
    case True
    then obtain z where ‹x = y * z› ..
    then have ‹degree (x div y) < degree (x div y * y)›
      using ‹y ≠ 0› ‹x ≠ 0› ‹degree y > 0› by (simp add: degree-mult-eq)
    with ‹y dvd x› show ?thesis
```

by *simp*
next
case *False*
with ‹$y \neq 0$› **have** ‹*degree* (*x mod y*) < *degree y*›
by (*rule degree-mod-less-degree*)
with ‹$y \neq 0$› ‹*x div y* ≠ *0*› **have** ‹*degree* (*x mod y*) < *degree* (*x div y* ∗ *y*)›
by (*simp add*: *degree-mult-eq*)
then have ‹*degree* (*x div y* ∗ *y* + *x mod y*) = *degree* (*x div y* ∗ *y*)›
by (*rule degree-add-eq-left*)
with ‹$y \neq 0$› ‹*x div y* ≠ *0*› ‹*degree y* > *0*› **show** *?thesis*
by (*simp add*: *degree-mult-eq*)
qed
qed

lemma *degree-mod-less′*: *b* ≠ *0* ⟹ *a mod b* ≠ *0* ⟹ *degree* (*a mod b*) < *degree b*
by (*rule degree-mod-less-degree*) *auto*

lemma *degree-mod-less*: *y* ≠ *0* ⟹ *x mod y* = *0* ∨ *degree* (*x mod y*) < *degree y*
using *degree-mod-less′* **by** *blast*

lemma *div-smult-left*: ‹*smult a x div y* = *smult a* (*x div y*)› (**is** *?Q*)
and *mod-smult-left*: ‹*smult a x mod y* = *smult a* (*x mod y*)› (**is** *?R*)
for *x y* :: ‹′*a*::*field poly*›
proof −
have ‹(*smult a x div y*, *smult a x mod y*) = (*smult a* (*x div y*), *smult a* (*x mod y*))›
proof (*cases* ‹*a* = *0*›)
case *True*
then show *?thesis*
by *simp*
next
case *False*
show *?thesis*
by (*rule euclidean-relation-polyI*)
(*use False* **in** ‹*simp-all add*: *dvd-smult-iff degree-mod-less-degree flip*: *smult-add-right*›)
qed
then show *?Q* **and** *?R*
by *simp-all*
qed

lemma *poly-div-minus-left* [*simp*]: (− *x*) *div y* = − (*x div y*)
for *x y* :: ′*a*::*field poly*
using *div-smult-left* [*of* − *1*::′*a*] **by** *simp*

lemma *poly-mod-minus-left* [*simp*]: (− *x*) *mod y* = − (*x mod y*)
for *x y* :: ′*a*::*field poly*
using *mod-smult-left* [*of* − *1*::′*a*] **by** *simp*

lemma *poly-div-add-left*: ‹(*x* + *y*) *div z* = *x div z* + *y div z*› (**is** *?Q*)

and *poly-mod-add-left*: ‹$(x + y)$ *mod* $z = x$ *mod* $z + y$ *mod* z› (**is** *?R*)
for *x y z* :: ‹′*a*::*field poly*›
proof −
have ‹$((x + y)$ *div* z, $(x + y)$ *mod* $z) = (x$ *div* $z + y$ *div* z, x *mod* $z + y$ *mod* $z)$›
proof (*induction rule*: *euclidean-relation-polyI*)
case *by0*
then show *?case* **by** *simp*
next
case *divides*
then obtain *w* **where** ‹$x + y = z * w$›
by *blast*
then have *y*: ‹$y = z * w - x$›
by (*simp add*: *algebra-simps*)
from ‹$z \neq 0$› **show** *?case*
using *mod-mult-self4* [*of z w* ‹− *x*›] *div-mult-self4* [*of z w* ‹− *x*›]
by (*simp add*: *algebra-simps y*)
next
case *euclidean-relation*
then have ‹*degree* $(x$ *mod* $z + y$ *mod* $z)$ < *degree* z›
using *degree-mod-less-degree* [*of z x*] *degree-mod-less-degree* [*of z y*]
dvd-add-right-iff [*of z x y*] *dvd-add-left-iff* [*of z y x*]
by (*cases* ‹*z dvd x* ∨ *z dvd y*›) (*auto intro*: *degree-add-less*)
moreover have ‹$x + y = (x$ *div* $z + y$ *div* $z) * z + (x$ *mod* $z + y$ *mod* $z)$›
by (*simp add*: *algebra-simps*)
ultimately show *?case*
by *simp*
qed
then show *?Q* **and** *?R*
by *simp-all*
qed

lemma *poly-div-diff-left*: $(x - y)$ *div* $z = x$ *div* $z - y$ *div* z
for *x y z* :: ′*a*::*field poly*
by (*simp only*: *diff-conv-add-uminus poly-div-add-left poly-div-minus-left*)

lemma *poly-mod-diff-left*: $(x - y)$ *mod* $z = x$ *mod* $z - y$ *mod* z
for *x y z* :: ′*a*::*field poly*
by (*simp only*: *diff-conv-add-uminus poly-mod-add-left poly-mod-minus-left*)

lemma *div-smult-right*: ‹*x div smult a y* = *smult* (*inverse a*) (*x div y*)› (**is** *?Q*)
and *mod-smult-right*: ‹*x mod smult a y* = (*if a* = *0 then x else x mod y*)› (**is** *?R*)
proof −
have ‹(*x div smult a y*, *x mod smult a y*) = (*smult* (*inverse a*) (*x div y*), (*if a* = *0 then x else x mod y*))›
proof (*induction rule*: *euclidean-relation-polyI*)
case *by0*
then show *?case* **by** *auto*
next
case *divides*

moreover define *w* **where** ‹*w* = *x div y*›
ultimately have ‹*x* = *y* ∗ *w*›
by (*simp add*: *smult-dvd-iff*)
with *divides* **show** *?case*
by *simp*
next
case *euclidean-relation*
then show *?case*
by (*simp add*: *smult-dvd-iff degree-mod-less-degree*)
qed
then show *?Q* **and** *?R*
by *simp-all*
qed

lemma *mod-mult-unit-eq*:
‹*x mod* (*z* ∗ *y*) = *x mod y*›
if ‹*is-unit z*›
for *x y z* :: ‹′*a*::*field poly*›
proof (*cases* ‹*y* = *0*›)
case *True*
then show *?thesis*
by *simp*
next
case *False*
moreover have ‹*z* ≠ *0*›
using *that* **by** *auto*
moreover define *a* **where** ‹*a* = *lead-coeff z*›
ultimately have ‹*z* = [:*a*:]› ‹*a* ≠ *0*›
using *that monom-0* [*of a*] **by** (*simp-all add*: *is-unit-monom-trivial*)
then show *?thesis*
by (*simp add*: *mod-smult-right*)
qed

lemma *poly-div-minus-right* [*simp*]: *x div* (− *y*) = − (*x div y*)
for *x y* :: ′*a*::*field poly*
using *div-smult-right* [*of* - − *1*::′*a*] **by** (*simp add*: *nonzero-inverse-minus-eq*)

lemma *poly-mod-minus-right* [*simp*]: *x mod* (− *y*) = *x mod y*
for *x y* :: ′*a*::*field poly*
using *mod-smult-right* [*of* - − *1*::′*a*] **by** *simp*

lemma *poly-div-mult-right*: ‹*x div* (*y* ∗ *z*) = (*x div y*) *div z*› (**is** *?Q*)
and *poly-mod-mult-right*: ‹*x mod* (*y* ∗ *z*) = *y* ∗ (*x div y mod z*) + *x mod y*› (**is** *?R*)
for *x y z* :: ‹′*a*::*field poly*›
proof −
have ‹(*x div* (*y* ∗ *z*), *x mod* (*y* ∗ *z*)) = ((*x div y*) *div z*, *y* ∗ (*x div y mod z*) + *x mod y*)›
proof (*induction rule*: *euclidean-relation-polyI*)

case *by0*
then show *?case* **by** *auto*
next
case *divides*
then show *?case* **by** *auto*
next
case *euclidean-relation*
then have ‹*y* ≠ *0*› ‹*z* ≠ *0*›
by *simp-all*
with ‹¬ *y* ∗ *z dvd x*› **have** ‹*degree* (*y* ∗ (*x div y mod z*) + *x mod y*) < *degree* (*y* ∗ *z*)›
using *degree-mod-less-degree* [*of y x*] *degree-mod-less-degree* [*of z* ‹*x div y*›]
degree-add-eq-left [*of* ‹*x mod y*› ‹*y* ∗ (*x div y mod z*)›]
by (*cases* ‹*z dvd x div y*›; *cases* ‹*y dvd x*›)
(*auto simp add*: *degree-mult-eq not-dvd-imp-mod-neq-0 dvd-div-iff-mult*)
moreover have ‹*x* = *x div y div z* ∗ (*y* ∗ *z*) + (*y* ∗ (*x div y mod z*) + *x mod y*)›
by (*simp add*: *field-simps flip*: *distrib-left*)
ultimately show *?case*
by *simp*
qed
then show *?Q* **and** *?R*
by *simp-all*
qed

lemma *dvd-pCons-imp-dvd-pCons-mod*:
‹*y dvd pCons a* (*x mod y*)› **if** ‹*y dvd pCons a x*›
proof −
have ‹*pCons a x* = *pCons a* (*x div y* ∗ *y* + *x mod y*)›
by *simp*
also have ‹. . . = *pCons 0* (*x div y* ∗ *y*) + *pCons a* (*x mod y*)›
by *simp*
also have ‹*pCons 0* (*x div y* ∗ *y*) = (*x div y* ∗ *monom 1* (*Suc 0*)) ∗ *y*›
by (*simp add*: *monom-Suc*)
finally show ‹*y dvd pCons a* (*x mod y*)›
using ‹*y dvd pCons a x*› **by** *simp*
qed

lemma *degree-less-if-less-eqI*:
‹*degree x* < *degree y*› **if** ‹*degree x* ≤ *degree y*› ‹*coeff x* (*degree y*) = *0*› ‹*x* ≠ *0*›
proof (*cases* ‹*degree x* = *degree y*›)
case *True*
with ‹*coeff x* (*degree y*) = *0*› **have** ‹*lead-coeff x* = *0*›
by *simp*
then have ‹*x* = *0*›
by *simp*
with ‹*x* ≠ *0*› **show** *?thesis*
by *simp*
next

case *False*
with ‹*degree x* ≤ *degree y*› **show** *?thesis*
by *simp*
qed

lemma *div-pCons-eq*:
‹*pCons a p div q* = (*if q* = *0 then 0 else pCons* (*coeff* (*pCons a* (*p mod q*)) (*degree q*) / *lead-coeff q*) (*p div q*))› (**is** *?Q*)
and *mod-pCons-eq*:
‹*pCons a p mod q* = (*if q* = *0 then pCons a p else pCons a* (*p mod q*) − *smult* (*coeff* (*pCons a* (*p mod q*)) (*degree q*) / *lead-coeff q*) *q*)› (**is** *?R*)
for *x y* :: ‹*'a::field poly*›
proof −
have ‹*?Q*› **and** ‹*?R*› **if** ‹*q* = *0*›
using *that* **by** *simp-all*
moreover have ‹*?Q*› **and** ‹*?R*› **if** ‹*q* ≠ *0*›
proof −
define *b* **where** ‹*b* = *coeff* (*pCons a* (*p mod q*)) (*degree q*) / *lead-coeff q*›
have ‹(*pCons a p div q*, *pCons a p mod q*) =
(*pCons b* (*p div q*), (*pCons a* (*p mod q*) − *smult b q*))› (**is** ‹- = (*?q*, *?r*)›)
proof (*induction rule*: *euclidean-relation-polyI*)
case *by0*
with ‹*q* ≠ *0*› **show** *?case* **by** *simp*
next
case *divides*
show *?case*
proof (*cases* ‹*pCons a* (*p mod q*) = *0*›)
case *True*
then show *?thesis*
by (*auto simp add*: *b-def*)
next
case *False*
have ‹*q dvd pCons a* (*p mod q*)›
using ‹*q dvd pCons a p*› **by** (*rule dvd-pCons-imp-dvd-pCons-mod*)
then obtain *s* **where** ∗: ‹*pCons a* (*p mod q*) = *q* ∗ *s*› **..**
with *False* **have** ‹*s* ≠ *0*›
by *auto*
from ‹*q* ≠ *0*› **have** ‹*degree* (*pCons a* (*p mod q*)) ≤ *degree q*›
by (*auto simp add*: *Suc-le-eq intro*: *degree-mod-less-degree*)
moreover from ‹*s* ≠ *0*› **have** ‹*degree q* ≤ *degree* (*pCons a* (*p mod q*))›
by (*simp add*: *degree-mult-right-le* ∗)
ultimately have ‹*degree* (*pCons a* (*p mod q*)) = *degree q*›
by (*rule order.antisym*)
with ‹*s* ≠ *0*› ‹*q* ≠ *0*› **have** ‹*degree s* = *0*›
by (*simp add*: ∗ *degree-mult-eq*)
then obtain *c* **where** ‹*s* = [:*c*:]›
by (*rule degree-eq-zeroE*)
also have ‹*c* = *b*›
using ‹*q* ≠ *0*› **by** (*simp add*: *b-def* ∗ ‹*s* = [:*c*:]›)

```
      finally have ‹smult b q = pCons a (p mod q)›
        by (simp add: ∗)
      then show ?thesis
        by simp
    qed
  next
    case euclidean-relation
    then have ‹degree q > 0›
      using is-unit-iff-degree by blast
    from ‹q ≠ 0› have ‹degree (pCons a (p mod q)) ≤ degree q›
      by (auto simp add: Suc-le-eq intro: degree-mod-less-degree)
    moreover have ‹degree (smult b q) ≤ degree q›
      by (rule degree-smult-le)
    ultimately have ‹degree (pCons a (p mod q) − smult b q) ≤ degree q›
      by (rule degree-diff-le)
    moreover have ‹coeff (pCons a (p mod q) − smult b q) (degree q) = 0›
      using ‹degree q > 0› by (auto simp add: b-def)
    ultimately have ‹degree (pCons a (p mod q) − smult b q) < degree q›
      using ‹degree q > 0›
      by (cases ‹pCons a (p mod q) = smult b q›)
        (auto intro: degree-less-if-less-eqI)
    then show ?case
      by simp
  qed
  with ‹q ≠ 0› show ?Q and ?R
    by (simp-all add: b-def)
 qed
 ultimately show ?Q and ?R
   by simp-all
qed

lemma div-mod-fold-coeffs:
 (p div q, p mod q) =
   (if q = 0 then (0, p)
    else
     fold-coeffs
      (λa (s, r).
        let b = coeff (pCons a r) (degree q) / coeff q (degree q)
        in (pCons b s, pCons a r − smult b q)) p (0, 0))
 by (rule sym, induct p) (auto simp: div-pCons-eq mod-pCons-eq Let-def)

lemma mod-pCons:
 fixes a :: 'a::field
   and x y :: 'a::field poly
 assumes y: y ≠ 0
 defines b ≡ coeff (pCons a (x mod y)) (degree y) / coeff y (degree y)
 shows (pCons a x) mod y = pCons a (x mod y) − smult b y
 unfolding b-def
 by (simp add: mod-pCons-eq)
```

4.30.4 List-based versions for fast implementation

```
fun minus-poly-rev-list :: 'a :: group-add list ⇒ 'a list ⇒ 'a list
  where
    minus-poly-rev-list (x # xs) (y # ys) = (x − y) # (minus-poly-rev-list xs ys)
  | minus-poly-rev-list xs [] = xs
  | minus-poly-rev-list [] (y # ys) = []

fun pseudo-divmod-main-list ::
  'a::comm-ring-1 ⇒ 'a list ⇒ 'a list ⇒ 'a list ⇒ nat ⇒ 'a list × 'a list
  where
    pseudo-divmod-main-list lc q r d (Suc n) =
      (let
        rr = map ((∗) lc) r;
        a = hd r;
        qqq = cCons a (map ((∗) lc) q);
        rrr = tl (if a = 0 then rr else minus-poly-rev-list rr (map ((∗) a) d))
       in pseudo-divmod-main-list lc qqq rrr d n)
  | pseudo-divmod-main-list lc q r d 0 = (q, r)

fun pseudo-mod-main-list :: 'a::comm-ring-1 ⇒ 'a list ⇒ 'a list ⇒ nat ⇒ 'a list
  where
    pseudo-mod-main-list lc r d (Suc n) =
      (let
        rr = map ((∗) lc) r;
        a = hd r;
        rrr = tl (if a = 0 then rr else minus-poly-rev-list rr (map ((∗) a) d))
       in pseudo-mod-main-list lc rrr d n)
  | pseudo-mod-main-list lc r d 0 = r


fun divmod-poly-one-main-list ::
    'a::comm-ring-1 list ⇒ 'a list ⇒ 'a list ⇒ nat ⇒ 'a list × 'a list
  where
    divmod-poly-one-main-list q r d (Suc n) =
      (let
        a = hd r;
        qqq = cCons a q;
        rr = tl (if a = 0 then r else minus-poly-rev-list r (map ((∗) a) d))
       in divmod-poly-one-main-list qqq rr d n)
  | divmod-poly-one-main-list q r d 0 = (q, r)

fun mod-poly-one-main-list :: 'a::comm-ring-1 list ⇒ 'a list ⇒ nat ⇒ 'a list
  where
    mod-poly-one-main-list r d (Suc n) =
      (let
        a = hd r;
        rr = tl (if a = 0 then r else minus-poly-rev-list r (map ((∗) a) d))
       in mod-poly-one-main-list rr d n)
  | mod-poly-one-main-list r d 0 = r
```

definition *pseudo-divmod-list* :: *'a::comm-ring-1 list* ⇒ *'a list* ⇒ *'a list* × *'a list*
where *pseudo-divmod-list p q* =
(*if q* = [] *then* ([], *p*)
else
(*let rq* = *rev q*;
(*qu*,*re*) = *pseudo-divmod-main-list* (*hd rq*) [] (*rev p*) *rq* (*1* + *length p* − *length q*)
in (*qu*, *rev re*)))

definition *pseudo-mod-list* :: *'a::comm-ring-1 list* ⇒ *'a list* ⇒ *'a list*
where *pseudo-mod-list p q* =
(*if q* = [] *then p*
else
(*let*
rq = *rev q*;
re = *pseudo-mod-main-list* (*hd rq*) (*rev p*) *rq* (*1* + *length p* − *length q*)
in rev re))

lemma *minus-zero-does-nothing*: *minus-poly-rev-list x* (*map* ((∗) *0*) *y*) = *x*
for *x* :: *'a::ring list*
by (*induct x y rule*: *minus-poly-rev-list.induct*) *auto*

lemma *length-minus-poly-rev-list* [*simp*]: *length* (*minus-poly-rev-list xs ys*) = *length xs*
by (*induct xs ys rule*: *minus-poly-rev-list.induct*) *auto*

lemma *if-0-minus-poly-rev-list*:
(*if a* = *0 then x else minus-poly-rev-list x* (*map* ((∗) *a*) *y*)) =
minus-poly-rev-list x (*map* ((∗) *a*) *y*)
for *a* :: *'a::ring*
by(*cases a* = *0*) (*simp-all add*: *minus-zero-does-nothing*)

lemma *Poly-append*: *Poly* (*a* @ *b*) = *Poly a* + *monom 1* (*length a*) ∗ *Poly b*
for *a* :: *'a::comm-semiring-1 list*
by (*induct a*) (*auto simp*: *monom-0 monom-Suc*)

lemma *minus-poly-rev-list*: *length p* ≥ *length q* ⟹
Poly (*rev* (*minus-poly-rev-list* (*rev p*) (*rev q*))) =
Poly p − *monom 1* (*length p* − *length q*) ∗ *Poly q*
for *p q* :: *'a* :: *comm-ring-1 list*
proof (*induct rev p rev q arbitrary*: *p q rule*: *minus-poly-rev-list.induct*)
case (*1 x xs y ys*)
then have *length* (*rev q*) ≤ *length* (*rev p*)
by *simp*
from *this*[*folded 1*(*2*,*3*)] **have** *ys-xs*: *length ys* ≤ *length xs*
by *simp*
then have ∗: *Poly* (*rev* (*minus-poly-rev-list xs ys*)) =
Poly (*rev xs*) − *monom 1* (*length xs* − *length ys*) ∗ *Poly* (*rev ys*)

by (*subst 1.hyps*(*1*)[*of rev xs rev ys*, *unfolded rev-rev-ident length-rev*]) *auto*
have *Poly p* − *monom 1* (*length p* − *length q*) ∗ *Poly q* =
Poly (*rev* (*rev p*)) − *monom 1* (*length* (*rev* (*rev p*)) − *length* (*rev* (*rev q*))) ∗ *Poly* (*rev* (*rev q*))
by *simp*
also have ... =
Poly (*rev* (*x* # *xs*)) − *monom 1* (*length* (*x* # *xs*) − *length* (*y* # *ys*)) ∗ *Poly* (*rev* (*y* # *ys*))
unfolding *1*(*2*,*3*) **by** *simp*
also from *ys-xs* **have** ... =
Poly (*rev xs*) + *monom x* (*length xs*) −
(*monom 1* (*length xs* − *length ys*) ∗ *Poly* (*rev ys*) + *monom y* (*length xs*))
by (*simp add*: *Poly-append distrib-left mult-monom smult-monom*)
also have ... = *Poly* (*rev* (*minus-poly-rev-list xs ys*)) + *monom* (*x* − *y*) (*length xs*)
unfolding ∗ *diff-monom*[*symmetric*] **by** *simp*
finally show *?case*
by (*simp add*: *1*(*2*,*3*)[*symmetric*] *smult-monom Poly-append*)
qed *auto*

lemma *smult-monom-mult*: *smult a* (*monom b n* ∗ *f*) = *monom* (*a* ∗ *b*) *n* ∗ *f*
using *smult-monom* [*of a - n*] **by** (*metis mult-smult-left*)

lemma *head-minus-poly-rev-list*:
length d ≤ *length r* ⟹ *d* ≠ [] ⟹
hd (*minus-poly-rev-list* (*map* ((∗) (*last d*)) *r*) (*map* ((∗) (*hd r*)) (*rev d*))) = *0*
for *d r* :: ′*a*::*comm-ring list*
proof (*induct r*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons a rs*)
then show *?case* **by** (*cases rev d*) (*simp-all add*: *ac-simps*)
qed

lemma *Poly-map*: *Poly* (*map* ((∗) *a*) *p*) = *smult a* (*Poly p*)
proof (*induct p*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons x xs*)
then show *?case* **by** (*cases Poly xs* = *0*) *auto*
qed

lemma *last-coeff-is-hd*: *xs* ≠ [] ⟹ *coeff* (*Poly xs*) (*length xs* − *1*) = *hd* (*rev xs*)
by (*simp-all add*: *hd-conv-nth rev-nth nth-default-nth nth-append*)

lemma *pseudo-divmod-main-list-invar*:
assumes *leading-nonzero*: *last d* ≠ *0*

and *lc*: *last d* = *lc*
and *d* ≠ []
and *pseudo-divmod-main-list lc q* (*rev r*) (*rev d*) *n* = (*q'*, *rev r'*)
and *n* = *1* + *length r* − *length d*
shows *pseudo-divmod-main lc* (*monom 1 n* ∗ *Poly q*) (*Poly r*) (*Poly d*) (*length r* − *1*) *n* =
(*Poly q'*, *Poly r'*)
using *assms*(*4*−)
proof (*induct n arbitrary*: *r q*)
case (*Suc n*)
from *Suc.prems* **have** ∗: ¬ *Suc* (*length r*) ≤ *length d*
by *simp*
with ‹*d* ≠ []› **have** *r* ≠ []
using *Suc-leI length-greater-0-conv list.size*(*3*) **by** *fastforce*
let *?a* = (*hd* (*rev r*))
let *?rr* = *map* ((∗) *lc*) (*rev r*)
let *?rrr* = *rev* (*tl* (*minus-poly-rev-list ?rr* (*map* ((∗) *?a*) (*rev d*))))
let *?qq* = *cCons ?a* (*map* ((∗) *lc*) *q*)
from ∗ *Suc*(*3*) **have** *n*: *n* = (*1* + *length r* − *length d* − *1*)
by *simp*
from ∗ **have** *rr-val*:(*length ?rrr*) = (*length r* − *1*)
by *auto*
with ‹*r* ≠ []› ∗ **have** *rr-smaller*: (*1* + *length r* − *length d* − *1*) = (*1* + *length ?rrr* − *length d*)
by *auto*
from ∗ **have** *id*: *Suc* (*length r*) − *length d* = *Suc* (*length r* − *length d*)
by *auto*
from *Suc.prems* ∗
have *pseudo-divmod-main-list lc ?qq* (*rev ?rrr*) (*rev d*) (*1* + *length r* − *length d* − *1*) = (*q'*, *rev r'*)
by (*simp add*: *Let-def if-0-minus-poly-rev-list id*)
with *n* **have** *v*: *pseudo-divmod-main-list lc ?qq* (*rev ?rrr*) (*rev d*) *n* = (*q'*, *rev r'*)
by *auto*
from ∗ **have** *sucrr*:*Suc* (*length r*) − *length d* = *Suc* (*length r* − *length d*)
using *Suc-diff-le not-less-eq-eq* **by** *blast*
from *Suc*(*3*) ‹*r* ≠ []› **have** *n-ok* : *n* = *1* + (*length ?rrr*) − *length d*
by *simp*
have *cong*: ⋀*x1 x2 x3 x4 y1 y2 y3 y4*. *x1* = *y1* ⟹ *x2* = *y2* ⟹ *x3* = *y3* ⟹ *x4* = *y4* ⟹
pseudo-divmod-main lc x1 x2 x3 x4 n = *pseudo-divmod-main lc y1 y2 y3 y4 n*
by *simp*
have *hd-rev*: *coeff* (*Poly r*) (*length r* − *Suc 0*) = *hd* (*rev r*)
using *last-coeff-is-hd*[*OF* ‹*r* ≠ []›] **by** *simp*
show *?case*
unfolding *Suc.hyps*(*1*)[*OF v n-ok*, *symmetric*] *pseudo-divmod-main.simps Let-def*
proof (*rule cong*[*OF - - refl*], *goal-cases*)
case *1*
show *?case*
by (*simp add*: *monom-Suc hd-rev*[*symmetric*] *smult-monom Poly-map*)

next
case *2*
show *?case*
proof (*subst Poly-on-rev-starting-with-0*, *goal-cases*)
show *hd* (*minus-poly-rev-list* (*map* ((∗) *lc*) (*rev r*)) (*map* ((∗) (*hd* (*rev r*))) (*rev d*))) = *0*
by (*fold lc*, *subst head-minus-poly-rev-list*, *insert* ∗ ‹*d* ≠ []›, *auto*)
from ∗ **have** *length d* ≤ *length r*
by *simp*
then show *smult lc* (*Poly r*) − *monom* (*coeff* (*Poly r*) (*length r* − *1*)) *n* ∗ *Poly d* =
Poly (*rev* (*minus-poly-rev-list* (*map* ((∗) *lc*) (*rev r*)) (*map* ((∗) (*hd* (*rev r*))) (*rev d*))))
by (*fold rev-map*) (*auto simp add*: *n smult-monom-mult Poly-map hd-rev* [*symmetric*]
minus-poly-rev-list)
qed
qed *simp*
qed *simp*

lemma *pseudo-divmod-impl* [*code*]:
pseudo-divmod f g = *map-prod poly-of-list poly-of-list* (*pseudo-divmod-list* (*coeffs f*) (*coeffs g*))
for *f g* :: *′a*::*comm-ring-1 poly*
proof (*cases g* = *0*)
case *False*
then have *last* (*coeffs g*) ≠ *0*
and *last* (*coeffs g*) = *lead-coeff g*
and *coeffs g* ≠ []
by (*simp-all add*: *last-coeffs-eq-coeff-degree*)
moreover obtain *q r* **where** *qr*: *pseudo-divmod-main-list*
(*last* (*coeffs g*)) (*rev* [])
(*rev* (*coeffs f*)) (*rev* (*coeffs g*))
(*1* + *length* (*coeffs f*) −
length (*coeffs g*)) = (*q*, *rev* (*rev r*))
by *force*
ultimately have (*Poly q*, *Poly* (*rev r*)) = *pseudo-divmod-main* (*lead-coeff g*) *0 f g*
(*length* (*coeffs f*) − *Suc 0*) (*Suc* (*length* (*coeffs f*)) − *length* (*coeffs g*))
by (*subst pseudo-divmod-main-list-invar* [*symmetric*]) *auto*
moreover have *pseudo-divmod-main-list*
(*hd* (*rev* (*coeffs g*))) []
(*rev* (*coeffs f*)) (*rev* (*coeffs g*))
(*1* + *length* (*coeffs f*) −
length (*coeffs g*)) = (*q*, *r*)
by (*metis hd-rev qr rev.simps*(*1*) *rev-swap*)
ultimately show *?thesis*
by (*simp add*: *degree-eq-length-coeffs pseudo-divmod-def pseudo-divmod-list-def*)
next

case *True*
then show *?thesis*
by (*auto simp add*: *pseudo-divmod-def pseudo-divmod-list-def*)
qed

lemma *pseudo-mod-main-list*:
snd (*pseudo-divmod-main-list l q xs ys n*) = *pseudo-mod-main-list l xs ys n*
by (*induct n arbitrary*: *l q xs ys*) (*auto simp*: *Let-def*)

lemma *pseudo-mod-impl*[*code*]: *pseudo-mod f g* = *poly-of-list* (*pseudo-mod-list* (*coeffs f*) (*coeffs g*))
proof −
have *snd-case*: $\bigwedge$*f g p*. *snd* ((λ(*x*,*y*). (*f x*, *g y*)) *p*) = *g* (*snd p*)
by *auto*
show *?thesis*
unfolding *pseudo-mod-def pseudo-divmod-impl pseudo-divmod-list-def pseudo-mod-list-def Let-def*
by (*simp add*: *snd-case pseudo-mod-main-list*)
qed

4.30.5 Improved Code-Equations for Polynomial (Pseudo) Division

lemma *pdivmod-via-pseudo-divmod*:
‹(*f div g*, *f mod g*) =
(*if g* = *0 then* (*0*, *f*)
else
let
ilc = *inverse* (*lead-coeff g*);
h = *smult ilc g*;
(*q*,*r*) = *pseudo-divmod f h*
in (*smult ilc q*, *r*))›
(**is** ‹*?l* = *?r*›)
proof (*cases* ‹*g* = *0*›)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
define *ilc* **where** ‹*ilc* = *inverse* (*lead-coeff g*)›
define *h* **where** ‹*h* = *smult ilc g*›
from *False* **have** ‹*lead-coeff h* = *1*›
and ‹*ilc* ≠ *0*›
by (*auto simp*: *h-def ilc-def*)
define *q r* **where** ‹*q* = *f div h*› **and** ‹*r* = *f mod h*›
with ‹*lead-coeff h* = *1*› **have** *p*: ‹*pseudo-divmod f h* = (*q*, *r*)›
by (*simp add*: *pseudo-divmod-eq-div-mod*)
from ‹*ilc* ≠ *0*› **have** ‹(*f div g*, *f mod g*) = (*smult ilc q*, *r*)›
by (*auto simp*: *h-def div-smult-right mod-smult-right q-def r-def*)
also have ‹(*smult ilc q*, *r*) = *?r*›

using ‹*g* ≠ *0*› **by** (*auto simp*: *Let-def p simp flip*: *h-def ilc-def*)
finally show *?thesis* **.**
qed

lemma *pdivmod-via-pseudo-divmod-list*:
(*f div g*, *f mod g*) =
(*let cg* = *coeffs g in*
if cg = [] *then* (*0*, *f*)
else
let
cf = *coeffs f*;
ilc = *inverse* (*last cg*);
ch = *map* ((∗) *ilc*) *cg*;
(*q*, *r*) = *pseudo-divmod-main-list 1* [] (*rev cf*) (*rev ch*) (*1* + *length cf* − *length cg*)
in (*poly-of-list* (*map* ((∗) *ilc*) *q*), *poly-of-list* (*rev r*)))
proof −
note *d* = *pdivmod-via-pseudo-divmod pseudo-divmod-impl pseudo-divmod-list-def*
show *?thesis*
proof (*cases g* = *0*)
case *True*
with *d* **show** *?thesis* **by** *auto*
next
case *False*
define *ilc* **where** *ilc* = *inverse* (*coeff g* (*degree g*))
from *False* **have** *ilc*: *ilc* ≠ *0*
by (*auto simp*: *ilc-def*)
with *False* **have** *id*: *g* = *0* ⟷ *False coeffs g* = [] ⟷ *False*
last (*coeffs g*) = *coeff g* (*degree g*)
coeffs (*smult ilc g*) = [] ⟷ *False*
by (*auto simp*: *last-coeffs-eq-coeff-degree*)
have *id2*: *hd* (*rev* (*coeffs* (*smult ilc g*))) = *1*
by (*subst hd-rev*, *insert id ilc*, *auto simp*: *coeffs-smult*, *subst last-map*, *auto simp*: *id ilc-def*)
have *id3*: *length* (*coeffs* (*smult ilc g*)) = *length* (*coeffs g*)
rev (*coeffs* (*smult ilc g*)) = *rev* (*map* ((∗) *ilc*) (*coeffs g*))
unfolding *coeffs-smult* **using** *ilc* **by** *auto*
obtain *q r* **where** *pair*:
pseudo-divmod-main-list 1 [] (*rev* (*coeffs f*)) (*rev* (*map* ((∗) *ilc*) (*coeffs g*)))
(*1* + *length* (*coeffs f*) − *length* (*coeffs g*)) = (*q*, *r*)
by *force*
show *?thesis*
unfolding *d Let-def id if-False ilc-def*[*symmetric*] *map-prod-def*[*symmetric*] *id2*
unfolding *id3 pair map-prod-def split*
by (*auto simp*: *Poly-map*)
qed
qed

lemma *pseudo-divmod-main-list-1*: *pseudo-divmod-main-list 1* = *divmod-poly-one-main-list*
proof (*intro ext*, *goal-cases*)
  case (*1 q r d n*)
  have ∗: *map* ((∗) *1*) *xs* = *xs* **for** *xs* :: ′*a list*
    by (*induct xs*) *auto*
  show *?case*
    by (*induct n arbitrary*: *q r d*) (*auto simp*: ∗ *Let-def*)
qed

fun *divide-poly-main-list* :: ′*a*::*idom-divide* ⇒ ′*a list* ⇒ ′*a list* ⇒ ′*a list* ⇒ *nat* ⇒ ′*a list*
  where
    divide-poly-main-list lc q r d (*Suc n*) =
      (*let*
        cr = *hd r*
        in if cr = *0 then divide-poly-main-list lc* (*cCons cr q*) (*tl r*) *d n else let*
        a = *cr div lc*;
        qq = *cCons a q*;
        rr = *minus-poly-rev-list r* (*map* ((∗) *a*) *d*)
       in if hd rr = *0 then divide-poly-main-list lc qq* (*tl rr*) *d n else* [])
  | *divide-poly-main-list lc q r d 0* = *q*

lemma *divide-poly-main-list-simp* [*simp*]:
  divide-poly-main-list lc q r d (*Suc n*) =
    (*let*
      cr = *hd r*;
      a = *cr div lc*;
      qq = *cCons a q*;
      rr = *minus-poly-rev-list r* (*map* ((∗) *a*) *d*)
     in if hd rr = *0 then divide-poly-main-list lc qq* (*tl rr*) *d n else* [])
  by (*simp add*: *Let-def minus-zero-does-nothing*)

declare *divide-poly-main-list.simps*(*1*)[*simp del*]

definition *divide-poly-list* :: ′*a*::*idom-divide poly* ⇒ ′*a poly* ⇒ ′*a poly*
  where *divide-poly-list f g* =
    (*let cg* = *coeffs g in*
      if cg = [] *then g*
      else
        let
          cf = *coeffs f*;
          cgr = *rev cg*
         in poly-of-list (*divide-poly-main-list* (*hd cgr*) [] (*rev cf*) *cgr* (*1* + *length cf* − *length cg*)))

lemmas *pdivmod-via-divmod-list* = *pdivmod-via-pseudo-divmod-list*[*unfolded pseudo-divmod-main-list-1*]

lemma *mod-poly-one-main-list*: *snd* (*divmod-poly-one-main-list q r d n*) = *mod-poly-one-main-list r d n*

by (*induct n arbitrary*: *q r d*) (*auto simp*: *Let-def*)

lemma *mod-poly-code* [*code*]:
f mod g =
(*let cg* = *coeffs g in*
if cg = [] *then f*
else
let
cf = *coeffs f*;
ilc = *inverse* (*last cg*);
ch = *map* ((∗) *ilc*) *cg*;
r = *mod-poly-one-main-list* (*rev cf*) (*rev ch*) (*1* + *length cf* − *length cg*)
in poly-of-list (*rev r*))
(**is** - = *?rhs*)
proof −
have *snd* (*f div g*, *f mod g*) = *?rhs*
unfolding *pdivmod-via-divmod-list Let-def mod-poly-one-main-list* [*symmetric*, *of* - - - *Nil*]
by (*auto split*: *prod.splits*)
then show *?thesis* **by** *simp*
qed

definition *div-field-poly-impl* :: ′*a* :: *field poly* ⇒ ′*a poly* ⇒ ′*a poly*
where *div-field-poly-impl f g* =
(*let cg* = *coeffs g in*
if cg = [] *then 0*
else
let
cf = *coeffs f*;
ilc = *inverse* (*last cg*);
ch = *map* ((∗) *ilc*) *cg*;
q = *fst* (*divmod-poly-one-main-list* [] (*rev cf*) (*rev ch*) (*1* + *length cf* − *length cg*))
in poly-of-list ((*map* ((∗) *ilc*) *q*)))

We do not declare the following lemma as code equation, since then polynomial division on non-fields will no longer be executable. However, a code-unfold is possible, since *div-field-poly-impl* is a bit more efficient than the generic polynomial division.

lemma *div-field-poly-impl*[*code-unfold*]: (*div*) = *div-field-poly-impl*
proof (*intro ext*)
fix *f g* :: ′*a poly*
have *fst* (*f div g*, *f mod g*) = *div-field-poly-impl f g*
unfolding *div-field-poly-impl-def pdivmod-via-divmod-list Let-def*
by (*auto split*: *prod.splits*)
then show *f div g* = *div-field-poly-impl f g*
by *simp*
qed

lemma *divide-poly-main-list*:
  assumes *lc0*: *lc* ≠ *0*
    and *lc*: *last d* = *lc*
    and *d*: *d* ≠ []
    and *n* = (*1* + *length r* − *length d*)
  shows *Poly* (*divide-poly-main-list lc q* (*rev r*) (*rev d*) *n*) =
    divide-poly-main lc (*monom 1 n* ∗ *Poly q*) (*Poly r*) (*Poly d*) (*length r* − *1*) *n*
  using *assms*(*4*−)
proof (*induct n arbitrary*: *r q*)
  case (*Suc n*)
  from *Suc.prems* **have** *ifCond*: ¬ *Suc* (*length r*) ≤ *length d*
    by *simp*
  with *d* **have** *r*: *r* ≠ []
    using *Suc-leI length-greater-0-conv list.size*(*3*) **by** *fastforce*
  then obtain *rr lcr* **where** *r*: *r* = *rr* @ [*lcr*]
    by (*cases r rule*: *rev-cases*) *auto*
  from *d lc* **obtain** *dd* **where** *d*: *d* = *dd* @ [*lc*]
    by (*cases d rule*: *rev-cases*) *auto*
  from *Suc*(*2*) *ifCond* **have** *n*: *n* = *1* + *length rr* − *length d*
    by (*auto simp*: *r*)
  from *ifCond* **have** *len*: *length dd* ≤ *length rr*
    by (*simp add*: *r d*)
  show *?case*
  proof (*cases lcr div lc* ∗ *lc* = *lcr*)
    case *False*
    with *r d* **show** *?thesis*
      unfolding *Suc*(*2*)[*symmetric*]
      by (*auto simp add*: *Let-def nth-default-append*)
  next
    case *True*
    with *r d* **have** *id*:
      ?thesis ⟷
        Poly (*divide-poly-main-list lc* (*cCons* (*lcr div lc*) *q*)
          (*rev* (*rev* (*minus-poly-rev-list* (*rev rr*) (*rev* (*map* ((∗) (*lcr div lc*)) *dd*)))))
(*rev d*) *n*) =
          divide-poly-main lc
            (*monom 1* (*Suc n*) ∗ *Poly q* + *monom* (*lcr div lc*) *n*)
            (*Poly r* − *monom* (*lcr div lc*) *n* ∗ *Poly d*)
            (*Poly d*) (*length rr* − *1*) *n*
      by (*cases r rule*: *rev-cases*; *cases d rule*: *rev-cases*)
        (*auto simp add*: *Let-def rev-map nth-default-append*)
    have *cong*: ⋀*x1 x2 x3 x4 y1 y2 y3 y4*. *x1* = *y1* ⟹ *x2* = *y2* ⟹ *x3* = *y3* ⟹
x4 = *y4* ⟹
        divide-poly-main lc x1 x2 x3 x4 n = *divide-poly-main lc y1 y2 y3 y4 n*
      by *simp*
    show *?thesis*
      unfolding *id*
    proof (*subst Suc*(*1*), *simp add*: *n*,
        subst minus-poly-rev-list, *force simp*: *len*, *rule cong*[*OF - - refl*], *goal-cases*)

case *2*
have *monom lcr* (*length rr*) = *monom* (*lcr div lc*) (*length rr* − *length dd*) ∗ *monom lc* (*length dd*)
by (*simp add*: *mult-monom len True*)
then show *?case* **unfolding** *r d Poly-append n ring-distribs*
by (*auto simp*: *Poly-map smult-monom smult-monom-mult*)
qed (*auto simp*: *len monom-Suc smult-monom*)
qed
qed *simp*

lemma *divide-poly-list*[*code*]: *f div g* = *divide-poly-list f g*
proof −
note *d* = *divide-poly-def divide-poly-list-def*
show *?thesis*
proof (*cases g* = *0*)
case *True*
show *?thesis* **by** (*auto simp*: *d True*)
next
case *False*
then obtain *cg lcg* **where** *cg*: *coeffs g* = *cg* @ [*lcg*]
by (*cases coeffs g rule*: *rev-cases*) *auto*
with *False* **have** *id*: (*g* = *0*) = *False* (*cg* @ [*lcg*] = []) = *False*
by *auto*
from *cg False* **have** *lcg*: *coeff g* (*degree g*) = *lcg*
using *last-coeffs-eq-coeff-degree last-snoc* **by** *force*
with *False* **have** *lcg* ≠ *0* **by** *auto*
from *cg Poly-coeffs* [*of g*] **have** *ltp*: *Poly* (*cg* @ [*lcg*]) = *g*
by *auto*
show *?thesis*
unfolding *d cg Let-def id if-False poly-of-list-def*
by (*subst divide-poly-main-list*, *insert False cg* ‹*lcg* ≠ *0*›)
(*auto simp*: *lcg ltp*, *simp add*: *degree-eq-length-coeffs*)
qed
qed

4.31 Primality and irreducibility in polynomial rings

lemma *prod-mset-const-poly*: $(\prod x{\in}\#A.\ [{:}f\ x{:}]) = [{:}prod\text{-}mset\ (image\text{-}mset\ f\ A){:}]$
by (*induct A*) (*simp-all add*: *ac-simps*)

lemma *irreducible-const-poly-iff*:
fixes *c* :: *'a* :: {*comm-semiring-1*,*semiring-no-zero-divisors*}
shows *irreducible* [:*c*:] ⟷ *irreducible c*
proof
assume *A*: *irreducible c*
show *irreducible* [:*c*:]
proof (*rule irreducibleI*)
fix *a b* **assume** *ab*: [:*c*:] = *a* ∗ *b*
hence *degree* [:*c*:] = *degree* (*a* ∗ *b*) **by** (*simp only*:)

```
    also from A ab have a ≠ 0 b ≠ 0 by auto
    hence degree (a * b) = degree a + degree b by (simp add: degree-mult-eq)
    finally have degree a = 0 degree b = 0 by auto
   then obtain a' b' where ab': a = [:a':] b = [:b':] by (auto elim!: degree-eq-zeroE)
    from ab have coeff [:c:] 0 = coeff (a * b) 0 by (simp only: )
    hence c = a' * b' by (simp add: ab' mult-ac)
    from A and this have a' dvd 1 ∨ b' dvd 1 by (rule irreducibleD)
    with ab' show a dvd 1 ∨ b dvd 1
      by (auto simp add: is-unit-const-poly-iff)
  qed (insert A, auto simp: irreducible-def is-unit-poly-iff)
next
  assume A: irreducible [:c:]
  then have c ≠ 0 and ¬ c dvd 1
    by (auto simp add: irreducible-def is-unit-const-poly-iff)
  then show irreducible c
  proof (rule irreducibleI)
    fix a b assume ab: c = a * b
    hence [:c:] = [:a:] * [:b:] by (simp add: mult-ac)
    from A and this have [:a:] dvd 1 ∨ [:b:] dvd 1 by (rule irreducibleD)
    then show a dvd 1 ∨ b dvd 1
      by (auto simp add: is-unit-const-poly-iff)
  qed
qed

lemma lift-prime-elem-poly:
  assumes prime-elem (c :: 'a :: semidom)
  shows   prime-elem [:c:]
proof (rule prime-elemI)
  fix a b assume *: [:c:] dvd a * b
  from * have dvd: c dvd coeff (a * b) n for n
    by (subst (asm) const-poly-dvd-iff) blast
  {
    define m where m = (GREATEST m. ¬c dvd coeff b m)
    assume ¬[:c:] dvd b
    hence A: ∃ i. ¬c dvd coeff b i by (subst (asm) const-poly-dvd-iff) blast
    have B: ⋀i. ¬c dvd coeff b i ⟹ i ≤ degree b
      by (auto intro: le-degree)
   have coeff-m: ¬c dvd coeff b m unfolding m-def by (rule GreatestI-ex-nat[OF
A B])
    have i ≤ m if ¬c dvd coeff b i for i
      unfolding m-def by (metis (mono-tags, lifting) B Greatest-le-nat that)
    hence dvd-b: c dvd coeff b i if i > m for i using that by force

    have c dvd coeff a i for i
    proof (induction i rule: nat-descend-induct[of degree a])
      case (base i)
      thus ?case by (simp add: coeff-eq-0)
    next
      case (descend i)
```

```
      let ?A = {..i+m} − {i}
      have c dvd coeff (a ∗ b) (i + m) by (rule dvd)
      also have coeff (a ∗ b) (i + m) = (∑ k≤i + m. coeff a k ∗ coeff b (i + m
− k))
        by (simp add: coeff-mult)
      also have {..i+m} = insert i ?A by auto
      also have (∑ k∈. . . . coeff a k ∗ coeff b (i + m − k)) =
                coeff a i ∗ coeff b m + (∑ k∈?A. coeff a k ∗ coeff b (i + m − k))
        (is - = - + ?S)
        by (subst sum.insert) simp-all
      finally have eq: c dvd coeff a i ∗ coeff b m + ?S .
      moreover have c dvd ?S
      proof (rule dvd-sum)
        fix k assume k: k ∈ {..i+m} − {i}
        show c dvd coeff a k ∗ coeff b (i + m − k)
        proof (cases k < i)
          case False
          with k have c dvd coeff a k by (intro descend.IH) simp
          thus ?thesis by simp
        next
          case True
          hence c dvd coeff b (i + m − k) by (intro dvd-b) simp
          thus ?thesis by simp
        qed
      qed
      ultimately have c dvd coeff a i ∗ coeff b m
        by (simp add: dvd-add-left-iff)
      with assms coeff-m show c dvd coeff a i
        by (simp add: prime-elem-dvd-mult-iff)
    qed
    hence [:c:] dvd a by (subst const-poly-dvd-iff) blast
  }
  then show [:c:] dvd a ∨ [:c:] dvd b by blast
next
  from assms show [:c:] ≠ 0 and ¬ [:c:] dvd 1
    by (simp-all add: prime-elem-def is-unit-const-poly-iff)
qed

lemma prime-elem-const-poly-iff:
  fixes c :: 'a :: semidom
  shows   prime-elem [:c:] ⟷ prime-elem c
proof
  assume A: prime-elem [:c:]
  show prime-elem c
  proof (rule prime-elemI)
    fix a b assume c dvd a ∗ b
    hence [:c:] dvd [:a:] ∗ [:b:] by (simp add: mult-ac)
    from A and this have [:c:] dvd [:a:] ∨ [:c:] dvd [:b:] by (rule prime-elem-dvd-multD)
    thus c dvd a ∨ c dvd b by simp
```

qed (*insert A*, *auto simp*: *prime-elem-def is-unit-poly-iff*)
qed (*auto intro*: *lift-prime-elem-poly*)

4.32 Content and primitive part of a polynomial

definition *content* :: *'a::semiring-gcd poly* ⇒ *'a*
where *content p* = *gcd-list* (*coeffs p*)

lemma *content-eq-fold-coeffs* [*code*]: *content p* = *fold-coeffs gcd p 0*
by (*simp add*: *content-def Gcd-fin.set-eq-fold fold-coeffs-def foldr-fold fun-eq-iff ac-simps*)

lemma *content-0* [*simp*]: *content 0* = *0*
by (*simp add*: *content-def*)

lemma *content-1* [*simp*]: *content 1* = *1*
by (*simp add*: *content-def*)

lemma *content-const* [*simp*]: *content* [:*c*:] = *normalize c*
by (*simp add*: *content-def cCons-def*)

lemma *const-poly-dvd-iff-dvd-content*: [:*c*:] *dvd p* ⟷ *c dvd content p*
for *c* :: *'a::semiring-gcd*
proof (*cases p* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
have [:*c*:] *dvd p* ⟷ (∀ *n*. *c dvd coeff p n*)
by (*rule const-poly-dvd-iff*)
also have ... ⟷ (∀ *a*∈*set* (*coeffs p*). *c dvd a*)
proof *safe*
fix *n* :: *nat*
assume ∀ *a*∈*set* (*coeffs p*). *c dvd a*
then show *c dvd coeff p n*
by (*cases n* ≤ *degree p*) (*auto simp*: *coeff-eq-0 coeffs-def split*: *if-splits*)
qed (*auto simp*: *coeffs-def simp del*: *upt-Suc split*: *if-splits*)
also have ... ⟷ *c dvd content p*
by (*simp add*: *content-def dvd-Gcd-fin-iff dvd-mult-unit-iff*)
finally show *?thesis* **.**
qed

lemma *content-dvd* [*simp*]: [:*content p*:] *dvd p*
by (*subst const-poly-dvd-iff-dvd-content*) *simp-all*

lemma *content-dvd-coeff* [*simp*]: *content p dvd coeff p n*
proof (*cases p* = *0*)
case *True*
then show *?thesis*

by *simp*
next
case *False*
then show *?thesis*
by (*cases n ≤ degree p*)
(*auto simp add*: *content-def not-le coeff-eq-0 coeff-in-coeffs intro*: *Gcd-fin-dvd*)
qed

lemma *content-dvd-coeffs*: *c* ∈ *set* (*coeffs p*) $\Longrightarrow$ *content p dvd c*
by (*simp add*: *content-def Gcd-fin-dvd*)

lemma *normalize-content* [*simp*]: *normalize* (*content p*) = *content p*
by (*simp add*: *content-def*)

lemma *is-unit-content-iff* [*simp*]: *is-unit* (*content p*) ⟷ *content p* = *1*
proof
assume *is-unit* (*content p*)
then have *normalize* (*content p*) = *1* **by** (*simp add*: *is-unit-normalize del*: *normalize-content*)
then show *content p* = *1* **by** *simp*
qed *auto*

lemma *content-smult* [*simp*]:
fixes *c* :: *'a* :: {*normalization-semidom-multiplicative*, *semiring-gcd*}
shows *content* (*smult c p*) = *normalize c* ∗ *content p*
by (*simp add*: *content-def coeffs-smult Gcd-fin-mult normalize-mult*)

lemma *content-eq-zero-iff* [*simp*]: *content p* = *0* ⟷ *p* = *0*
by (*auto simp*: *content-def simp*: *poly-eq-iff coeffs-def*)

definition *primitive-part* :: *'a* :: *semiring-gcd poly* ⇒ *'a poly*
where *primitive-part p* = *map-poly* (*λx. x div content p*) *p*

lemma *primitive-part-0* [*simp*]: *primitive-part 0* = *0*
by (*simp add*: *primitive-part-def*)

lemma *content-times-primitive-part* [*simp*]: *smult* (*content p*) (*primitive-part p*) = *p*
for *p* :: *'a* :: *semiring-gcd poly*
proof (*cases p* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then show *?thesis*
unfolding *primitive-part-def*
by (*auto simp*: *smult-conv-map-poly map-poly-map-poly o-def content-dvd-coeffs intro*: *map-poly-idI*)
qed

lemma *primitive-part-eq-0-iff* [*simp*]: *primitive-part p* = *0* ⟷ *p* = *0*
proof (*cases p* = *0*)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  then have *primitive-part p* = *map-poly* (*λx. x div content p*) *p*
    by (*simp add*: *primitive-part-def*)
  also from *False* **have** ... = *0* ⟷ *p* = *0*
    by (*intro map-poly-eq-0-iff*) (*auto simp*: *dvd-div-eq-0-iff content-dvd-coeffs*)
  finally show *?thesis*
    using *False* **by** *simp*
qed

lemma *content-primitive-part* [*simp*]:
  fixes *p* :: *'a* :: {*normalization-semidom-multiplicative*, *semiring-gcd*} *poly*
  assumes *p* ≠ *0*
  shows *content* (*primitive-part p*) = *1*
proof −
  have *p* = *smult* (*content p*) (*primitive-part p*)
    by *simp*
  also have *content* ... = *content* (*primitive-part p*) ∗ *content p*
    by (*simp del*: *content-times-primitive-part add*: *ac-simps*)
  finally have *1* ∗ *content p* = *content* (*primitive-part p*) ∗ *content p*
    by *simp*
  then have *1* ∗ *content p div content p* = *content* (*primitive-part p*) ∗ *content p div content p*
    by *simp*
  with *assms* **show** *?thesis*
    by *simp*
qed

lemma *content-decompose*:
  obtains *p′* :: *'a* :: {*normalization-semidom-multiplicative*, *semiring-gcd*} *poly*
  where *p* = *smult* (*content p*) *p′ content p′* = *1*
proof (*cases p* = *0*)
  case *True*
  then have *p* = *smult* (*content p*) *1 content 1* = *1*
    by *simp-all*
  then show *?thesis* **..**
next
  case *False*
  then have *p* = *smult* (*content p*) (*primitive-part p*) *content* (*primitive-part p*) = *1*
    by *simp-all*
  then show *?thesis* **..**
qed

lemma *content-dvd-contentI* [*intro*]: *p dvd q* $\Longrightarrow$ *content p dvd content q*
  using *const-poly-dvd-iff-dvd-content content-dvd dvd-trans* **by** *blast*

lemma *primitive-part-const-poly* [*simp*]: *primitive-part* [:*x*:] = [:*unit-factor x*:]
  by (*simp add*: *primitive-part-def map-poly-pCons*)

lemma *primitive-part-prim*: *content p* = *1* $\Longrightarrow$ *primitive-part p* = *p*
  by (*auto simp*: *primitive-part-def*)

lemma *degree-primitive-part* [*simp*]: *degree* (*primitive-part p*) = *degree p*
proof (*cases p* = *0*)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  have *p* = *smult* (*content p*) (*primitive-part p*)
    by *simp*
  also from *False* **have** *degree* ... = *degree* (*primitive-part p*)
    by (*subst degree-smult-eq*) *simp-all*
  finally show *?thesis* **..**
qed

lemma *smult-content-normalize-primitive-part* [*simp*]:
  fixes *p* :: *'a* :: {*normalization-semidom-multiplicative*, *semiring-gcd*, *idom-divide*} *poly*
  shows *smult* (*content p*) (*normalize* (*primitive-part p*)) = *normalize p*
proof −
  have *smult* (*content p*) (*normalize* (*primitive-part p*)) =
      normalize ([:*content p*:] ∗ *primitive-part p*)
    by (*subst normalize-mult*) (*simp-all add*: *normalize-const-poly*)
  also have [:*content p*:] ∗ *primitive-part p* = *p* **by** *simp*
  finally show *?thesis* **.**
qed

context
begin

private

lemma *content-1-mult*:
  fixes *f g* :: *'a* :: {*semiring-gcd*, *factorial-semiring*} *poly*
  assumes *content f* = *1 content g* = *1*
  shows *content* (*f* ∗ *g*) = *1*
proof (*cases f* ∗ *g* = *0*)
  case *False*
  from *assms* **have** *f* ≠ *0 g* ≠ *0* **by** *auto*

  hence *f* ∗ *g* ≠ *0* **by** *auto*
  {

assume ¬*is-unit* (*content* (*f* ∗ *g*))
with *False* **have** ∃ *p*. *p dvd content* (*f* ∗ *g*) ∧ *prime p*
by (*intro prime-divisor-exists*) *simp-all*
then obtain *p* **where** *p dvd content* (*f* ∗ *g*) *prime p* **by** *blast*
from ‹*p dvd content* (*f* ∗ *g*)› **have** [:*p*:] *dvd f* ∗ *g*
by (*simp add*: *const-poly-dvd-iff-dvd-content*)
moreover from ‹*prime p*› **have** *prime-elem* [:*p*:] **by** (*simp add*: *lift-prime-elem-poly*)
ultimately have [:*p*:] *dvd f* ∨ [:*p*:] *dvd g*
by (*simp add*: *prime-elem-dvd-mult-iff*)
with *assms* **have** *is-unit p* **by** (*simp add*: *const-poly-dvd-iff-dvd-content*)
with ‹*prime p*› **have** *False* **by** *simp*
}
hence *is-unit* (*content* (*f* ∗ *g*)) **by** *blast*
hence *normalize* (*content* (*f* ∗ *g*)) = *1* **by** (*simp add*: *is-unit-normalize del*: *normalize-content*)
thus *?thesis* **by** *simp*
qed (*insert assms*, *auto*)

lemma *content-mult*:
fixes *p q* :: *'a* :: {*factorial-semiring*, *semiring-gcd*, *normalization-semidom-multiplicative*} *poly*
shows *content* (*p* ∗ *q*) = *content p* ∗ *content q*
proof (*cases p* ∗ *q* = *0*)
case *False*
then have *p* ≠ *0* **and** *q* ≠ *0*
by *simp-all*
then have ∗: *content* (*primitive-part p* ∗ *primitive-part q*) = *1*
by (*auto intro*: *content-1-mult*)
have *p* ∗ *q* = *smult* (*content p*) (*primitive-part p*) ∗ *smult* (*content q*) (*primitive-part q*)
by *simp*
also have ... = *smult* (*content p* ∗ *content q*) (*primitive-part p* ∗ *primitive-part q*)
by (*metis mult.commute mult-smult-right smult-smult*)
with ∗ **show** *?thesis*
by (*simp add*: *normalize-mult*)
next
case *True*
then show *?thesis*
by *auto*
qed

end

lemma *primitive-part-mult*:
fixes *p q* :: *'a* :: {*factorial-semiring*, *semiring-Gcd*, *ring-gcd*, *idom-divide*, *normalization-semidom-multiplicative*} *poly*
shows *primitive-part* (*p* ∗ *q*) = *primitive-part p* ∗ *primitive-part q*
proof −

have *primitive-part* (*p* ∗ *q*) = *p* ∗ *q* *div* [:*content* (*p* ∗ *q*):]
by (*simp add*: *primitive-part-def div-const-poly-conv-map-poly*)
also have ... = (*p* *div* [:*content* *p*:]) ∗ (*q* *div* [:*content* *q*:])
by (*subst div-mult-div-if-dvd*) (*simp-all add*: *content-mult mult-ac*)
also have ... = *primitive-part p* ∗ *primitive-part q*
by (*simp add*: *primitive-part-def div-const-poly-conv-map-poly*)
finally show *?thesis* **.**
qed

lemma *primitive-part-smult*:
fixes *p* :: *'a* :: {*factorial-semiring*, *semiring-Gcd*, *ring-gcd*, *idom-divide*, *normalization-semidom-multiplicative*} *poly*
shows *primitive-part* (*smult a p*) = *smult* (*unit-factor a*) (*primitive-part p*)
proof −
have *smult a p* = [:*a*:] ∗ *p* **by** *simp*
also have *primitive-part* ... = *smult* (*unit-factor a*) (*primitive-part p*)
by (*subst primitive-part-mult*) *simp-all*
finally show *?thesis* **.**
qed

lemma *primitive-part-dvd-primitive-partI* [*intro*]:
fixes *p q* :: *'a* :: {*factorial-semiring*, *semiring-Gcd*, *ring-gcd*, *idom-divide*, *normalization-semidom-multiplicative*} *poly*
shows *p dvd q* ⟹ *primitive-part p dvd primitive-part q*
by (*auto elim*!: *dvdE simp*: *primitive-part-mult*)

lemma *content-prod-mset*:
fixes *A* :: *'a* :: {*factorial-semiring*, *semiring-Gcd*, *normalization-semidom-multiplicative*} *poly multiset*
shows *content* (*prod-mset A*) = *prod-mset* (*image-mset content A*)
by (*induction A*) (*simp-all add*: *content-mult mult-ac*)

lemma *content-prod-eq-1-iff*:
fixes *p q* :: *'a* :: {*factorial-semiring*, *semiring-Gcd*, *normalization-semidom-multiplicative*} *poly*
shows *content* (*p* ∗ *q*) = *1* ⟷ *content p* = *1* ∧ *content q* = *1*
proof *safe*
assume *A*: *content* (*p* ∗ *q*) = *1*
{
fix *p q* :: *'a poly* **assume** *content p* ∗ *content q* = *1*
hence *1* = *content p* ∗ *content q* **by** *simp*
hence *content p dvd 1* **by** (*rule dvdI*)
hence *content p* = *1* **by** *simp*
} **note** *B* = *this*
from *A B*[*of p q*] *B* [*of q p*] **show** *content p* = *1* *content q* = *1*
by (*simp-all add*: *content-mult mult-ac*)
qed (*auto simp*: *content-mult*)

4.33 A typeclass for algebraically closed fields

Since the required sort constraints are not available inside the class, we have to resort to a somewhat awkward way of writing the definition of algebraically closed fields:

class *alg-closed-field* = *field* +
  assumes *alg-closed*: $n > 0 \Longrightarrow f\ n \neq 0 \Longrightarrow \exists\, x.\ (\sum k{\le}n.\ f\ k * x\ \hat{}\ k) = 0$

We can then however easily show the equivalence to the proper definition:

lemma *alg-closed-imp-poly-has-root*:
  assumes $degree\ (p :: 'a :: alg\text{-}closed\text{-}field\ poly) > 0$
  shows $\exists\, x.\ poly\ p\ x = 0$
proof −
  have $\exists\, x.\ (\sum k{\le}degree\ p.\ coeff\ p\ k * x\ \hat{}\ k) = 0$
    using *assms* **by** (*intro alg-closed*) *auto*
  thus *?thesis*
    by (*simp add*: *poly-altdef*)
qed

lemma *alg-closedI* [*Pure.intro*]:
  assumes $\bigwedge p :: 'a\ poly.\ degree\ p > 0 \Longrightarrow lead\text{-}coeff\ p = 1 \Longrightarrow \exists\, x.\ poly\ p\ x = 0$
  shows *OFCLASS*(*'a* :: *field*, *alg-closed-field-class*)
proof
  fix $n :: nat$ **and** $f :: nat \Rightarrow 'a$
  assume n: $n > 0\ f\ n \neq 0$
  define p **where** $p = Abs\text{-}poly\ (\lambda k.\ if\ k \le n\ then\ f\ k\ else\ 0)$
  have *coeff-p*: $coeff\ p\ k = (if\ k \le n\ then\ f\ k\ else\ 0)$ **for** k
  proof −
    have $eventually\ (\lambda k.\ k > n)\ cofinite$
      by (*auto simp*: *MOST-nat*)
    hence $eventually\ (\lambda k.\ (if\ k \le n\ then\ f\ k\ else\ 0) = 0)\ cofinite$
      by *eventually-elim auto*
    thus *?thesis*
      unfolding *p-def* **by** (*subst Abs-poly-inverse*) *auto*
  qed

  from n **have** $degree\ p \ge n$
    by (*intro le-degree*) (*auto simp*: *coeff-p*)
  moreover have $degree\ p \le n$
    by (*intro degree-le*) (*auto simp*: *coeff-p*)
  ultimately have *deg-p*: $degree\ p = n$
    by *linarith*
  from *deg-p* **and** n **have** [*simp*]: $p \neq 0$
    by *auto*

  define p' **where** $p' = smult\ (inverse\ (lead\text{-}coeff\ p))\ p$
  have *deg-p'*: $degree\ p' = degree\ p$
    by (*auto simp*: *p'-def*)
  have *lead-coeff-p'* [*simp*]: $lead\text{-}coeff\ p' = 1$

```
    by (auto simp: p′-def)

  from deg-p and deg-p′ and n have degree p′ > 0
    by simp
  from assms[OF this] obtain x where poly p′ x = 0
    by auto
  hence poly p x = 0
    by (simp add: p′-def)
  also have poly p x = (∑ k≤n. f k ∗ x ^ k)
    unfolding poly-altdef by (intro sum.cong) (auto simp: deg-p coeff-p)
  finally show ∃ x. (∑ k≤n. f k ∗ x ^ k) = 0 ..
qed

lemma (in alg-closed-field) nth-root-exists:
  assumes n > 0
  shows   ∃ y. y ^ n = (x :: ′a)
proof −
  define f where f = (λi. if i = 0 then −x else if i = n then 1 else 0)
  have ∃ x. (∑ k≤n. f k ∗ x ^ k) = 0
    by (rule alg-closed) (use assms in ‹auto simp: f-def›)
  also have (λx. ∑ k≤n. f k ∗ x ^ k) = (λx. ∑ k∈{0,n}. f k ∗ x ^ k)
    by (intro ext sum.mono-neutral-right) (auto simp: f-def)
  finally show ∃ y. y ^ n = x
    using assms by (simp add: f-def)
qed
```

We can now prove by induction that every polynomial of degree n splits into a product of n linear factors:

```
lemma alg-closed-imp-factorization:
  fixes p :: ′a :: alg-closed-field poly
  assumes p ≠ 0
  shows ∃ A. size A = degree p ∧ p = smult (lead-coeff p) (∏ x∈#A. [:−x, 1:])
  using assms
proof (induction degree p arbitrary: p rule: less-induct)
  case (less p)
  show ?case
  proof (cases degree p = 0)
    case True
    thus ?thesis
      by (intro exI[of - {#}]) (auto elim!: degree-eq-zeroE)
  next
    case False
    then obtain x where x: poly p x = 0
      using alg-closed-imp-poly-has-root by blast
    hence [:−x, 1:] dvd p
      using poly-eq-0-iff-dvd by blast
    then obtain q where p-eq: p = [:−x, 1:] ∗ q
      by (elim dvdE)
    have q ≠ 0
```

```
      using less.prems p-eq by auto
    moreover from this have deg: degree p = Suc (degree q)
      unfolding p-eq by (subst degree-mult-eq) auto
    ultimately obtain A where A: size A = degree q q = smult (lead-coeff q)
(∏ x∈#A. [:−x, 1:])
      using less.hyps[of q] by auto
    have smult (lead-coeff p) (∏ y∈#add-mset x A. [:− y, 1:]) =
          [:− x, 1:] ∗ smult (lead-coeff q) (∏ y∈#A. [:− y, 1:])
      unfolding p-eq lead-coeff-mult by simp
    also note A(2) [symmetric]
    also note p-eq [symmetric]
    finally show ?thesis using A(1)
      by (intro exI[of - add-mset x A]) (auto simp: deg)
  qed
qed
```

As an alternative characterisation of algebraic closure, one can also say that any polynomial of degree at least 2 splits into non-constant factors:

```
lemma alg-closed-imp-reducible:
  assumes degree (p :: 'a :: alg-closed-field poly) > 1
  shows   ¬irreducible p
proof −
  have degree p > 0
    using assms by auto
  then obtain z where z: poly p z = 0
    using alg-closed-imp-poly-has-root[of p] by blast
  then have dvd: [:−z, 1:] dvd p
    by (subst dvd-iff-poly-eq-0) auto
  then obtain q where q: p = [:−z, 1:] ∗ q
    by (erule dvdE)
  have [simp]: q ≠ 0
    using assms q by auto

  show ?thesis
  proof (rule reducible-polyI)
    show p = [:−z, 1:] ∗ q
      by fact
  next
    have degree p = degree ([:−z, 1:] ∗ q)
      by (simp only: q)
    also have ... = degree q + 1
      by (subst degree-mult-eq) auto
    finally show degree q > 0
      using assms by linarith
  qed auto
qed
```

When proving algebraic closure through reducibility, we can assume w.l.o.g. that the polynomial is monic and has a non-zero constant coefficient:

lemma *alg-closedI-reducible*:
  assumes $\bigwedge p$ :: *'a poly. degree p > 1* $\Longrightarrow$ *lead-coeff p = 1* $\Longrightarrow$ *coeff p 0* $\neq$ *0* $\Longrightarrow$
        ¬*irreducible p*
  shows *OFCLASS('a :: field, alg-closed-field-class)*
proof
  fix *p* :: *'a poly* **assume** *p*: *degree p > 0 lead-coeff p = 1*
  show ∃ *x. poly p x = 0*
  proof (*cases coeff p 0 = 0*)
    case *True*
    hence *poly p 0 = 0*
      by (*simp add*: *poly-0-coeff-0*)
    thus *?thesis* **by** *blast*
  next
    case *False*
    from *p* **and** *this* **show** *?thesis*
    proof (*induction degree p arbitrary*: *p rule*: *less-induct*)
      case (*less p*)
      show *?case*
      proof (*cases degree p = 1*)
        case *True*
        then obtain *a b* **where** *p*: *p* = [:*a*, *b*:]
          by (*cases p*) (*auto split*: *if-splits elim*!: *degree-eq-zeroE*)
        from *True* **have** [*simp*]: *b* ≠ *0*
          by (*auto simp*: *p*)
        have *poly p* (−*a*/*b*) = *0*
          by (*auto simp*: *p*)
        thus *?thesis* **by** *blast*
      next
        case *False*
        hence *degree p > 1*
          using *less.prems* **by** *auto*
        from *assms*[*OF* ‹*degree p > 1*› ‹*lead-coeff p = 1*› ‹*coeff p 0* ≠ *0*›]
        have ¬*irreducible p* **by** *auto*
        then obtain *r s* **where** *rs*: *degree r > 0 degree s > 0 p = r* ∗ *s*
          using *less.prems* **unfolding** *irreducible-def*
          by (*metis is-unit-iff-degree mult-not-zero zero-less-iff-neq-zero*)
        hence *coeff r 0* ≠ *0*
          using ‹*coeff p 0* ≠ *0*› **by** (*auto simp*: *coeff-mult-0*)

        define *r'* **where** *r'* = *smult* (*inverse* (*lead-coeff r*)) *r*
        have [*simp*]: *degree r'* = *degree r*
          by (*simp add*: *r'-def*)
        have *lc*: *lead-coeff r'* = *1*
          using *rs* **by** (*auto simp*: *r'-def*)
        have *nz*: *coeff r' 0* ≠ *0*
          using ‹*coeff r 0* ≠ *0*› **by** (*auto simp*: *r'-def*)

        have *degree r < degree r + degree s*
          using *rs* **by** *linarith*

also have ... = *degree* (*r* ∗ *s*)
  using *rs*(*3*) *less.prems* **by** (*subst degree-mult-eq*) *auto*
also have *r* ∗ *s* = *p*
  using *rs*(*3*) **by** *simp*
finally have ∃ *x*. *poly r′ x* = *0*
  by (*intro less*) (*use lc rs nz* **in** *auto*)
thus *?thesis*
  using *rs*(*3*) **by** (*auto simp*: *r′-def*)
qed
qed
qed
qed

Using a clever Tschirnhausen transformation mentioned e.g. in the article by Nowak [1], we can also assume w.l.o.g. that the coefficient a_{n-1} is zero.

lemma *alg-closedI-reducible-coeff-deg-minus-one-eq-0*:
  assumes ⋀*p* :: ′*a poly*. *degree p* > *1* ⟹ *lead-coeff p* = *1* ⟹ *coeff p* (*degree p* − *1*) = *0* ⟹
      coeff p 0 ≠ *0* ⟹ ¬*irreducible p*
  shows *OFCLASS*(′*a* :: *field-char-0*, *alg-closed-field-class*)
proof (*rule alg-closedI-reducible*, *goal-cases*)
  case (*1 p*)
  define *n* **where** [*simp*]: *n* = *degree p*
  define *a* **where** *a* = *coeff p* (*n* − *1*)
  define *r* **where** *r* = [: −*a* / *of-nat n*, *1* :]
  define *s* **where** *s* = [: *a* / *of-nat n*, *1* :]
  define *q* **where** *q* = *pcompose p r*

  have *n* > *0*
    using *1* **by** *simp*
  have *r-altdef*: *r* = *monom 1 1* + [:−*a* / *of-nat n*:]
    by (*simp add*: *r-def monom-altdef*)
  have *deg-q*: *degree q* = *n*
    by (*simp add*: *q-def r-def degree-pcompose*)
  have *lc-q*: *lead-coeff q* = *1*
    unfolding *q-def* **using** *1* **by** (*subst lead-coeff-comp*) (*simp-all add*: *r-def*)
  have *q* ≠ *0*
    using *1 deg-q* **by** *auto*

  have *coeff q* (*n* − *1*) =
      (∑ *i*≤*n*. ∑ *k*≤*i*. *coeff p i* ∗ (*of-nat* (*i choose k*) ∗
        ((−*a* / *of-nat n*) ^ (*i* − *k*) ∗ (*if k* = *n* − *1 then 1 else 0*))))
    unfolding *q-def pcompose-altdef poly-altdef r-altdef*
    by (*simp-all add*: *degree-map-poly coeff-map-poly coeff-sum binomial-ring sum-distrib-left poly-const-pow*
        sum-distrib-right mult-ac monom-power coeff-monom-mult of-nat-poly
  cong: *if-cong*)
  also have ... = (∑ *i*≤*n*. ∑ *k*∈(*if i* ≥ *n* − *1 then* {*n*−*1*} *else* {}).
        coeff p i ∗ (*of-nat* (*i choose k*) ∗ (−*a* / *of-nat n*) ^ (*i* − *k*)))

by (*rule sum.cong* [*OF refl*], *rule sum.mono-neutral-cong-right*) (*auto split*: *if-splits*)
also have ... = ($\sum$ *i*∈{*n*−*1*,*n*}. $\sum$ *k*∈(*if i* ≥ *n* − *1 then* {*n*−*1*} *else* {}).
coeff p i ∗ (*of-nat* (*i choose k*) ∗ (−*a* / *of-nat n*) ^ (*i* − *k*)))
by (*rule sum.mono-neutral-right*) *auto*
also have ... = *a* − *of-nat* (*n choose* (*n* − *1*)) ∗ *a* / *of-nat n*
using *1* **by** (*simp add*: *a-def*)
also have *n choose* (*n* − *1*) = *n*
using ‹*n* > *0*› **by** (*subst binomial-symmetric*) *auto*
also have *a* − *of-nat n* ∗ *a* / *of-nat n* = *0*
using ‹*n* > *0*› **by** *simp*
finally have *coeff q* (*n* − *1*) = *0* **.**

show *?case*
proof (*cases coeff q 0* = *0*)
case *True*
hence *poly p* (− (*a* / *of-nat* (*degree p*))) = *0*
by (*auto simp*: *q-def r-def*)
thus *?thesis*
by (*rule root-imp-reducible-poly*) (*use 1* **in** *auto*)
next
case *False*
hence ¬*irreducible q*
using *assms*[*of q*] **and** *lc-q* **and** *1* **and** ‹*coeff q* (*n* − *1*) = *0*›
by (*auto simp*: *deg-q*)
then obtain *u v* **where** *uv*: *degree u* > *0 degree v* > *0 q* = *u* ∗ *v*
using ‹*q* ≠ *0*› *1 deg-q* **unfolding** *irreducible-def*
by (*metis degree-mult-eq-0 is-unit-iff-degree n-def neq0-conv not-one-less-zero*)

have *p* = *pcompose q s*
by (*simp add*: *q-def r-def s-def pcompose-pCons flip*: *pcompose-assoc*)
also have *q* = *u* ∗ *v*
by *fact*
finally have *p* = *pcompose u s* ∗ *pcompose v s*
by (*simp add*: *pcompose-mult*)
moreover have *degree* (*pcompose u s*) > *0 degree* (*pcompose v s*) > *0*
using *uv* **by** (*simp-all add*: *s-def degree-pcompose*)
ultimately show ¬*irreducible p*
using *1* **by** (*intro reducible-polyI*)
qed
qed

As a consequence of the full factorisation lemma proven above, we can also show that any polynomial with at least two different roots splits into two non-constant coprime factors:

lemma *alg-closed-imp-poly-splits-coprime*:
assumes *degree* (*p* :: ′*a* :: {*alg-closed-field*} *poly*) > *1*
assumes *poly p x* = *0 poly p y* = *0 x* ≠ *y*
obtains *r s* **where** *degree r* > *0 degree s* > *0 coprime r s p* = *r* ∗ *s*

```
proof -
  define n where n = order x p
  have n > 0
    using assms by (metis degree-0 gr0I n-def not-one-less-zero order-root)
  have [:-x, 1:] ^ n dvd p
    unfolding n-def by (simp add: order-1)
  then obtain q where p-eq: p = [:-x, 1:] ^ n * q
    by (elim dvdE)
  from assms have [simp]: q ≠ 0
    by (auto simp: p-eq)
  have order x p = n + Polynomial.order x q
    unfolding p-eq by (subst order-mult) (auto simp: order-power-n-n)
  hence Polynomial.order x q = 0
    by (simp add: n-def)
  hence poly q x ≠ 0
    by (simp add: order-root)

  show ?thesis
  proof (rule that)
    show coprime ([:-x, 1:] ^ n) q
    proof (rule coprimeI)
      fix d
      assume d: d dvd [:-x, 1:] ^ n d dvd q
      have degree d = 0
      proof (rule ccontr)
        assume ¬(degree d = 0)
        then obtain z where z: poly d z = 0
          using alg-closed-imp-poly-has-root by blast
        moreover from this and d(1) have poly ([:-x, 1:] ^ n) z = 0
          using dvd-trans poly-eq-0-iff-dvd by blast
        ultimately have poly d x = 0
          by auto
        with d(2) have poly q x = 0
          using dvd-trans poly-eq-0-iff-dvd by blast
        with ‹poly q x ≠ 0› show False by contradiction
      qed
      thus is-unit d using d
        by (metis ‹q ≠ 0› dvd-0-left is-unit-iff-degree)
    qed
  next
    have poly q y = 0
      using ‹poly p y = 0› ‹x ≠ y› by (auto simp: p-eq)
    with ‹q ≠ 0› show degree q > 0
      using order-degree order-gt-0-iff order-less-le-trans by blast
  qed (use ‹n > 0› in ‹simp-all add: p-eq degree-power-eq›)
qed

no-notation cCons (infixr ‹##› 65)
```

end

5 A formalization of formal power series

theory *Formal-Power-Series*
imports
  Complex-Main
  Euclidean-Algorithm
  Primes
begin

5.1 The type of formal power series

typedef *'a fps* = {*f* :: *nat* ⇒ *'a*. *True*}
  morphisms *fps-nth Abs-fps*
  by *simp*

notation *fps-nth* (**infixl** ‹$› *75*)

lemma *expand-fps-eq*: *p* = *q* ⟷ (∀ *n*. *p* $ *n* = *q* $ *n*)
  by (*simp add*: *fps-nth-inject* [*symmetric*] *fun-eq-iff*)

lemmas *fps-eq-iff* = *expand-fps-eq*

lemma *fps-ext*: (⋀*n*. *p* $ *n* = *q* $ *n*) ⟹ *p* = *q*
  by (*simp add*: *expand-fps-eq*)

lemma *fps-nth-Abs-fps* [*simp*]: *Abs-fps f* $ *n* = *f n*
  by (*simp add*: *Abs-fps-inverse*)

Definition of the basic elements 0 and 1 and the basic operations of addition, negation and multiplication.

instantiation *fps* :: (*zero*) *zero*
begin
  definition *fps-zero-def*: *0* = *Abs-fps* (λ*n*. *0*)
  instance ..
end

lemma *fps-zero-nth* [*simp*]: *0* $ *n* = *0*
  unfolding *fps-zero-def* **by** *simp*

lemma *fps-nonzero-nth*: *f* ≠ *0* ⟷ (∃ *n*. *f* $ *n* ≠ *0*)
  by (*simp add*: *expand-fps-eq*)

lemma *fps-nonzero-nth-minimal*: *f* ≠ *0* ⟷ (∃ *n*. *f* $ *n* ≠ *0* ∧ (∀ *m* < *n*. *f* $ *m* = *0*))
  (**is** *?lhs* ⟷ *?rhs*)
proof
  let *?n* = *LEAST n*. *f* $ *n* ≠ *0*

show *?rhs* **if** *?lhs*
proof −
from *that* **have** $\exists n.\ f\ \$\ n \neq 0$
by (*simp add*: *fps-nonzero-nth*)
then have $f\ \$\ ?n \neq 0$
by (*rule LeastI-ex*)
moreover have $\forall m<?n.\ f\ \$\ m = 0$
by (*auto dest*: *not-less-Least*)
ultimately show *?thesis* **by** *metis*
qed
qed (*auto simp*: *expand-fps-eq*)

lemma *fps-nonzeroI*: $f\$n \neq 0 \Longrightarrow f \neq 0$
by *auto*

instantiation *fps* :: ({*one*, *zero*}) *one*
begin
definition *fps-one-def*: $1 = Abs\text{-}fps\ (\lambda n.\ if\ n = 0\ then\ 1\ else\ 0)$
instance ..
end

lemma *fps-one-nth* [*simp*]: $1\ \$\ n = (if\ n = 0\ then\ 1\ else\ 0)$
unfolding *fps-one-def* **by** *simp*

instantiation *fps* :: (*plus*) *plus*
begin
definition *fps-plus-def*: $(+) = (\lambda f\ g.\ Abs\text{-}fps\ (\lambda n.\ f\ \$\ n + g\ \$\ n))$
instance ..
end

lemma *fps-add-nth* [*simp*]: $(f + g)\ \$\ n = f\ \$\ n + g\ \$\ n$
unfolding *fps-plus-def* **by** *simp*

instantiation *fps* :: (*minus*) *minus*
begin
definition *fps-minus-def*: $(-) = (\lambda f\ g.\ Abs\text{-}fps\ (\lambda n.\ f\ \$\ n - g\ \$\ n))$
instance ..
end

lemma *fps-sub-nth* [*simp*]: $(f - g)\ \$\ n = f\ \$\ n - g\ \$\ n$
unfolding *fps-minus-def* **by** *simp*

instantiation *fps* :: (*uminus*) *uminus*
begin
definition *fps-uminus-def*: $uminus = (\lambda f.\ Abs\text{-}fps\ (\lambda n.\ -\ (f\ \$\ n)))$
instance ..
end

lemma *fps-neg-nth* [*simp*]: $(-\ f)\ \$\ n = -\ (f\ \$\ n)$

unfolding *fps-uminus-def* **by** *simp*

lemma *fps-neg-0* [*simp*]: $-(0::'a::group\text{-}add\ fps) = 0$
by (*rule iffD2*, *rule fps-eq-iff*, *auto*)

instantiation *fps* :: ({*comm-monoid-add*, *times*}) *times*
begin
definition *fps-times-def*: $(*) = (\lambda f\ g.\ Abs\text{-}fps\ (\lambda n.\ \sum i{=}0..n.\ f\ \$\ i * g\ \$\ (n - i)))$
instance ..
end

lemma *fps-mult-nth*: $(f * g)\ \$\ n = (\sum i{=}0..n.\ f\$i * g\$(n - i))$
unfolding *fps-times-def* **by** *simp*

lemma *fps-mult-nth-0* [*simp*]: $(f * g)\ \$\ 0 = f\ \$\ 0 * g\ \$\ 0$
unfolding *fps-times-def* **by** *simp*

lemma *fps-mult-nth-1*: $(f * g)\ \$\ 1 = f\$0 * g\$1 + f\$1 * g\$0$
by (*simp add*: *fps-mult-nth*)

lemma *fps-mult-nth-1'* [*simp*]: $(f * g)\ \$\ Suc\ 0 = f\$0 * g\$Suc\ 0 + f\$Suc\ 0 * g\$0$
by (*simp add*: *fps-mult-nth*)

lemmas *mult-nth-0* = *fps-mult-nth-0*
lemmas *mult-nth-1* = *fps-mult-nth-1*

instance *fps* :: ({*comm-monoid-add*, *mult-zero*}) *mult-zero*
proof
fix *a* :: *'a fps*
show $0 * a = 0$ **by** (*simp add*: *fps-ext fps-mult-nth*)
show $a * 0 = 0$ **by** (*simp add*: *fps-ext fps-mult-nth*)
qed

declare *atLeastAtMost-iff* [*presburger*]
declare *Bex-def* [*presburger*]
declare *Ball-def* [*presburger*]

lemma *mult-delta-left*:
fixes *x y* :: *'a::mult-zero*
shows (*if b then x else 0*) $*$ *y* = (*if b then x* $*$ *y else 0*)
by *simp*

lemma *mult-delta-right*:
fixes *x y* :: *'a::mult-zero*
shows *x* $*$ (*if b then y else 0*) = (*if b then x* $*$ *y else 0*)
by *simp*

lemma *fps-one-mult*:

fixes *f* :: *'a*::{*comm-monoid-add*, *mult-zero*, *monoid-mult*} *fps*
shows *1* ∗ *f* = *f*
and *f* ∗ *1* = *f*
by (*simp-all add*: *fps-ext fps-mult-nth mult-delta-left mult-delta-right*)

5.2 Subdegrees

definition *subdegree* :: (*'a*::*zero*) *fps* ⇒ *nat* **where**
subdegree f = (*if f* = *0 then 0 else LEAST n*. *f*$*n* ≠ *0*)

lemma *subdegreeI*:
assumes *f* $ *d* ≠ *0* **and** ⋀*i*. *i* < *d* ⟹ *f* $ *i* = *0*
shows *subdegree f* = *d*
by (*smt* (*verit*) *LeastI-ex assms fps-zero-nth linorder-cases not-less-Least subdegree-def*)

lemma *nth-subdegree-nonzero* [*simp*,*intro*]: *f* ≠ *0* ⟹ *f* $ *subdegree f* ≠ *0*
using *fps-nonzero-nth-minimal subdegreeI* **by** *blast*

lemma *nth-less-subdegree-zero* [*dest*]: *n* < *subdegree f* ⟹ *f* $ *n* = *0*
by (*metis fps-nonzero-nth-minimal fps-zero-nth subdegreeI*)

lemma *subdegree-geI*:
assumes *f* ≠ *0* ⋀*i*. *i* < *n* ⟹ *f*$*i* = *0*
shows *subdegree f* ≥ *n*
by (*meson assms leI nth-subdegree-nonzero*)

lemma *subdegree-greaterI*:
assumes *f* ≠ *0* ⋀*i*. *i* ≤ *n* ⟹ *f*$*i* = *0*
shows *subdegree f* > *n*
by (*meson assms leI nth-subdegree-nonzero*)

lemma *subdegree-leI*:
f $ *n* ≠ *0* ⟹ *subdegree f* ≤ *n*
using *linorder-not-less* **by** *blast*

lemma *subdegree-0* [*simp*]: *subdegree 0* = *0*
by (*simp add*: *subdegree-def*)

lemma *subdegree-1* [*simp*]: *subdegree 1* = *0*
by (*metis fps-one-nth nth-subdegree-nonzero subdegree-0*)

lemma *subdegree-eq-0-iff*: *subdegree f* = *0* ⟷ *f* = *0* ∨ *f* $ *0* ≠ *0*
using *nth-subdegree-nonzero subdegree-leI* **by** *fastforce*

lemma *subdegree-eq-0* [*simp*]: *f* $ *0* ≠ *0* ⟹ *subdegree f* = *0*
by (*simp add*: *subdegree-eq-0-iff*)

lemma *nth-subdegree-zero-iff* [*simp*]: *f* $ *subdegree f* = *0* ⟷ *f* = *0*

by (*cases f* = *0*) *auto*

lemma *fps-nonzero-subdegree-nonzeroI*: *subdegree f* > *0* $\Longrightarrow$ *f* $\neq$ *0*
by *auto*

lemma *subdegree-uminus* [*simp*]:
subdegree (−(*f*::(′*a*::*group-add*) *fps*)) = *subdegree f*
proof (*cases f*=*0*)
case *False* **thus** *?thesis* **by** (*force intro*: *subdegreeI*)
qed *simp*

lemma *subdegree-minus-commute* [*simp*]:
fixes *f* :: ′*a*::*group-add fps*
shows *subdegree* (*f*−*g*) = *subdegree* (*g* − *f*)
proof (*cases g*−*f*=*0*)
case *True* **then show** *?thesis*
by (*metis fps-sub-nth nth-subdegree-nonzero right-minus-eq*)
next
case *False* **show** *?thesis*
using *nth-subdegree-nonzero*[*OF False*] **by** (*fastforce intro*: *subdegreeI*)
qed

lemma *subdegree-add-ge*′:
fixes *f g* :: ′*a*::*monoid-add fps*
assumes *f* + *g* $\neq$ *0*
shows *subdegree* (*f* + *g*) $\geq$ *min* (*subdegree f*) (*subdegree g*)
using *assms*
by (*force intro*: *subdegree-geI*)

lemma *subdegree-add-ge*:
assumes *f* $\neq$ −(*g* :: (′*a* :: *group-add*) *fps*)
shows *subdegree* (*f* + *g*) $\geq$ *min* (*subdegree f*) (*subdegree g*)
proof (*rule subdegree-add-ge*′)
have *f* + *g* = *0* $\Longrightarrow$ *False*
proof−
assume *fg*: *f* + *g* = *0*
have $\bigwedge$*n*. *f* \$ *n* = − *g* \$ *n*
by (*metis add-eq-0-iff equation-minus-iff fg fps-add-nth fps-neg-nth fps-zero-nth*)
with *assms* **show** *False* **by** (*auto intro*: *fps-ext*)
qed
thus *f* + *g* $\neq$ *0* **by** *fast*
qed

lemma *subdegree-add-eq1*:
assumes *f* $\neq$ *0*
and *subdegree f* < *subdegree* (*g* :: ′*a*::*monoid-add fps*)
shows *subdegree* (*f* + *g*) = *subdegree f*
using *assms* **by**(*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-add-eq2*:
assumes $g \neq 0$
and *subdegree g* < *subdegree* (*f* :: ′*a* :: *monoid-add fps*)
shows *subdegree* (*f* + *g*) = *subdegree g*
using *assms* **by** (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-diff-eq1*:
assumes $f \neq 0$
and *subdegree f* < *subdegree* (*g* :: ′*a* :: *group-add fps*)
shows *subdegree* (*f* − *g*) = *subdegree f*
using *assms* **by** (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-diff-eq1-cancel*:
assumes $f \neq 0$
and *subdegree f* < *subdegree* (*g* :: ′*a* :: *cancel-comm-monoid-add fps*)
shows *subdegree* (*f* − *g*) = *subdegree f*
using *assms* **by** (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-diff-eq2*:
assumes $g \neq 0$
and *subdegree g* < *subdegree* (*f* :: ′*a* :: *group-add fps*)
shows *subdegree* (*f* − *g*) = *subdegree g*
using *assms* **by** (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-diff-ge* [*simp*]:
assumes *f* ≠ (*g* :: ′*a* :: *group-add fps*)
shows *subdegree* (*f* − *g*) ≥ *min* (*subdegree f*) (*subdegree g*)
proof−
have *f* ≠ − (− *g*)
using *assms expand-fps-eq* **by** *fastforce*
moreover have *f* + − *g* = *f* − *g* **by** (*simp add*: *fps-ext*)
ultimately show *?thesis*
using *subdegree-add-ge*[*of f* −*g*] **by** *simp*
qed

lemma *subdegree-diff-ge′*:
fixes *f g* :: ′*a* :: *comm-monoid-diff fps*
assumes $f - g \neq 0$
shows *subdegree* (*f* − *g*) ≥ *subdegree f*
using *assms* **by** (*auto intro*: *subdegree-geI simp*: *nth-less-subdegree-zero*)

lemma *nth-subdegree-mult-left* [*simp*]:
fixes *f g* :: (′*a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows (*f* ∗ *g*) $ (*subdegree f*) = *f* $ *subdegree f* ∗ *g* $ *0*
by (*cases subdegree f*) (*simp-all add*: *fps-mult-nth nth-less-subdegree-zero*)

lemma *nth-subdegree-mult-right* [*simp*]:
fixes *f g* :: (′*a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows (*f* ∗ *g*) $ (*subdegree g*) = *f* $ *0* ∗ *g* $ *subdegree g*

by (*cases subdegree g*) (*simp-all add*: *fps-mult-nth nth-less-subdegree-zero sum.atLeast-Suc-atMost*)

lemma *nth-subdegree-mult* [*simp*]:
fixes *f g* :: (*'a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows $(f * g)$ \$ (*subdegree f* + *subdegree g*) = *f* \$ *subdegree f* ∗ *g* \$ *subdegree g*
proof−
let *?n* = *subdegree f* + *subdegree g*
have $(f * g)$ \$ *?n* = (∑ *i*=*0*..*?n*. *f*\$*i* ∗ *g*\$(*?n*−*i*))
by (*simp add*: *fps-mult-nth*)
also have ... = (∑ *i*=*0*..*?n*. *if i* = *subdegree f then f*\$*i* ∗ *g*\$(*?n*−*i*) *else 0*)
proof (*intro sum.cong*)
fix *x* **assume** *x*: *x* ∈ {*0*..*?n*}
hence *x* = *subdegree f* ∨ *x* < *subdegree f* ∨ *?n* − *x* < *subdegree g* **by** *auto*
thus *f* \$ *x* ∗ *g* \$ (*?n* − *x*) = (*if x* = *subdegree f then f* \$ *x* ∗ *g* \$ (*?n* − *x*) *else 0*)
by (*elim disjE conjE*) *auto*
qed *auto*
also have ... = *f* \$ *subdegree f* ∗ *g* \$ *subdegree g* **by** *simp*
finally show *?thesis* **.**
qed

lemma *fps-mult-nth-eq0*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
assumes *n* < *subdegree f* + *subdegree g*
shows (*f*∗*g*) \$ *n* = *0*
proof−
have ⋀*i*. *i*∈{*0*..*n*} ⟹ *f*\$*i* ∗ *g*\$(*n* − *i*) = *0*
proof−
fix *i* **assume** *i*: *i*∈{*0*..*n*}
show *f*\$*i* ∗ *g*\$(*n* − *i*) = *0*
proof (*cases i* < *subdegree f* ∨ *n* − *i* < *subdegree g*)
case *False* **with** *assms i* **show** *?thesis* **by** *auto*
qed (*auto simp*: *nth-less-subdegree-zero*)
qed
thus (*f* ∗ *g*) \$ *n* = *0* **by** (*simp add*: *fps-mult-nth*)
qed

lemma *fps-mult-subdegree-ge*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
assumes *f*∗*g* ≠ *0*
shows *subdegree* (*f*∗*g*) ≥ *subdegree f* + *subdegree g*
using *assms fps-mult-nth-eq0*
by (*intro subdegree-geI*) *simp*

lemma *subdegree-mult'*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
assumes *f* \$ *subdegree f* ∗ *g* \$ *subdegree g* ≠ *0*
shows *subdegree* (*f*∗*g*) = *subdegree f* + *subdegree g*

proof −
  from *assms* **have** $(f * g)$ \$ $(subdegree\ f + subdegree\ g) \neq 0$ **by** *simp*
  hence $f*g \neq 0$ **by** *fastforce*
  hence *subdegree* $(f*g) \geq$ *subdegree f* + *subdegree g* **using** *fps-mult-subdegree-ge*
by *fast*
  moreover from *assms* **have** *subdegree* $(f*g) \leq$ *subdegree f* + *subdegree g*
    by (*intro subdegree-leI*) *simp*
  ultimately show *?thesis* **by** *simp*
qed

lemma *subdegree-mult* [*simp*]:
  fixes *f g* :: *'a* :: {*semiring-no-zero-divisors*} *fps*
  assumes $f \neq 0$ $g \neq 0$
  shows *subdegree* $(f * g)$ = *subdegree f* + *subdegree g*
  using *assms*
  by (*intro subdegree-mult'*) *simp*

lemma *fps-mult-nth-conv-upto-subdegree-left*:
  fixes *f g* :: (*'a* :: {*mult-zero,comm-monoid-add*}) *fps*
  shows $(f * g)$ \$ $n = (\sum i{=}subdegree\ f..n.\ f$ \$ $i * g$ \$ $(n - i))$
proof (*cases subdegree f* $\leq n$)
  case *True*
  hence $\{0..n\} = \{0..{<}subdegree\ f\} \cup \{subdegree\ f..n\}$ **by** *auto*
  moreover have $\{0..{<}subdegree\ f\} \cap \{subdegree\ f..n\} = \{\}$ **by** *auto*
  ultimately show *?thesis*
    using *nth-less-subdegree-zero*[*of - f*]
    by (*simp add*: *fps-mult-nth sum.union-disjoint*)
qed (*simp add*: *fps-mult-nth nth-less-subdegree-zero*)

lemma *fps-mult-nth-conv-upto-subdegree-right*:
  fixes *f g* :: (*'a* :: {*mult-zero,comm-monoid-add*}) *fps*
  shows $(f * g)$ \$ $n = (\sum i{=}0..n - subdegree\ g.\ f$ \$ $i * g$ \$ $(n - i))$
proof −
  have $\{0..n\} = \{0..n - subdegree\ g\} \cup \{n - subdegree\ g{<}..n\}$ **by** *auto*
  moreover have $\{0..n - subdegree\ g\} \cap \{n - subdegree\ g{<}..n\} = \{\}$ **by** *auto*
  moreover have $\forall i \in \{n - subdegree\ g{<}..n\}.\ g$ \$ $(n - i) = 0$
    using *nth-less-subdegree-zero*[*of - g*] **by** *auto*
  ultimately show *?thesis* **by** (*simp add*: *fps-mult-nth sum.union-disjoint*)
qed

lemma *fps-mult-nth-conv-inside-subdegrees*:
  fixes *f g* :: (*'a* :: {*mult-zero,comm-monoid-add*}) *fps*
  shows $(f * g)$ \$ $n = (\sum i{=}subdegree\ f..n - subdegree\ g.\ f$ \$ $i * g$ \$ $(n - i))$
proof (*cases subdegree f* $\leq n -$ *subdegree g*)
  case *True*
  hence $\{subdegree\ f..n\} = \{subdegree\ f..n - subdegree\ g\} \cup \{n - subdegree\ g{<}..n\}$
    by *auto*
  moreover have $\{subdegree\ f..n - subdegree\ g\} \cap \{n - subdegree\ g{<}..n\} = \{\}$
by *auto*

moreover have ∀ *i*∈{*n* − *subdegree g*<..*n*}. *f* $ *i* ∗ *g* $ (*n* − *i*) = *0*
  using *nth-less-subdegree-zero*[*of - g*] **by** *auto*
ultimately show *?thesis*
  using *fps-mult-nth-conv-upto-subdegree-left*[*of f g n*]
  by (*simp add*: *sum.union-disjoint*)
next
  case *False*
  hence *1*: *subdegree f* > *n* − *subdegree g* **by** *simp*
  show *?thesis*
  proof (*cases f*∗*g* = *0*)
    case *False*
    with *1* **have** *n* < *subdegree* (*f*∗*g*) **using** *fps-mult-subdegree-ge*[*of f g*] **by** *simp*
    with *1* **show** *?thesis* **by** *auto*
  qed (*simp add*: *1*)
qed

lemma *fps-mult-nth-outside-subdegrees*:
  fixes *f g* :: (′*a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
  shows *n* < *subdegree f* ⟹ (*f* ∗ *g*) $ *n* = *0*
  and *n* < *subdegree g* ⟹ (*f* ∗ *g*) $ *n* = *0*
  by (*auto simp*: *fps-mult-nth-conv-inside-subdegrees*)

5.3 Ring structure

instance *fps* :: (*semigroup-add*) *semigroup-add*
proof
  fix *a b c* :: ′*a fps*
  show *a* + *b* + *c* = *a* + (*b* + *c*)
    by (*simp add*: *fps-ext add.assoc*)
qed

instance *fps* :: (*ab-semigroup-add*) *ab-semigroup-add*
proof
  fix *a b* :: ′*a fps*
  show *a* + *b* = *b* + *a*
    by (*simp add*: *fps-ext add.commute*)
qed

instance *fps* :: (*monoid-add*) *monoid-add*
proof
  fix *a* :: ′*a fps*
  show *0* + *a* = *a* **by** (*simp add*: *fps-ext*)
  show *a* + *0* = *a* **by** (*simp add*: *fps-ext*)
qed

instance *fps* :: (*comm-monoid-add*) *comm-monoid-add*
proof
  fix *a* :: ′*a fps*
  show *0* + *a* = *a* **by** (*simp add*: *fps-ext*)

qed

instance *fps* :: (*cancel-semigroup-add*) *cancel-semigroup-add*
proof
  fix *a b c* :: *'a fps*
  show *b* = *c* **if** *a* + *b* = *a* + *c*
    using *that* **by** (*simp add*: *expand-fps-eq*)
  show *b* = *c* **if** *b* + *a* = *c* + *a*
    using *that* **by** (*simp add*: *expand-fps-eq*)
qed

instance *fps* :: (*cancel-ab-semigroup-add*) *cancel-ab-semigroup-add*
proof
  fix *a b c* :: *'a fps*
  show *a* + *b* − *a* = *b*
    by (*simp add*: *expand-fps-eq*)
  show *a* − *b* − *c* = *a* − (*b* + *c*)
    by (*simp add*: *expand-fps-eq diff-diff-eq*)
qed

instance *fps* :: (*cancel-comm-monoid-add*) *cancel-comm-monoid-add* **..**

instance *fps* :: (*group-add*) *group-add*
proof
  fix *a b* :: *'a fps*
  show − *a* + *a* = *0* **by** (*simp add*: *fps-ext*)
  show *a* + − *b* = *a* − *b* **by** (*simp add*: *fps-ext*)
qed

instance *fps* :: (*ab-group-add*) *ab-group-add*
proof
  fix *a b* :: *'a fps*
  show − *a* + *a* = *0* **by** (*simp add*: *fps-ext*)
  show *a* − *b* = *a* + − *b* **by** (*simp add*: *fps-ext*)
qed

instance *fps* :: (*zero-neq-one*) *zero-neq-one*
  by *standard* (*simp add*: *expand-fps-eq*)

lemma *fps-mult-assoc-lemma*:
  fixes *k* :: *nat*
    and *f* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *'a*::*comm-monoid-add*
  shows (∑ *j*=*0*..*k*. ∑ *i*=*0*..*j*. *f i* (*j* − *i*) (*n* − *j*)) =
        (∑ *j*=*0*..*k*. ∑ *i*=*0*..*k* − *j*. *f j i* (*n* − *j* − *i*))
  by (*induct k*) (*simp-all add*: *Suc-diff-le sum.distrib add.assoc*)

instance *fps* :: (*semiring-0*) *semiring-0*
proof
  fix *a b c* :: *'a fps*

show $(a + b) * c = a * c + b * c$
  by (*simp add*: *expand-fps-eq fps-mult-nth distrib-right sum.distrib*)
show $a * (b + c) = a * b + a * c$
  by (*simp add*: *expand-fps-eq fps-mult-nth distrib-left sum.distrib*)
show $(a * b) * c = a * (b * c)$
proof (*rule fps-ext*)
  fix *n* :: *nat*
  have $(\sum j=0..n. \sum i=0..j. a\$i * b\$(j - i) * c\$(n - j)) =$
    $(\sum j=0..n. \sum i=0..n - j. a\$j * b\$i * c\$(n - j - i))$
    by (*rule fps-mult-assoc-lemma*)
  then show $((a * b) * c) \$ n = (a * (b * c)) \$ n$
    by (*simp add*: *fps-mult-nth sum-distrib-left sum-distrib-right mult.assoc*)
qed
qed

instance *fps* :: (*semiring-0-cancel*) *semiring-0-cancel* **..**

lemma *fps-mult-commute-lemma*:
fixes *n* :: *nat*
  and f :: *nat* ⇒ *nat* ⇒ *'a*::*comm-monoid-add*
shows $(\sum i=0..n. f\ i\ (n - i)) = (\sum i=0..n. f\ (n - i)\ i)$
by (*rule sum.reindex-bij-witness*[**where** i=(−) n **and** j=(−) n]) *auto*

instance *fps* :: (*comm-semiring-0*) *comm-semiring-0*
proof
fix *a b c* :: *'a fps*
show $a * b = b * a$
proof (*rule fps-ext*)
  fix *n* :: *nat*
  have $(\sum i=0..n. a\$i * b\$(n - i)) = (\sum i=0..n. a\$(n - i) * b\$i)$
    by (*rule fps-mult-commute-lemma*)
  then show $(a * b) \$ n = (b * a) \$ n$
    by (*simp add*: *fps-mult-nth mult.commute*)
qed
qed (*simp add*: *distrib-right*)

instance *fps* :: (*comm-semiring-0-cancel*) *comm-semiring-0-cancel* **..**

instance *fps* :: (*semiring-1*) *semiring-1*
proof
fix *a* :: *'a fps*
show $1 * a = a\ a * 1 = a$ **by** (*simp-all add*: *fps-one-mult*)
qed

instance *fps* :: (*comm-semiring-1*) *comm-semiring-1*
by *standard simp*

instance *fps* :: (*semiring-1-cancel*) *semiring-1-cancel* **..**

lemma *fps-square-nth*: $(f\hat{\ }2)$ \$ $n = (\sum k{\le}n.\ f$ \$ $k * f$ \$ $(n - k))$
  by (*simp add*: *power2-eq-square fps-mult-nth atLeast0AtMost*)

lemma *fps-sum-nth*: *sum f S* \$ *n* = *sum* (λ*k*. (*f k*) \$ *n*) *S*
proof (*cases finite S*)
  case *True*
  then show *?thesis* **by** (*induct set*: *finite*) *auto*
next
  case *False*
  then show *?thesis* **by** *simp*
qed

definition *fps-const c* = *Abs-fps* (λ*n*. *if n* = *0 then c else 0*)

lemma *fps-nth-fps-const* [*simp*]: *fps-const c* \$ *n* = (*if n* = *0 then c else 0*)
  unfolding *fps-const-def* **by** *simp*

lemma *fps-const-0-eq-0* [*simp*]: *fps-const 0* = *0*
  by (*simp add*: *fps-ext*)

lemma *fps-const-nonzero-eq-nonzero*: *c* ≠ *0* ⟹ *fps-const c* ≠ *0*
  using *fps-nonzeroI*[*of fps-const c 0*] **by** *simp*

lemma *fps-const-eq-0-iff* [*simp*]: *fps-const c* = *0* ⟷ *c* = *0*
  by (*auto simp*: *fps-eq-iff*)

lemma *fps-const-1-eq-1* [*simp*]: *fps-const 1* = *1*
  by (*simp add*: *fps-ext*)

lemma *fps-const-eq-1-iff* [*simp*]: *fps-const c* = *1* ⟷ *c* = *1*
  by (*auto simp*: *fps-eq-iff*)

lemma *subdegree-fps-const* [*simp*]: *subdegree* (*fps-const c*) = *0*
  by (*cases c* = *0*) (*auto intro*!: *subdegreeI*)

lemma *fps-const-neg* [*simp*]: − (*fps-const* (*c*::*'a*::*group-add*)) = *fps-const* (− *c*)
  by (*simp add*: *fps-ext*)

lemma *fps-const-add* [*simp*]: *fps-const* (*c*::*'a*::*monoid-add*) + *fps-const d* = *fps-const* (*c* + *d*)
  by (*simp add*: *fps-ext*)

lemma *fps-const-add-left*: *fps-const* (*c*::*'a*::*monoid-add*) + *f* =
    Abs-fps (λ*n*. *if n* = *0 then c* + *f*\$*0 else f*\$*n*)
  by (*simp add*: *fps-ext*)

lemma *fps-const-add-right*: *f* + *fps-const* (*c*::*'a*::*monoid-add*) =
    Abs-fps (λ*n*. *if n* = *0 then f*\$*0* + *c else f*\$*n*)

by (*simp add*: *fps-ext*)

lemma *fps-const-sub* [*simp*]: *fps-const* (*c*::'*a*::*group-add*) − *fps-const* *d* = *fps-const* (*c* − *d*)
by (*simp add*: *fps-ext*)

lemmas *fps-const-minus* = *fps-const-sub*

lemma *fps-const-mult*[*simp*]:
fixes *c d* :: '*a*::{*comm-monoid-add*,*mult-zero*}
shows *fps-const c* ∗ *fps-const d* = *fps-const* (*c* ∗ *d*)
by (*simp add*: *fps-eq-iff fps-mult-nth sum.neutral*)

lemma *fps-const-mult-left*:
fps-const (*c*::'*a*::{*comm-monoid-add*,*mult-zero*}) ∗ *f* = *Abs-fps* (λ*n*. *c* ∗ *f*$*n*)
unfolding *fps-eq-iff fps-mult-nth*
by (*simp add*: *fps-const-def mult-delta-left*)

lemma *fps-const-mult-right*:
f ∗ *fps-const* (*c*::'*a*::{*comm-monoid-add*,*mult-zero*}) = *Abs-fps* (λ*n*. *f*$*n* ∗ *c*)
unfolding *fps-eq-iff fps-mult-nth*
by (*simp add*: *fps-const-def mult-delta-right*)

lemma *fps-mult-left-const-nth* [*simp*]:
(*fps-const* (*c*::'*a*::{*comm-monoid-add*,*mult-zero*}) ∗ *f*)$*n* = *c*∗ *f*$*n*
by (*simp add*: *fps-mult-nth mult-delta-left*)

lemma *fps-mult-right-const-nth* [*simp*]:
(*f* ∗ *fps-const* (*c*::'*a*::{*comm-monoid-add*,*mult-zero*}))$*n* = *f*$*n* ∗ *c*
by (*simp add*: *fps-mult-nth mult-delta-right*)

lemma *fps-const-power* [*simp*]: *fps-const c* ^ *n* = *fps-const* (*c*^*n*)
by (*induct n*) *auto*

instance *fps* :: (*ring*) *ring* ..

instance *fps* :: (*comm-ring*) *comm-ring* ..

instance *fps* :: (*ring-1*) *ring-1* ..

instance *fps* :: (*comm-ring-1*) *comm-ring-1* ..

instance *fps* :: (*semiring-no-zero-divisors*) *semiring-no-zero-divisors*
proof
fix *a b* :: '*a fps*
assume *a* ≠ *0* **and** *b* ≠ *0*
hence (*a* ∗ *b*) $ (*subdegree a* + *subdegree b*) ≠ *0* **by** *simp*
thus *a* ∗ *b* ≠ *0* **using** *fps-nonzero-nth* **by** *fast*

qed

instance *fps* :: (*semiring-1-no-zero-divisors*) *semiring-1-no-zero-divisors* **..**

instance *fps* :: ({*cancel-semigroup-add*,*semiring-no-zero-divisors-cancel*})
semiring-no-zero-divisors-cancel
proof
fix *a b c* :: *'a fps*
show $(a * c = b * c) = (c = 0 \lor a = b)$
proof
assume *ab*: $a * c = b * c$
have $c \neq 0 \Longrightarrow a = b$
proof (*rule fps-ext*)
fix *n*
assume *c*: $c \neq 0$
show *a* $ *n* = *b* $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*)
with *ab c* **show** *?case*
using *fps-mult-nth-conv-upto-subdegree-right*[*of a c subdegree c* + *n*]
fps-mult-nth-conv-upto-subdegree-right[*of b c subdegree c* + *n*]
by (*cases n*) *auto*
qed
qed
thus $c = 0 \lor a = b$ **by** *fast*
qed *auto*
show $(c * a = c * b) = (c = 0 \lor a = b)$
proof
assume *ab*: $c * a = c * b$
have $c \neq 0 \Longrightarrow a = b$
proof (*rule fps-ext*)
fix *n*
assume *c*: $c \neq 0$
show *a* $ *n* = *b* $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*)
moreover have ∀ *i*∈{*Suc* (*subdegree c*)..*subdegree c* + *n*}. *subdegree c* + *n* − *i* < *n* **by** *auto*
ultimately show *?case*
using *ab c fps-mult-nth-conv-upto-subdegree-left*[*of c a subdegree c* + *n*]
fps-mult-nth-conv-upto-subdegree-left[*of c b subdegree c* + *n*]
by (*simp add*: *sum.atLeast-Suc-atMost*)
qed
qed
thus $c = 0 \lor a = b$ **by** *fast*
qed *auto*
qed

instance *fps* :: (*ring-no-zero-divisors*) *ring-no-zero-divisors* **..**

instance *fps* :: (*ring-1-no-zero-divisors*) *ring-1-no-zero-divisors* **..**

instance *fps* :: (*idom*) *idom* **..**

lemma *fps-of-nat*: *fps-const* (*of-nat c*) = *of-nat c*
  by (*induction c*) (*simp-all add*: *fps-const-add* [*symmetric*] *del*: *fps-const-add*)

lemma *fps-of-int*: *fps-const* (*of-int c*) = *of-int c*
  by (*induction c*) (*simp-all add*: *fps-const-minus* [*symmetric*] *fps-of-nat fps-const-neg* [*symmetric*]
      del: *fps-const-minus fps-const-neg*)

lemma *semiring-char-fps* [*simp*]: *CHAR*(′*a* :: *comm-semiring-1 fps*) = *CHAR*(′*a*)
  by (*rule CHAR-eqI*) (*auto simp flip*: *fps-of-nat simp*: *of-nat-eq-0-iff-char-dvd*)

instance *fps* :: ({*semiring-prime-char*,*comm-semiring-1*}) *semiring-prime-char*
  by (*rule semiring-prime-charI*) *auto*
instance *fps* :: ({*comm-semiring-prime-char*,*comm-semiring-1*}) *comm-semiring-prime-char*
  by *standard*
instance *fps* :: ({*comm-ring-prime-char*,*comm-semiring-1*}) *comm-ring-prime-char*
  by *standard*
instance *fps* :: ({*idom-prime-char*,*comm-semiring-1*}) *idom-prime-char*
  by *standard*

lemma *fps-numeral-fps-const*: *numeral k* = *fps-const* (*numeral k*)
  by (*induct k*) (*simp-all only*: *numeral.simps fps-const-1-eq-1 fps-const-add* [*symmetric*])

lemmas *numeral-fps-const* = *fps-numeral-fps-const*

lemma *neg-numeral-fps-const*:
  (− *numeral k* :: ′*a* :: *ring-1 fps*) = *fps-const* (− *numeral k*)
  by (*simp add*: *numeral-fps-const*)

lemma *fps-numeral-nth*: *numeral n* $ *i* = (*if i* = *0 then numeral n else 0*)
  by (*simp add*: *numeral-fps-const*)

lemma *fps-numeral-nth-0* [*simp*]: *numeral n* $ *0* = *numeral n*
  by (*simp add*: *numeral-fps-const*)

lemma *subdegree-numeral* [*simp*]: *subdegree* (*numeral n*) = *0*
  by (*simp add*: *numeral-fps-const*)

lemma *fps-nth-of-nat* [*simp*]:
  (*of-nat c*) $ *n* = (*if n*=*0 then of-nat c else 0*)
  by (*simp add*: *fps-of-nat*[*symmetric*])

lemma *fps-nth-of-int* [*simp*]:
  (*of-int c*) $ *n* = (*if n*=*0 then of-int c else 0*)

by (*simp add*: *fps-of-int*[*symmetric*])

lemma *fps-mult-of-nat-nth* [*simp*]:
shows (*of-nat k* ∗ *f*) $ *n* = *of-nat k* ∗ *f*$*n*
and (*f* ∗ *of-nat k*) $ *n* = *f*$*n* ∗ *of-nat k*
by (*simp-all add*: *fps-of-nat*[*symmetric*])

lemma *fps-mult-of-int-nth* [*simp*]:
shows (*of-int k* ∗ *f*) $ *n* = *of-int k* ∗ *f*$*n*
and (*f* ∗ *of-int k*) $ *n* = *f*$*n* ∗ *of-int k*
by (*simp-all add*: *fps-of-int*[*symmetric*])

lemma *numeral-neq-fps-zero* [*simp*]: (*numeral f* :: ′*a* :: *field-char-0 fps*) ≠ *0*
proof
assume *numeral f* = (*0* :: ′*a fps*)
from *arg-cong*[*of - - λF. F* $ *0*, *OF this*] **show** *False* **by** *simp*
qed

lemma *subdegree-power-ge*:
f^*n* ≠ *0* ⟹ *subdegree* (*f*^*n*) ≥ *n* ∗ *subdegree f*
proof (*induct n*)
case (*Suc n*) **thus** *?case* **using** *fps-mult-subdegree-ge* **by** *fastforce*
qed *simp*

lemma *fps-pow-nth-below-subdegree*:
k < *n* ∗ *subdegree f* ⟹ (*f*^*n*) $ *k* = *0*
proof (*cases f*^*n* = *0*)
case *False*
assume *k* < *n* ∗ *subdegree f*
with *False* **have** *k* < *subdegree* (*f*^*n*) **using** *subdegree-power-ge*[*of f n*] **by** *simp*
thus (*f*^*n*) $ *k* = *0* **by** *auto*
qed *simp*

lemma *fps-pow-base* [*simp*]:
(*f* ^ *n*) $ (*n* ∗ *subdegree f*) = (*f* $ *subdegree f*) ^ *n*
proof (*induct n*)
case (*Suc n*)
show *?case*
proof (*cases Suc n* ∗ *subdegree f* < *subdegree f* + *subdegree* (*f*^*n*))
case *True* **with** *Suc* **show** *?thesis*
by (*auto simp*: *fps-mult-nth-eq0 distrib-right*)
next
case *False*
hence ∀ *i*∈{*Suc* (*subdegree f*)..*Suc n* ∗ *subdegree f* − *subdegree* (*f* ^ *n*)}.
f ^ *n* $ (*Suc n* ∗ *subdegree f* − *i*) = *0*
by (*auto simp*: *fps-pow-nth-below-subdegree*)
with *False Suc* **show** *?thesis*
using *fps-mult-nth-conv-inside-subdegrees*[*of f f*^*n Suc n* ∗ *subdegree f*]
sum.atLeast-Suc-atMost[*of*

subdegree f
Suc n ∗ subdegree f − subdegree (f ^ n)
λi. f $ i ∗ f ^ n $ (Suc n ∗ subdegree f − i)
]
by *simp*
qed
qed *simp*

lemma *subdegree-power-eqI*:
fixes *f* :: *'a::semiring-1 fps*
shows *(f $ subdegree f) ^ n ≠ 0 ⟹ subdegree (f ^ n) = n ∗ subdegree f*
proof (*induct n*)
case (*Suc n*)
from *Suc* **have** *1*: *subdegree (f ^ n) = n ∗ subdegree f* **by** *fastforce*
with *Suc(2)* **have** *f $ subdegree f ∗ f ^ n $ subdegree (f ^ n) ≠ 0* **by** *simp*
with *1* **show** *?case* **using** *subdegree-mult'[of f f^n]* **by** *simp*
qed *simp*

lemma *subdegree-power* [*simp*]:
subdegree ((f :: ('a :: semiring-1-no-zero-divisors) fps) ^ n) = n ∗ subdegree f
by (*cases f = 0*; *induction n*) *simp-all*

lemma *minus-one-power-iff*: *(− (1::'a::ring-1)) ^ n = (if even n then 1 else − 1)*
by (*induct n*) *auto*

definition *fps-X = Abs-fps (λn. if n = 1 then 1 else 0)*

lemma *subdegree-fps-X* [*simp*]: *subdegree (fps-X :: ('a :: zero-neq-one) fps) = 1*
by (*auto intro!*: *subdegreeI simp*: *fps-X-def*)

lemma *fps-X-mult-nth* [*simp*]:
fixes *f* :: *'a::{comm-monoid-add,mult-zero,monoid-mult} fps*
shows *(fps-X ∗ f) $ n = (if n = 0 then 0 else f $ (n − 1))*
proof (*cases n*)
case (*Suc m*)
moreover have *(fps-X ∗ f) $ Suc m = f $ (Suc m − 1)*
proof (*cases m*)
case *0* **thus** *?thesis* **using** *fps-mult-nth-1[of fps-X f]* **by** (*simp add*: *fps-X-def*)
next
case (*Suc k*) **thus** *?thesis* **by** (*simp add*: *fps-mult-nth fps-X-def sum.atLeast-Suc-atMost*)
qed
ultimately show *?thesis* **by** *simp*
qed (*simp add*: *fps-X-def*)

lemma *fps-X-mult-right-nth* [*simp*]:
fixes *a* :: *'a::{comm-monoid-add,mult-zero,monoid-mult} fps*
shows *(a ∗ fps-X) $ n = (if n = 0 then 0 else a $ (n − 1))*
proof (*cases n*)

case (*Suc m*)
moreover have (*a* ∗ *fps-X*) $ *Suc m* = *a* $ (*Suc m* − *1*)
proof (*cases m*)
case *0* **thus** *?thesis* **using** *fps-mult-nth-1*[*of a fps-X*] **by** (*simp add*: *fps-X-def*)
next
case (*Suc k*)
hence (*a* ∗ *fps-X*) $ *Suc m* = (∑ *i*=*0*..*k*. *a*$*i* ∗ *fps-X*$(*Suc m* − *i*)) + *a*$(*Suc k*)
by (*simp add*: *fps-mult-nth fps-X-def*)
moreover have ∀ *i*∈{*0*..*k*}. *a*$*i* ∗ *fps-X*$(*Suc m* − *i*) = *0* **by** (*auto simp*: *Suc fps-X-def*)
ultimately show *?thesis* **by** (*simp add*: *Suc*)
qed
ultimately show *?thesis* **by** *simp*
qed (*simp add*: *fps-X-def*)

lemma *fps-mult-fps-X-commute*:
fixes *a* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *fps-X* ∗ *a* = *a* ∗ *fps-X*
by (*simp add*: *fps-eq-iff*)

lemma *fps-mult-fps-X-power-commute*: *fps-X* ^ *k* ∗ *a* = *a* ∗ *fps-X* ^ *k*
proof (*induct k*)
case (*Suc k*)
hence *fps-X* ^ *Suc k* ∗ *a* = *a* ∗ *fps-X* ∗ *fps-X* ^ *k*
by (*simp add*: *mult.assoc fps-mult-fps-X-commute*[*symmetric*])
thus *?case* **by** (*simp add*: *mult.assoc*)
qed *simp*

lemma *fps-subdegree-mult-fps-X*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
assumes *f* ≠ *0*
shows *subdegree* (*fps-X* ∗ *f*) = *subdegree f* + *1*
and *subdegree* (*f* ∗ *fps-X*) = *subdegree f* + *1*
proof−
show *subdegree* (*fps-X* ∗ *f*) = *subdegree f* + *1*
proof (*intro subdegreeI*)
fix *i* :: *nat* **assume** *i*: *i* < *subdegree f* + *1*
show (*fps-X* ∗ *f*) $ *i* = *0*
proof (*cases i*=*0*)
case *False* **with** *i* **show** *?thesis* **by** (*simp add*: *nth-less-subdegree-zero*)
next
case *True* **thus** *?thesis* **using** *fps-X-mult-nth*[*of f i*] **by** *simp*
qed
qed (*simp add*: *assms*)
thus *subdegree* (*f* ∗ *fps-X*) = *subdegree f* + *1*
by (*simp add*: *fps-mult-fps-X-commute*)
qed

lemma *fps-mult-fps-X-nonzero*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
assumes *f* ≠ *0*
shows *fps-X* ∗ *f* ≠ *0*
and *f* ∗ *fps-X* ≠ *0*
using *assms fps-subdegree-mult-fps-X*[*of f*]
fps-nonzero-subdegree-nonzeroI[*of fps-X* ∗ *f*]
fps-nonzero-subdegree-nonzeroI[*of f* ∗ *fps-X*]
by *auto*

lemma *fps-mult-fps-X-power-nonzero*:
assumes *f* ≠ *0*
shows *fps-X* ^ *n* ∗ *f* ≠ *0*
and *f* ∗ *fps-X* ^ *n* ≠ *0*
proof −
show *fps-X* ^ *n* ∗ *f* ≠ *0*
by (*induct n*) (*simp-all add*: *assms mult.assoc fps-mult-fps-X-nonzero*(*1*))
thus *f* ∗ *fps-X* ^ *n* ≠ *0*
by (*simp add*: *fps-mult-fps-X-power-commute*)
qed

lemma *fps-X-power-iff*: *fps-X* ^ *n* = *Abs-fps* (λ*m*. *if m* = *n then 1 else 0*)
by (*induction n*) (*auto simp*: *fps-eq-iff*)

lemma *fps-X-nth*[*simp*]: *fps-X*$*n* = (*if n* = *1 then 1 else 0*)
by (*simp add*: *fps-X-def*)

lemma *fps-X-power-nth*[*simp*]: (*fps-X*^*k*) $*n* = (*if n* = *k then 1 else 0*)
by (*simp add*: *fps-X-power-iff*)

lemma *fps-X-power-subdegree*: *subdegree* (*fps-X*^*n*) = *n*
by (*auto intro*: *subdegreeI*)

lemma *fps-X-power-mult-nth*:
(*fps-X*^*k* ∗ *f*) $ *n* = (*if n* < *k then 0 else f* $ (*n* − *k*))
by (*cases n*<*k*)
(*simp-all add*: *fps-mult-nth-conv-upto-subdegree-left fps-X-power-subdegree sum.atLeast-Suc-atMost*)

lemma *fps-X-power-mult-right-nth*:
(*f* ∗ *fps-X*^*k*) $ *n* = (*if n* < *k then 0 else f* $ (*n* − *k*))
using *fps-mult-fps-X-power-commute*[*of k f*] *fps-X-power-mult-nth*[*of k f*] **by** *simp*

lemma *fps-subdegree-mult-fps-X-power*:
assumes *f* ≠ *0*
shows *subdegree* (*fps-X* ^ *n* ∗ *f*) = *subdegree f* + *n*
and *subdegree* (*f* ∗ *fps-X* ^ *n*) = *subdegree f* + *n*
proof −
from *assms* **show** *subdegree* (*fps-X* ^ *n* ∗ *f*) = *subdegree f* + *n*

by (*induct n*)
(*simp-all add*: *algebra-simps fps-subdegree-mult-fps-X*(*1*) *fps-mult-fps-X-power-nonzero*(*1*))
thus *subdegree* (*f* ∗ *fps-X* ^ *n*) = *subdegree f* + *n*
by (*simp add*: *fps-mult-fps-X-power-commute*)
qed

lemma *fps-mult-fps-X-plus-1-nth*:
((*1*+*fps-X*)∗*a*) $*n* = (*if n* = *0 then* (*a*$*n* :: ′*a*::*semiring-1*) *else a*$*n* + *a*$(*n* − *1*))
proof (*cases n*)
case *0*
then show *?thesis*
by (*simp add*: *fps-mult-nth*)
next
case (*Suc m*)
have ((*1* + *fps-X*)∗*a*) $ *n* = *sum* (λ*i*. (*1* + *fps-X*) $ *i* ∗ *a* $ (*n* − *i*)) {*0*..*n*}
by (*simp add*: *fps-mult-nth*)
also have . . . = *sum* (λ*i*. (*1*+*fps-X*)$*i* ∗ *a*$(*n*−*i*)) {*0*.. *1*}
unfolding *Suc* **by** (*rule sum.mono-neutral-right*) *auto*
also have . . . = (*if n* = *0 then a*$*n else a*$*n* + *a*$(*n* − *1*))
by (*simp add*: *Suc*)
finally show *?thesis* **.**
qed

lemma *fps-mult-right-fps-X-plus-1-nth*:
fixes *a* :: ′*a* :: *semiring-1 fps*
shows (*a*∗(*1*+*fps-X*)) $ *n* = (*if n* = *0 then a*$*n else a*$*n* + *a*$(*n* − *1*))
using *fps-mult-fps-X-plus-1-nth*
by (*simp add*: *distrib-left fps-mult-fps-X-commute distrib-right*)

lemma *fps-X-neq-fps-const* [*simp*]: (*fps-X* :: ′*a* :: *zero-neq-one fps*) ≠ *fps-const c*
proof
assume (*fps-X*::′*a fps*) = *fps-const* (*c*::′*a*)
hence *fps-X*$*1* = (*fps-const* (*c*::′*a*))$*1* **by** (*simp only*:)
thus *False* **by** *auto*
qed

lemma *fps-X-neq-zero* [*simp*]: (*fps-X* :: ′*a* :: *zero-neq-one fps*) ≠ *0*
by (*simp only*: *fps-const-0-eq-0*[*symmetric*] *fps-X-neq-fps-const*) *simp*

lemma *fps-X-neq-one* [*simp*]: (*fps-X* :: ′*a* :: *zero-neq-one fps*) ≠ *1*
by (*simp only*: *fps-const-1-eq-1*[*symmetric*] *fps-X-neq-fps-const*) *simp*

lemma *fps-X-neq-numeral* [*simp*]: *fps-X* ≠ *numeral c*
by (*simp only*: *numeral-fps-const fps-X-neq-fps-const*) *simp*

lemma *fps-X-pow-eq-fps-X-pow-iff* [*simp*]: *fps-X* ^ *m* = *fps-X* ^ *n* ⟷ *m* = *n*
proof
assume (*fps-X* :: ′*a fps*) ^ *m* = *fps-X* ^ *n*

hence (*fps-X* :: *'a fps*) ^ *m* \$ *m* = *fps-X* ^ *n* \$ *m* **by** (*simp only*:)
thus *m* = *n* **by** (*simp split*: *if-split-asm*)
qed *simp-all*

5.4 Shifting and slicing

definition *fps-shift* :: *nat* ⇒ *'a fps* ⇒ *'a fps* **where**
fps-shift n f = *Abs-fps* (*λi. f* \$ (*i* + *n*))

lemma *fps-shift-nth* [*simp*]: *fps-shift n f* \$ *i* = *f* \$ (*i* + *n*)
by (*simp add*: *fps-shift-def*)

lemma *fps-shift-0* [*simp*]: *fps-shift 0 f* = *f*
by (*intro fps-ext*) (*simp add*: *fps-shift-def*)

lemma *fps-shift-zero* [*simp*]: *fps-shift n 0* = *0*
by (*intro fps-ext*) (*simp add*: *fps-shift-def*)

lemma *fps-shift-one*: *fps-shift n 1* = (*if n* = *0 then 1 else 0*)
by (*intro fps-ext*) (*simp add*: *fps-shift-def*)

lemma *fps-shift-fps-const*: *fps-shift n* (*fps-const c*) = (*if n* = *0 then fps-const c else 0*)
by (*intro fps-ext*) (*simp add*: *fps-shift-def*)

lemma *fps-shift-numeral*: *fps-shift n* (*numeral c*) = (*if n* = *0 then numeral c else 0*)
by (*simp add*: *numeral-fps-const fps-shift-fps-const*)

lemma *fps-shift-fps-X* [*simp*]:
n ≥ *1* ⟹ *fps-shift n fps-X* = (*if n* = *1 then 1 else 0*)
by (*intro fps-ext*) (*auto simp*: *fps-X-def*)

lemma *fps-shift-fps-X-power* [*simp*]:
n ≤ *m* ⟹ *fps-shift n* (*fps-X* ^ *m*) = *fps-X* ^ (*m* − *n*)
by (*intro fps-ext*) *auto*

lemma *fps-shift-subdegree* [*simp*]:
n ≤ *subdegree f* ⟹ *subdegree* (*fps-shift n f*) = *subdegree f* − *n*
by (*cases f=0*) (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *fps-shift-fps-shift*:
fps-shift (*m* + *n*) *f* = *fps-shift m* (*fps-shift n f*)
by (*rule fps-ext*) (*simp add*: *add-ac*)

lemma *fps-shift-fps-shift-reorder*:
fps-shift m (*fps-shift n f*) = *fps-shift n* (*fps-shift m f*)
using *fps-shift-fps-shift*[*of m n f*] *fps-shift-fps-shift*[*of n m f*] **by** (*simp add*: *add.commute*)

lemma *fps-shift-rev-shift*:
  $m \leq n \Longrightarrow$ *fps-shift n* (*Abs-fps* (λk. *if* $k<m$ *then 0 else f* \$ $(k-m)$)) = *fps-shift* $(n-m)$ *f*
  $m > n \Longrightarrow$ *fps-shift n* (*Abs-fps* (λk. *if* $k<m$ *then 0 else f* \$ $(k-m)$)) =
    Abs-fps (λk. *if* $k<m-n$ *then 0 else f* \$ $(k-(m-n))$)
proof −
  assume $m \leq n$
  thus *fps-shift n* (*Abs-fps* (λk. *if* $k<m$ *then 0 else f* \$ $(k-m)$)) = *fps-shift* $(n-m)$ *f*
    by (*intro fps-ext*) *auto*
next
  assume *mn*: $m > n$
  hence $\bigwedge k$. $k \geq m-n \Longrightarrow k+n-m = k - (m-n)$ **by** *auto*
  thus
    fps-shift n (*Abs-fps* (λk. *if* $k<m$ *then 0 else f* \$ $(k-m)$)) =
      Abs-fps (λk. *if* $k<m-n$ *then 0 else f* \$ $(k-(m-n))$)
    by (*intro fps-ext*) *auto*
qed

lemma *fps-shift-add*:
  fps-shift n $(f + g)$ = *fps-shift n f* + *fps-shift n g*
  by (*simp add*: *fps-eq-iff*)

lemma *fps-shift-diff*:
  fps-shift n $(f - g)$ = *fps-shift n f* − *fps-shift n g*
  by (*auto intro*: *fps-ext*)

lemma *fps-shift-uminus*:
  fps-shift n $(-f)$ = − *fps-shift n f*
  by (*auto intro*: *fps-ext*)

lemma *fps-shift-mult*:
  assumes $n \leq$ *subdegree* (*g* :: *'b* :: {*comm-monoid-add*, *mult-zero*} *fps*)
  shows *fps-shift n* $(h*g) = h *$ *fps-shift n g*
proof−
  have *case1*: $\bigwedge$*a b*::*'b fps*. *1* $\leq$ *subdegree b* $\Longrightarrow$ *fps-shift 1* $(a*b) = a *$ *fps-shift 1 b*
  proof (*rule fps-ext*)
    fix *a b* :: *'b fps*
    and *n* :: *nat*
    assume *b*: *1* $\leq$ *subdegree b*
    have $\bigwedge i$. $i \leq n \Longrightarrow n + 1 - i = (n-i) + 1$
      by (*simp add*: *algebra-simps*)
    with *b* **show** *fps-shift 1* $(a*b)$ \$ n = $(a *$ *fps-shift 1 b*) \$ n
      by (*simp add*: *fps-mult-nth nth-less-subdegree-zero*)
  qed
  have $n \leq$ *subdegree g* $\Longrightarrow$ *fps-shift n* $(h*g) = h *$ *fps-shift n g*
  proof (*induct n*)

case (*Suc n*)
have *fps-shift* (*Suc n*) (*h*∗*g*) = *fps-shift 1* (*fps-shift n* (*h*∗*g*))
by (*simp add*: *fps-shift-fps-shift*[*symmetric*])
also have ... = *h* ∗ (*fps-shift 1* (*fps-shift n g*))
using *Suc case1* **by** *force*
finally show *?case* **by** (*simp add*: *fps-shift-fps-shift*[*symmetric*])
qed *simp*
with *assms* **show** *?thesis* **by** *fast*
qed

lemma *fps-shift-mult-right-noncomm*:
assumes $n \leq$ *subdegree* (*g* :: ′*b* :: {*comm-monoid-add*, *mult-zero*} *fps*)
shows *fps-shift n* (*g*∗*h*) = *fps-shift n g* ∗ *h*
proof−
have *case1*: ⋀*a b*::′*b fps*. *1* ≤ *subdegree a* ⟹ *fps-shift 1* (*a*∗*b*) = *fps-shift 1 a* ∗ *b*
proof (*rule fps-ext*)
fix *a b* :: ′*b fps*
and *n* :: *nat*
assume *1* ≤ *subdegree a*
hence *fps-shift 1* (*a*∗*b*) $ *n* = (∑ *i*=*Suc 0*..*Suc n*. *a*$*i* ∗ *b*$(*n*+*1*−*i*))
using *sum.atLeast-Suc-atMost*[*of 0 n*+*1* λ*i*. *a*$*i* ∗ *b*$(*n*+*1*−*i*)]
by (*simp add*: *fps-mult-nth nth-less-subdegree-zero*)
thus *fps-shift 1* (*a*∗*b*) $ *n* = (*fps-shift 1 a* ∗ *b*) $ *n*
using *sum.shift-bounds-cl-Suc-ivl*[*of* λ*i*. *a*$*i* ∗ *b*$(*n*+*1*−*i*) *0 n*]
by (*simp add*: *fps-mult-nth*)
qed
have *n* ≤ *subdegree g* ⟹ *fps-shift n* (*g*∗*h*) = *fps-shift n g* ∗ *h*
proof (*induct n*)
case (*Suc n*)
have *fps-shift* (*Suc n*) (*g*∗*h*) = *fps-shift 1* (*fps-shift n* (*g*∗*h*))
by (*simp add*: *fps-shift-fps-shift*[*symmetric*])
also have ... = (*fps-shift 1* (*fps-shift n g*)) ∗ *h*
using *Suc case1* **by** *force*
finally show *?case* **by** (*simp add*: *fps-shift-fps-shift*[*symmetric*])
qed *simp*
with *assms* **show** *?thesis* **by** *fast*
qed

lemma *fps-shift-mult-right*:
assumes *n* ≤ *subdegree* (*g* :: ′*b* :: *comm-semiring-0 fps*)
shows *fps-shift n* (*g*∗*h*) = *h* ∗ *fps-shift n g*
by (*simp add*: *assms fps-shift-mult-right-noncomm mult.commute*)

lemma *fps-shift-mult-both*:
fixes *f g* :: ′*a*::{*comm-monoid-add*, *mult-zero*} *fps*
assumes *m* ≤ *subdegree f n* ≤ *subdegree g*
shows *fps-shift m f* ∗ *fps-shift n g* = *fps-shift* (*m*+*n*) (*f*∗*g*)
using *assms*

by (*simp add*: *fps-shift-mult fps-shift-mult-right-noncomm fps-shift-fps-shift*)

lemma *fps-shift-subdegree-zero-iff* [*simp*]:
fps-shift (*subdegree f*) *f* = *0* ⟷ *f* = *0*
by (*subst* (*1*) *nth-subdegree-zero-iff*[*symmetric*], *cases f* = *0*)
(*simp-all del*: *nth-subdegree-zero-iff*)

lemma *fps-shift-times-fps-X*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *1* ≤ *subdegree f* ⟹ *fps-shift 1 f* ∗ *fps-X* = *f*
by (*intro fps-ext*) (*simp add*: *nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X'* [*simp*]:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *fps-shift 1* (*f* ∗ *fps-X*) = *f*
by (*intro fps-ext*) (*simp add*: *nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X''*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *1* ≤ *n* ⟹ *fps-shift n* (*f* ∗ *fps-X*) = *fps-shift* (*n* − *1*) *f*
by (*intro fps-ext*) (*simp add*: *nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X-power*:
n ≤ *subdegree f* ⟹ *fps-shift n f* ∗ *fps-X* ^ *n* = *f*
by (*intro fps-ext*) (*simp add*: *fps-X-power-mult-right-nth nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X-power'* [*simp*]:
fps-shift n (*f* ∗ *fps-X*^*n*) = *f*
by (*intro fps-ext*) (*simp add*: *fps-X-power-mult-right-nth nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X-power''*:
m ≤ *n* ⟹ *fps-shift n* (*f* ∗ *fps-X*^*m*) = *fps-shift* (*n* − *m*) *f*
by (*intro fps-ext*) (*simp add*: *fps-X-power-mult-right-nth nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X-power'''*:
m > *n* ⟹ *fps-shift n* (*f* ∗ *fps-X*^*m*) = *f* ∗ *fps-X*^(*m* − *n*)
proof (*cases f*=*0*)
case *False*
assume *m*: *m*>*n*
hence *m* = *n* + (*m*−*n*) **by** *auto*
with *False m* **show** *?thesis*
using *power-add*[*of fps-X*::*'a fps n m*−*n*]
fps-shift-mult-right-noncomm[*of n f* ∗ *fps-X*^*n fps-X*^(*m*−*n*)]
by (*simp add*: *mult.assoc fps-subdegree-mult-fps-X-power*(*2*))
qed *simp*

lemma *subdegree-decompose*:
f = *fps-shift* (*subdegree f*) *f* ∗ *fps-X* ^ *subdegree f*
by (*rule fps-ext*) (*auto simp*: *fps-X-power-mult-right-nth*)

lemma *subdegree-decompose'*:
$n \leq$ *subdegree* $f \Longrightarrow f =$ *fps-shift* $n\ f *$ *fps-X*^n
by (*rule fps-ext*) (*auto simp*: *fps-X-power-mult-right-nth intro*!: *nth-less-subdegree-zero*)

instantiation *fps* :: (*zero*) *unit-factor*
begin
definition *fps-unit-factor-def* [*simp*]:
unit-factor $f =$ *fps-shift* (*subdegree* f) f
instance ..
end

lemma *fps-unit-factor-zero-iff*: *unit-factor* (f::'a::*zero fps*) $= 0 \longleftrightarrow f = 0$
by *simp*

lemma *fps-unit-factor-nth-0*: $f \neq 0 \Longrightarrow$ *unit-factor* f \$ $0 \neq 0$
by *simp*

lemma *fps-X-unit-factor*: *unit-factor* (*fps-X* :: 'a :: *zero-neq-one fps*) = *1*
by (*intro fps-ext*) *auto*

lemma *fps-X-power-unit-factor*: *unit-factor* (*fps-X* ^ n) = *1*
proof−
define X :: 'a *fps* **where** $X \equiv$ *fps-X*
hence *unit-factor* (X^n) = *fps-shift* n (X^n)
by (*simp add*: *fps-X-power-subdegree*)
moreover have *fps-shift* n (X^n) = *1*
by (*auto intro*: *fps-ext simp*: *fps-X-power-iff X-def*)
ultimately show *?thesis* **by** (*simp add*: *X-def*)
qed

lemma *fps-unit-factor-decompose*:
$f =$ *unit-factor* $f *$ *fps-X* ^ *subdegree* f
by (*simp add*: *subdegree-decompose*)

lemma *fps-unit-factor-decompose'*:
$f =$ *fps-X* ^ *subdegree* $f *$ *unit-factor* f
using *fps-unit-factor-decompose* **by** (*simp add*: *fps-mult-fps-X-power-commute*)

lemma *fps-unit-factor-uminus*:
unit-factor ($-f$) = − *unit-factor* (f::'a::*group-add fps*)
by (*simp add*: *fps-shift-uminus*)

lemma *fps-unit-factor-shift*:
assumes $n \leq$ *subdegree* f
shows *unit-factor* (*fps-shift* $n\ f$) = *unit-factor* f
by (*simp add*: *assms fps-shift-fps-shift*[*symmetric*])

lemma *fps-unit-factor-mult-fps-X*:

fixes *f* :: *'a*::{*comm-monoid-add*,*monoid-mult*,*mult-zero*} *fps*
shows *unit-factor* (*fps-X* ∗ *f*) = *unit-factor f*
and *unit-factor* (*f* ∗ *fps-X*) = *unit-factor f*
proof −
show *unit-factor* (*fps-X* ∗ *f*) = *unit-factor f*
by (*cases f=0*) (*auto intro*: *fps-ext simp*: *fps-subdegree-mult-fps-X*(*1*))
thus *unit-factor* (*f* ∗ *fps-X*) = *unit-factor f* **by** (*simp add*: *fps-mult-fps-X-commute*)
qed

lemma *fps-unit-factor-mult-fps-X-power*:
shows *unit-factor* (*fps-X* ^ *n* ∗ *f*) = *unit-factor f*
and *unit-factor* (*f* ∗ *fps-X* ^ *n*) = *unit-factor f*
proof −
show *unit-factor* (*fps-X* ^ *n* ∗ *f*) = *unit-factor f*
proof (*induct n*)
case (*Suc m*) **thus** *?case*
using *fps-unit-factor-mult-fps-X*(*1*)[*of fps-X* ^ *m* ∗ *f*] **by** (*simp add*: *mult.assoc*)
qed *simp*
thus *unit-factor* (*f* ∗ *fps-X* ^ *n*) = *unit-factor f*
by (*simp add*: *fps-mult-fps-X-power-commute*)
qed

lemma *fps-unit-factor-mult-unit-factor*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
shows *unit-factor* (*f* ∗ *unit-factor g*) = *unit-factor* (*f* ∗ *g*)
and *unit-factor* (*unit-factor f* ∗ *g*) = *unit-factor* (*f* ∗ *g*)
proof −
show *unit-factor* (*f* ∗ *unit-factor g*) = *unit-factor* (*f* ∗ *g*)
proof (*cases f*∗*g* = *0*)
case *False* **thus** *?thesis*
using *fps-mult-subdegree-ge*[*of f g*] *fps-unit-factor-shift*[*of subdegree g f*∗*g*]
by (*simp add*: *fps-shift-mult*)
next
case *True*
moreover have *f* ∗ *unit-factor g* = *fps-shift* (*subdegree g*) (*f*∗*g*)
by (*simp add*: *fps-shift-mult*)
ultimately show *?thesis* **by** *simp*
qed
show *unit-factor* (*unit-factor f* ∗ *g*) = *unit-factor* (*f* ∗ *g*)
proof (*cases f*∗*g* = *0*)
case *False* **thus** *?thesis*
using *fps-mult-subdegree-ge*[*of f g*] *fps-unit-factor-shift*[*of subdegree f f*∗*g*]
by (*simp add*: *fps-shift-mult-right-noncomm*)
next
case *True*
moreover have *unit-factor f* ∗ *g* = *fps-shift* (*subdegree f*) (*f*∗*g*)
by (*simp add*: *fps-shift-mult-right-noncomm*)
ultimately show *?thesis* **by** *simp*
qed

qed

lemma *fps-unit-factor-mult-both-unit-factor*:
  fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
  shows *unit-factor* (*unit-factor f* ∗ *unit-factor g*) = *unit-factor* (*f* ∗ *g*)
  using *fps-unit-factor-mult-unit-factor*(*1*)[*of unit-factor f g*]
        fps-unit-factor-mult-unit-factor(*2*)[*of f g*]
  by *simp*

lemma *fps-unit-factor-mult'*:
  fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
  assumes *f* $ *subdegree f* ∗ *g* $ *subdegree g* ≠ *0*
  shows *unit-factor* (*f* ∗ *g*) = *unit-factor f* ∗ *unit-factor g*
  using *assms*
  by (*simp add*: *subdegree-mult' fps-shift-mult-both*)

lemma *fps-unit-factor-mult*:
  fixes *f g* :: *'a*::*semiring-no-zero-divisors fps*
  shows *unit-factor* (*f* ∗ *g*) = *unit-factor f* ∗ *unit-factor g*
  using *fps-unit-factor-mult'*[*of f g*]
  by (*cases f*=*0* ∨ *g*=*0*) *auto*

definition *fps-cutoff n f* = *Abs-fps* (*λi*. *if i* < *n then f*$*i else 0*)

lemma *fps-cutoff-nth* [*simp*]: *fps-cutoff n f* $ *i* = (*if i* < *n then f*$*i else 0*)
  unfolding *fps-cutoff-def* **by** *simp*

lemma *fps-cutoff-zero-iff*: *fps-cutoff n f* = *0* ⟷ (*f* = *0* ∨ *n* ≤ *subdegree f*)
proof
  assume *A*: *fps-cutoff n f* = *0*
  thus *f* = *0* ∨ *n* ≤ *subdegree f*
  proof (*cases f* = *0*)
    assume *f* ≠ *0*
    with *A* **have** *n* ≤ *subdegree f*
      by (*intro subdegree-geI*) (*simp-all add*: *fps-eq-iff split*: *if-split-asm*)
    thus *?thesis* **..**
  qed *simp*
qed (*auto simp*: *fps-eq-iff intro*: *nth-less-subdegree-zero*)

lemma *fps-cutoff-0* [*simp*]: *fps-cutoff 0 f* = *0*
  by (*simp add*: *fps-eq-iff*)

lemma *fps-cutoff-zero* [*simp*]: *fps-cutoff n 0* = *0*
  by (*simp add*: *fps-eq-iff*)

lemma *fps-cutoff-one*: *fps-cutoff n 1* = (*if n* = *0 then 0 else 1*)
  by (*simp add*: *fps-eq-iff*)

lemma *fps-cutoff-fps-const*: *fps-cutoff n* (*fps-const c*) = (*if n* = *0 then 0 else*

fps-const c)
  by (*simp add*: *fps-eq-iff*)

lemma *fps-cutoff-numeral*: *fps-cutoff n* (*numeral c*) = (*if n* = *0 then 0 else numeral c*)
  by (*simp add*: *numeral-fps-const fps-cutoff-fps-const*)

lemma *fps-shift-cutoff*:
  fps-shift n f ∗ *fps-X*^*n* + *fps-cutoff n f* = *f*
  by (*simp add*: *fps-eq-iff fps-X-power-mult-right-nth*)

lemma *fps-shift-cutoff'*:
  fps-X^*n* ∗ *fps-shift n f* + *fps-cutoff n f* = *f*
  by (*simp add*: *fps-eq-iff fps-X-power-mult-nth*)

lemma *fps-cutoff-left-mult-nth*:
  k < *n* ⟹ (*fps-cutoff n f* ∗ *g*) $ *k* = (*f* ∗ *g*) $ *k*
  by (*simp add*: *fps-mult-nth*)

lemma *fps-cutoff-right-mult-nth*:
  assumes *k* < *n*
  shows (*f* ∗ *fps-cutoff n g*) $ *k* = (*f* ∗ *g*) $ *k*
proof−
  from *assms* **have** ∀ *i*∈{*0*..*k*}. *fps-cutoff n g* $ (*k* − *i*) = *g* $ (*k* − *i*) **by** *auto*
  thus *?thesis* **by** (*simp add*: *fps-mult-nth*)
qed

5.5 Metrizability

instantiation *fps* :: ({*minus*,*zero*}) *dist*
begin

definition
  dist-fps-def: *dist* (*a* :: '*a fps*) *b* = (*if a* = *b then 0 else inverse* (*2* ^ *subdegree* (*a* − *b*)))

lemma *dist-fps-ge0*: *dist* (*a* :: '*a fps*) *b* ≥ *0*
  by (*simp add*: *dist-fps-def*)

instance ..

end

instantiation *fps* :: (*group-add*) *metric-space*
begin

definition *uniformity-fps-def* [*code del*]:
  (*uniformity* :: ('*a fps* × '*a fps*) *filter*) = (*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-fps-def′* [*code del*]:
open (*U* :: ′*a fps set*) ⟷ (∀ *x*∈*U*. *eventually* (λ(*x*′, *y*). *x*′ = *x* ⟶ *y* ∈ *U*) *uniformity*)

lemma *dist-fps-sym*: *dist* (*a* :: ′*a fps*) *b* = *dist b a*
by (*simp add*: *dist-fps-def*)

instance
proof
show *th*: *dist a b* = *0* ⟷ *a* = *b* **for** *a b* :: ′*a fps*
by (*simp add*: *dist-fps-def split*: *if-split-asm*)
then have *th*′[*simp*]: *dist a a* = *0* **for** *a* :: ′*a fps* **by** *simp*

fix *a b c* :: ′*a fps*
consider *a* = *b* | *c* = *a* ∨ *c* = *b* | *a* ≠ *b a* ≠ *c b* ≠ *c* **by** *blast*
then show *dist a b* ≤ *dist a c* + *dist b c*
proof *cases*
case *1*
then show *?thesis* **by** (*simp add*: *dist-fps-def*)
next
case *2*
then show *?thesis*
by (*cases c* = *a*) (*simp-all add*: *th dist-fps-sym*)
next
case *neq*: *3*
have *False* **if** *dist a b* > *dist a c* + *dist b c*
proof −
let *?n* = *subdegree* (*a* − *b*)
from *neq* **have** *dist a b* > *0 dist b c* > *0* **and** *dist a c* > *0* **by** (*simp-all add*: *dist-fps-def*)
with *that* **have** *dist a b* > *dist a c* **and** *dist a b* > *dist b c* **by** *simp-all*
with *neq* **have** *?n* < *subdegree* (*a* − *c*) **and** *?n* < *subdegree* (*b* − *c*)
by (*simp-all add*: *dist-fps-def field-simps*)
hence (*a* − *c*) $ *?n* = *0* **and** (*b* − *c*) $ *?n* = *0*
by (*simp-all only*: *nth-less-subdegree-zero*)
hence (*a* − *b*) $ *?n* = *0* **by** *simp*
moreover from *neq* **have** (*a* − *b*) $ *?n* ≠ *0* **by** (*intro nth-subdegree-nonzero*) *simp-all*
ultimately show *False* **by** *contradiction*
qed
thus *?thesis* **by** (*auto simp add*: *not-le*[*symmetric*])
qed
qed (*rule open-fps-def′ uniformity-fps-def*)+

end

declare *uniformity-Abort*[**where** ′*a*=′*a* :: *group-add fps*, *code*]

lemma *open-fps-def*: *open* (*S* :: *'a*::*group-add fps set*) = (∀ *a* ∈ *S*. ∃ *r*. *r* >*0* ∧ {*y*. *dist y a* < *r*} ⊆ *S*)
  unfolding *open-dist subset-eq* **by** *simp*

The infinite sums and justification of the notation in textbooks.

lemma *reals-power-lt-ex*:
  fixes *x y* :: *real*
  assumes *xp*: *x* > *0*
    and *y1*: *y* > *1*
  shows ∃ *k*>*0*. (*1*/*y*)^*k* < *x*
proof −
  have *yp*: *y* > *0*
    using *y1* **by** *simp*
  from *reals-Archimedean2*[*of max 0* (− *log y x*) + *1*]
  obtain *k* :: *nat* **where** *k*: *real k* > *max 0* (− *log y x*) + *1*
    by *blast*
  from *k* **have** *kp*: *k* > *0*
    by *simp*
  from *k* **have** *real k* > − *log y x*
    by *simp*
  then have *ln y* ∗ *real k* > − *ln x*
    unfolding *log-def*
    using *ln-gt-zero-iff*[*OF yp*] *y1*
    by (*simp add*: *minus-divide-left field-simps del*: *minus-divide-left*[*symmetric*])
  then have *ln y* ∗ *real k* + *ln x* > *0*
    by *simp*
  then have *exp* (*real k* ∗ *ln y* + *ln x*) > *exp 0*
    by (*simp add*: *ac-simps*)
  then have *y* ^ *k* ∗ *x* > *1*
    unfolding *exp-zero exp-add exp-of-nat-mult exp-ln* [*OF xp*] *exp-ln* [*OF yp*]
    by *simp*
  then have *x* > (*1* / *y*)^*k* **using** *yp*
    by (*simp add*: *field-simps*)
  then show *?thesis*
    using *kp* **by** *blast*
qed

lemma *fps-sum-rep-nth*: (*sum* (λ*i*. *fps-const*(*a*$*i*)∗*fps-X*^*i*) {*0*..*m*})$*n* = (*if n* ≤ *m then a*$*n else 0*)
  by (*simp add*: *fps-sum-nth if-distrib cong del*: *if-weak-cong*)

lemma *fps-notation*: (λ*n*. *sum* (λ*i*. *fps-const*(*a*$*i*) ∗ *fps-X*^*i*) {*0*..*n*}) ⟶ *a*
  (**is** *?s* ⟶ *a*)
proof −
  have ∃ *n0*. ∀ *n* ≥ *n0*. *dist* (*?s n*) *a* < *r* **if** *r* > *0* **for** *r*
  proof −
    obtain *n0* **where** *n0*: (*1*/*2*)^*n0* < *r n0* > *0*
      using *reals-power-lt-ex*[*OF* ‹*r* > *0*›, *of 2*] **by** *auto*
    show *?thesis*

proof −
have *dist* (*?s n*) *a* < *r* **if** *nn0*: *n* ≥ *n0* **for** *n*
proof −
from *that* **have** *thnn0*: (*1*/*2*)^*n* ≤ (*1*/*2* :: *real*)^*n0*
by (*simp add*: *field-split-simps*)
show *?thesis*
proof (*cases ?s n* = *a*)
case *True*
then show *?thesis*
unfolding *dist-eq-0-iff* [*of ?s n a*, *symmetric*]
using ‹*r* > *0*› **by** (*simp del*: *dist-eq-0-iff*)
next
case *False*
from *False* **have** *dth*: *dist* (*?s n*) *a* = (*1*/*2*)^*subdegree* (*?s n* − *a*)
by (*simp add*: *dist-fps-def field-simps*)
from *False* **have** *kn*: *subdegree* (*?s n* − *a*) > *n*
by (*intro subdegree-greaterI*) (*simp-all add*: *fps-sum-rep-nth*)
then have *dist* (*?s n*) *a* < (*1*/*2*)^*n*
by (*simp add*: *field-simps dist-fps-def*)
also have ... ≤ (*1*/*2*)^*n0*
using *nn0* **by** (*simp add*: *field-split-simps*)
also have ... < *r*
using *n0* **by** *simp*
finally show *?thesis* **.**
qed
qed
then show *?thesis* **by** *blast*
qed
qed
then show *?thesis*
unfolding *lim-sequentially* **by** *blast*
qed

5.6 Division

declare *sum.cong*[*fundef-cong*]

fun *fps-left-inverse-constructor* ::
'a::{*comm-monoid-add*,*times*,*uminus*} *fps* ⇒ *'a* ⇒ *nat* ⇒ *'a*
where
fps-left-inverse-constructor f a 0 = *a*
| *fps-left-inverse-constructor f a* (*Suc n*) =
− *sum* (λ*i*. *fps-left-inverse-constructor f a i* ∗ *f*$(*Suc n* − *i*)) {*0*..*n*} ∗ *a*

— This will construct a left inverse for f in case that $x * f \$ 0 = 1$
abbreviation *fps-left-inverse* ≡ (λ*f x*. *Abs-fps* (*fps-left-inverse-constructor f x*))

fun *fps-right-inverse-constructor* ::
'a::{*comm-monoid-add*,*times*,*uminus*} *fps* ⇒ *'a* ⇒ *nat* ⇒ *'a*

where
fps-right-inverse-constructor f a 0 = *a*
| *fps-right-inverse-constructor f a n* =
− *a* ∗ *sum* (λ*i*. *f*$*i* ∗ *fps-right-inverse-constructor f a* (*n* − *i*)) {*1*..*n*}

— This will construct a right inverse for f in case that *f* $ *0* ∗ *y* = *1*
abbreviation *fps-right-inverse* ≡ (λ*f y*. *Abs-fps* (*fps-right-inverse-constructor f y*))

instantiation *fps* :: ({*comm-monoid-add*,*inverse*,*times*,*uminus*}) *inverse*
begin

— For backwards compatibility.
abbreviation *natfun-inverse*:: ′*a fps* ⇒ *nat* ⇒ ′*a*
where *natfun-inverse f* ≡ *fps-right-inverse-constructor f* (*inverse* (*f*$*0*))

definition *fps-inverse-def*: *inverse f* = *Abs-fps* (*natfun-inverse f*)
— With scalars from a (possibly non-commutative) ring, this defines a right inverse. Furthermore, if scalars are of class *mult-zero* and satisfy condition *inverse 0* = *0*, then this will evaluate to zero when the zeroth term is zero.

definition *fps-divide-def*: *f div g* = *fps-shift* (*subdegree g*) (*f* ∗ *inverse* (*unit-factor g*))
— If scalars are of class *mult-zero* and satisfy condition *inverse 0* = *0*, then div by zero will equal zero.

instance ..

end

lemma *fps-lr-inverse-0-iff*:
(*fps-left-inverse f x*) $ *0* = *0* ⟷ *x* = *0*
(*fps-right-inverse f x*) $ *0* = *0* ⟷ *x* = *0*
by *auto*

lemma *fps-inverse-0-iff* ′: (*inverse f*) $ *0* = *0* ⟷ *inverse* (*f* $ *0*) = *0*
by (*simp add*: *fps-inverse-def fps-lr-inverse-0-iff*(*2*))

lemma *fps-inverse-0-iff*[*simp*]: (*inverse f*) $ *0* = (*0*::′*a*::*division-ring*) ⟷ *f* $ *0* = *0*
by (*simp add*: *fps-inverse-0-iff* ′)

lemma *fps-lr-inverse-nth-0*:
(*fps-left-inverse f x*) $ *0* = *x* (*fps-right-inverse f x*) $ *0* = *x*
by *auto*

lemma *fps-inverse-nth-0* [*simp*]: (*inverse f*) $ *0* = *inverse* (*f* $ *0*)
by (*simp add*: *fps-inverse-def*)

lemma *fps-lr-inverse-starting0*:

fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*} *fps*
and *g* :: *'b*::{*ab-group-add*,*mult-zero*} *fps*
shows *fps-left-inverse f 0* = *0*
and *fps-right-inverse g 0* = *0*
proof−
show *fps-left-inverse f 0* = *0*
proof (*rule fps-ext*)
fix *n* **show** *fps-left-inverse f 0* \$ *n* = *0* \$ *n*
by (*cases n*) (*simp-all add*: *fps-inverse-def*)
qed
show *fps-right-inverse g 0* = *0*
proof (*rule fps-ext*)
fix *n* **show** *fps-right-inverse g 0* \$ *n* = *0* \$ *n*
by (*cases n*) (*simp-all add*: *fps-inverse-def*)
qed
qed

lemma *fps-lr-inverse-eq0-imp-starting0*:
fps-left-inverse f x = *0* ⟹ *x* = *0*
fps-right-inverse f x = *0* ⟹ *x* = *0*
proof−
assume *A*: *fps-left-inverse f x* = *0*
have *0* = *fps-left-inverse f x* \$ *0* **by** (*subst A*) *simp*
thus *x* = *0* **by** *simp*
next
assume *A*: *fps-right-inverse f x* = *0*
have *0* = *fps-right-inverse f x* \$ *0* **by** (*subst A*) *simp*
thus *x* = *0* **by** *simp*
qed

lemma *fps-lr-inverse-eq-0-iff*:
fixes *x* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
and *y* :: *'b*::{*ab-group-add*,*mult-zero*}
shows *fps-left-inverse f x* = *0* ⟷ *x* = *0*
and *fps-right-inverse g y* = *0* ⟷ *y* = *0*
using *fps-lr-inverse-starting0 fps-lr-inverse-eq0-imp-starting0*
by *auto*

lemma *fps-inverse-eq-0-iff'*:
fixes *f* :: *'a*::{*ab-group-add*,*inverse*,*mult-zero*} *fps*
shows *inverse f* = *0* ⟷ *inverse* (*f* \$ *0*) = *0*
by (*simp add*: *fps-inverse-def fps-lr-inverse-eq-0-iff*(*2*))

lemma *fps-inverse-eq-0-iff*[*simp*]: *inverse f* = (*0*:: (*'a*::*division-ring*) *fps*) ⟷ *f* \$ *0* = *0*
using *fps-inverse-eq-0-iff'*[*of f*] **by** *simp*

lemmas *fps-inverse-eq-0'* = *iffD2*[*OF fps-inverse-eq-0-iff'*]
lemmas *fps-inverse-eq-0* = *iffD2*[*OF fps-inverse-eq-0-iff*]

lemma *fps-const-lr-inverse*:
  fixes *a* :: *'a*::{*ab-group-add*,*mult-zero*}
  and *b* :: *'b*::{*comm-monoid-add*,*mult-zero*,*uminus*}
  shows *fps-left-inverse* (*fps-const a*) *x* = *fps-const x*
  and *fps-right-inverse* (*fps-const b*) *y* = *fps-const y*
proof−
  show *fps-left-inverse* (*fps-const a*) *x* = *fps-const x*
  proof (*rule fps-ext*)
    fix *n* **show** *fps-left-inverse* (*fps-const a*) *x* $ *n* = *fps-const x* $ *n*
      by (*cases n*) *auto*
  qed
  show *fps-right-inverse* (*fps-const b*) *y* = *fps-const y*
  proof (*rule fps-ext*)
    fix *n* **show** *fps-right-inverse* (*fps-const b*) *y* $ *n* = *fps-const y* $ *n*
      by (*cases n*) *auto*
  qed
qed

lemma *fps-const-inverse*:
  fixes *a* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}
  shows *inverse* (*fps-const a*) = *fps-const* (*inverse a*)
  unfolding *fps-inverse-def*
  by (*simp add*: *fps-const-lr-inverse*(*2*))

lemma *fps-lr-inverse-zero*:
  fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*}
  and *y* :: *'b*::{*comm-monoid-add*,*mult-zero*,*uminus*}
  shows *fps-left-inverse 0 x* = *fps-const x*
  and *fps-right-inverse 0 y* = *fps-const y*
  using *fps-const-lr-inverse*[*of 0*]
  by *simp-all*

lemma *fps-inverse-zero-conv-fps-const*:
  inverse (*0*::*'a*::{*comm-monoid-add*,*mult-zero*,*uminus*,*inverse*} *fps*) = *fps-const* (*inverse 0*)
  using *fps-lr-inverse-zero*(*2*)[*of inverse* (*0*::*'a*)] **by** (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-zero'*:
  assumes *inverse* (*0*::*'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}) = *0*
  shows *inverse* (*0*::*'a fps*) = *0*
  by (*simp add*: *assms fps-inverse-zero-conv-fps-const*)

lemma *fps-inverse-zero* [*simp*]:
  inverse (*0*::*'a*::*division-ring fps*) = *0*
  by (*rule fps-inverse-zero'*[*OF inverse-zero*])

lemma *fps-lr-inverse-one*:
  fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*,*one*}

and *y* :: *'b*::{*comm-monoid-add*,*mult-zero*,*uminus*,*one*}
shows *fps-left-inverse 1 x* = *fps-const x*
and *fps-right-inverse 1 y* = *fps-const y*
using *fps-const-lr-inverse*[*of 1*]
by *simp-all*

lemma *fps-lr-inverse-one-one*:
fps-left-inverse 1 1 = (*1*::*'a*::{*ab-group-add*,*mult-zero*,*one*} *fps*)
fps-right-inverse 1 1 = (*1*::*'b*::{*comm-monoid-add*,*mult-zero*,*uminus*,*one*} *fps*)
by (*simp-all add*: *fps-lr-inverse-one*)

lemma *fps-inverse-one'*:
assumes *inverse* (*1*::*'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*one*}) = *1*
shows *inverse* (*1* :: *'a fps*) = *1*
using *assms fps-lr-inverse-one-one*(*2*)
by (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-one* [*simp*]: *inverse* (*1* :: *'a* :: *division-ring fps*) = *1*
by (*rule fps-inverse-one'*[*OF inverse-1*])

lemma *fps-lr-inverse-minus*:
fixes *f* :: *'a*::*ring-1 fps*
shows *fps-left-inverse* (−*f*) (−*x*) = − *fps-left-inverse f x*
and *fps-right-inverse* (−*f*) (−*x*) = − *fps-right-inverse f x*
proof−

show *fps-left-inverse* (−*f*) (−*x*) = − *fps-left-inverse f x*
proof (*intro fps-ext*)
fix *n* **show** *fps-left-inverse* (−*f*) (−*x*) $ *n* = − *fps-left-inverse f x* $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*) **thus** *?case* **by** (*cases n*) (*simp-all add*: *sum-negf algebra-simps*)
qed
qed

show *fps-right-inverse* (−*f*) (−*x*) = − *fps-right-inverse f x*
proof (*intro fps-ext*)
fix *n* **show** *fps-right-inverse* (−*f*) (−*x*) $ *n* = − *fps-right-inverse f x* $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*) **show** *?case*
proof (*cases n*)
case (*Suc m*)
with *1* **have**
∀ *i*∈{*1*..*Suc m*}. *fps-right-inverse* (−*f*) (−*x*) $ (*Suc m* − *i*) =
− *fps-right-inverse f x* $ (*Suc m* − *i*)
by *auto*
with *Suc* **show** *?thesis* **by** (*simp add*: *sum-negf algebra-simps*)
qed *simp*
qed
qed

qed

lemma *fps-inverse-minus* [*simp*]: *inverse* (−*f*) = −*inverse* (*f* :: ′*a* :: *division-ring fps*)
  by (*simp add*: *fps-inverse-def fps-lr-inverse-minus*(*2*))

lemma *fps-left-inverse*:
  fixes *f* :: ′*a*::*ring-1 fps*
  assumes *f0*: *x* ∗ *f*$*0* = *1*
  shows *fps-left-inverse f x* ∗ *f* = *1*
proof (*rule fps-ext*)
  fix *n* **show** (*fps-left-inverse f x* ∗ *f*) $ *n* = *1* $ *n*
    by (*cases n*) (*simp-all add*: *f0 fps-mult-nth mult.assoc*)
qed

lemma *fps-right-inverse*:
  fixes *f* :: ′*a*::*ring-1 fps*
  assumes *f0*: *f*$*0* ∗ *y* = *1*
  shows *f* ∗ *fps-right-inverse f y* = *1*
proof (*rule fps-ext*)
  fix *n*
  show (*f* ∗ *fps-right-inverse f y*) $ *n* = *1* $ *n*
  proof (*cases n*)
    case (*Suc k*)
    moreover from *Suc* **have** *fps-right-inverse f y* $ *n* =
          − *y* ∗ *sum* (λ*i*. *f*$*i* ∗ *fps-right-inverse-constructor f y* (*n* − *i*)) {*1*..*n*}
      by *simp*
    hence
      (*f* ∗ *fps-right-inverse f y*) $ *n* =
        − *1* ∗ *sum* (λ*i*. *f*$*i* ∗ *fps-right-inverse-constructor f y* (*n* − *i*)) {*1*..*n*} +
        sum (λ*i*. *f*$*i* ∗ (*fps-right-inverse-constructor f y* (*n* − *i*))) {*1*..*n*}
     by (*simp add*: *fps-mult-nth sum.atLeast-Suc-atMost mult.assoc f0*[*symmetric*])
    thus (*f* ∗ *fps-right-inverse f y*) $ *n* = *1* $ *n* **by** (*simp add*: *Suc*)
  qed (*simp add*: *f0 fps-inverse-def*)
qed

It is possible in a ring for an element to have a left inverse but not a right inverse, or vice versa. But when an element has both, they must be the same.

lemma *fps-left-inverse-eq-fps-right-inverse*:
  fixes *f* :: ′*a*::*ring-1 fps*
  assumes *f0*: *x* ∗ *f*$*0* = *1 f* $ *0* ∗ *y* = *1*
  — These assumptions imply that *x* equals *y*, but no need to assume that.
  shows *fps-left-inverse f x* = *fps-right-inverse f y*
proof−
  from *f0*(*2*) **have** *f* ∗ *fps-right-inverse f y* = *1*
      by (*simp add*: *fps-right-inverse*)
  hence *fps-left-inverse f x* ∗ *f* ∗ *fps-right-inverse f y* = *fps-left-inverse f x*

by (*simp add*: *mult.assoc*)
moreover from *f0*(*1*) **have**
fps-left-inverse f x ∗ *f* ∗ *fps-right-inverse f y* = *fps-right-inverse f y*
by (*simp add*: *fps-left-inverse*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fps-left-inverse-eq-fps-right-inverse-comm*:
fixes *f* :: ′*a*::*comm-ring-1 fps*
assumes *f0*: *x* ∗ *f*\$*0* = *1*
shows *fps-left-inverse f x* = *fps-right-inverse f x*
using *assms fps-left-inverse-eq-fps-right-inverse*[*of x f x*]
by (*simp add*: *mult.commute*)

lemma *fps-left-inverse*′:
fixes *f* :: ′*a*::*ring-1 fps*
assumes *x* ∗ *f*\$*0* = *1 f*\$*0* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *fps-right-inverse f y* ∗ *f* = *1*
using *assms fps-left-inverse-eq-fps-right-inverse*[*of x f y*] *fps-left-inverse*[*of x f*]
by *simp*

lemma *fps-right-inverse*′:
fixes *f* :: ′*a*::*ring-1 fps*
assumes *x* ∗ *f*\$*0* = *1 f*\$*0* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *f* ∗ *fps-left-inverse f x* = *1*
using *assms fps-left-inverse-eq-fps-right-inverse*[*of x f y*] *fps-right-inverse*[*of f y*]
by *simp*

lemma *inverse-mult-eq-1* [*intro*]:
assumes *f*\$*0* ≠ (*0*::′*a*::*division-ring*)
shows *inverse f* ∗ *f* = *1*
using *fps-left-inverse*′[*of inverse* (*f*\$*0*)]
by (*simp add*: *assms fps-inverse-def*)

lemma *inverse-mult-eq-1* ′:
assumes *f*\$*0* ≠ (*0*::′*a*::*division-ring*)
shows *f* ∗ *inverse f* = *1*
using *assms fps-right-inverse*
by (*force simp*: *fps-inverse-def*)

lemma *fps-mult-left-inverse-unit-factor*:
fixes *f* :: ′*a*::*ring-1 fps*
assumes *x* ∗ *f* \$ *subdegree f* = *1*
shows *fps-left-inverse* (*unit-factor f*) *x* ∗ *f* = *fps-X* ^ *subdegree f*
proof−
have
fps-left-inverse (*unit-factor f*) *x* ∗ *f* =

fps-left-inverse (*unit-factor f*) *x* ∗ *unit-factor f* ∗ *fps-X* ^ *subdegree f*
using *fps-unit-factor-decompose*[*of f*] **by** (*simp add*: *mult.assoc*)
with *assms* **show** *?thesis* **by** (*simp add*: *fps-left-inverse*)
qed

lemma *fps-mult-right-inverse-unit-factor*:
fixes *f* :: *'a*::*ring-1 fps*
assumes *f* $ *subdegree f* ∗ *y* = *1*
shows *f* ∗ *fps-right-inverse* (*unit-factor f*) *y* = *fps-X* ^ *subdegree f*
proof−
have
f ∗ *fps-right-inverse* (*unit-factor f*) *y* =
fps-X ^ *subdegree f* ∗ (*unit-factor f* ∗ *fps-right-inverse* (*unit-factor f*) *y*)
using *fps-unit-factor-decompose'*[*of f*] **by** (*simp add*: *mult.assoc*[*symmetric*])
with *assms* **show** *?thesis* **by** (*simp add*: *fps-right-inverse*)
qed

lemma *fps-mult-right-inverse-unit-factor-divring*:
(*f* :: *'a*::*division-ring fps*) ≠ *0* ⟹ *f* ∗ *inverse* (*unit-factor f*) = *fps-X* ^ *subdegree f*
using *fps-mult-right-inverse-unit-factor*[*of f*]
by (*simp add*: *fps-inverse-def*)

lemma *fps-left-inverse-idempotent-ring1*:
fixes *f* :: *'a*::*ring-1 fps*
assumes *x* ∗ *f*$*0* = *1* *y* ∗ *x* = *1*
— These assumptions imply *y* equals *f*$*0*, but no need to assume that.
shows *fps-left-inverse* (*fps-left-inverse f x*) *y* = *f*
proof−
from *assms*(*1*) **have**
fps-left-inverse (*fps-left-inverse f x*) *y* ∗ *fps-left-inverse f x* ∗ *f* =
fps-left-inverse (*fps-left-inverse f x*) *y*
by (*simp add*: *fps-left-inverse mult.assoc*)
moreover from *assms*(*2*) **have**
fps-left-inverse (*fps-left-inverse f x*) *y* ∗ *fps-left-inverse f x* = *1*
by (*simp add*: *fps-left-inverse*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fps-left-inverse-idempotent-comm-ring1*:
fixes *f* :: *'a*::*comm-ring-1 fps*
assumes *x* ∗ *f*$*0* = *1*
shows *fps-left-inverse* (*fps-left-inverse f x*) (*f*$*0*) = *f*
using *assms fps-left-inverse-idempotent-ring1*[*of x f f*$*0*]
by (*simp add*: *mult.commute*)

lemma *fps-right-inverse-idempotent-ring1*:
fixes *f* :: *'a*::*ring-1 fps*
assumes *f*$*0* ∗ *x* = *1* *x* ∗ *y* = *1*

— These assumptions imply *y* equals *f*$*0*, but no need to assume that.
shows *fps-right-inverse* (*fps-right-inverse f x*) *y* = *f*
proof−
from *assms*(*1*) **have** *f* ∗ (*fps-right-inverse f x* ∗ *fps-right-inverse* (*fps-right-inverse f x*) *y*) =
fps-right-inverse (*fps-right-inverse f x*) *y*
by (*simp add*: *fps-right-inverse mult.assoc*[*symmetric*])
moreover from *assms*(*2*) **have**
fps-right-inverse f x ∗ *fps-right-inverse* (*fps-right-inverse f x*) *y* = *1*
by (*simp add*: *fps-right-inverse*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fps-right-inverse-idempotent-comm-ring1*:
fixes *f* :: *'a*::*comm-ring-1 fps*
assumes *f*$*0* ∗ *x* = *1*
shows *fps-right-inverse* (*fps-right-inverse f x*) (*f*$*0*) = *f*
using *assms fps-right-inverse-idempotent-ring1*[*of f x f*$*0*]
by (*simp add*: *mult.commute*)

lemma *fps-inverse-idempotent*[*intro*, *simp*]:
f$*0* ≠ (*0*::*'a*::*division-ring*) ⟹ *inverse* (*inverse f*) = *f*
using *fps-right-inverse-idempotent-ring1*[*of f*]
by (*simp add*: *fps-inverse-def*)

lemma *fps-lr-inverse-unique-ring1*:
fixes *f g* :: *'a* :: *ring-1 fps*
assumes *fg*: *f* ∗ *g* = *1 g*$*0* ∗ *f*$*0* = *1*
shows *fps-left-inverse g* (*f*$*0*) = *f*
and *fps-right-inverse f* (*g*$*0*) = *g*
proof−

show *fps-left-inverse g* (*f*$*0*) = *f*
proof (*intro fps-ext*)
fix *n* **show** *fps-left-inverse g* (*f*$*0*) $ *n* = *f* $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*) **show** *?case*
proof (*cases n*)
case (*Suc k*)
hence ∀ *i*∈{*0*..*k*}. *fps-left-inverse g* (*f*$*0*) $ *i* = *f* $ *i* **using** *1* **by** *simp*
hence *fps-left-inverse g* (*f*$*0*) $ *Suc k* = *f* $ *Suc k* − *1* $ *Suc k* ∗ *f*$*0*
by (*simp add*: *fps-mult-nth fg*(*1*)[*symmetric*] *distrib-right mult.assoc fg*(*2*))
with *Suc* **show** *?thesis* **by** *simp*
qed *simp*
qed
qed

show *fps-right-inverse f* (*g*$*0*) = *g*
proof (*intro fps-ext*)

fix *n* **show** *fps-right-inverse f* (*g*$*0*) $ *n* = *g* $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*) **show** *?case*
proof (*cases n*)
case (*Suc k*)
hence ∀ *i*∈{*1*..*Suc k*}. *fps-right-inverse f* (*g*$*0*) $ (*Suc k* − *i*) = *g* $ (*Suc k* − *i*)
using *1* **by** *auto*
hence
fps-right-inverse f (*g*$*0*) $ *Suc k* = *1* ∗ *g* $ *Suc k* − *g*$*0* ∗ *1* $ *Suc k*
by (*simp add*: *fps-mult-nth fg*(*1*)[*symmetric*] *algebra-simps fg*(*2*)[*symmetric*] *sum.atLeast-Suc-atMost*)
with *Suc* **show** *?thesis* **by** *simp*
qed *simp*
qed
qed

qed

lemma *fps-lr-inverse-unique-divring*:
fixes *f g* :: ′*a* ::*division-ring fps*
assumes *fg*: *f* ∗ *g* = *1*
shows *fps-left-inverse g* (*f*$*0*) = *f*
and *fps-right-inverse f* (*g*$*0*) = *g*
proof−
from *fg* **have** *f*$*0* ∗ *g*$*0* = *1* **using** *fps-mult-nth-0*[*of f g*] **by** *simp*
hence *g*$*0* ∗ *f*$*0* = *1* **using** *inverse-unique*[*of f*$*0*] *left-inverse*[*of f*$*0*] **by** *force*
thus *fps-left-inverse g* (*f*$*0*) = *f fps-right-inverse f* (*g*$*0*) = *g*
using *fg fps-lr-inverse-unique-ring1* **by** *auto*
qed

lemma *fps-inverse-unique*:
fixes *f g* :: ′*a* :: *division-ring fps*
assumes *fg*: *f* ∗ *g* = *1*
shows *inverse f* = *g*
proof −
from *fg* **have** *if0*: *inverse* (*f*$*0*) = *g*$*0 f*$*0* ≠ *0*
using *inverse-unique*[*of f*$*0*] *fps-mult-nth-0*[*of f g*] **by** *auto*
with *fg* **have** *fps-right-inverse f* (*g*$*0*) = *g*
using *left-inverse*[*of f*$*0*] **by** (*intro fps-lr-inverse-unique-ring1*(*2*)) *simp-all*
with *if0*(*1*) **show** *?thesis* **by** (*simp add*: *fps-inverse-def*)
qed

lemma *inverse-fps-numeral*:
inverse (*numeral n* :: (′*a* :: *field-char-0*) *fps*) = *fps-const* (*inverse* (*numeral n*))
by (*intro fps-inverse-unique fps-ext*) (*simp-all add*: *fps-numeral-nth*)

lemma *inverse-fps-of-nat*:
inverse (*of-nat n* :: ′*a* :: {*semiring-1*,*times*,*uminus*,*inverse*} *fps*) =

fps-const (*inverse* (*of-nat n*))
by (*simp add*: *fps-of-nat fps-const-inverse*[*symmetric*])

lemma *fps-lr-inverse-mult-ring1*:
fixes *f g* :: *'a::ring-1 fps*
assumes *x*: *x* ∗ *f*\$*0* = *1 f*\$*0* ∗ *x* = *1*
and *y*: *y* ∗ *g*\$*0* = *1 g*\$*0* ∗ *y* = *1*
shows *fps-left-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fps-left-inverse g y* ∗ *fps-left-inverse f x*
and *fps-right-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fps-right-inverse g y* ∗ *fps-right-inverse f x*
proof −
define *h* **where** *h* ≡ *fps-left-inverse g y* ∗ *fps-left-inverse f x*
hence *h0*: *h*\$*0* = *y*∗*x* **by** *simp*
have *fps-left-inverse* (*f*∗*g*) (*h*\$*0*) = *h*
proof (*intro fps-lr-inverse-unique-ring1*(*1*))
from *h-def*
have *h* ∗ (*f* ∗ *g*) = *fps-left-inverse g y* ∗ (*fps-left-inverse f x* ∗ *f*) ∗ *g*
by (*simp add*: *mult.assoc*)
thus *h* ∗ (*f* ∗ *g*) = *1*
using *fps-left-inverse*[*OF x*(*1*)] *fps-left-inverse*[*OF y*(*1*)] **by** *simp*
from *h-def* **have** (*f*∗*g*)\$*0* ∗ *h*\$*0* = *f*\$*0* ∗ *1* ∗ *x*
by (*simp add*: *mult.assoc y*(*2*)[*symmetric*])
with *x*(*2*) **show** (*f* ∗ *g*) \$ *0* ∗ *h* \$ *0* = *1* **by** *simp*
qed
with *h-def*
show *fps-left-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fps-left-inverse g y* ∗ *fps-left-inverse f x*
by *simp*
next
define *h* **where** *h* ≡ *fps-right-inverse g y* ∗ *fps-right-inverse f x*
hence *h0*: *h*\$*0* = *y*∗*x* **by** *simp*
have *fps-right-inverse* (*f*∗*g*) (*h*\$*0*) = *h*
proof (*intro fps-lr-inverse-unique-ring1*(*2*))
from *h-def*
have *f* ∗ *g* ∗ *h* = *f* ∗ (*g* ∗ *fps-right-inverse g y*) ∗ *fps-right-inverse f x*
by (*simp add*: *mult.assoc*)
thus *f* ∗ *g* ∗ *h* = *1*
using *fps-right-inverse*[*OF x*(*2*)] *fps-right-inverse*[*OF y*(*2*)] **by** *simp*
from *h-def* **have** *h*\$*0* ∗ (*f*∗*g*)\$*0* = *y* ∗ *1* ∗ *g*\$*0*
by (*simp add*: *mult.assoc x*(*1*)[*symmetric*])
with *y*(*1*) **show** *h*\$*0* ∗ (*f*∗*g*)\$*0* = *1* **by** *simp*
qed
with *h-def*
show *fps-right-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fps-right-inverse g y* ∗ *fps-right-inverse f x*
by *simp*
qed

lemma *fps-lr-inverse-mult-divring*:
fixes *f g* :: *'a::division-ring fps*

shows *fps-left-inverse* (*f* ∗ *g*) (*inverse* ((*f*∗*g*)\$*0*)) =
fps-left-inverse g (*inverse* (*g*\$*0*)) ∗ *fps-left-inverse f* (*inverse* (*f*\$*0*))
and *fps-right-inverse* (*f* ∗ *g*) (*inverse* ((*f*∗*g*)\$*0*)) =
fps-right-inverse g (*inverse* (*g*\$*0*)) ∗ *fps-right-inverse f* (*inverse* (*f*\$*0*))
proof−
show *fps-left-inverse* (*f* ∗ *g*) (*inverse* ((*f*∗*g*)\$*0*)) =
fps-left-inverse g (*inverse* (*g*\$*0*)) ∗ *fps-left-inverse f* (*inverse* (*f*\$*0*))
proof (*cases f*\$*0* = *0* ∨ *g*\$*0* = *0*)
case *True*
hence *fps-left-inverse* (*f* ∗ *g*) (*inverse* ((*f*∗*g*)\$*0*)) = *0*
by (*simp add*: *fps-lr-inverse-eq-0-iff* (*1*))
moreover from *True* **have**
fps-left-inverse g (*inverse* (*g*\$*0*)) ∗ *fps-left-inverse f* (*inverse* (*f*\$*0*)) = *0*
by (*auto simp*: *fps-lr-inverse-eq-0-iff* (*1*))
ultimately show *?thesis* **by** *simp*
next
case *False*
hence *fps-left-inverse* (*f* ∗ *g*) (*inverse* (*g*\$*0*) ∗ *inverse* (*f*\$*0*)) =
fps-left-inverse g (*inverse* (*g*\$*0*)) ∗ *fps-left-inverse f* (*inverse* (*f*\$*0*))
by (*intro fps-lr-inverse-mult-ring1* (*1*)) *simp-all*
with *False* **show** *?thesis* **by** (*simp add*: *nonzero-inverse-mult-distrib*)
qed
show *fps-right-inverse* (*f* ∗ *g*) (*inverse* ((*f*∗*g*)\$*0*)) =
fps-right-inverse g (*inverse* (*g*\$*0*)) ∗ *fps-right-inverse f* (*inverse* (*f*\$*0*))
proof (*cases f*\$*0* = *0* ∨ *g*\$*0* = *0*)
case *True*
from *True* **have** *fps-right-inverse* (*f* ∗ *g*) (*inverse* ((*f*∗*g*)\$*0*)) = *0*
by (*simp add*: *fps-lr-inverse-eq-0-iff* (*2*))
moreover from *True* **have**
fps-right-inverse g (*inverse* (*g*\$*0*)) ∗ *fps-right-inverse f* (*inverse* (*f*\$*0*)) = *0*
by (*auto simp*: *fps-lr-inverse-eq-0-iff* (*2*))
ultimately show *?thesis* **by** *simp*
next
case *False*
hence *fps-right-inverse* (*f* ∗ *g*) (*inverse* (*g*\$*0*) ∗ *inverse* (*f*\$*0*)) =
fps-right-inverse g (*inverse* (*g*\$*0*)) ∗ *fps-right-inverse f* (*inverse* (*f*\$*0*))
by (*intro fps-lr-inverse-mult-ring1* (*2*)) *simp-all*
with *False* **show** *?thesis* **by** (*simp add*: *nonzero-inverse-mult-distrib*)
qed
qed

lemma *fps-inverse-mult-divring*:
inverse (*f* ∗ *g*) = *inverse g* ∗ *inverse* (*f* :: *'a*::*division-ring fps*)
using *fps-lr-inverse-mult-divring*(*2*) **by** (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-mult*: *inverse* (*f* ∗ *g* :: *'a*::*field fps*) = *inverse f* ∗ *inverse g*
by (*simp add*: *fps-inverse-mult-divring*)

lemma *fps-lr-inverse-gp-ring1*:

fixes *ones ones-inv* :: *'a* :: *ring-1 fps*
defines *ones* ≡ *Abs-fps* (*λn. 1*)
and *ones-inv* ≡ *Abs-fps* (*λn. if n=0 then 1 else if n=1 then − 1 else 0*)
shows *fps-left-inverse ones 1* = *ones-inv*
and *fps-right-inverse ones 1* = *ones-inv*
proof−
show *fps-left-inverse ones 1* = *ones-inv*
proof (*rule fps-ext*)
fix *n*
show *fps-left-inverse ones 1* $ *n* = *ones-inv* $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*) **show** *?case*
proof (*cases n*)
case (*Suc m*)
have *m*: *n* = *Suc m* **by** *fact*
moreover have *fps-left-inverse ones 1* $ *Suc m* = *ones-inv* $ *Suc m*
proof (*cases m*)
case (*Suc k*) **thus** *?thesis*
using *Suc m 1* **by** (*simp add*: *ones-def ones-inv-def sum.atLeast-Suc-atMost*)
qed (*simp add*: *ones-def ones-inv-def*)
ultimately show *?thesis* **by** *simp*
qed (*simp add*: *ones-inv-def*)
qed
qed
moreover have *fps-right-inverse ones 1* = *fps-left-inverse ones 1*
by (*auto intro*: *fps-left-inverse-eq-fps-right-inverse*[*symmetric*] *simp*: *ones-def*)
ultimately show *fps-right-inverse ones 1* = *ones-inv* **by** *simp*
qed

lemma *fps-lr-inverse-gp-ring1'*:
fixes *ones* :: *'a* :: *ring-1 fps*
defines *ones* ≡ *Abs-fps* (*λn. 1*)
shows *fps-left-inverse ones 1* = *1* − *fps-X*
and *fps-right-inverse ones 1* = *1* − *fps-X*
proof−
define *ones-inv* :: *'a* :: *ring-1 fps*
where *ones-inv* ≡ *Abs-fps* (*λn. if n=0 then 1 else if n=1 then − 1 else 0*)
hence *fps-left-inverse ones 1* = *ones-inv*
and *fps-right-inverse ones 1* = *ones-inv*
using *ones-def fps-lr-inverse-gp-ring1* **by** *auto*
thus *fps-left-inverse ones 1* = *1* − *fps-X*
and *fps-right-inverse ones 1* = *1* − *fps-X*
by (*auto intro*: *fps-ext simp*: *ones-inv-def*)
qed

lemma *fps-inverse-gp*:
inverse (*Abs-fps*(*λn.* (*1*::*'a*::*division-ring*))) =
Abs-fps (*λn. if n= 0 then 1 else if n=1 then − 1 else 0*)
using *fps-lr-inverse-gp-ring1*(*2*) **by** (*simp add*: *fps-inverse-def*)

```
lemma fps-inverse-gp': inverse (Abs-fps (λn. 1::'a::division-ring)) = 1 − fps-X
  by (simp add: fps-inverse-def fps-lr-inverse-gp-ring1'(2))

lemma fps-lr-inverse-one-minus-fps-X:
  fixes   ones :: 'a :: ring-1 fps
  defines ones ≡ Abs-fps (λn. 1)
  shows fps-left-inverse (1 − fps-X) 1 = ones
  and   fps-right-inverse (1 − fps-X) 1 = ones
proof−
  have fps-left-inverse ones 1 = 1 − fps-X
    using fps-lr-inverse-gp-ring1'(1) by (simp add: ones-def)
  thus fps-left-inverse (1 − fps-X) 1 = ones
    using fps-left-inverse-idempotent-ring1[of 1 ones 1] by (simp add: ones-def)
  have fps-right-inverse ones 1 = 1 − fps-X
    using fps-lr-inverse-gp-ring1'(2) by (simp add: ones-def)
  thus fps-right-inverse (1 − fps-X) 1 = ones
    using fps-right-inverse-idempotent-ring1[of ones 1 1] by (simp add: ones-def)
qed

lemma fps-inverse-one-minus-fps-X:
  fixes   ones :: 'a :: division-ring fps
  defines ones ≡ Abs-fps (λn. 1)
  shows   inverse (1 − fps-X) = ones
  by      (simp add: fps-inverse-def assms fps-lr-inverse-one-minus-fps-X(2))

lemma fps-lr-one-over-one-minus-fps-X-squared:
  shows   fps-left-inverse ((1 − fps-X)^2) (1::'a::ring-1) = Abs-fps (λn. of-nat
(n+1))
          fps-right-inverse ((1 − fps-X)^2) (1::'a) = Abs-fps (λn. of-nat (n+1))
proof−
  define f invf2 :: 'a fps
    where f ≡ (1 − fps-X)
    and   invf2 ≡ Abs-fps (λn. of-nat (n+1))

  have f2-nth-simps:
    f^2 $ 1 = − of-nat 2 f^2 $ 2 = 1 ⋀n. n>2 ⟹ f^2 $ n = 0
     by (simp-all add: power2-eq-square f-def fps-mult-nth sum.atLeast-Suc-atMost)

  show fps-left-inverse (f^2) 1 = invf2
  proof (intro fps-ext)
    fix n show fps-left-inverse (f^2) 1 $ n = invf2 $ n
    proof (induct n rule: nat-less-induct)
      case (1 t)
      hence induct-assm:
        ⋀m. m < t ⟹ fps-left-inverse (f²) 1 $ m = invf2 $ m
        by fast
      show ?case
      proof (cases t)
```

```
      case (Suc m)
      have m: t = Suc m by fact
      moreover have fps-left-inverse (f^2) 1 $ Suc m = invf2 $ Suc m
      proof (cases m)
        case 0 thus ?thesis using f2-nth-simps(1) by (simp add: invf2-def)
      next
        case (Suc l)
        have l: m = Suc l by fact
        moreover have fps-left-inverse (f^2) 1 $ Suc (Suc l) = invf2 $ Suc (Suc
l)
        proof (cases l)
       case 0 thus ?thesis using f2-nth-simps(1,2) by (simp add: Suc-1[symmetric]
invf2-def)
        next
          case (Suc k)
          from Suc l m
            have A: fps-left-inverse (f²) 1 $ Suc (Suc k) = invf2 $ Suc (Suc k)
            and  B: fps-left-inverse (f²) 1 $ Suc k = invf2 $ Suc k
            using induct-assm[of Suc k] induct-assm[of Suc (Suc k)]
            by    auto
          have times2: ⋀a::nat. 2*a = a + a by simp
          have ∀ i∈{0..k}. (f^2)$(Suc (Suc (Suc k)) − i) = 0
            using f2-nth-simps(3) by auto
          hence
            fps-left-inverse (f^2) 1 $ Suc (Suc (Suc k)) =
              fps-left-inverse (f²) 1 $ Suc (Suc k) ∗ of-nat 2 −
              fps-left-inverse (f²) 1 $ Suc k
            using sum.ub-add-nat f2-nth-simps(1,2) by simp
          also have ... = of-nat (2 ∗ Suc (Suc (Suc k))) − of-nat (Suc (Suc k))
            by (subst A, subst B) (simp add: invf2-def mult.commute)
          also have ... = of-nat (Suc (Suc (Suc k)) + 1)
            by (subst times2[of Suc (Suc (Suc k))]) simp
          finally have
             fps-left-inverse (f^2) 1 $ Suc (Suc (Suc k)) = invf2 $ Suc (Suc (Suc
k))
             by (simp add: invf2-def)
          with Suc show ?thesis by simp
        qed
        ultimately show ?thesis by simp
      qed
      ultimately show ?thesis by simp
    qed (simp add: invf2-def)
  qed
qed

moreover have fps-right-inverse (f^2) 1 = fps-left-inverse (f^2) 1
  by (auto
      intro: fps-left-inverse-eq-fps-right-inverse[symmetric]
      simp: f-def power2-eq-square
```

```
      )
  ultimately show fps-right-inverse (f^2) 1 = invf2
    by simp

qed

lemma fps-one-over-one-minus-fps-X-squared':
  assumes inverse (1::'a::{ring-1,inverse}) = 1
  shows   inverse ((1 - fps-X)^2 :: 'a fps) = Abs-fps (λn. of-nat (n+1))
  using   assms fps-lr-one-over-one-minus-fps-X-squared(2)
  by      (simp add: fps-inverse-def power2-eq-square)

lemma fps-one-over-one-minus-fps-X-squared:
  inverse ((1 - fps-X)^2 :: 'a :: division-ring fps) = Abs-fps (λn. of-nat (n+1))
  by (rule fps-one-over-one-minus-fps-X-squared'[OF inverse-1])

lemma fps-lr-inverse-fps-X-plus1:
  fps-left-inverse (1 + fps-X) (1::'a::ring-1) = Abs-fps (λn. (-1)^n)
  fps-right-inverse (1 + fps-X) (1::'a) = Abs-fps (λn. (-1)^n)
proof-

  show fps-left-inverse (1 + fps-X) (1::'a) = Abs-fps (λn. (-1)^n)
  proof (rule fps-ext)
    fix n show fps-left-inverse (1 + fps-X) (1::'a) $ n = Abs-fps (λn. (-1)^n) $
n
    proof (induct n rule: nat-less-induct)
      case (1 n) show ?case
      proof (cases n)
        case (Suc m)
        have m: n = Suc m by fact
        from Suc 1 have
          A: fps-left-inverse (1 + fps-X) (1::'a) $ n =
                - (∑ i=0..m. (- 1)^i * (1 + fps-X) $ (Suc m - i))
          by simp
        show ?thesis
        proof (cases m)
          case (Suc l)
          have ∀ i∈{0..l}. ((1::'a fps) + fps-X) $ (Suc (Suc l) - i) = 0 by auto
          with Suc A m show ?thesis by simp
        qed (simp add: m A)
      qed simp
    qed
  qed

  moreover have
    fps-right-inverse (1 + fps-X) (1::'a) = fps-left-inverse (1 + fps-X) 1
    by (intro fps-left-inverse-eq-fps-right-inverse[symmetric]) simp-all
  ultimately show fps-right-inverse (1 + fps-X) (1::'a) = Abs-fps (λn. (-1)^n)
by simp
```

qed

lemma *fps-inverse-fps-X-plus1′*:
  assumes *inverse* (*1*::′*a*::{*ring-1*,*inverse*}) = *1*
  shows *inverse* (*1* + *fps-X*) = *Abs-fps* (λ*n*. (− (*1*::′*a*)) ^ *n*)
  using *assms fps-lr-inverse-fps-X-plus1*(*2*)
  by (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-fps-X-plus1*:
  inverse (*1* + *fps-X*) = *Abs-fps* (λ*n*. (− (*1*::′*a*::*division-ring*)) ^ *n*)
  by (*rule fps-inverse-fps-X-plus1′*[*OF inverse-1*])

lemma *subdegree-lr-inverse*:
  fixes *x* :: ′*a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
  and *y* :: ′*b*::{*ab-group-add*,*mult-zero*}
  shows *subdegree* (*fps-left-inverse f x*) = *0*
  and *subdegree* (*fps-right-inverse g y*) = *0*
proof−
  show *subdegree* (*fps-left-inverse f x*) = *0*
    using *fps-lr-inverse-eq-0-iff*(*1*) *subdegree-eq-0-iff* **by** *fastforce*
  show *subdegree* (*fps-right-inverse g y*) = *0*
    using *fps-lr-inverse-eq-0-iff*(*2*) *subdegree-eq-0-iff* **by** *fastforce*
qed

lemma *subdegree-inverse* [*simp*]:
  fixes *f* :: ′*a*::{*ab-group-add*,*inverse*,*mult-zero*} *fps*
  shows *subdegree* (*inverse f*) = *0*
  using *subdegree-lr-inverse*(*2*)
  by (*simp add*: *fps-inverse-def*)

lemma *fps-div-zero* [*simp*]:
  0 div (*g* :: ′*a* :: {*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fps*) = *0*
  by (*simp add*: *fps-divide-def*)

lemma *fps-div-by-zero′*:
  fixes *g* :: ′*a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fps*
  assumes *inverse* (*0*::′*a*) = *0*
  shows *g div 0* = *0*
  by (*simp add*: *fps-divide-def assms fps-inverse-zero′*)

lemma *fps-div-by-zero* [*simp*]: (*g*::′*a*::*division-ring fps*) *div 0* = *0*
  by (*rule fps-div-by-zero′*[*OF inverse-zero*])

lemma *fps-divide-unit′*: *subdegree g* = *0* ⟹ *f div g* = *f* ∗ *inverse g*
  by (*simp add*: *fps-divide-def*)

lemma *fps-divide-unit*: *g*$*0* ≠ *0* ⟹ *f div g* = *f* ∗ *inverse g*
  by (*intro fps-divide-unit′*) (*simp add*: *subdegree-eq-0-iff*)

lemma *fps-divide-nth-0'*:
subdegree (*g*::'*a*::*division-ring fps*) = *0* ⟹ (*f div g*) $ *0* = *f* $ *0* / (*g* $ *0*)
by (*simp add*: *fps-divide-unit' divide-inverse*)

lemma *fps-divide-nth-0* [*simp*]:
g $ *0* ≠ *0* ⟹ (*f div g*) $ *0* = *f* $ *0* / (*g* $ *0* :: - :: *division-ring*)
by (*simp add*: *fps-divide-nth-0'*)

lemma *fps-divide-nth-below*:
fixes *f g* :: '*a*::{*comm-monoid-add*,*uminus*,*mult-zero*,*inverse*} *fps*
shows *n* < *subdegree f* − *subdegree g* ⟹ (*f div g*) $ *n* = *0*
by (*simp add*: *fps-divide-def fps-mult-nth-eq0*)

lemma *fps-divide-nth-base*:
fixes *f g* :: '*a*::*division-ring fps*
assumes *subdegree g* ≤ *subdegree f*
shows (*f div g*) $ (*subdegree f* − *subdegree g*) = *f* $ *subdegree f* ∗ *inverse* (*g* $ *subdegree g*)
by (*simp add*: *assms fps-divide-def fps-divide-unit'*)

lemma *fps-divide-subdegree-ge*:
fixes *f g* :: '*a*::{*comm-monoid-add*,*uminus*,*mult-zero*,*inverse*} *fps*
assumes *f* / *g* ≠ *0*
shows *subdegree* (*f* / *g*) ≥ *subdegree f* − *subdegree g*
by (*intro subdegree-geI*) (*simp-all add*: *assms fps-divide-nth-below*)

lemma *fps-divide-subdegree*:
fixes *f g* :: '*a*::*division-ring fps*
assumes *f* ≠ *0* *g* ≠ *0* *subdegree g* ≤ *subdegree f*
shows *subdegree* (*f* / *g*) = *subdegree f* − *subdegree g*
proof (*intro antisym*)
from *assms* **have** *1*: (*f div g*) $ (*subdegree f* − *subdegree g*) ≠ *0*
using *fps-divide-nth-base*[*of g f*] **by** *simp*
thus *subdegree* (*f* / *g*) ≤ *subdegree f* − *subdegree g* **by** (*intro subdegree-leI*) *simp*
from *1* **have** *f* / *g* ≠ *0* **by** (*auto intro*: *fps-nonzeroI*)
thus *subdegree f* − *subdegree g* ≤ *subdegree* (*f* / *g*) **by** (*rule fps-divide-subdegree-ge*)
qed

lemma *fps-divide-shift-numer*:
fixes *f g* :: '*a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
assumes *n* ≤ *subdegree f*
shows *fps-shift n f* / *g* = *fps-shift n* (*f*/*g*)
using *assms fps-shift-mult-right-noncomm*[*of n f inverse* (*unit-factor g*)]
fps-shift-fps-shift-reorder[*of subdegree g n f* ∗ *inverse* (*unit-factor g*)]
by (*simp add*: *fps-divide-def*)

lemma *fps-divide-shift-denom*:
fixes *f g* :: '*a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*

assumes *n* ≤ *subdegree g subdegree g* ≤ *subdegree f*
shows *f* / *fps-shift n g* = *Abs-fps* (λ*k*. *if k*<*n then 0 else* (*f*/*g*) $ (*k*−*n*))
proof (*intro fps-ext*)
fix *k*
from *assms*(*1*) **have** *LHS*:
(*f* / *fps-shift n g*) $ *k* = (*f* ∗ *inverse* (*unit-factor g*)) $ (*k* + (*subdegree g* − *n*))
using *fps-unit-factor-shift*[*of n g*]
by (*simp add*: *fps-divide-def*)
show (*f* / *fps-shift n g*) $ *k* = *Abs-fps* (λ*k*. *if k*<*n then 0 else* (*f*/*g*) $ (*k*−*n*)) $ *k*
proof (*cases k*<*n*)
case *True* **with** *assms LHS* **show** *?thesis* **using** *fps-mult-nth-eq0*[*of - f*] **by** *simp*
next
case *False*
hence (*f*/*g*) $ (*k*−*n*) = (*f* ∗ *inverse* (*unit-factor g*)) $ ((*k*−*n*) + *subdegree g*)
by (*simp add*: *fps-divide-def*)
with *False LHS assms*(*1*) **show** *?thesis* **by** *auto*
qed
qed

lemma *fps-divide-unit-factor-numer*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
shows *unit-factor f* / *g* = *fps-shift* (*subdegree f*) (*f*/*g*)
by (*simp add*: *fps-divide-shift-numer*)

lemma *fps-divide-unit-factor-denom*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
assumes *subdegree g* ≤ *subdegree f*
shows
f / *unit-factor g* = *Abs-fps* (λ*k*. *if k*<*subdegree g then 0 else* (*f*/*g*) $ (*k*−*subdegree g*))
by (*simp add*: *assms fps-divide-shift-denom*)

lemma *fps-divide-unit-factor-both'*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
assumes *subdegree g* ≤ *subdegree f*
shows *unit-factor f* / *unit-factor g* = *fps-shift* (*subdegree f* − *subdegree g*) (*f* / *g*)
using *assms fps-divide-unit-factor-numer*[*of f unit-factor g*]
fps-divide-unit-factor-denom[*of g f*]
fps-shift-rev-shift(*1*)[*of subdegree g subdegree f f*/*g*]
by *simp*

lemma *fps-divide-unit-factor-both*:
fixes *f g* :: *'a*::*division-ring fps*
assumes *subdegree g* ≤ *subdegree f*
shows *unit-factor f* / *unit-factor g* = *unit-factor* (*f* / *g*)
using *assms fps-divide-unit-factor-both'*[*of g f*] *fps-divide-subdegree*[*of f g*]
by (*cases f*=*0* ∨ *g*=*0*) *auto*

lemma *fps-divide-self*:
  (*f*::*'a*::*division-ring fps*) ≠ *0* ⟹ *f* / *f* = *1*
  using *fps-mult-right-inverse-unit-factor-divring*[*of f*]
  by (*simp add*: *fps-divide-def*)

lemma *fps-divide-add*:
  fixes *f g h* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fps*
  shows (*f* + *g*) / *h* = *f* / *h* + *g* / *h*
  by (*simp add*: *fps-divide-def algebra-simps fps-shift-add*)

lemma *fps-divide-diff*:
  fixes *f g h* :: *'a*::{*ring*,*inverse*} *fps*
  shows (*f* − *g*) / *h* = *f* / *h* − *g* / *h*
  by (*simp add*: *fps-divide-def algebra-simps fps-shift-diff*)

lemma *fps-divide-uminus*:
  fixes *f g h* :: *'a*::{*ring*,*inverse*} *fps*
  shows (− *f*) / *g* = − (*f* / *g*)
  by (*simp add*: *fps-divide-def algebra-simps fps-shift-uminus*)

lemma *fps-divide-uminus'*:
  fixes *f g h* :: *'a*::*division-ring fps*
  shows *f* / (− *g*) = − (*f* / *g*)
  by (*simp add*: *fps-divide-def fps-unit-factor-uminus fps-shift-uminus*)

lemma *fps-divide-times*:
  fixes *f g h* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fps*
  assumes *subdegree h* ≤ *subdegree g*
  shows (*f* ∗ *g*) / *h* = *f* ∗ (*g* / *h*)
  using *assms fps-mult-subdegree-ge*[*of g inverse* (*unit-factor h*)]
    fps-shift-mult[*of subdegree h g* ∗ *inverse* (*unit-factor h*) *f*]
  by (*fastforce simp add*: *fps-divide-def mult.assoc*)

lemma *fps-divide-times2*:
  fixes *f g h* :: *'a*::{*comm-semiring-0*,*inverse*,*uminus*} *fps*
  assumes *subdegree h* ≤ *subdegree f*
  shows (*f* ∗ *g*) / *h* = (*f* / *h*) ∗ *g*
  using *assms fps-divide-times*[*of h f g*]
  by (*simp add*: *mult.commute*)

lemma *fps-times-divide-eq*:
  fixes *f g* :: *'a*::*field fps*
  assumes *g* ≠ *0* **and** *subdegree f* ≥ *subdegree g*
  shows *f div g* ∗ *g* = *f*
  using *assms fps-divide-times2*[*of g f g*]
  by (*simp add*: *fps-divide-times fps-divide-self*)

lemma *fps-divide-times-eq*:

(*g* :: *'a*::*division-ring fps*) ≠ *0* ⟹ (*f* ∗ *g*) *div g* = *f*
by (*simp add*: *fps-divide-times fps-divide-self*)

lemma *fps-divide-by-mult'*:
fixes *f g h* :: *'a* :: *division-ring fps*
assumes *subdegree h* ≤ *subdegree f*
shows *f* / (*g* ∗ *h*) = *f* / *h* / *g*
proof (*cases f*=*0* ∨ *g*=*0* ∨ *h*=*0*)
case *False* **with** *assms* **show** *?thesis*
using *fps-unit-factor-mult*[*of g h*]
by (*auto simp*:
fps-divide-def fps-shift-fps-shift fps-inverse-mult-divring mult.assoc
fps-shift-mult-right-noncomm
)
qed *auto*

lemma *fps-divide-by-mult*:
fixes *f g h* :: *'a* :: *field fps*
assumes *subdegree g* ≤ *subdegree f*
shows *f* / (*g* ∗ *h*) = *f* / *g* / *h*
proof−
have *f* / (*g* ∗ *h*) = *f* / (*h* ∗ *g*) **by** (*simp add*: *mult.commute*)
also have ... = *f* / *g* / *h* **using** *fps-divide-by-mult'*[*OF assms*] **by** *simp*
finally show *?thesis* **by** *simp*
qed

lemma *fps-divide-cancel*:
fixes *f g h* :: *'a* :: *division-ring fps*
shows *h* ≠ *0* ⟹ (*f* ∗ *h*) *div* (*g* ∗ *h*) = *f div g*
by (*cases f*=*0*)
(*auto simp*: *fps-divide-by-mult' fps-divide-times-eq*)

lemma *fps-divide-1'*:
fixes *a* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows *a* / *1* = *a*
using *assms fps-inverse-one' fps-one-mult*(*2*)[*of a*]
by (*force simp*: *fps-divide-def*)

lemma *fps-divide-1* [*simp*]: (*a* :: *'a*::*division-ring fps*) / *1* = *a*
by (*rule fps-divide-1'*[*OF inverse-1*])

lemma *fps-divide-X'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / *fps-X* = *fps-shift 1 f*
using *assms fps-one-mult*(*2*)[*of f*]

by (*simp add*: *fps-divide-def fps-X-unit-factor fps-inverse-one′*)

lemma *fps-divide-X* [*simp*]: *a* / *fps-X* = *fps-shift 1* (*a*::′*a*::*division-ring fps*)
by (*rule fps-divide-X′*[*OF inverse-1*])

lemma *fps-divide-X-power′*:
fixes *f* :: ′*a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::′*a*) = *1*
shows *f* / (*fps-X* ^ *n*) = *fps-shift n f*
using *fps-inverse-one′*[*OF assms*] *fps-one-mult*(*2*)[*of f*]
by (*simp add*: *fps-divide-def fps-X-power-subdegree*)

lemma *fps-divide-X-power* [*simp*]: *a* / (*fps-X* ^ *n*) = *fps-shift n* (*a*::′*a*::*division-ring fps*)
by (*rule fps-divide-X-power′*[*OF inverse-1*])

lemma *fps-divide-shift-denom-conv-times-fps-X-power*:
fixes *f g* :: ′*a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *n* ≤ *subdegree g subdegree g* ≤ *subdegree f*
shows *f* / *fps-shift n g* = *f* / *g* ∗ *fps-X* ^ *n*
using *assms*
by (*intro fps-ext*) (*simp-all add*: *fps-divide-shift-denom fps-X-power-mult-right-nth*)

lemma *fps-divide-unit-factor-denom-conv-times-fps-X-power*:
fixes *f g* :: ′*a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *subdegree g* ≤ *subdegree f*
shows *f* / *unit-factor g* = *f* / *g* ∗ *fps-X* ^ *subdegree g*
by (*simp add*: *assms fps-divide-shift-denom-conv-times-fps-X-power*)

lemma *fps-shift-altdef′*:
fixes *f* :: ′*a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::′*a*) = *1*
shows *fps-shift n f* = *f div fps-X*^*n*
using *assms*
by (*simp add*:
fps-divide-def fps-X-power-subdegree fps-X-power-unit-factor fps-inverse-one′
)

lemma *fps-shift-altdef*:
fps-shift n f = (*f* :: ′*a* :: *division-ring fps*) *div fps-X*^*n*
by (*rule fps-shift-altdef′*[*OF inverse-1*])

lemma *fps-div-fps-X-power-nth′*:
fixes *f* :: ′*a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::′*a*) = *1*
shows (*f div fps-X*^*n*) $ *k* = *f* $ (*k* + *n*)
using *assms*
by (*simp add*: *fps-shift-altdef′* [*symmetric*])

lemma *fps-div-fps-X-power-nth*: ((*f* :: *'a* :: *division-ring fps*) *div fps-X^n*) $ *k* = *f* $ (*k* + *n*)
by (*rule fps-div-fps-X-power-nth'*[*OF inverse-1*])

lemma *fps-div-fps-X-nth'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows (*f div fps-X*) $ *k* = *f* $ *Suc k*
using *assms fps-div-fps-X-power-nth'*[*of f 1*]
by *simp*

lemma *fps-div-fps-X-nth*: ((*f* :: *'a* :: *division-ring fps*) *div fps-X*) $ *k* = *f* $ *Suc k*
by (*rule fps-div-fps-X-nth'*[*OF inverse-1*])

lemma *divide-fps-const'*:
fixes *c* :: *'a* :: {*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*}
shows *f* / *fps-const c* = *f* ∗ *fps-const* (*inverse c*)
by (*simp add*: *fps-divide-def fps-const-inverse*)

lemma *divide-fps-const* [*simp*]:
fixes *c* :: *'a* :: {*comm-semiring-0*,*inverse*,*uminus*}
shows *f* / *fps-const c* = *fps-const* (*inverse c*) ∗ *f*
by (*simp add*: *divide-fps-const' mult.commute*)

lemma *fps-const-divide*: *fps-const* (*x* :: - :: *division-ring*) / *fps-const y* = *fps-const* (*x* / *y*)
by (*simp add*: *fps-divide-def fps-const-inverse divide-inverse*)

lemma *fps-numeral-divide-divide*:
x / *numeral b* / *numeral c* = (*x* / *numeral* (*b* ∗ *c*) :: *'a* :: *field fps*)
by (*simp add*: *fps-divide-by-mult*[*symmetric*])

lemma *fps-numeral-mult-divide*:
numeral b ∗ *x* / *numeral c* = (*numeral b* / *numeral c* ∗ *x* :: *'a* :: *field fps*)
by (*simp add*: *fps-divide-times2*)

lemmas *fps-numeral-simps* =
fps-numeral-divide-divide fps-numeral-mult-divide inverse-fps-numeral neg-numeral-fps-const

lemma *fps-is-left-unit-iff-zeroth-is-left-unit*:
fixes *f* :: *'a* :: *ring-1 fps*
shows (∃ *g*. *1* = *f* ∗ *g*) ⟷ (∃ *k*. *1* = *f*$*0* ∗ *k*)
proof
assume ∃ *g*. *1* = *f* ∗ *g*
then obtain *g* **where** *1* = *f* ∗ *g* **by** *fast*
hence *1* = *f*$*0* ∗ *g*$*0* **using** *fps-mult-nth-0*[*of f g*] **by** *simp*
thus ∃ *k*. *1* = *f*$*0* ∗ *k* **by** *auto*
next
assume ∃ *k*. *1* = *f*$*0* ∗ *k*

then obtain *k* **where** *1 = f$0 ∗ k* **by** *fast*
hence *1 = f ∗ fps-right-inverse f k*
using *fps-right-inverse* **by** *simp*
thus *∃ g. 1 = f ∗ g* **by** *fast*
qed

lemma *fps-is-right-unit-iff-zeroth-is-right-unit*:
fixes *f :: 'a :: ring-1 fps*
shows *(∃ g. 1 = g ∗ f) ⟷ (∃ k. 1 = k ∗ f$0)*
proof
assume *∃ g. 1 = g ∗ f*
then obtain *g* **where** *1 = g ∗ f* **by** *fast*
hence *1 = g$0 ∗ f$0* **using** *fps-mult-nth-0[of g f]* **by** *simp*
thus *∃ k. 1 = k ∗ f$0* **by** *auto*
next
assume *∃ k. 1 = k ∗ f$0*
then obtain *k* **where** *1 = k ∗ f$0* **by** *fast*
hence *1 = fps-left-inverse f k ∗ f*
using *fps-left-inverse* **by** *simp*
thus *∃ g. 1 = g ∗ f* **by** *fast*
qed

lemma *fps-is-unit-iff [simp]*: *(f :: 'a :: field fps) dvd 1 ⟷ f $ 0 ≠ 0*
proof
assume *f dvd 1*
then obtain *g* **where** *1 = f ∗ g* **by** *(elim dvdE)*
from *this[symmetric]* **have** *(f∗g) $ 0 = 1* **by** *simp*
thus *f $ 0 ≠ 0* **by** *auto*
next
assume *A*: *f $ 0 ≠ 0*
thus *f dvd 1* **by** *(simp add: inverse-mult-eq-1[OF A, symmetric])*
qed

lemma *subdegree-eq-0-left*:
fixes *f :: 'a::{comm-monoid-add,zero-neq-one,mult-zero} fps*
assumes *∃ g. 1 = f ∗ g*
shows *subdegree f = 0*
proof *(intro subdegree-eq-0)*
from *assms* **obtain** *g* **where** *1 = f ∗ g* **by** *fast*
hence *f$0 ∗ g$0 = 1* **using** *fps-mult-nth-0[of f g]* **by** *simp*
thus *f$0 ≠ 0* **by** *auto*
qed

lemma *subdegree-eq-0-right*:
fixes *f :: 'a::{comm-monoid-add,zero-neq-one,mult-zero} fps*
assumes *∃ g. 1 = g ∗ f*
shows *subdegree f = 0*
proof *(intro subdegree-eq-0)*
from *assms* **obtain** *g* **where** *1 = g ∗ f* **by** *fast*

hence *g$0 ∗ f$0 = 1* **using** *fps-mult-nth-0[of g f]* **by** *simp*
thus *f$0 ≠ 0* **by** *auto*
qed

lemma *subdegree-eq-0′* [*simp*]: (*f* :: ′*a* :: *field fps*) *dvd 1* ⟹ *subdegree f = 0*
by *simp*

lemma *fps-dvd1-left-trivial-unit-factor*:
fixes *f* :: ′*a*::{*comm-monoid-add*, *zero-neq-one*, *mult-zero*} *fps*
assumes ∃ *g*. *1 = f ∗ g*
shows *unit-factor f = f*
using *assms subdegree-eq-0-left*
by *fastforce*

lemma *fps-dvd1-right-trivial-unit-factor*:
fixes *f* :: ′*a*::{*comm-monoid-add*, *zero-neq-one*, *mult-zero*} *fps*
assumes ∃ *g*. *1 = g ∗ f*
shows *unit-factor f = f*
using *assms subdegree-eq-0-right*
by *fastforce*

lemma *fps-dvd1-trivial-unit-factor*:
(*f* :: ′*a*::*comm-semiring-1 fps*) *dvd 1* ⟹ *unit-factor f = f*
unfolding *dvd-def* **by** (*rule fps-dvd1-left-trivial-unit-factor*) *simp*

lemma *fps-unit-dvd-left*:
fixes *f* :: ′*a* :: *division-ring fps*
assumes *f* $ *0* ≠ *0*
shows ∃ *g*. *1 = f ∗ g*
using *assms fps-is-left-unit-iff-zeroth-is-left-unit right-inverse*
by *fastforce*

lemma *fps-unit-dvd-right*:
fixes *f* :: ′*a* :: *division-ring fps*
assumes *f* $ *0* ≠ *0*
shows ∃ *g*. *1 = g ∗ f*
using *assms fps-is-right-unit-iff-zeroth-is-right-unit left-inverse*
by *fastforce*

lemma *fps-unit-dvd* [*simp*]: (*f* $ *0* :: ′*a* :: *field*) ≠ *0* ⟹ *f dvd g*
using *fps-unit-dvd-left dvd-trans*[*of f 1*] **by** *simp*

lemma *dvd-left-imp-subdegree-le*:
fixes *f g* :: ′*a*::{*comm-monoid-add*,*mult-zero*} *fps*
assumes ∃ *k*. *g = f ∗ k g ≠ 0*
shows *subdegree f ≤ subdegree g*
using *assms fps-mult-subdegree-ge*
by *fastforce*

lemma *dvd-right-imp-subdegree-le*:
  fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
  assumes $\exists k.\ g = k * f$ $g \neq 0$
  shows *subdegree f* $\leq$ *subdegree g*
  using *assms fps-mult-subdegree-ge*
  by *fastforce*

lemma *dvd-imp-subdegree-le*:
  f dvd g $\Longrightarrow$ $g \neq 0$ $\Longrightarrow$ *subdegree f* $\leq$ *subdegree g*
  using *dvd-left-imp-subdegree-le* **by** *fast*

lemma *subdegree-le-imp-dvd-left-ring1*:
  fixes *f g* :: *'a* :: *ring-1 fps*
  assumes $\exists y.$ *f* \$ *subdegree f* $* y = 1$ *subdegree f* $\leq$ *subdegree g*
  shows $\exists k.\ g = f * k$
proof−
  define *h* :: *'a fps* **where** *h* $\equiv$ *fps-X* ^ (*subdegree g* − *subdegree f*)
  from *assms*(*1*) **obtain** *y* **where** *f* \$ *subdegree f* $* y = 1$ **by** *fast*
  hence *unit-factor f* \$ *0* $* y = 1$ **by** *simp*
  from *this* **obtain** *k* **where** *1* = *unit-factor f* $*$ *k*
    using *fps-is-left-unit-iff-zeroth-is-left-unit*[*of unit-factor f*] **by** *auto*
  hence *fps-X* ^ *subdegree f* = *fps-X* ^ *subdegree f* $*$ *unit-factor f* $*$ *k*
    by (*simp add*: *mult.assoc*)
  moreover have *fps-X* ^ *subdegree f* $*$ *unit-factor f* = *f*
    by (*rule fps-unit-factor-decompose'*[*symmetric*])
  ultimately have
    fps-X ^ (*subdegree f* + (*subdegree g* − *subdegree f*)) = *f* $*$ *k* $*$ *h*
    by (*simp add*: *power-add h-def*)
  hence *g* = *f* $*$ (*k* $*$ *h* $*$ *unit-factor g*)
    using *fps-unit-factor-decompose'*[*of g*]
    by (*simp add*: *assms*(*2*) *mult.assoc*)
  thus *?thesis* **by** *fast*
qed

lemma *subdegree-le-imp-dvd-left-divring*:
  fixes *f g* :: *'a* :: *division-ring fps*
  assumes $f \neq 0$ *subdegree f* $\leq$ *subdegree g*
  shows $\exists k.\ g = f * k$
proof (*intro subdegree-le-imp-dvd-left-ring1*)
  from *assms*(*1*) **have** *f* \$ *subdegree f* $\neq 0$ **by** *simp*
  thus $\exists y.$ *f* \$ *subdegree f* $* y = 1$ **using** *right-inverse* **by** *blast*
qed (*rule assms*(*2*))

lemma *subdegree-le-imp-dvd-right-ring1*:
  fixes *f g* :: *'a* :: *ring-1 fps*
  assumes $\exists x.\ x * f$ \$ *subdegree f* $= 1$ *subdegree f* $\leq$ *subdegree g*
  shows $\exists k.\ g = k * f$
proof−
  define *h* :: *'a fps* **where** *h* $\equiv$ *fps-X* ^ (*subdegree g* − *subdegree f*)

from *assms*(*1*) **obtain** *x* **where** *x* ∗ *f* $ *subdegree f* = *1* **by** *fast*
hence *x* ∗ *unit-factor f* $ *0* = *1* **by** *simp*
from *this* **obtain** *k* **where** *1* = *k* ∗ *unit-factor f*
using *fps-is-right-unit-iff-zeroth-is-right-unit*[*of unit-factor f*] **by** *auto*
hence *fps-X* ^ *subdegree f* = *k* ∗ (*unit-factor f* ∗ *fps-X* ^ *subdegree f*)
by (*simp add*: *mult.assoc*[*symmetric*])
moreover have *unit-factor f* ∗ *fps-X* ^ *subdegree f* = *f*
by (*rule fps-unit-factor-decompose*[*symmetric*])
ultimately have *fps-X* ^ (*subdegree g* − *subdegree f* + *subdegree f*) = *h* ∗ *k* ∗ *f*
by (*simp add*: *power-add h-def mult.assoc*)
hence *g* = *unit-factor g* ∗ *h* ∗ *k* ∗ *f*
using *fps-unit-factor-decompose*[*of g*]
by (*simp add*: *assms*(*2*) *mult.assoc*)
thus *?thesis* **by** *fast*
qed

lemma *subdegree-le-imp-dvd-right-divring*:
fixes *f g* :: *'a* :: *division-ring fps*
assumes *f* ≠ *0 subdegree f* ≤ *subdegree g*
shows ∃ *k*. *g* = *k* ∗ *f*
proof (*intro subdegree-le-imp-dvd-right-ring1*)
from *assms*(*1*) **have** *f* $ *subdegree f* ≠ *0* **by** *simp*
thus ∃ *x*. *x* ∗ *f* $ *subdegree f* = *1* **using** *left-inverse* **by** *blast*
qed (*rule assms*(*2*))

lemma *fps-dvd-iff*:
assumes (*f* :: *'a* :: *field fps*) ≠ *0 g* ≠ *0*
shows *f dvd g* ⟷ *subdegree f* ≤ *subdegree g*
proof
assume *subdegree f* ≤ *subdegree g*
with *assms* **show** *f dvd g*
using *subdegree-le-imp-dvd-left-divring*
by (*auto intro*: *dvdI*)
qed (*simp add*: *assms dvd-imp-subdegree-le*)

lemma *subdegree-div'*:
fixes *p q* :: *'a*::*division-ring fps*
assumes ∃ *k*. *p* = *k* ∗ *q*
shows *subdegree* (*p div q*) = *subdegree p* − *subdegree q*
proof (*cases p* = *0*)
case *False*
from *assms*(*1*) **obtain** *k* **where** *k*: *p* = *k* ∗ *q* **by** *blast*
with *False* **have** *subdegree* (*p div q*) = *subdegree k* **by** (*simp add*: *fps-divide-times-eq*)
moreover have *k* $ *subdegree k* ∗ *q* $ *subdegree q* ≠ *0*
proof
assume *k* $ *subdegree k* ∗ *q* $ *subdegree q* = *0*
hence *k* $ *subdegree k* ∗ *q* $ *subdegree q* ∗ *inverse* (*q* $ *subdegree q*) = *0* **by** *simp*
with *False k* **show** *False* **by** (*simp add*: *mult.assoc*)

qed
ultimately show *?thesis* by (*simp add*: *k subdegree-mult'*)
qed *simp*

lemma *subdegree-div*:
fixes *p q* :: *'a* :: *field fps*
assumes *q dvd p*
shows *subdegree* (*p div q*) = *subdegree p* − *subdegree q*
using *assms*
unfolding *dvd-def*
by (*auto intro*: *subdegree-div'*)

lemma *subdegree-div-unit'*:
fixes *p q* :: *'a* :: {*ab-group-add*,*mult-zero*,*inverse*} *fps*
assumes *q* $ *0* ≠ *0 p* $ *subdegree p* ∗ *inverse* (*q* $ *0*) ≠ *0*
shows *subdegree* (*p div q*) = *subdegree p*
using *assms subdegree-mult'*[*of p inverse q*]
by (*auto simp add*: *fps-divide-unit*)

lemma *subdegree-div-unit''*:
fixes *p q* :: *'a* :: {*ring-no-zero-divisors*,*inverse*} *fps*
assumes *q* $ *0* ≠ *0 inverse* (*q* $ *0*) ≠ *0*
shows *subdegree* (*p div q*) = *subdegree p*
by (*cases p* = *0*) (*auto intro*: *subdegree-div-unit' simp*: *assms*)

lemma *subdegree-div-unit*:
fixes *p q* :: *'a* :: *division-ring fps*
assumes *q* $ *0* ≠ *0*
shows *subdegree* (*p div q*) = *subdegree p*
by (*intro subdegree-div-unit''*) (*simp-all add*: *assms*)

instantiation *fps* :: ({*comm-semiring-1*,*inverse*,*uminus*}) *modulo*
begin

definition *fps-mod-def*:
f mod g = (*if g* = *0 then f else*
let h = *unit-factor g in fps-cutoff* (*subdegree g*) (*f* ∗ *inverse h*) ∗ *h*)

instance ..

end

lemma *fps-mod-zero* [*simp*]:
(*f*::*'a*::{*comm-semiring-1*,*inverse*,*uminus*} *fps*) *mod 0* = *f*
by (*simp add*: *fps-mod-def*)

lemma *fps-mod-eq-zero*:
assumes *g* ≠ *0* **and** *subdegree f* ≥ *subdegree g*
shows *f mod g* = *0*

proof (*cases f* ∗ *inverse* (*unit-factor g*) = *0*)
  case *False*
  have *fps-cutoff* (*subdegree g*) (*f* ∗ *inverse* (*unit-factor g*)) = *0*
    using *False assms*(*2*) *fps-mult-subdegree-ge fps-cutoff-zero-iff* **by** *force*
  with *assms*(*1*) **show** *?thesis* **by** (*simp add*: *fps-mod-def Let-def*)
qed (*simp add*: *assms fps-mod-def*)

lemma *fps-mod-unit* [*simp*]: *g$0* ≠ *0* ⟹ *f mod g* = *0*
  by (*intro fps-mod-eq-zero*) *auto*

lemma *subdegree-mod*:
  assumes *subdegree* (*f*::′*a*::*field fps*) < *subdegree g*
  shows *subdegree* (*f mod g*) = *subdegree f*
proof (*cases f* = *0*)
  case *False*
  with *assms* **show** *?thesis*
    by (*intro subdegreeI*)
       (*auto simp*: *inverse-mult-eq-1 fps-mod-def Let-def fps-cutoff-left-mult-nth mult.assoc*)
qed (*simp add*: *fps-mod-def*)

instance *fps* :: (*field*) *idom-modulo*
proof

  fix *f g* :: ′*a fps*

  define *n* **where** *n* = *subdegree g*
  define *h* **where** *h* = *f* ∗ *inverse* (*unit-factor g*)

  show *f div g* ∗ *g* + *f mod g* = *f*
  proof (*cases g* = *0*)
    case *False*
    with *n-def h-def* **have**
     f div g ∗ *g* + *f mod g* = (*fps-shift n h* ∗ *fps-X* ^ *n* + *fps-cutoff n h*) ∗ *unit-factor g*
      by (*simp add*: *fps-divide-def fps-mod-def Let-def subdegree-decompose algebra-simps*)
    with *False* **show** *?thesis*
      by (*simp add*: *fps-shift-cutoff h-def inverse-mult-eq-1*)
  qed *auto*

qed (*rule fps-divide-times-eq*, *simp-all add*: *fps-divide-def*)

instantiation *fps* :: (*field*) *normalization-semidom-multiplicative*
begin

definition *fps-normalize-def* [*simp*]:
  normalize f = (*if f* = *0 then 0 else fps-X* ^ *subdegree f*)

instance proof
  fix *f g* :: *'a fps*
  assume *is-unit f*
  thus *unit-factor* (*f* ∗ *g*) = *f* ∗ *unit-factor g*
    using *fps-unit-factor-mult*[*of f g*] **by** *simp*
next
  fix *f g* :: *'a fps*
  show *unit-factor f* ∗ *normalize f* = *f*
    by (*simp add*: *fps-shift-times-fps-X-power*)
next
  fix *f g* :: *'a fps*
  show *unit-factor* (*f* ∗ *g*) = *unit-factor f* ∗ *unit-factor g*
    using *fps-unit-factor-mult*[*of f g*] **by** *simp*
qed (*simp-all add*: *fps-divide-def Let-def*)

end

5.7 Euclidean division

instantiation *fps* :: (*field*) *euclidean-ring-cancel*
begin

definition *fps-euclidean-size-def*:
  euclidean-size f = (*if f* = *0 then 0 else 2* ^ *subdegree f*)

instance proof
  fix *f g* :: *'a fps* **assume** [*simp*]: *g* ≠ *0*
  show *euclidean-size f* ≤ *euclidean-size* (*f* ∗ *g*)
    by (*cases f* = *0*) (*simp-all add*: *fps-euclidean-size-def*)
  show *euclidean-size* (*f mod g*) < *euclidean-size g*
  proof (*cases f* = *0*)
    case *True*
    then show *?thesis*
      by (*simp add*: *fps-euclidean-size-def*)
  next
    case *False*
    then show *?thesis*
      using *le-less-linear*[*of subdegree g subdegree f*]
      by (*force simp add*: *fps-mod-eq-zero fps-euclidean-size-def subdegree-mod*)
  qed
next
  fix *f g h* :: *'a fps* **assume** [*simp*]: *h* ≠ *0*
  show (*h* ∗ *f*) *div* (*h* ∗ *g*) = *f div g*
    by (*simp add*: *fps-divide-cancel mult.commute*)
  show (*f* + *g* ∗ *h*) *div h* = *g* + *f div h*
    by (*simp add*: *fps-divide-add fps-divide-times-eq*)
qed (*simp add*: *fps-euclidean-size-def*)

end

instance *fps* :: (*field*) *normalization-euclidean-semiring* **..**

instantiation *fps* :: (*field*) *euclidean-ring-gcd*
begin
definition *fps-gcd-def*: (*gcd* :: *'a fps* ⇒ -) = *Euclidean-Algorithm.gcd*
definition *fps-lcm-def*: (*lcm* :: *'a fps* ⇒ -) = *Euclidean-Algorithm.lcm*
definition *fps-Gcd-def*: (*Gcd* :: *'a fps set* ⇒ -) = *Euclidean-Algorithm.Gcd*
definition *fps-Lcm-def*: (*Lcm* :: *'a fps set* ⇒ -) = *Euclidean-Algorithm.Lcm*
instance by *standard* (*simp-all add*: *fps-gcd-def fps-lcm-def fps-Gcd-def fps-Lcm-def*)
end

lemma *fps-gcd*:
  assumes [*simp*]: *f* ≠ *0 g* ≠ *0*
  shows *gcd f g* = *fps-X* ^ *min* (*subdegree f*) (*subdegree g*)
proof −
  let *?m* = *min* (*subdegree f*) (*subdegree g*)
  show *gcd f g* = *fps-X* ^ *?m*
  proof (*rule sym*, *rule gcdI*)
    fix *d* **assume** *d dvd f d dvd g*
    thus *d dvd fps-X* ^ *?m* **by** (*cases d* = *0*) (*simp-all add*: *fps-dvd-iff*)
  qed (*simp-all add*: *fps-dvd-iff*)
qed

lemma *fps-gcd-altdef*: *gcd f g* =
  (*if f* = *0* ∧ *g* = *0 then 0 else*
   if f = *0 then fps-X* ^ *subdegree g else*
   if g = *0 then fps-X* ^ *subdegree f else*
    fps-X ^ *min* (*subdegree f*) (*subdegree g*))
  by (*simp add*: *fps-gcd*)

lemma *fps-lcm*:
  assumes [*simp*]: *f* ≠ *0 g* ≠ *0*
  shows *lcm f g* = *fps-X* ^ *max* (*subdegree f*) (*subdegree g*)
proof −
  let *?m* = *max* (*subdegree f*) (*subdegree g*)
  show *lcm f g* = *fps-X* ^ *?m*
  proof (*rule sym*, *rule lcmI*)
    fix *d* **assume** *f dvd d g dvd d*
    thus *fps-X* ^ *?m dvd d* **by** (*cases d* = *0*) (*simp-all add*: *fps-dvd-iff*)
  qed (*simp-all add*: *fps-dvd-iff*)
qed

lemma *fps-lcm-altdef*: *lcm f g* =
  (*if f* = *0* ∨ *g* = *0 then 0 else fps-X* ^ *max* (*subdegree f*) (*subdegree g*))
  by (*simp add*: *fps-lcm*)

lemma *fps-Gcd*:
  assumes *A* − {*0*} ≠ {}

shows *Gcd A = fps-X ^ (INF f∈A−{0}. subdegree f)*
proof (*rule sym*, *rule GcdI*)
fix *f* **assume** *f ∈ A*
thus *fps-X ^ (INF f∈A − {0}. subdegree f) dvd f*
by (*cases f = 0*) (*auto simp*: *fps-dvd-iff intro*!: *cINF-lower*)
next
fix *d* **assume** *d*: ⋀*f*. *f ∈ A ⟹ d dvd f*
from *assms* **obtain** *f* **where** *f ∈ A − {0}* **by** *auto*
with *d*[*of f*] **have** [*simp*]: *d ≠ 0* **by** *auto*
from *d assms* **have** *subdegree d ≤ (INF f∈A−{0}. subdegree f)*
by (*intro cINF-greatest*) (*simp-all add*: *fps-dvd-iff*[*symmetric*])
with *d assms* **show** *d dvd fps-X ^ (INF f∈A−{0}. subdegree f)* **by** (*simp add*: *fps-dvd-iff*)
qed *simp-all*

lemma *fps-Gcd-altdef*: *Gcd A =*
(*if A ⊆ {0} then 0 else fps-X ^ (INF f∈A−{0}. subdegree f)*)
using *fps-Gcd* **by** *auto*

lemma *fps-Lcm*:
assumes *A ≠ {} 0 ∉ A bdd-above* (*subdegree'A*)
shows *Lcm A = fps-X ^ (SUP f∈A. subdegree f)*
proof (*rule sym*, *rule LcmI*)
fix *f* **assume** *f ∈ A*
moreover from *assms*(*3*) **have** *bdd-above* (*subdegree ' A*) **by** *auto*
ultimately show *f dvd fps-X ^ (SUP f∈A. subdegree f)* **using** *assms*(*2*)
by (*cases f = 0*) (*auto simp*: *fps-dvd-iff intro*!: *cSUP-upper*)
next
fix *d* **assume** *d*: ⋀*f*. *f ∈ A ⟹ f dvd d*
from *assms* **obtain** *f* **where** *f*: *f ∈ A f ≠ 0* **by** *auto*
show *fps-X ^ (SUP f∈A. subdegree f) dvd d*
proof (*cases d = 0*)
assume *d ≠ 0*
moreover from *d* **have** ⋀*f*. *f ∈ A ⟹ f ≠ 0 ⟹ f dvd d* **by** *blast*
ultimately have *subdegree d ≥ (SUP f∈A. subdegree f)* **using** *assms*
by (*intro cSUP-least*) (*auto simp*: *fps-dvd-iff*)
with ‹*d ≠ 0*› **show** *?thesis* **by** (*simp add*: *fps-dvd-iff*)
qed *simp-all*
qed *simp-all*

lemma *fps-Lcm-altdef*:
Lcm A =
(*if 0 ∈ A ∨ ¬bdd-above (subdegree'A) then 0 else*
if A = {} then 1 else fps-X ^ (SUP f∈A. subdegree f))
proof (*cases bdd-above* (*subdegree'A*))
assume *unbounded*: *¬bdd-above* (*subdegree'A*)
have *Lcm A = 0*
proof (*rule ccontr*)
assume *Lcm A ≠ 0*

from *unbounded* **obtain** *f* **where** *f*: $f \in A$ *subdegree* (*Lcm A*) < *subdegree f*
unfolding *bdd-above-def* **by** (*auto simp*: *not-le*)
moreover from *f* **and** ‹*Lcm A* ≠ *0*› **have** *subdegree f* ≤ *subdegree* (*Lcm A*)
by (*intro dvd-imp-subdegree-le dvd-Lcm*) *simp-all*
ultimately show *False* **by** *simp*
qed
with *unbounded* **show** *?thesis* **by** *simp*
qed (*simp-all add*: *fps-Lcm Lcm-eq-0-I*)

5.8 Formal Derivatives

definition *fps-deriv f* = *Abs-fps* (*λn. of-nat* (*n* + *1*) ∗ *f* \$ (*n* + *1*))

lemma *fps-deriv-nth*[*simp*]: *fps-deriv f* \$ *n* = *of-nat* (*n* + *1*) ∗ *f* \$ (*n* + *1*)
by (*simp add*: *fps-deriv-def*)

lemma *fps-0th-higher-deriv*:
(*fps-deriv* ^^ *n*) *f* \$ *0* = *fact n* ∗ *f* \$ *n*
by (*induction n arbitrary*: *f*)
(*simp-all add*: *funpow-Suc-right mult-of-nat-commute algebra-simps del*: *funpow.simps*)

lemma *fps-deriv-mult*[*simp*]:
fps-deriv (*f* ∗ *g*) = *f* ∗ *fps-deriv g* + *fps-deriv f* ∗ *g*
proof (*intro fps-ext*)
fix *n*
have *LHS*: *fps-deriv* (*f* ∗ *g*) \$ *n* = (∑ *i*=*0*..*Suc n*. *of-nat* (*n*+*1*) ∗ *f*\$*i* ∗ *g*\$(*Suc n* − *i*))
by (*simp add*: *fps-mult-nth sum-distrib-left algebra-simps*)

have ∀ *i*∈{*1*..*n*}. *n* − (*i* − *1*) = *n* − *i* + *1* **by** *auto*
moreover have
(∑ *i*=*0*..*n*. *of-nat* (*i*+*1*) ∗ *f*\$(*i*+*1*) ∗ *g*\$(*n* − *i*)) =
(∑ *i*=*1*..*Suc n*. *of-nat i* ∗ *f*\$*i* ∗ *g*\$(*n* − (*i* − *1*)))
by (*intro sum.reindex-bij-witness*[**where** *i*=*λx*. *x*−*1* **and** *j*=*λx*. *x*+*1*]) *auto*
ultimately have
(*f* ∗ *fps-deriv g* + *fps-deriv f* ∗ *g*) \$ *n* =
of-nat (*Suc n*) ∗ *f*\$*0* ∗ *g*\$(*Suc n*) +
(∑ *i*=*1*..*n*. (*of-nat* (*n* − *i* + *1*) + *of-nat i*) ∗ *f* \$ *i* ∗ *g* \$ (*n* − *i* + *1*)) +
of-nat (*Suc n*) ∗ *f*\$(*Suc n*) ∗ *g*\$*0*
by (*simp add*: *fps-mult-nth algebra-simps mult-of-nat-commute sum.atLeast-Suc-atMost sum.distrib*)
moreover have
∀ *i*∈{*1*..*n*}.
(*of-nat* (*n* − *i* + *1*) + *of-nat i*) ∗ *f* \$ *i* ∗ *g* \$ (*n* − *i* + *1*) =
of-nat (*n* + *1*) ∗ *f* \$ *i* ∗ *g* \$ (*Suc n* − *i*)
proof
fix *i* **assume** *i*: *i* ∈ {*1*..*n*}
from *i* **have** *of-nat* (*n* − *i* + *1*) + (*of-nat i* :: ′*a*) = *of-nat* (*n* + *1*)

using *of-nat-add*[*of n−i+1 i*,*symmetric*] **by** *simp*
moreover from *i* **have** *Suc n − i = n − i + 1* **by** *auto*
ultimately show (*of-nat* (*n − i + 1*) + *of-nat i*) ∗ *f* $ *i* ∗ *g* $ (*n − i + 1*) =
of-nat (*n + 1*) ∗ *f* $ *i* ∗ *g* $ (*Suc n − i*)
by *simp*
qed
ultimately have
(*f* ∗ *fps-deriv g* + *fps-deriv f* ∗ *g*) $ *n* =
(∑ *i=0..Suc n. of-nat* (*Suc n*) ∗ *f* $ *i* ∗ *g* $ (*Suc n − i*))
by (*simp add*: *sum.atLeast-Suc-atMost*)
with *LHS* **show** *fps-deriv* (*f* ∗ *g*) $ *n* = (*f* ∗ *fps-deriv g* + *fps-deriv f* ∗ *g*) $ *n*
by *simp*
qed

lemma *fps-deriv-fps-X*[*simp*]: *fps-deriv fps-X = 1*
by (*simp add*: *fps-deriv-def fps-X-def fps-eq-iff*)

lemma *fps-deriv-neg*[*simp*]:
fps-deriv (− (*f*:: ′*a*::*ring-1 fps*)) = − (*fps-deriv f*)
by (*simp add*: *fps-eq-iff fps-deriv-def*)

lemma *fps-deriv-add*[*simp*]: *fps-deriv* (*f* + *g*) = *fps-deriv f* + *fps-deriv g*
by (*auto intro*: *fps-ext simp*: *algebra-simps*)

lemma *fps-deriv-sub*[*simp*]:
fps-deriv ((*f*:: ′*a*::*ring-1 fps*) − *g*) = *fps-deriv f* − *fps-deriv g*
using *fps-deriv-add* [*of f* − *g*] **by** *simp*

lemma *fps-deriv-const*[*simp*]: *fps-deriv* (*fps-const c*) = *0*
by (*simp add*: *fps-ext fps-deriv-def fps-const-def*)

lemma *fps-deriv-of-nat* [*simp*]: *fps-deriv* (*of-nat n*) = *0*
by (*simp add*: *fps-of-nat* [*symmetric*])

lemma *fps-deriv-of-int* [*simp*]: *fps-deriv* (*of-int n*) = *0*
by (*simp add*: *fps-of-int* [*symmetric*])

lemma *fps-deriv-numeral* [*simp*]: *fps-deriv* (*numeral n*) = *0*
by (*simp add*: *numeral-fps-const*)

lemma *fps-deriv-mult-const-left*[*simp*]:
fps-deriv (*fps-const c* ∗ *f*) = *fps-const c* ∗ *fps-deriv f*
by *simp*

lemma *fps-deriv-linear*[*simp*]:
fps-deriv (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) =
fps-const a ∗ *fps-deriv f* + *fps-const b* ∗ *fps-deriv g*
by *simp*

lemma *fps-deriv-0*[*simp*]: *fps-deriv 0* = *0*
  by (*simp add*: *fps-deriv-def fps-eq-iff*)

lemma *fps-deriv-1*[*simp*]: *fps-deriv 1* = *0*
  by (*simp add*: *fps-deriv-def fps-eq-iff*)

lemma *fps-deriv-mult-const-right*[*simp*]:
  fps-deriv (*f* ∗ *fps-const c*) = *fps-deriv f* ∗ *fps-const c*
  by *simp*

lemma *fps-deriv-sum*:
  fps-deriv (*sum f S*) = *sum* (λ*i*. *fps-deriv* (*f i*)) *S*
proof (*cases finite S*)
  case *False*
  then show *?thesis* **by** *simp*
next
  case *True*
  show *?thesis* **by** (*induct rule*: *finite-induct* [*OF True*]) *simp-all*
qed

lemma *fps-deriv-eq-0-iff* [*simp*]:
  fps-deriv f = *0* ⟷ *f* = *fps-const* (*f*\$*0* :: ′*a*::{*semiring-no-zero-divisors*,*semiring-char-0*})
proof
  assume *f*: *fps-deriv f* = *0*
  show *f* = *fps-const* (*f*\$*0*)
  proof (*intro fps-ext*)
    fix *n* **show** *f* \$ *n* = *fps-const* (*f*\$*0*) \$ *n*
    proof (*cases n*)
      case (*Suc m*)
      have (*of-nat* (*Suc m*) :: ′*a*) ≠ *0* **by** (*rule of-nat-neq-0*)
      with *f Suc* **show** *?thesis* **using** *fps-deriv-nth*[*of f*] **by** *auto*
    qed *simp*
  qed
next
  show *f* = *fps-const* (*f*\$*0*) ⟹ *fps-deriv f* = *0* **using** *fps-deriv-const*[*of f*\$*0*] **by** *simp*
qed

lemma *fps-deriv-eq-iff*:
  fixes *f g* :: ′*a*::{*ring-1-no-zero-divisors*,*semiring-char-0*} *fps*
  shows *fps-deriv f* = *fps-deriv g* ⟷ (*f* = *fps-const*(*f*\$*0* − *g*\$*0*) + *g*)
proof −
  have *fps-deriv f* = *fps-deriv g* ⟷ *fps-deriv* (*f* − *g*) = *0*
    using *fps-deriv-sub*[*of f g*]
    by *simp*
  also have . . . ⟷ *f* − *g* = *fps-const* ((*f* − *g*) \$ *0*)
    unfolding *fps-deriv-eq-0-iff* **..**
  finally show *?thesis*
    by (*simp add*: *field-simps*)

qed

lemma *fps-deriv-eq-iff-ex*:
  fixes *f g* :: *'a*::{*ring-1-no-zero-divisors*,*semiring-char-0*} *fps*
  shows (*fps-deriv f* = *fps-deriv g*) ⟷ (∃ *c*. *f* = *fps-const c* + *g*)
  by (*auto simp*: *fps-deriv-eq-iff*)

fun *fps-nth-deriv* :: *nat* ⇒ *'a*::*semiring-1 fps* ⇒ *'a fps*
where
  fps-nth-deriv 0 f = *f*
| *fps-nth-deriv* (*Suc n*) *f* = *fps-nth-deriv n* (*fps-deriv f*)

lemma *fps-nth-deriv-commute*: *fps-nth-deriv* (*Suc n*) *f* = *fps-deriv* (*fps-nth-deriv n f*)
  by (*induct n arbitrary*: *f*) *auto*

lemma *fps-nth-deriv-linear*[*simp*]:
  fps-nth-deriv n (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) =
    fps-const a ∗ *fps-nth-deriv n f* + *fps-const b* ∗ *fps-nth-deriv n g*
  by (*induct n arbitrary*: *f g*) *auto*

lemma *fps-nth-deriv-neg*[*simp*]:
  fps-nth-deriv n (− (*f* :: *'a*::*ring-1 fps*)) = − (*fps-nth-deriv n f*)
  by (*induct n arbitrary*: *f*) *simp-all*

lemma *fps-nth-deriv-add*[*simp*]:
  fps-nth-deriv n ((*f* :: *'a*::*ring-1 fps*) + *g*) = *fps-nth-deriv n f* + *fps-nth-deriv n g*
  using *fps-nth-deriv-linear*[*of n 1 f 1 g*] **by** *simp*

lemma *fps-nth-deriv-sub*[*simp*]:
  fps-nth-deriv n ((*f* :: *'a*::*ring-1 fps*) − *g*) = *fps-nth-deriv n f* − *fps-nth-deriv n g*
  using *fps-nth-deriv-add* [*of n f* − *g*] **by** *simp*

lemma *fps-nth-deriv-0*[*simp*]: *fps-nth-deriv n 0* = *0*
  by (*induct n*) *simp-all*

lemma *fps-nth-deriv-1*[*simp*]: *fps-nth-deriv n 1* = (*if n* = *0 then 1 else 0*)
  by (*induct n*) *simp-all*

lemma *fps-nth-deriv-const*[*simp*]:
  fps-nth-deriv n (*fps-const c*) = (*if n* = *0 then fps-const c else 0*)
  by (*cases n*) *simp-all*

lemma *fps-nth-deriv-mult-const-left*[*simp*]:
  fps-nth-deriv n (*fps-const c* ∗ *f*) = *fps-const c* ∗ *fps-nth-deriv n f*
  using *fps-nth-deriv-linear*[*of n c f 0 0*] **by** *simp*

lemma *fps-nth-deriv-mult-const-right*[*simp*]:

fps-nth-deriv n (*f* ∗ *fps-const c*) = *fps-nth-deriv n f* ∗ *fps-const c*
by (*induct n arbitrary*: *f*) *auto*

lemma *fps-nth-deriv-sum*:
fps-nth-deriv n (*sum f S*) = *sum* (λ*i*. *fps-nth-deriv n* (*f i* :: ′*a*::*ring-1 fps*)) *S*
proof (*cases finite S*)
case *True*
show *?thesis* **by** (*induct rule*: *finite-induct* [*OF True*]) *simp-all*
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *fps-deriv-maclauren-0*:
(*fps-nth-deriv k* (*f* :: ′*a*::*comm-semiring-1 fps*)) $ *0* = *of-nat* (*fact k*) ∗ *f* $ *k*
by (*induct k arbitrary*: *f*) (*simp-all add*: *field-simps*)

lemma *fps-deriv-lr-inverse*:
fixes *x y* :: ′*a*::*ring-1*
assumes *x* ∗ *f*$*0* = *1 f*$*0* ∗ *y* = *1*
— These assumptions imply *x* equals *y*, but no need to assume that.
shows *fps-deriv* (*fps-left-inverse f x*) =
− *fps-left-inverse f x* ∗ *fps-deriv f* ∗ *fps-left-inverse f x*
and *fps-deriv* (*fps-right-inverse f y*) =
− *fps-right-inverse f y* ∗ *fps-deriv f* ∗ *fps-right-inverse f y*
proof−

define *L* **where** *L* ≡ *fps-left-inverse f x*
hence *fps-deriv* (*L* ∗ *f*) = *0* **using** *fps-left-inverse*[*OF assms*(*1*)] **by** *simp*
with *assms* **show** *fps-deriv L* = − *L* ∗ *fps-deriv f* ∗ *L*
using *fps-right-inverse*′[*OF assms*]
by (*simp add*: *minus-unique mult.assoc L-def*)

define *R* **where** *R* ≡ *fps-right-inverse f y*
hence *fps-deriv* (*f* ∗ *R*) = *0* **using** *fps-right-inverse*[*OF assms*(*2*)] **by** *simp*
hence *1*: *f* ∗ *fps-deriv R* + *fps-deriv f* ∗ *R* = *0* **by** *simp*
have *R* ∗ *f* ∗ *fps-deriv R* = − *R* ∗ *fps-deriv f* ∗ *R*
using *iffD2*[*OF eq-neg-iff-add-eq-0*, *OF 1*] **by** (*simp add*: *mult.assoc*)
thus *fps-deriv R* = − *R* ∗ *fps-deriv f* ∗ *R*
using *fps-left-inverse*′[*OF assms*] **by** (*simp add*: *R-def*)

qed

lemma *fps-deriv-lr-inverse-comm*:
fixes *x* :: ′*a*::*comm-ring-1*
assumes *x* ∗ *f*$*0* = *1*
shows *fps-deriv* (*fps-left-inverse f x*) = − *fps-deriv f* ∗ (*fps-left-inverse f x*)^2
and *fps-deriv* (*fps-right-inverse f x*) = − *fps-deriv f* ∗ (*fps-right-inverse f x*)^2
using *assms fps-deriv-lr-inverse*[*of x f x*]

by (*simp-all add*: *mult.commute power2-eq-square*)

lemma *fps-inverse-deriv-divring*:
fixes *a* :: *'a*::*division-ring fps*
assumes *a*$*0* ≠ *0*
shows *fps-deriv* (*inverse a*) = − *inverse a* ∗ *fps-deriv a* ∗ *inverse a*
using *assms fps-deriv-lr-inverse*(*2*)[*of inverse* (*a*$*0*) *a inverse* (*a*$*0*)]
by (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-deriv*:
fixes *a* :: *'a*::*field fps*
assumes *a*$*0* ≠ *0*
shows *fps-deriv* (*inverse a*) = − *fps-deriv a* ∗ (*inverse a*)^2
using *assms fps-deriv-lr-inverse-comm*(*2*)[*of inverse* (*a*$*0*) *a*]
by (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-deriv'*:
fixes *a* :: *'a*::*field fps*
assumes *a0*: *a* $ *0* ≠ *0*
shows *fps-deriv* (*inverse a*) = − *fps-deriv a* / *a*^{2}
using *fps-inverse-deriv*[*OF a0*] *a0*
by (*simp add*: *fps-divide-unit power2-eq-square fps-inverse-mult*)

lemma *fps-divide-deriv*:
assumes *b dvd* (*a* :: *'a* :: *field fps*)
shows *fps-deriv* (*a* / *b*) = (*fps-deriv a* ∗ *b* − *a* ∗ *fps-deriv b*) / *b*^*2*
proof −
have *eq-divide-imp*: *c* ≠ *0* ⟹ *a* ∗ *c* = *b* ⟹ *a* = *b div c* **for** *a b c* :: *'a* :: *field fps*
by (*drule sym*) (*simp add*: *mult.assoc*)
from *assms* **have** *a* = *a* / *b* ∗ *b* **by** *simp*
also have *fps-deriv* (*a* / *b* ∗ *b*) = *fps-deriv* (*a* / *b*) ∗ *b* + *a* / *b* ∗ *fps-deriv b* **by** *simp*
finally have *fps-deriv* (*a* / *b*) ∗ *b*^*2* = *fps-deriv a* ∗ *b* − *a* ∗ *fps-deriv b* **using** *assms*
by (*simp add*: *power2-eq-square algebra-simps*)
thus *?thesis* **by** (*cases b* = *0*) (*simp-all add*: *eq-divide-imp*)
qed

lemma *fps-nth-deriv-fps-X*[*simp*]: *fps-nth-deriv n fps-X* = (*if n* = *0 then fps-X else if n*=*1 then 1 else 0*)
by (*cases n*) *simp-all*

5.9 Powers

lemma *fps-power-zeroth*: (*a*^*n*) $ *0* = (*a*$*0*)^*n*
by (*induct n*) *auto*

lemma *fps-power-zeroth-eq-one*: $a\$0 = 1 \Longrightarrow a\hat{\ }n\ \$\ 0 = 1$
 by (*simp add*: *fps-power-zeroth*)

lemma *fps-power-first*:
 fixes *a* :: *'a::comm-semiring-1 fps*
 shows $(a\hat{\ }n)\ \$\ 1 = of\text{-}nat\ n * (a\$0)\hat{\ }(n-1) * a\$1$
proof (*cases n*)
 case (*Suc m*)
 have $(a\ \hat{\ }\ Suc\ m)\ \$\ 1 = of\text{-}nat\ (Suc\ m) * (a\$0)\hat{\ }(Suc\ m - 1) * a\$1$
 proof (*induct m*)
  case (*Suc k*)
  hence $(a\ \hat{\ }\ Suc\ (Suc\ k))\ \$\ 1 =$
   $a\$0 * of\text{-}nat\ (Suc\ k) * (a\ \$\ 0)\hat{\ }k * a\$1 + a\$1 * ((a\$0)\hat{\ }(Suc\ k))$
  using *fps-mult-nth-1*[*of a*] **by** (*simp add*: *fps-power-zeroth*[*symmetric*] *mult.assoc*)
  thus *?case* **by** (*simp add*: *algebra-simps*)
 qed *simp*
 with *Suc* **show** *?thesis* **by** *simp*
qed *simp*

lemma *fps-power-first-eq*: $a\ \$\ 0 = 1 \Longrightarrow a\hat{\ }n\ \$\ 1 = of\text{-}nat\ n * a\$1$
proof (*induct n*)
 case (*Suc n*)
 show *?case* **unfolding** *power-Suc fps-mult-nth*
  using *Suc.hyps*[*OF* ‹$a\$0 = 1$›] ‹$a\$0 = 1$› *fps-power-zeroth-eq-one*[*OF* ‹$a\$0=1$›]
  by (*simp add*: *algebra-simps*)
qed *simp*

lemma *fps-power-first-eq'*:
 assumes $a\ \$\ 1 = 1$
 shows $a\hat{\ }n\ \$\ 1 = of\text{-}nat\ n * (a\$0)\hat{\ }(n-1)$
proof (*cases n*)
 case (*Suc m*)
 from *assms* **have** $(a\ \hat{\ }\ Suc\ m)\ \$\ 1 = of\text{-}nat\ (Suc\ m) * (a\$0)\hat{\ }(Suc\ m - 1)$
  using *fps-mult-nth-1*[*of a*]
  by (*induct m*)
   (*simp-all add*: *algebra-simps mult-of-nat-commute fps-power-zeroth*)
 with *Suc* **show** *?thesis* **by** *simp*
qed *simp*

lemmas *startsby-one-power* = *fps-power-zeroth-eq-one*

lemma *startsby-zero-power*: $a\ \$\ 0 = 0 \Longrightarrow n > 0 \Longrightarrow a\hat{\ }n\ \$0 = 0$
 by (*simp add*: *fps-power-zeroth zero-power*)

lemma *startsby-power*: $a\ \$0 = v \Longrightarrow a\hat{\ }n\ \$0 = v\hat{\ }n$
 by (*simp add*: *fps-power-zeroth*)

lemma *startsby-nonzero-power*:
 fixes *a* :: *'a::semiring-1-no-zero-divisors fps*

shows *a* \$ *0* ≠ *0* ⟹ *a*^*n* \$ *0* ≠ *0*
by (*simp add*: *startsby-power*)

lemma *startsby-zero-power-iff* [*simp*]:
a^*n* \$*0* = (*0*::′*a*::*semiring-1-no-zero-divisors*) ⟷ *n* ≠ *0* ∧ *a*\$*0* = *0*
proof
show *a* ^ *n* \$ *0* = *0* ⟹ *n* ≠ *0* ∧ *a* \$ *0* = *0*
proof
assume *a*: *a*^*n* \$ *0* = *0*
thus *a* \$ *0* = *0* **using** *startsby-nonzero-power* **by** *auto*
have *n* = *0* ⟹ *a*^*n* \$ *0* = *1* **by** *simp*
with *a* **show** *n* ≠ *0* **by** *fastforce*
qed
show *n* ≠ *0* ∧ *a* \$ *0* = *0* ⟹ *a* ^ *n* \$ *0* = *0*
by (*cases n*) *auto*
qed

lemma *startsby-zero-power-prefix*:
assumes *a0*: *a* \$ *0* = *0*
shows ∀ *n* < *k*. *a* ^ *k* \$ *n* = *0*
proof (*induct k rule*: *nat-less-induct*, *clarify*)
case (*1 k*)
fix *j* :: *nat* **assume** *j*: *j* < *k*
show *a* ^ *k* \$ *j* = *0*
proof (*cases k*)
case *0* **with** *j* **show** *?thesis* **by** *simp*
next
case (*Suc i*)
with *1 j* **have** ∀ *m*∈{*0*<..*j*}. *a* ^ *i* \$ (*j* − *m*) = *0* **by** *auto*
with *Suc a0* **show** *?thesis* **by** (*simp add*: *fps-mult-nth sum.atLeast-Suc-atMost*)
qed
qed

lemma *startsby-zero-sum-depends*:
assumes *a0*: *a* \$*0* = *0*
and *kn*: *n* ≥ *k*
shows *sum* (λ*i*. (*a* ^ *i*)\$*k*) {*0* .. *n*} = *sum* (λ*i*. (*a* ^ *i*)\$*k*) {*0* .. *k*}
proof (*intro strip sum.mono-neutral-right*)
show ⋀*i*. *i* ∈ {*0*..*n*} − {*0*..*k*} ⟹ *a* ^ *i* \$ *k* = *0*
by (*simp add*: *a0 startsby-zero-power-prefix*)
qed (*use kn* **in** *auto*)

lemma *startsby-zero-power-nth-same*:
assumes *a0*: *a*\$*0* = *0*
shows *a*^*n* \$ *n* = (*a*\$*1*) ^ *n*
proof (*induct n*)
case (*Suc n*)
have ∀ *i*∈{*Suc 1*..*Suc n*}. *a* ^ *n* \$ (*Suc n* − *i*) = *0*
using *a0 startsby-zero-power-prefix*[*of a n*] **by** *auto*

thus *?case*
using *a0 Suc sum.atLeast-Suc-atMost*[*of 0 Suc n* $\lambda i.$ *a* \$ *i* $*$ *a* ^ *n* \$ (*Suc n* − *i*)]
sum.atLeast-Suc-atMost[*of 1 Suc n* $\lambda i.$ *a* \$ *i* $*$ *a* ^ *n* \$ (*Suc n* − *i*)]
by (*simp add*: *fps-mult-nth*)
qed *simp*

lemma *fps-lr-inverse-power*:
fixes *a* :: *'a::ring-1 fps*
assumes *x* $*$ *a\$0* = *1 a\$0* $*$ *x* = *1*
shows *fps-left-inverse* (*a*^*n*) (*x*^*n*) = *fps-left-inverse a x* ^ *n*
and *fps-right-inverse* (*a*^*n*) (*x*^*n*) = *fps-right-inverse a x* ^ *n*
proof−

from *assms* **have** *xn*: $\bigwedge$*n*. *x*^*n* $*$ (*a*^*n* \$ *0*) = *1* $\bigwedge$*n*. (*a*^*n* \$ *0*) $*$ *x*^*n* = *1*
by (*simp-all add*: *left-right-inverse-power fps-power-zeroth*)

show *fps-left-inverse* (*a*^*n*) (*x*^*n*) = *fps-left-inverse a x* ^ *n*
proof (*induct n*)
case *0*
then show *?case* **by** (*simp add*: *fps-lr-inverse-one-one*(*1*))
next
case (*Suc n*)
with *assms* **show** *?case*
using *xn fps-lr-inverse-mult-ring1*(*1*)[*of x a x*^*n a*^*n*]
by (*simp add*: *power-Suc2*[*symmetric*])
qed

moreover have *fps-right-inverse* (*a*^*n*) (*x*^*n*) = *fps-left-inverse* (*a*^*n*) (*x*^*n*)
using *xn* **by** (*intro fps-left-inverse-eq-fps-right-inverse*[*symmetric*])
moreover have *fps-right-inverse a x* = *fps-left-inverse a x*
using *assms* **by** (*intro fps-left-inverse-eq-fps-right-inverse*[*symmetric*])
ultimately show *fps-right-inverse* (*a*^*n*) (*x*^*n*) = *fps-right-inverse a x* ^ *n*
by *simp*

qed

lemma *fps-inverse-power*:
fixes *a* :: *'a::division-ring fps*
shows *inverse* (*a*^*n*) = *inverse a* ^ *n*
proof (*cases n=0 a\$0* = *0 rule*: *case-split*[*case-product case-split*])
case *False-True*
hence *LHS*: *inverse* (*a*^*n*) = *0* **and** *RHS*: *inverse a* ^ *n* = *0*
by (*simp-all add*: *startsby-zero-power*)
show *?thesis* **using** *trans-sym*[*OF LHS RHS*] **by** *fast*
next
case *False-False*
from *False-False*(*2*) **show** *?thesis*
by (*simp add*:

fps-inverse-def fps-power-zeroth power-inverse fps-lr-inverse-power(*2*)[*symmetric*]
)
qed *auto*

lemma *fps-deriv-power′*:
fixes *a* :: ′*a*::*comm-semiring-1 fps*
shows *fps-deriv* (*a* ^ *n*) = (*of-nat n*) ∗ *fps-deriv a* ∗ *a* ^ (*n* − *1*)
proof (*cases n*)
case (*Suc m*)
moreover have *fps-deriv* (*a*^*Suc m*) = *of-nat* (*Suc m*) ∗ *fps-deriv a* ∗ *a*^*m*
by (*induct m*) (*simp-all add*: *algebra-simps*)
ultimately show *?thesis* **by** *simp*
qed *simp*

lemma *fps-deriv-power*:
fixes *a* :: ′*a*::*comm-semiring-1 fps*
shows *fps-deriv* (*a* ^ *n*) = *fps-const* (*of-nat n*) ∗ *fps-deriv a* ∗ *a* ^ (*n* − *1*)
by (*simp add*: *fps-deriv-power′ fps-of-nat*)

5.10 Integration

definition *fps-integral* :: ′*a*::{*semiring-1*,*inverse*} *fps* ⇒ ′*a* ⇒ ′*a fps*
where *fps-integral a a0* =
Abs-fps (λ*n*. *if n*=*0 then a0 else inverse* (*of-nat n*) ∗ *a*$(*n* − *1*))

abbreviation *fps-integral0 a* ≡ *fps-integral a 0*

lemma *fps-integral-nth-0-Suc* [*simp*]:
fixes *a* :: ′*a*::{*semiring-1*,*inverse*} *fps*
shows *fps-integral a a0* $ *0* = *a0*
and *fps-integral a a0* $ *Suc n* = *inverse* (*of-nat* (*Suc n*)) ∗ *a* $ *n*
by (*auto simp*: *fps-integral-def*)

lemma *fps-integral-conv-plus-const*:
fps-integral a a0 = *fps-integral a 0* + *fps-const a0*
unfolding *fps-integral-def* **by** (*intro fps-ext*) *simp*

lemma *fps-deriv-fps-integral*:
fixes *a* :: ′*a*::{*division-ring*,*ring-char-0*} *fps*
shows *fps-deriv* (*fps-integral a a0*) = *a*
proof (*intro fps-ext*)
fix *n*
have (*of-nat* (*Suc n*) :: ′*a*) ≠ *0* **by** (*rule of-nat-neq-0*)
hence *of-nat* (*Suc n*) ∗ *inverse* (*of-nat* (*Suc n*) :: ′*a*) = *1* **by** *simp*
moreover have
fps-deriv (*fps-integral a a0*) $ *n* = *of-nat* (*Suc n*) ∗ *inverse* (*of-nat* (*Suc n*)) ∗
a $ *n*
by (*simp add*: *mult.assoc*)
ultimately show *fps-deriv* (*fps-integral a a0*) $ *n* = *a* $ *n* **by** *simp*

qed

lemma *fps-integral0-deriv*:
fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fps*
shows *fps-integral0* (*fps-deriv a*) = *a* − *fps-const* (*a*\$*0*)
proof (*intro fps-ext*)
fix *n*
show *fps-integral0* (*fps-deriv a*) \$ *n* = (*a* − *fps-const* (*a*\$*0*)) \$ *n*
proof (*cases n*)
case (*Suc m*)
have (*of-nat* (*Suc m*) :: *'a*) ≠ *0* **by** (*rule of-nat-neq-0*)
hence *inverse* (*of-nat* (*Suc m*) :: *'a*) ∗ *of-nat* (*Suc m*) = *1* **by** *simp*
moreover have
fps-integral0 (*fps-deriv a*) \$ *Suc m* =
inverse (*of-nat* (*Suc m*)) ∗ *of-nat* (*Suc m*) ∗ *a* \$ (*Suc m*)
by (*simp add*: *mult.assoc*)
ultimately show *?thesis* **using** *Suc* **by** *simp*
qed *simp*
qed

lemma *fps-integral-deriv*:
fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fps*
shows *fps-integral* (*fps-deriv a*) (*a*\$*0*) = *a*
using *fps-integral-conv-plus-const*[*of fps-deriv a a*\$*0*]
by (*simp add*: *fps-integral0-deriv*)

lemma *fps-integral0-zero*:
fps-integral0 (*0*::*'a*::{*semiring-1*,*inverse*} *fps*) = *0*
by (*intro fps-ext*) (*simp add*: *fps-integral-def*)

lemma *fps-integral0-fps-const'*:
fixes *c* :: *'a*::{*semiring-1*,*inverse*}
assumes *inverse* (*1*::*'a*) = *1*
shows *fps-integral0* (*fps-const c*) = *fps-const c* ∗ *fps-X*
proof (*intro fps-ext*)
fix *n*
show *fps-integral0* (*fps-const c*) \$ *n* = (*fps-const c* ∗ *fps-X*) \$ *n*
by (*cases n*) (*simp-all add*: *assms mult-delta-right*)
qed

lemma *fps-integral0-fps-const*:
fixes *c* :: *'a*::*division-ring*
shows *fps-integral0* (*fps-const c*) = *fps-const c* ∗ *fps-X*
by (*rule fps-integral0-fps-const'*[*OF inverse-1*])

lemma *fps-integral0-one'*:
assumes *inverse* (*1*::*'a*::{*semiring-1*,*inverse*}) = *1*
shows *fps-integral0* (*1*::*'a fps*) = *fps-X*
using *assms fps-integral0-fps-const'*[*of 1*::*'a*]

by *simp*

lemma *fps-integral0-one*:
fps-integral0 (*1*::*'a*::*division-ring fps*) = *fps-X*
by (*rule fps-integral0-one'*[*OF inverse-1*])

lemma *fps-integral0-fps-const-mult-left*:
fixes *a* :: *'a*::*division-ring fps*
shows *fps-integral0* (*fps-const c* ∗ *a*) = *fps-const c* ∗ *fps-integral0 a*
proof (*intro fps-ext*)
fix *n*
show *fps-integral0* (*fps-const c* ∗ *a*) $ *n* = (*fps-const c* ∗ *fps-integral0 a*) $ *n*
using *mult-inverse-of-nat-commute*[*of n c*, *symmetric*]
mult.assoc[*of inverse* (*of-nat n*) *c a*$(*n*−*1*)]
mult.assoc[*of c inverse* (*of-nat n*) *a*$(*n*−*1*)]
by (*simp add*: *fps-integral-def*)
qed

lemma *fps-integral0-fps-const-mult-right*:
fixes *a* :: *'a*::{*semiring-1*,*inverse*} *fps*
shows *fps-integral0* (*a* ∗ *fps-const c*) = *fps-integral0 a* ∗ *fps-const c*
by (*intro fps-ext*) (*simp add*: *fps-integral-def algebra-simps*)

lemma *fps-integral0-neg*:
fixes *a* :: *'a*::{*ring-1*,*inverse*} *fps*
shows *fps-integral0* (−*a*) = − *fps-integral0 a*
using *fps-integral0-fps-const-mult-right*[*of a* −*1*]
by (*simp add*: *fps-const-neg*[*symmetric*])

lemma *fps-integral0-add*:
fps-integral0 (*a*+*b*) = *fps-integral0 a* + *fps-integral0 b*
by (*intro fps-ext*) (*simp add*: *fps-integral-def algebra-simps*)

lemma *fps-integral0-linear*:
fixes *a b* :: *'a*::*division-ring*
shows *fps-integral0* (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) =
fps-const a ∗ *fps-integral0 f* + *fps-const b* ∗ *fps-integral0 g*
by (*simp add*: *fps-integral0-add fps-integral0-fps-const-mult-left*)

lemma *fps-integral0-linear2*:
fps-integral0 (*f* ∗ *fps-const a* + *g* ∗ *fps-const b*) =
fps-integral0 f ∗ *fps-const a* + *fps-integral0 g* ∗ *fps-const b*
by (*simp add*: *fps-integral0-add fps-integral0-fps-const-mult-right*)

lemma *fps-integral-linear*:
fixes *a b a0 b0* :: *'a*::*division-ring*
shows
fps-integral (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) (*a*∗*a0* + *b*∗*b0*) =
fps-const a ∗ *fps-integral f a0* + *fps-const b* ∗ *fps-integral g b0*

```
  using fps-integral-conv-plus-const[of
          fps-const a ∗ f + fps-const b ∗ g
          a∗a0 + b∗b0
        ]
        fps-integral-conv-plus-const[of f a0] fps-integral-conv-plus-const[of g b0]
  by    (simp add: fps-integral0-linear algebra-simps)

lemma fps-integral0-sub:
  fixes a b :: 'a::{ring-1,inverse} fps
  shows fps-integral0 (a−b) = fps-integral0 a − fps-integral0 b
  using fps-integral0-linear2[of a 1 b −1]
  by    (simp add: fps-const-neg[symmetric])

lemma fps-integral0-of-nat:
  fps-integral0 (of-nat n :: 'a::division-ring fps) = of-nat n ∗ fps-X
  using fps-integral0-fps-const[of of-nat n :: 'a] by (simp add: fps-of-nat)

lemma fps-integral0-sum:
  fps-integral0 (sum f S) = sum (λi. fps-integral0 (f i)) S
proof (cases finite S)
  case True show ?thesis
    by (induct rule: finite-induct [OF True])
       (simp-all add: fps-integral0-zero fps-integral0-add)
qed (simp add: fps-integral0-zero)

lemma fps-integral0-by-parts:
  fixes a b :: 'a::{division-ring,ring-char-0} fps
  shows
    fps-integral0 (a ∗ b) =
      a ∗ fps-integral0 b − fps-integral0 (fps-deriv a ∗ fps-integral0 b)
proof−
  have fps-integral0 (fps-deriv (a ∗ fps-integral0 b)) = a ∗ fps-integral0 b
    using fps-integral0-deriv[of (a ∗ fps-integral0 b)] by simp
  moreover have
    fps-integral0 (a ∗ b) =
      fps-integral0 (fps-deriv (a ∗ fps-integral0 b)) −
      fps-integral0 (fps-deriv a ∗ fps-integral0 b)
    by (auto simp: fps-deriv-fps-integral fps-integral0-sub[symmetric])
  ultimately show ?thesis by simp
qed

lemma fps-integral0-fps-X:
  fps-integral0 (fps-X::'a::{semiring-1,inverse} fps) =
    fps-const (inverse (of-nat 2)) ∗ fps-X^2
  by (intro fps-ext) (auto simp: fps-integral-def)

lemma fps-integral0-fps-X-power:
  fps-integral0 ((fps-X::'a::{semiring-1,inverse} fps) ^ n) =
           fps-const (inverse (of-nat (Suc n))) ∗ fps-X ^ Suc n
```

proof (*intro fps-ext*)
fix *k* **show**
fps-integral0 ((*fps-X*::′*a fps*) ^ *n*) $ *k* =
(*fps-const* (*inverse* (*of-nat* (*Suc n*))) ∗ *fps-X* ^ *Suc n*) $ *k*
by (*cases k*) *simp-all*
qed

5.11 Composition

definition *fps-compose* :: ′*a*::*semiring-1 fps* ⇒ ′*a fps* ⇒ ′*a fps* (**infixl** ‹*oo*› *55*)
where *a oo b* = *Abs-fps* (λ*n*. *sum* (λ*i*. *a*$*i* ∗ (*b*^*i*$*n*)) {*0*..*n*})

lemma *fps-compose-nth*: (*a oo b*)$*n* = *sum* (λ*i*. *a*$*i* ∗ (*b*^*i*$*n*)) {*0*..*n*}
by (*simp add*: *fps-compose-def*)

lemma *fps-compose-nth-0* [*simp*]: (*f oo g*) $ *0* = *f* $ *0*
by (*simp add*: *fps-compose-nth*)

lemma *fps-compose-fps-X*[*simp*]: *a oo fps-X* = (*a* :: ′*a*::*comm-ring-1 fps*)
by (*simp add*: *fps-ext fps-compose-def mult-delta-right*)

lemma *fps-const-compose*[*simp*]: *fps-const* (*a*::′*a*::*comm-ring-1*) *oo b* = *fps-const a*
by (*simp add*: *fps-eq-iff fps-compose-nth mult-delta-left*)

lemma *numeral-compose*[*simp*]: (*numeral k* :: ′*a*::*comm-ring-1 fps*) *oo b* = *numeral k*
unfolding *numeral-fps-const* **by** *simp*

lemma *neg-numeral-compose*[*simp*]: (− *numeral k* :: ′*a*::*comm-ring-1 fps*) *oo b* = − *numeral k*
unfolding *neg-numeral-fps-const* **by** *simp*

lemma *fps-X-fps-compose-startby0*[*simp*]: *a*$*0* = *0* ⟹ *fps-X oo a* = (*a* :: ′*a*::*comm-ring-1 fps*)
by (*simp add*: *fps-eq-iff fps-compose-def mult-delta-left not-le*)

5.12 Rules from Herbert Wilf's Generatingfunctionology

5.12.1 Rule 1

lemma *fps-power-mult-eq-shift*:
fps-X^*Suc k* ∗ *Abs-fps* (λ*n*. *a* (*n* + *Suc k*)) =
Abs-fps a − *sum* (λ*i*. *fps-const* (*a i* :: ′*a*::*comm-ring-1*) ∗ *fps-X*^*i*) {*0* .. *k*}
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* $ *n* = *?rhs* $ *n* **for** *n* :: *nat*
proof −
have *?lhs* $ *n* = (*if n* < *Suc k then 0 else a n*)
unfolding *fps-X-power-mult-nth* **by** *auto*
also have . . . = *?rhs* $ *n*

proof (*induct k*)
case *0*
then show *?case*
by (*simp add*: *fps-sum-nth*)
next
case (*Suc k*)
have (*Abs-fps a* − *sum* (*λi. fps-const* (*a i* :: *'a*) ∗ *fps-X^i*) {*0* .. *Suc k*})$*n* =
(*Abs-fps a* − *sum* (*λi. fps-const* (*a i* :: *'a*) ∗ *fps-X^i*) {*0* .. *k*} −
fps-const (*a* (*Suc k*)) ∗ *fps-X^ Suc k*) $ *n*
by (*simp add*: *field-simps*)
also have ... = (*if n* < *Suc k then 0 else a n*) − (*fps-const* (*a* (*Suc k*)) ∗ *fps-X^ Suc k*)$*n*
using *Suc.hyps*[*symmetric*] **unfolding** *fps-sub-nth* **by** *simp*
also have ... = (*if n* < *Suc* (*Suc k*) *then 0 else a n*)
unfolding *fps-X-power-mult-right-nth*
by (*simp add*: *not-less le-less-Suc-eq*)
finally show *?case*
by *simp*
qed
finally show *?thesis* **.**
qed
then show *?thesis*
by (*simp add*: *fps-eq-iff*)
qed

5.12.2 Rule 2

definition *fps-XD* = (∗) *fps-X* ∘ *fps-deriv*

lemma *fps-XD-add*[*simp*]:*fps-XD* (*a* + *b*) = *fps-XD a* + *fps-XD* (*b* :: *'a*::*comm-ring-1 fps*)
by (*simp add*: *fps-XD-def field-simps*)

lemma *fps-XD-mult-const*[*simp*]:*fps-XD* (*fps-const* (*c*::*'a*::*comm-ring-1*) ∗ *a*) = *fps-const c* ∗ *fps-XD a*
by (*simp add*: *fps-XD-def field-simps*)

lemma *fps-XD-linear*[*simp*]: *fps-XD* (*fps-const c* ∗ *a* + *fps-const d* ∗ *b*) =
fps-const c ∗ *fps-XD a* + *fps-const d* ∗ *fps-XD* (*b* :: *'a*::*comm-ring-1 fps*)
by *simp*

lemma *fps-XDN-linear*:
(*fps-XD ^^ n*) (*fps-const c* ∗ *a* + *fps-const d* ∗ *b*) =
fps-const c ∗ (*fps-XD ^^ n*) *a* + *fps-const d* ∗ (*fps-XD ^^ n*) (*b* :: *'a*::*comm-ring-1 fps*)
by (*induct n*) *simp-all*

lemma *fps-mult-fps-X-deriv-shift*: *fps-X*∗ *fps-deriv a* = *Abs-fps* (*λn. of-nat n*∗ *a*$*n*)
by (*simp add*: *fps-eq-iff*)

lemma *fps-mult-fps-XD-shift*:
(*fps-XD* ^^ *k*) (*a* :: *'a::comm-ring-1 fps*) = *Abs-fps* (λ*n*. (*of-nat n* ^ *k*) ∗ *a*$*n*)
by (*induct k arbitrary*: *a*) (*simp-all add*: *fps-XD-def fps-eq-iff field-simps del*: *One-nat-def*)

5.12.3 Rule 3

Rule 3 is trivial and is given by `fps_times_def`.

5.12.4 Rule 5 — summation and "division" by $1 - X$

lemma *fps-divide-fps-X-minus1-sum-lemma*:
a = ((*1::'a::ring-1 fps*) − *fps-X*) ∗ *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0..n*})
proof (*rule fps-ext*)
define *f g* :: *'a fps*
where *f* ≡ *1* − *fps-X*
and *g* ≡ *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0..n*})
fix *n* **show** *a* $ *n*= (*f* ∗ *g*) $ *n*
proof (*cases n*)
case (*Suc m*)
hence (*f* ∗ *g*) $ *n* = *g* $ *Suc m* − *g* $ *m*
using *fps-mult-nth*[*of f g Suc m*]
sum.atLeast-Suc-atMost[*of 0 Suc m* λ*i*. *f* $ *i* ∗ *g* $ (*Suc m* − *i*)]
sum.atLeast-Suc-atMost[*of 1 Suc m* λ*i*. *f* $ *i* ∗ *g* $ (*Suc m* − *i*)]
by (*simp add*: *f-def*)
with *Suc* **show** *?thesis* **by** (*simp add*: *g-def*)
qed (*simp add*: *f-def g-def*)
qed

lemma *fps-divide-fps-X-minus1-sum-ring1*:
assumes *inverse 1* = (*1::'a::*{*ring-1,inverse*})
shows *a* /((*1::'a fps*) − *fps-X*) = *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0..n*})
proof−
from *assms* **have** *a* /((*1::'a fps*) − *fps-X*) = *a* ∗ *Abs-fps* (λ*n*. *1*)
by (*simp add*: *fps-divide-def fps-inverse-def fps-lr-inverse-one-minus-fps-X*(*2*))
thus *?thesis* **by** (*auto intro*: *fps-ext simp*: *fps-mult-nth*)
qed

lemma *fps-divide-fps-X-minus1-sum*:
a /((*1::'a::division-ring fps*) − *fps-X*) = *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0..n*})
using *fps-divide-fps-X-minus1-sum-ring1*[*of a*] **by** *simp*

5.12.5 Rule 4 in its more general form

This generalizes Rule 3 for an arbitrary finite product of FPS, also the relevant instance of powers of a FPS.

definition *natpermute n k* = {*l* :: *nat list*. *length l* = *k* ∧ *sum-list l* = *n*}

lemma *natlist-trivial-1*: *natpermute n 1* = {[*n*]}
proof −
  have ⟦*length xs* = *1*; *n* = *sum-list xs*⟧ ⟹ *xs* = [*sum-list xs*] **for** *xs*
    by (*cases xs*) *auto*
  then show *?thesis*
    by (*auto simp add*: *natpermute-def*)
qed

lemma *natlist-trivial-Suc0* [*simp*]: *natpermute n* (*Suc 0*) = {[*n*]}
  using *natlist-trivial-1* **by** *force*

lemma *append-natpermute-less-eq*:
  assumes *xs* @ *ys* ∈ *natpermute n k*
  shows *sum-list xs* ≤ *n*
    and *sum-list ys* ≤ *n*
proof −
  from *assms* **have** *sum-list* (*xs* @ *ys*) = *n*
    by (*simp add*: *natpermute-def*)
  then have *sum-list xs* + *sum-list ys* = *n*
    by *simp*
  then show *sum-list xs* ≤ *n* **and** *sum-list ys* ≤ *n*
    by *simp-all*
qed

lemma *natpermute-split*:
  assumes *h* ≤ *k*
  shows *natpermute n k* =
    (⋃ *m* ∈{*0*..*n*}. {*l1* @ *l2* |*l1 l2*. *l1* ∈ *natpermute m h* ∧ *l2* ∈ *natpermute* (*n* − *m*) (*k* − *h*)})
  (**is** *?L* = *?R* **is** - = (⋃ *m* ∈{*0*..*n*}. *?S m*))
proof
  show *?R* ⊆ *?L*
  proof
    fix *l*
    assume *l*: *l* ∈ *?R*
    from *l* **obtain** *m xs ys* **where** *h*: *m* ∈ {*0*..*n*}
      and *xs*: *xs* ∈ *natpermute m h*
      and *ys*: *ys* ∈ *natpermute* (*n* − *m*) (*k* − *h*)
      and *leq*: *l* = *xs*@*ys* **by** *blast*
    from *xs* **have** *xs'*: *sum-list xs* = *m*
      by (*simp add*: *natpermute-def*)
    from *ys* **have** *ys'*: *sum-list ys* = *n* − *m*
      by (*simp add*: *natpermute-def*)
    show *l* ∈ *?L* **using** *leq xs ys h*
      using *assms* **by** (*force simp add*: *natpermute-def*)
  qed
  show *?L* ⊆ *?R*
  proof

fix *l*
assume *l*: *l* ∈ *natpermute n k*
let *?xs* = *take h l*
let *?ys* = *drop h l*
let *?m* = *sum-list ?xs*
from *l* **have** *ls*: *sum-list* (*?xs* @ *?ys*) = *n*
by (*simp add*: *natpermute-def*)
have *xs*: *?xs* ∈ *natpermute ?m h* **using** *l assms*
by (*simp add*: *natpermute-def*)
have *l-take-drop*: *sum-list l* = *sum-list* (*take h l* @ *drop h l*)
by *simp*
then have *ys*: *?ys* ∈ *natpermute* (*n* − *?m*) (*k* − *h*)
using *l assms ls* **by** (*auto simp add*: *natpermute-def simp del*: *append-take-drop-id*)
from *ls* **have** *m*: *?m* ∈ {*0*..*n*}
by (*simp add*: *l-take-drop del*: *append-take-drop-id*)
have *sum-list* (*take h l*) ≤ *sum-list l*
using *l-take-drop ls m* **by** *presburger*
with *xs ys ls l* **show** *l* ∈ *?R*
by *simp* (*metis append-take-drop-id m*)
qed
qed

lemma *natpermute-0*: *natpermute n 0* = (*if n* = *0 then* {[]} *else* {})
by (*auto simp add*: *natpermute-def*)

lemma *natpermute-0'*[*simp*]: *natpermute 0 k* = (*if k* = *0 then* {[]} *else* {*replicate k 0*})
by (*auto simp add*: *set-replicate-conv-if natpermute-def replicate-length-same*)

lemma *natpermute-finite*: *finite* (*natpermute n k*)
proof (*induct k arbitrary*: *n*)
case *0*
then show *?case*
by (*simp add*: *natpermute-0*)
next
case (*Suc k*)
then show *?case*
using *natpermute-split* [*of k Suc k*] *finite-UN-I* **by** *simp*
qed

lemma *natpermute-contain-maximal*:
{*xs* ∈ *natpermute n* (*k* + *1*). *n* ∈ *set xs*} = (⋃ *i*∈{*0* .. *k*}. {(*replicate* (*k* + *1*) *0*) [*i*:=*n*]})
(**is** *?A* = *?B*)
proof
show *?A* ⊆ *?B*
proof
fix *xs*
assume *xs* ∈ *?A*

```
then have H: xs ∈ natpermute n (k + 1) and n: n ∈ set xs
  by blast+
then obtain i where i: i ∈ {0.. k} xs!i = n
  unfolding in-set-conv-nth by (auto simp add: less-Suc-eq-le natpermute-def)
have eqs: ({0..k} − {i}) ∪ {i} = {0..k}
  using i by auto
have f: finite({0..k} − {i}) finite {i}
  by auto
have d: ({0..k} − {i}) ∩ {i} = {}
  using i by auto
from H have n = sum (nth xs) {0..k}
 by (auto simp add: natpermute-def atLeastLessThanSuc-atLeastAtMost sum-list-sum-nth)
also have ... = n + sum (nth xs) ({0..k} − {i})
  unfolding sum.union-disjoint[OF f d, unfolded eqs] using i by simp
finally have zxs: ∀ j∈ {0..k} − {i}. xs!j = 0
  by auto
from H have xsl: length xs = k+1
  by (simp add: natpermute-def)
from i have i': i < length (replicate (k+1) 0)   i < k+1
  unfolding length-replicate by presburger+
have xs = (replicate (k+1) 0) [i := n]
proof (rule nth-equalityI)
  show length xs = length ((replicate (k + 1) 0)[i := n])
    by (metis length-list-update length-replicate xsl)
  show xs ! j = (replicate (k + 1) 0)[i := n] ! j if j < length xs for j
  proof (cases j = i)
    case True
    then show ?thesis
      by (metis i'(1) i(2) nth-list-update)
  next
    case False
    with that show ?thesis
      by (simp add: xsl zxs del: replicate.simps split: nat.split)
  qed
qed
then show xs ∈ ?B using i by blast
qed
show ?B ⊆ ?A
proof
  fix xs
  assume xs ∈ ?B
  then obtain i where i: i ∈ {0..k} and xs: xs = (replicate (k + 1) 0) [i:=n]
    by auto
  have nxs: n ∈ set xs
    unfolding xs using set-update-memI i
   by (metis Suc-eq-plus1 atLeast0AtMost atMost-iff le-simps(2) length-replicate)
  have xsl: length xs = k + 1
    by (simp only: xs length-replicate length-list-update)
  have sum-list xs = sum (nth xs) {0..<k+1}
```

unfolding *sum-list-sum-nth xsl* **..**
also have ... = *sum* (λ*j*. *if j* = *i then n else 0*) {*0*..< *k*+*1*}
by (*rule sum.cong*) (*simp-all add*: *xs del*: *replicate.simps*)
also have ... = *n* **using** *i* **by** *simp*
finally have *xs* ∈ *natpermute n* (*k* + *1*)
using *xsl* **unfolding** *natpermute-def mem-Collect-eq* **by** *blast*
then show *xs* ∈ *?A*
using *nxs* **by** *blast*
qed
qed

The general form.

lemma *fps-prod-nth*:
fixes *m* :: *nat*
and *a* :: *nat* ⇒ *'a*::*comm-ring-1 fps*
shows (*prod a* {*0* .. *m*}) $ *n* =
sum (λ*v*. *prod* (λ*j*. (*a j*) $ (*v*!*j*)) {*0*..*m*}) (*natpermute n* (*m*+*1*))
(**is** *?P m n*)
proof (*induct m arbitrary*: *n rule*: *nat-less-induct*)
fix *m n* **assume** *H*: ∀ *m'* < *m*. ∀ *n*. *?P m' n*
show *?P m n*
proof (*cases m*)
case *0*
then show *?thesis*
by *simp*
next
case (*Suc k*)
then have *km*: *k* < *m* **by** *arith*
have *u0*: {*0* .. *k*} ∪ {*m*} = {*0*..*m*}
using *Suc* **by** (*simp add*: *set-eq-iff*) *presburger*
have *f0*: *finite* {*0* .. *k*} *finite* {*m*} **by** *auto*
have *d0*: {*0* .. *k*} ∩ {*m*} = {} **using** *Suc* **by** *auto*
have (*prod a* {*0* .. *m*}) $ *n* = (*prod a* {*0* .. *k*} ∗ *a m*) $ *n*
unfolding *prod.union-disjoint*[*OF f0 d0*, *unfolded u0*] **by** *simp*
also have ... = (∑ *i* = *0*..*n*. (∑ *v*∈*natpermute i* (*k* + *1*).
(∏ *j* = *0*..*k*. *a j* $ *v* ! *j*) ∗ *a m* $ (*n* − *i*)))
unfolding *fps-mult-nth H*[*rule-format*, *OF km*] *sum-distrib-right* **..**
also have ... = (∑ *i* = *0*..*n*.
∑ *v*∈(λ*l1*. *l1* @ [*n* − *i*]) ‘ *natpermute i* (*Suc k*).
(∏ *j* = *0*..*k*. *a j* $ *v* ! *j*) ∗ *a* (*Suc k*) $ *v* ! *Suc k*)
by (*intro sum.cong* [*OF refl sym*] *sum.reindex-cong*) (*auto simp*: *inj-on-def natpermute-def nth-append Suc*)
also have ... = (∑ *v*∈(⋃ *x*∈{*0*..*n*}. {*l1* @ [*n* − *x*] |*l1*. *l1* ∈ *natpermute x* (*Suc k*)}).
(∏ *j* = *0*..*k*. *a j* $ *v* ! *j*) ∗ *a* (*Suc k*) $ *v* ! *Suc k*)
by (*subst sum.UNION-disjoint*) (*auto simp add*: *natpermute-finite setcompr-eq-image*)
also have ... = (∑ *v*∈*natpermute n* (*m* + *1*). ∏ *j*∈{*0*..*m*}. *a j* $ *v* ! *j*)
using *natpermute-split*[*of m m* + *1*] **by** (*simp add*: *Suc*)
finally show *?thesis* **.**

  qed
qed

The special form for powers.

lemma *fps-power-nth-Suc*:
  fixes *m* :: *nat*
    and *a* :: *'a::comm-ring-1 fps*
  shows $(a \,\widehat{}\, Suc\ m)\$n = sum\ (\lambda v.\ prod\ (\lambda j.\ a\ \$\ (v!j))\ \{0..m\})\ (natpermute\ n\ (m+1))$
proof −
  have *th0*: $a\widehat{}Suc\ m = prod\ (\lambda i.\ a)\ \{0..m\}$
    by (*simp add*: *prod-constant*)
  show *?thesis* **unfolding** *th0 fps-prod-nth* **..**
qed

lemma *fps-power-nth*:
  fixes *m* :: *nat*
    and *a* :: *'a::comm-ring-1 fps*
  shows $(a \,\widehat{}m)\$n =$
    $(if\ m{=}0\ then\ 1\$n\ else\ sum\ (\lambda v.\ prod\ (\lambda j.\ a\ \$\ (v!j))\ \{0..m - 1\})\ (natpermute\ n\ m))$
  by (*cases m*) (*simp-all add*: *fps-power-nth-Suc del*: *power-Suc*)

lemmas *fps-nth-power-0* = *fps-power-zeroth*

lemma *natpermute-max-card*:
  assumes *n0*: $n \neq 0$
  shows $card\ \{xs \in natpermute\ n\ (k + 1).\ n \in set\ xs\} = k + 1$
  unfolding *natpermute-contain-maximal*
proof −
  let $?A = \lambda i.\ \{(replicate\ (k + 1)\ 0)[i := n]\}$
  let $?K = \{0\ ..k\}$
  have *fK*: *finite ?K*
    by *simp*
  have *fAK*: $\forall i\in ?K.\ finite\ (?A\ i)$
    by *auto*
  have *d*: $\forall i\in\ ?K.\ \forall j\in\ ?K.\ i \neq j \longrightarrow$
    $\{(replicate\ (k + 1)\ 0)[i := n]\} \cap \{(replicate\ (k + 1)\ 0)[j := n]\} = \{\}$
  proof *clarify*
    fix *i j*
    assume *i*: $i \in\ ?K$ **and** *j*: $j \in\ ?K$ **and** *ij*: $i \neq j$
    have *False* **if** *eq*: $(replicate\ (k{+}1)\ 0)[i{:=}n] = (replicate\ (k{+}1)\ 0)[j{:=}\ n]$
    proof −
      have $(replicate\ (k{+}1)\ 0)\ [i{:=}n]\ !\ i = n$
        using *i* **by** (*simp del*: *replicate.simps*)
      moreover
      have $(replicate\ (k{+}1)\ 0)\ [j{:=}n]\ !\ i = 0$
        using *i ij* **by** (*simp del*: *replicate.simps*)
      ultimately show *?thesis*

using *eq n0* **by** (*simp del*: *replicate.simps*)
qed
then show $\{(replicate\ (k + 1)\ 0)[i := n]\} \cap \{(replicate\ (k + 1)\ 0)[j := n]\} = \{\}$
by *auto*
qed
from *card-UN-disjoint*[*OF fK fAK d*]
show $card\ (\bigcup i\in\{0..k\}.\ \{(replicate\ (k + 1)\ 0)[i := n]\}) = k + 1$
by *simp*
qed

lemma *fps-power-Suc-nth*:
fixes f :: *'a* :: *comm-ring-1 fps*
assumes k: $k > 0$
shows $(f\ \hat{}\ Suc\ m)\ \$\ k =$
$of\text{-}nat\ (Suc\ m) * (f\ \$\ k * (f\ \$\ 0)\ \hat{}\ m) +$
$(\sum v\in\{v\in natpermute\ k\ (m+1).\ k \notin set\ v\}.\ \prod j = 0..m.\ f\ \$\ v\ !\ j)$
proof −
define $A\ B$
where $A = \{v\in natpermute\ k\ (m+1).\ k \in set\ v\}$
and $B = \{v\in natpermute\ k\ (m+1).\ k \notin set\ v\}$
have [*simp*]: *finite A finite B* $A \cap B = \{\}$ **by** (*auto simp*: *A-def B-def natpermute-finite*)

from *natpermute-max-card*[*of k m*] *k* **have** *card-A*: $card\ A = m + 1$ **by** (*simp add*: *A-def*)
{
fix v **assume** v: $v \in A$
from v **have** [*simp*]: $length\ v = Suc\ m$ **by** (*simp add*: *A-def natpermute-def*)
from v **have** $\exists j.\ j \le m \wedge v\ !\ j = k$
by (*auto simp*: *set-conv-nth A-def natpermute-def less-Suc-eq-le*)
then obtain j **where** j: $j \le m\ v\ !\ j = k$ **by** *auto*

from v **have** $k = sum\text{-}list\ v$ **by** (*simp add*: *A-def natpermute-def*)
also have $\ldots = (\sum i=0..m.\ v\ !\ i)$
by (*simp add*: *sum-list-sum-nth atLeastLessThanSuc-atLeastAtMost del*: *sum.op-ivl-Suc*)
also from j **have** $\{0..m\} = insert\ j\ (\{0..m\}-\{j\})$ **by** *auto*
also from j **have** $(\sum i\in\ldots.\ v\ !\ i) = k + (\sum i\in\{0..m\}-\{j\}.\ v\ !\ i)$
by (*subst sum.insert*) *simp-all*
finally have $(\sum i\in\{0..m\}-\{j\}.\ v\ !\ i) = 0$ **by** *simp*
hence *zero*: $v\ !\ i = 0$ **if** $i \in \{0..m\}-\{j\}$ **for** i **using** *that*
by (*subst* (*asm*) *sum-eq-0-iff*) *auto*

from j **have** $\{0..m\} = insert\ j\ (\{0..m\} - \{j\})$ **by** *auto*
also from j **have** $(\prod i\in\ldots.\ f\ \$\ (v\ !\ i)) = f\ \$\ k * (\prod i\in\{0..m\} - \{j\}.\ f\ \$\ (v\ !\ i))$
by (*subst prod.insert*) *auto*
also have $(\prod i\in\{0..m\} - \{j\}.\ f\ \$\ (v\ !\ i)) = (\prod i\in\{0..m\} - \{j\}.\ f\ \$\ 0)$
by (*intro prod.cong*) (*simp-all add*: *zero*)

also from j **have** ... $= (f \$ 0) \hat{\ } m$ **by** (*subst prod-constant*) *simp-all*
finally have $(\prod j = 0..m.\ f \$ (v ! j)) = f \$ k * (f \$ 0) \hat{\ } m$ **.**
} **note** $A = this$

have $(f \hat{\ } Suc\ m) \$ k = (\sum v \in natpermute\ k\ (m + 1).\ \prod j = 0..m.\ f \$ v ! j)$
by (*rule fps-power-nth-Suc*)
also have *natpermute k* $(m+1) = A \cup B$ **unfolding** *A-def B-def* **by** *blast*
also have $(\sum v \in \ldots\ \prod j = 0..m.\ f \$ (v ! j)) =$
$(\sum v \in A.\ \prod j = 0..m.\ f \$ (v ! j)) + (\sum v \in B.\ \prod j = 0..m.\ f \$ (v ! j))$
by (*intro sum.union-disjoint*) *simp-all*
also have $(\sum v \in A.\ \prod j = 0..m.\ f \$ (v ! j)) = of\text{-}nat\ (Suc\ m) * (f \$ k * (f \$ 0)$
$\hat{\ } m)$
by (*simp add*: *A card-A*)
finally show *?thesis* **by** (*simp add*: *B-def*)
qed

lemma *fps-power-Suc-eqD*:
fixes *f g* :: *'a* :: {*idom*,*semiring-char-0*} *fps*
assumes $f \hat{\ } Suc\ m = g \hat{\ } Suc\ m$ $f \$ 0 = g \$ 0$ $f \$ 0 \neq 0$
shows $f = g$
proof (*rule fps-ext*)
fix *k* :: *nat*
show $f \$ k = g \$ k$
proof (*induction k rule*: *less-induct*)
case (*less k*)
show *?case*
proof (*cases* $k = 0$)
case *False*
let $?h = \lambda f.\ (\sum v \mid v \in natpermute\ k\ (m + 1) \wedge k \notin set\ v.\ \prod j = 0..m.\ f \$$
$v ! j)$
from *False fps-power-Suc-nth*[*of k f m*] *fps-power-Suc-nth*[*of k g m*]
have $f \$ k * (of\text{-}nat\ (Suc\ m) * (f \$ 0) \hat{\ } m) + ?h\ f =$
$g \$ k * (of\text{-}nat\ (Suc\ m) * (f \$ 0) \hat{\ } m) + ?h\ g$ **using** *assms*
by (*simp add*: *mult-ac del*: *power-Suc of-nat-Suc*)
also have $v ! i < k$ **if** $v \in \{v \in natpermute\ k\ (m+1).\ k \notin set\ v\}$ $i \leq m$ **for** *v i*
using *that elem-le-sum-list*[*of i v*] **unfolding** *natpermute-def*
by (*auto simp*: *set-conv-nth dest*!: *spec*[*of - i*])
hence $?h\ f = ?h\ g$
by (*intro sum.cong refl prod.cong less lessI*) (*simp add*: *natpermute-def*)
finally have $f \$ k * (of\text{-}nat\ (Suc\ m) * (f \$ 0) \hat{\ } m) = g \$ k * (of\text{-}nat\ (Suc$
$m) * (f \$ 0) \hat{\ } m)$
by *simp*
with *assms* **show** $f \$ k = g \$ k$
by (*subst* (*asm*) *mult-right-cancel*) (*auto simp del*: *of-nat-Suc*)
qed (*simp-all add*: *assms*)
qed
qed

lemma *fps-power-Suc-eqD'*:

```
  fixes f g :: 'a :: {idom,semiring-char-0} fps
  assumes f ^ Suc m = g ^ Suc m f $ subdegree f = g $ subdegree g
  shows   f = g
proof (cases f = 0)
  case False
  have Suc m * subdegree f = subdegree (f ^ Suc m)
    by (rule subdegree-power [symmetric])
  also have f ^ Suc m = g ^ Suc m by fact
  also have subdegree ... = Suc m * subdegree g by (rule subdegree-power)
  finally have [simp]: subdegree f = subdegree g
    by (subst (asm) Suc-mult-cancel1)
  have fps-shift (subdegree f) f * fps-X ^ subdegree f = f
    by (rule subdegree-decompose [symmetric])
  also have ... ^ Suc m = g ^ Suc m by fact
  also have g = fps-shift (subdegree g) g * fps-X ^ subdegree g
    by (rule subdegree-decompose)
  also have subdegree f = subdegree g by fact
  finally have fps-shift (subdegree g) f ^ Suc m = fps-shift (subdegree g) g ^ Suc
m
    by (simp add: algebra-simps power-mult-distrib del: power-Suc)
  hence fps-shift (subdegree g) f = fps-shift (subdegree g) g
    by (rule fps-power-Suc-eqD) (insert assms False, auto)
  with subdegree-decompose[of f] subdegree-decompose[of g] show ?thesis by simp
qed (insert assms, simp-all)

lemma fps-power-eqD':
  fixes f g :: 'a :: {idom,semiring-char-0} fps
  assumes f ^ m = g ^ m f $ subdegree f = g $ subdegree g m > 0
  shows   f = g
  using fps-power-Suc-eqD'[of f m−1 g] assms by simp

lemma fps-power-eqD:
  fixes f g :: 'a :: {idom,semiring-char-0} fps
  assumes f ^ m = g ^ m f $ 0 = g $ 0 f $ 0 ≠ 0 m > 0
  shows   f = g
  by (rule fps-power-eqD'[of f m g]) (insert assms, simp-all)

lemma fps-compose-inj-right:
  assumes a0: a$0 = (0::'a::idom)
    and a1: a$1 ≠ 0
  shows (b oo a = c oo a) ⟷ b = c
  (is ?lhs ⟷?rhs)
proof
  show ?lhs if ?rhs using that by simp
  show ?rhs if ?lhs
  proof −
    have b$n = c$n for n
    proof (induct n rule: nat-less-induct)
      fix n
```

assume *H*: ∀ *m*<*n*. *b*\$*m* = *c*\$*m*
show *b*\$*n* = *c*\$*n*
proof (*cases n*)
case *0*
from ‹*?lhs*› **have** (*b oo a*)\$*n* = (*c oo a*)\$*n*
by *simp*
then show *?thesis*
using *0* **by** (*simp add*: *fps-compose-nth*)
next
case (*Suc n1*)
have *f*: *finite* {*0* .. *n1*} *finite* {*n*} **by** *simp-all*
have *eq*: {*0* .. *n1*} ∪ {*n*} = {*0* .. *n*} **using** *Suc* **by** *auto*
have *d*: {*0* .. *n1*} ∩ {*n*} = {} **using** *Suc* **by** *auto*
have *seq*: (∑ *i* = *0*..*n1*. *b* \$ *i* ∗ *a* ^ *i* \$ *n*) = (∑ *i* = *0*..*n1*. *c* \$ *i* ∗ *a* ^ *i* \$ *n*)
using *H Suc* **by** *auto*
have *th0*: (*b oo a*) \$*n* = (∑ *i* = *0*..*n1*. *c* \$ *i* ∗ *a* ^ *i* \$ *n*) + *b*\$*n* ∗ (*a*\$*1*)^*n*
unfolding *fps-compose-nth sum.union-disjoint*[*OF f d*, *unfolded eq*] *seq*
using *startsby-zero-power-nth-same*[*OF a0*]
by *simp*
have *th1*: (*c oo a*) \$*n* = (∑ *i* = *0*..*n1*. *c* \$ *i* ∗ *a* ^ *i* \$ *n*) + *c*\$*n* ∗ (*a*\$*1*)^*n*
unfolding *fps-compose-nth sum.union-disjoint*[*OF f d*, *unfolded eq*]
using *startsby-zero-power-nth-same*[*OF a0*]
by *simp*
from ‹*?lhs*›[*unfolded fps-eq-iff*, *rule-format*, *of n*] *th0 th1 a1*
show *?thesis* **by** *auto*
qed
qed
then show *?rhs* **by** (*simp add*: *fps-eq-iff*)
qed
qed

5.13 Radicals

declare *prod.cong* [*fundef-cong*]

function *radical* :: (*nat* ⇒ ′*a* ⇒ ′*a*) ⇒ *nat* ⇒ ′*a*::*field fps* ⇒ *nat* ⇒ ′*a*
where
radical r 0 a 0 = *1*
| *radical r 0 a* (*Suc n*) = *0*
| *radical r* (*Suc k*) *a 0* = *r* (*Suc k*) (*a*\$*0*)
| *radical r* (*Suc k*) *a* (*Suc n*) =
(*a*\$ *Suc n* − *sum* (λ*xs*. *prod* (λ*j*. *radical r* (*Suc k*) *a* (*xs* ! *j*)) {*0*..*k*})
{*xs*. *xs* ∈ *natpermute* (*Suc n*) (*Suc k*) ∧ *Suc n* ∉ *set xs*}) /
(*of-nat* (*Suc k*) ∗ (*radical r* (*Suc k*) *a 0*)^*k*)
by *pat-completeness auto*

termination *radical*
proof

```
  let ?R = measure (λ(r, k, a, n). n)
  {
    show wf ?R by auto
  next
    fix r :: nat ⇒ 'a ⇒ 'a
    and a :: 'a fps
    and k n xs i
    assume xs: xs ∈ {xs ∈ natpermute (Suc n) (Suc k). Suc n ∉ set xs} and i: i
∈ {0..k}
    have False if c: Suc n ≤ xs ! i
    proof −
      from xs i have xs !i ≠ Suc n
        by (simp add: in-set-conv-nth natpermute-def)
      with c have c': Suc n < xs!i by arith
      have fths: finite {0 ..< i} finite {i} finite {i+1..<Suc k}
        by simp-all
      have d: {0 ..< i} ∩ ({i} ∪ {i+1 ..< Suc k}) = {} {i} ∩ {i+1..< Suc k} =
{}
        by auto
      have eqs: {0..<Suc k} = {0 ..< i} ∪ ({i} ∪ {i+1 ..< Suc k})
        using i by auto
      from xs have Suc n = sum-list xs
        by (simp add: natpermute-def)
      also have ... = sum (nth xs) {0..<Suc k} using xs
        by (simp add: natpermute-def sum-list-sum-nth)
      also have ... = xs!i + sum (nth xs) {0..<i} + sum (nth xs) {i+1..<Suc k}
       unfolding eqs  sum.union-disjoint[OF fths(1) finite-UnI[OF fths(2,3)] d(1)]
        unfolding sum.union-disjoint[OF fths(2) fths(3) d(2)]
        by simp
      finally show ?thesis using c' by simp
    qed
    then show ((r, Suc k, a, xs!i), r, Suc k, a, Suc n) ∈ ?R
      using not-less by auto
  next
    fix r :: nat ⇒ 'a ⇒ 'a
    and a :: 'a fps
    and k n
    show ((r, Suc k, a, 0), r, Suc k, a, Suc n) ∈ ?R by simp
  }
qed

definition fps-radical r n a = Abs-fps (radical r n a)

lemma radical-0 [simp]: ⋀n. 0 < n ⟹ radical r 0 a n = 0
  using radical.elims by blast

lemma fps-radical0[simp]: fps-radical r 0 a = 1
  by (auto simp add: fps-eq-iff fps-radical-def)
```

lemma *fps-radical-nth-0*[*simp*]: *fps-radical r n a* $ *0* = (*if n* = *0 then 1 else r n* (*a*$*0*))
  by (*cases n*) (*simp-all add*: *fps-radical-def*)

lemma *fps-radical-power-nth*[*simp*]:
  assumes *r*: (*r k* (*a*$*0*)) ^ *k* = *a*$*0*
  shows *fps-radical r k a* ^ *k* $ *0* = (*if k* = *0 then 1 else a*$*0*)
proof (*cases k*)
  case *0*
  then show *?thesis* **by** *simp*
next
  case (*Suc h*)
  have *eq1*: *fps-radical r k a* ^ *k* $ *0* = (∏ *j*∈{*0*..*h*}. *fps-radical r k a* $ (*replicate k 0*) ! *j*)
    unfolding *fps-power-nth Suc* **by** *simp*
  also have ... = (∏ *j*∈{*0*..*h*}. *r k* (*a*$*0*))
  proof (*rule prod.cong* [*OF refl*])
    show *fps-radical r k a* $ *replicate k 0* ! *j* = *r k* (*a* $ *0*) **if** *j* ∈ {*0*..*h*} **for** *j*
    proof −
      have *j* < *Suc h*
        using *that* **by** *presburger*
      then show *?thesis*
        by (*metis Suc fps-radical-nth-0 nth-replicate old.nat.distinct*(*2*))
    qed
  qed
  also have ... = *a*$*0*
    using *r Suc* **by** *simp*
  finally show *?thesis*
    using *Suc* **by** *simp*
qed

lemma *power-radical*:
  fixes *a*:: *'a*::*field-char-0 fps*
  assumes *a0*: *a*$*0* ≠ *0*
  shows (*r* (*Suc k*) (*a*$*0*)) ^ *Suc k* = *a*$*0* ⟷ (*fps-radical r* (*Suc k*) *a*) ^ (*Suc k*) = *a*
    (**is** *?lhs* ⟷ *?rhs*)
proof
  let *?r* = *fps-radical r* (*Suc k*) *a*
  show *?rhs* **if** *r0*: *?lhs*
  proof −
    from *a0 r0* **have** *r00*: *r* (*Suc k*) (*a*$*0*) ≠ *0* **by** *auto*
    have *?r* ^ *Suc k* $ *z* = *a*$*z* **for** *z*
    proof (*induct z rule*: *nat-less-induct*)
      fix *n*
      assume *H*: ∀ *m*<*n*. *?r* ^ *Suc k* $ *m* = *a*$*m*
      show *?r* ^ *Suc k* $ *n* = *a* $*n*
      proof (*cases n*)
        case *0*

then show *?thesis*
using *fps-radical-power-nth*[*of r Suc k a*, *OF r0*] **by** *simp*
next
case (*Suc n1*)
then have $n \neq 0$ **by** *simp*
let *?Pnk* = *natpermute n* (*k* + *1*)
let *?Pnkn* = {*xs* ∈ *?Pnk*. *n* ∈ *set xs*}
let *?Pnknn* = {*xs* ∈ *?Pnk*. *n* ∉ *set xs*}
have *eq*: *?Pnkn* ∪ *?Pnknn* = *?Pnk* **by** *blast*
have *d*: *?Pnkn* ∩ *?Pnknn* = {} **by** *blast*
have *f*: *finite ?Pnkn finite ?Pnknn*
using *finite-Un*[*of ?Pnkn ?Pnknn*, *unfolded eq*]
by (*metis natpermute-finite*)+
let *?f* = λ*v*. ∏*j*∈{*0*..*k*}. *?r* $ *v* ! *j*
have *sum ?f ?Pnkn* = *sum* (λ*v*. *?r* $ *n* ∗ *r* (*Suc k*) (*a* $ *0*) ^ *k*) *?Pnkn*
proof (*rule sum.cong*)
fix *v* **assume** *v*: *v* ∈ {*xs* ∈ *natpermute n* (*k* + *1*). *n* ∈ *set xs*}
let *?ths* = (∏*j*∈{*0*..*k*}. *fps-radical r* (*Suc k*) *a* $ *v* ! *j*) =
fps-radical r (*Suc k*) *a* $ *n* ∗ *r* (*Suc k*) (*a* $ *0*) ^ *k*
from *v* **obtain** *i* **where** *i*: *i* ∈ {*0*..*k*} *v* = (*replicate* (*k*+*1*) *0*) [*i*:= *n*]
unfolding *natpermute-contain-maximal* **by** *auto*
have (∏*j*∈{*0*..*k*}. *fps-radical r* (*Suc k*) *a* $ *v* ! *j*) =
(∏*j*∈{*0*..*k*}. *if j* = *i then fps-radical r* (*Suc k*) *a* $ *n else r* (*Suc k*)
(*a*$*0*))
using *i r0* **by** (*auto simp del*: *replicate.simps intro*: *prod.cong*)
also have ... = (*fps-radical r* (*Suc k*) *a* $ *n*) ∗ *r* (*Suc k*) (*a*$*0*) ^ *k*
using *i r0* **by** (*simp add*: *prod-gen-delta*)
finally show *?ths* **.**
qed *rule*
then have *sum ?f ?Pnkn* = *of-nat* (*k*+*1*) ∗ *?r* $ *n* ∗ *r* (*Suc k*) (*a* $ *0*) ^ *k*
by (*simp add*: *natpermute-max-card*[*OF* ‹$n \neq 0$›, *simplified*])
also have ... = *a*$*n* − *sum ?f ?Pnknn*
unfolding *Suc* **using** *r00 a0* **by** (*simp add*: *field-simps fps-radical-def del*:
of-nat-Suc)
finally have *fn*: *sum ?f ?Pnkn* = *a*$*n* − *sum ?f ?Pnknn* **.**
have (*?r* ^ *Suc k*)$*n* = *sum ?f ?Pnkn* + *sum ?f ?Pnknn*
unfolding *fps-power-nth-Suc sum.union-disjoint*[*OF f d*, *unfolded eq*] **..**
also have ... = *a*$*n* **unfolding** *fn* **by** *simp*
finally show *?thesis* **.**
qed
qed
then show *?thesis* **using** *r0* **by** (*simp add*: *fps-eq-iff*)
qed
show *?lhs* **if** *?rhs*
proof −
from *that* **have** ((*fps-radical r* (*Suc k*) *a*) ^(*Suc k*))$*0* = *a*$*0*
by *simp*
then show *?thesis*
unfolding *fps-power-nth-Suc*

```
      by (simp add: prod-constant del: replicate.simps)
  qed
qed

lemma radical-unique:
  assumes r0: (r (Suc k) (b$0)) ^ Suc k = b$0
    and a0: r (Suc k) (b$0 ::'a::field-char-0) = a$0
    and b0: b$0 ≠ 0
  shows a^(Suc k) = b ⟷ a = fps-radical r (Suc k) b
    (is ?lhs ⟷ ?rhs is - ⟷ a = ?r)
proof
  show ?lhs if ?rhs
    using that using power-radical[OF b0, of r k, unfolded r0] by simp
  show ?rhs if ?lhs
  proof −
    have r00: r (Suc k) (b$0) ≠ 0 using b0 r0 by auto
    have ceq: card {0..k} = Suc k by simp
    from a0 have a0r0: a$0 = ?r$0 by simp
    have a $ n = ?r $ n for n
    proof (induct n rule: nat-less-induct)
      fix n
      assume h: ∀ m<n. a$m = ?r $m
      show a$n = ?r $ n
      proof (cases n)
        case 0
        then show ?thesis using a0 by simp
      next
        case (Suc n1)
        have fK: finite {0..k} by simp
        have nz: n ≠ 0 using Suc by simp
        let ?Pnk = natpermute n (Suc k)
        let ?Pnkn = {xs ∈ ?Pnk. n ∈ set xs}
        let ?Pnknn = {xs ∈ ?Pnk. n ∉ set xs}
        have eq: ?Pnkn ∪ ?Pnknn = ?Pnk by blast
        have d: ?Pnkn ∩ ?Pnknn = {} by blast
        have f: finite ?Pnkn finite ?Pnknn
          using finite-Un[of ?Pnkn ?Pnknn, unfolded eq]
          by (metis natpermute-finite)+
        let ?f = λv. ∏j∈{0..k}. ?r $ v ! j
        let ?g = λv. ∏j∈{0..k}. a $ v ! j
        have sum ?g ?Pnkn = sum (λv. a $ n ∗ (?r$0)^k) ?Pnkn
        proof (rule sum.cong)
          fix v
          assume v: v ∈ {xs ∈ natpermute n (Suc k). n ∈ set xs}
          let ?ths = (∏j∈{0..k}. a $ v ! j) = a $ n ∗ (?r$0)^k
          from v obtain i where i: i ∈ {0..k} v = (replicate (k+1) 0) [i:= n]
            unfolding Suc-eq-plus1 natpermute-contain-maximal
            by (auto simp del: replicate.simps)
         have (∏j∈{0..k}. a $ v ! j) = (∏j∈{0..k}. if j = i then a $ n else r (Suc
```

```
k) (b$0))
      using i a0 by (auto simp del: replicate.simps intro: prod.cong)
    also have ... = a $ n * (?r $ 0)^k
      using i by (simp add: prod-gen-delta)
    finally show ?ths .
  qed rule
  then have th0: sum ?g ?Pnkn = of-nat (k+1) * a $ n * (?r $ 0)^k
    by (simp add: natpermute-max-card[OF nz, simplified])
  have th1: sum ?g ?Pnknn = sum ?f ?Pnknn
  proof (rule sum.cong, rule refl, rule prod.cong, simp)
    fix xs i
    assume xs: xs ∈ ?Pnknn and i: i ∈ {0..k}
    have False if c: n ≤ xs ! i
    proof −
      from xs i have xs ! i ≠ n
        by (simp add: in-set-conv-nth natpermute-def)
      with c have c': n < xs!i by arith
      have fths: finite {0 ..< i} finite {i} finite {i+1..<Suc k}
        by simp-all
       have d: {0 ..< i} ∩ ({i} ∪ {i+1 ..< Suc k}) = {} {i} ∩ {i+1..< Suc
k} = {}
        by auto
      have eqs: {0..<Suc k} = {0 ..< i} ∪ ({i} ∪ {i+1 ..< Suc k})
        using i by auto
      from xs have n = sum-list xs
        by (simp add: natpermute-def)
      also have ... = sum (nth xs) {0..<Suc k}
        using xs by (simp add: natpermute-def sum-list-sum-nth)
     also have ... = xs!i + sum (nth xs) {0..<i} + sum (nth xs) {i+1..<Suc
k}
        unfolding eqs  sum.union-disjoint[OF fths(1) finite-UnI[OF fths(2,3)]
d(1)]
        unfolding sum.union-disjoint[OF fths(2) fths(3) d(2)]
        by simp
      finally show ?thesis using c' by simp
    qed
    then have thn: xs!i < n by presburger
    from h[rule-format, OF thn] show a$(xs !i) = ?r$(xs!i) .
  qed
  have th00: ⋀x::'a. of-nat (Suc k) * (x * inverse (of-nat (Suc k))) = x
    by (simp add: field-simps del: of-nat-Suc)
  from ‹?lhs› have b$n = a^Suc k $ n
    by (simp add: fps-eq-iff)
  also have a ^ Suc k$n = sum ?g ?Pnkn + sum ?g ?Pnknn
    unfolding fps-power-nth-Suc
    using sum.union-disjoint[OF f d, unfolded Suc-eq-plus1[symmetric],
      unfolded eq, of ?g] by simp
  also have ... = of-nat (k+1) * a $ n * (?r $ 0)^k + sum ?f ?Pnknn
    unfolding th0 th1 ..
```

```
      finally have §: of-nat (k+1) * a $ n * (?r $ 0)^k = b$n - sum ?f ?Pnknn
        by simp
      have a$n = (b$n - sum ?f ?Pnknn) / (of-nat (k+1) * (?r $ 0)^k)
        apply (rule eq-divide-imp)
        using r00 § by (simp-all add: ac-simps del: of-nat-Suc)
      then show ?thesis
        unfolding fps-radical-def Suc
        by (simp del: of-nat-Suc)
    qed
  qed
  then show ?rhs by (simp add: fps-eq-iff)
 qed
qed


lemma radical-power:
 assumes r0: r (Suc k) ((a$0) ^ Suc k) = a$0
  and a0: (a$0 :: 'a::field-char-0) ≠ 0
 shows (fps-radical r (Suc k) (a ^ Suc k)) = a
proof -
 let ?ak = a^ Suc k
 have ak0: ?ak $ 0 = (a$0) ^ Suc k
  by (simp add: fps-nth-power-0 del: power-Suc)
 from r0 have th0: r (Suc k) (a ^ Suc k $ 0) ^ Suc k = a ^ Suc k $ 0
  using ak0 by auto
 from r0 ak0 have th1: r (Suc k) (a ^ Suc k $ 0) = a $ 0
  by auto
 from ak0 a0 have ak00: ?ak $ 0 ≠0
  by auto
 from radical-unique[of r k ?ak a, OF th0 th1 ak00] show ?thesis
  by metis
qed

lemma fps-deriv-radical':
 fixes a :: 'a::field-char-0 fps
 assumes r0: (r (Suc k) (a$0)) ^ Suc k = a$0
  and a0: a$0 ≠ 0
 shows fps-deriv (fps-radical r (Suc k) a) =
  fps-deriv a / ((of-nat (Suc k)) * (fps-radical r (Suc k) a) ^ k)
proof -
 let ?r = fps-radical r (Suc k) a
 let ?w = (of-nat (Suc k)) * ?r ^ k
 from a0 r0 have r0': r (Suc k) (a$0) ≠ 0
  by auto
 from r0' have w0: ?w $ 0 ≠ 0
  by (simp del: of-nat-Suc)
 note th0 = inverse-mult-eq-1[OF w0]
 let ?iw = inverse ?w
 from iffD1[OF power-radical[of a r], OF a0 r0]
```

```
  have fps-deriv (?r ^ Suc k) = fps-deriv a
    by simp
  then have fps-deriv ?r * ?w = fps-deriv a
    by (simp add: fps-deriv-power' ac-simps del: power-Suc)
  then have ?iw * fps-deriv ?r * ?w = ?iw * fps-deriv a
    by simp
  with a0 r0 have fps-deriv ?r * (?iw * ?w) = fps-deriv a / ?w
    by (subst fps-divide-unit) (auto simp del: of-nat-Suc)
  then show ?thesis unfolding th0 by simp
qed

lemma fps-deriv-radical:
  fixes a :: 'a::field-char-0 fps
  assumes r0: (r (Suc k) (a$0)) ^ Suc k = a$0
    and a0: a$0 ≠ 0
  shows fps-deriv (fps-radical r (Suc k) a) =
    fps-deriv a / (fps-const (of-nat (Suc k)) * (fps-radical r (Suc k) a) ^ k)
  using fps-deriv-radical'[of r k a, OF r0 a0]
  by (simp add: fps-of-nat[symmetric])

lemma radical-mult-distrib:
  fixes a :: 'a::field-char-0 fps
  assumes k: k > 0
    and ra0: r k (a $ 0) ^ k = a $ 0
    and rb0: r k (b $ 0) ^ k = b $ 0
    and a0: a $ 0 ≠ 0
    and b0: b $ 0 ≠ 0
  shows r k ((a * b) $ 0) = r k (a $ 0) * r k (b $ 0) ⟷
    fps-radical r k (a * b) = fps-radical r k a * fps-radical r k b
    (is ?lhs ⟷ ?rhs)
proof
  show ?rhs if r0': ?lhs
  proof -
    from r0' have r0: (r k ((a * b) $ 0)) ^ k = (a * b) $ 0
      by (simp add: fps-mult-nth ra0 rb0 power-mult-distrib)
    show ?thesis
    proof (cases k)
      case 0
      then show ?thesis using r0' by simp
    next
      case (Suc h)
      let ?ra = fps-radical r (Suc h) a
      let ?rb = fps-radical r (Suc h) b
       have th0: r (Suc h) ((a * b) $ 0) = (fps-radical r (Suc h) a * fps-radical r
(Suc h) b) $ 0
        using r0' Suc by (simp add: fps-mult-nth)
      have ab0: (a*b) $ 0 ≠ 0
        using a0 b0 by (simp add: fps-mult-nth)
      from radical-unique[of r h a*b fps-radical r (Suc h) a * fps-radical r (Suc h)
```

b, *OF r0*[*unfolded Suc*] *th0 ab0*, *symmetric*]
iffD1[*OF power-radical*[*of* - *r*], *OF a0 ra0*[*unfolded Suc*]] *iffD1*[*OF power-radical*[*of* - *r*], *OF b0 rb0*[*unfolded Suc*]] *Suc r0'*
show *?thesis*
by (*auto simp add*: *power-mult-distrib simp del*: *power-Suc*)
qed
qed
show *?lhs* **if** *?rhs*
proof −
from *that* **have** (*fps-radical r k* (*a* ∗ *b*)) $ *0* = (*fps-radical r k a* ∗ *fps-radical r k b*) $ *0*
by *simp*
then show *?thesis*
using *k* **by** (*simp add*: *fps-mult-nth*)
qed
qed

lemma *radical-divide*:
fixes *a* :: *'a*::*field-char-0 fps*
assumes *kp*: *k* > *0*
and *ra0*: (*r k* (*a* $ *0*)) ^ *k* = *a* $ *0*
and *rb0*: (*r k* (*b* $ *0*)) ^ *k* = *b* $ *0*
and *a0*: *a*$*0* ≠ *0*
and *b0*: *b*$*0* ≠ *0*
shows *r k* ((*a* $ *0*) / (*b*$*0*)) = *r k* (*a*$*0*) / *r k* (*b* $ *0*) ⟷
fps-radical r k (*a*/*b*) = *fps-radical r k a* / *fps-radical r k b*
(**is** *?lhs* = *?rhs*)
proof
let *?r* = *fps-radical r k*
from *kp* **obtain** *h* **where** *k*: *k* = *Suc h*
by (*cases k*) *auto*
have *ra0'*: *r k* (*a*$*0*) ≠ *0* **using** *a0 ra0 k* **by** *auto*
have *rb0'*: *r k* (*b*$*0*) ≠ *0* **using** *b0 rb0 k* **by** *auto*

show *?lhs* **if** *?rhs*
proof −
from *that* **have** *?r* (*a*/*b*) $ *0* = (*?r a* / *?r b*)$*0*
by *simp*
then show *?thesis*
using *k a0 b0 rb0'* **by** (*simp add*: *fps-divide-unit fps-mult-nth fps-inverse-def divide-inverse*)
qed
show *?rhs* **if** *?lhs*
proof −
from *a0 b0* **have** *ab0*[*simp*]: (*a*/*b*)$*0* = *a*$*0* / *b*$*0*
by (*simp add*: *fps-divide-def fps-mult-nth divide-inverse fps-inverse-def*)
have *th0*: *r k* ((*a*/*b*)$*0*) ^ *k* = (*a*/*b*)$*0*

by (*simp add*: ‹*?lhs*› *power-divide ra0 rb0*)
from *a0 b0 ra0′ rb0′ kp* ‹*?lhs*›
have *th1*: *r k* ((*a* / *b*) $ *0*) = (*fps-radical r k a* / *fps-radical r k b*) $ *0*
by (*simp add*: *fps-divide-unit fps-mult-nth fps-inverse-def divide-inverse*)
from *a0 b0 ra0′ rb0′ kp* **have** *ab0′*: (*a* / *b*) $ *0* ≠ *0*
by (*simp add*: *fps-divide-unit fps-mult-nth fps-inverse-def nonzero-imp-inverse-nonzero*)
note *tha*[*simp*] = *iffD1*[*OF power-radical*[**where** *r*=*r* **and** *k*=*h*], *OF a0 ra0*[*unfolded k*], *unfolded k*[*symmetric*]]
note *thb*[*simp*] = *iffD1*[*OF power-radical*[**where** *r*=*r* **and** *k*=*h*], *OF b0 rb0*[*unfolded k*], *unfolded k*[*symmetric*]]
from *b0 rb0′* **have** *th2*: (*?r a* / *?r b*)^*k* = *a*/*b*
by (*simp add*: *fps-divide-unit power-mult-distrib fps-inverse-power*[*symmetric*])

from *iffD1*[*OF radical-unique*[**where** *r*=*r* **and** *a*=*?r a* / *?r b* **and** *b*=*a*/*b* **and** *k*=*h*], *symmetric*, *unfolded k*[*symmetric*], *OF th0 th1 ab0′ th2*]
show *?thesis* **.**
qed
qed

lemma *radical-inverse*:
fixes *a* :: ′*a*::*field-char-0 fps*
assumes *k*: *k* > *0*
and *ra0*: *r k* (*a* $ *0*) ^ *k* = *a* $ *0*
and *r1*: (*r k 1*)^*k* = *1*
and *a0*: *a*$*0* ≠ *0*
shows *r k* (*inverse* (*a* $ *0*)) = *r k 1* / (*r k* (*a* $ *0*)) ⟷
fps-radical r k (*inverse a*) = *fps-radical r k 1* / *fps-radical r k a*
using *radical-divide*[**where** *k*=*k* **and** *r*=*r* **and** *a*=*1* **and** *b*=*a*, *OF k*] *ra0 r1 a0*
by (*simp add*: *divide-inverse fps-divide-def*)

5.14 Chain rule

lemma *fps-compose-deriv*:
fixes *a* :: ′*a*::*idom fps*
assumes *b0*: *b*$*0* = *0*
shows *fps-deriv* (*a oo b*) = ((*fps-deriv a*) *oo b*) ∗ *fps-deriv b*
proof −
have (*fps-deriv* (*a oo b*))$*n* = (((*fps-deriv a*) *oo b*) ∗ (*fps-deriv b*)) $*n* **for** *n*
proof −
have (*fps-deriv* (*a oo b*))$*n* = *sum* (λ*i*. *a* $ *i* ∗ (*fps-deriv* (*b*^*i*))$*n*) {*0*.. *Suc n*}
by (*simp add*: *fps-compose-def field-simps sum-distrib-left del*: *of-nat-Suc*)
also have ... = *sum* (λ*i*. *a*$*i* ∗ ((*fps-const* (*of-nat i*)) ∗ (*fps-deriv b* ∗ (*b*^(*i* − *1*))))$*n*) {*0*.. *Suc n*}
by (*simp add*: *field-simps fps-deriv-power del*: *fps-mult-left-const-nth of-nat-Suc*)
also have ... = *sum* (λ*i*. *of-nat i* ∗ *a*$*i* ∗ (((*b*^(*i* − *1*)) ∗ *fps-deriv b*))$*n*) {*0*.. *Suc n*}
unfolding *fps-mult-left-const-nth* **by** (*simp add*: *field-simps*)
also have ... = *sum* (λ*i*. *of-nat i* ∗ *a*$*i* ∗ (*sum* (λ*j*. (*b*^ (*i* − *1*))$*j* ∗ (*fps-deriv b*)$(*n* − *j*)) {*0*..*n*})) {*0*.. *Suc n*}

unfolding *fps-mult-nth* **..**
also have ... = *sum* (*λi. of-nat i* ∗ *a$i* ∗ (*sum* (*λj.* (*b* ^ (*i* − *1*))$*j* ∗ (*fps-deriv b*)$(*n* − *j*)) {*0..n*})) {*1.. Suc n*}
by (*intro sum.mono-neutral-right*) (*auto simp add*: *mult-delta-left not-le*)
also have ... = *sum* (*λi. of-nat* (*i* + *1*) ∗ *a*$(*i*+*1*) ∗ (*sum* (*λj.* (*b* ^ *i*)$*j* ∗ *of-nat* (*n* − *j* + *1*) ∗ *b*$(*n* − *j* + *1*)) {*0..n*})) {*0.. n*}
unfolding *fps-deriv-nth*
by (*rule sum.reindex-cong* [*of Suc*]) (*simp-all add*: *mult.assoc*)
finally have *th0*: (*fps-deriv* (*a oo b*))$*n* =
sum (*λi. of-nat* (*i* + *1*) ∗ *a*$(*i*+*1*) ∗ (*sum* (*λj.* (*b* ^ *i*)$*j* ∗ *of-nat* (*n* − *j* + *1*) ∗ *b*$(*n* − *j* + *1*)) {*0..n*})) {*0.. n*} **.**

have (((*fps-deriv a*) *oo b*) ∗ (*fps-deriv b*))$*n* = *sum* (*λi.* (*fps-deriv b*)$ (*n* − *i*) ∗ ((*fps-deriv a*) *oo b*)$*i*) {*0..n*}
unfolding *fps-mult-nth* **by** (*simp add*: *ac-simps*)
also have ... = *sum* (*λi. sum* (*λj. of-nat* (*n* − *i* +*1*) ∗ *b*$(*n* − *i* + *1*) ∗ *of-nat* (*j* + *1*) ∗ *a*$(*j*+*1*) ∗ (*b*^*j*)$*i*) {*0..n*}) {*0..n*}
unfolding *fps-deriv-nth fps-compose-nth sum-distrib-left mult.assoc*
by (*auto simp*: *subset-eq b0 startsby-zero-power-prefix sum.mono-neutral-left intro*: *sum.cong*)
also have ... = *sum* (*λi. of-nat* (*i* + *1*) ∗ *a*$(*i*+*1*) ∗ (*sum* (*λj.* (*b* ^ *i*)$*j* ∗ *of-nat* (*n* − *j* + *1*) ∗ *b*$(*n* − *j* + *1*)) {*0..n*})) {*0.. n*}
unfolding *sum-distrib-left*
by (*subst sum.swap*) (*force intro*: *sum.cong*)
finally show *?thesis*
unfolding *th0* **by** *simp*
qed
then show *?thesis* **by** (*simp add*: *fps-eq-iff*)
qed

lemma *fps-poly-sum-fps-X*:
assumes ∀ *i* > *n*. *a*$*i* = *0*
shows *a* = *sum* (*λi. fps-const* (*a*$*i*) ∗ *fps-X*^*i*) {*0..n*} (**is** *a* = *?r*)
proof −
have *a*$*i* = *?r*$*i* **for** *i*
unfolding *fps-sum-nth fps-mult-left-const-nth fps-X-power-nth*
by (*simp add*: *mult-delta-right assms*)
then show *?thesis*
unfolding *fps-eq-iff* **by** *blast*
qed

5.15 Compositional inverses

fun *compinv* :: *'a fps* ⇒ *nat* ⇒ *'a*::*field*
where
compinv a 0 = *fps-X*$*0*
| *compinv a* (*Suc n*) =
(*fps-X*$ *Suc n* − *sum* (*λi.* (*compinv a i*) ∗ (*a*^*i*)$*Suc n*) {*0* .. *n*}) / (*a*$*1*) ^ *Suc n*

definition *fps-inv a* = *Abs-fps* (*compinv a*)

lemma *fps-inv*:
  assumes *a0*: *a*\$*0* = *0*
    and *a1*: *a*\$*1* ≠ *0*
  shows *fps-inv a oo a* = *fps-X*
proof −
  let *?i* = *fps-inv a oo a*
  have *?i* \$*n* = *fps-X*\$*n* **for** *n*
  proof (*induct n rule*: *nat-less-induct*)
    fix *n*
    assume *h*: ∀ *m*<*n*. *?i*\$*m* = *fps-X*\$*m*
    show *?i* \$ *n* = *fps-X*\$*n*
    proof (*cases n*)
      case *0*
      then show *?thesis* **using** *a0*
        by (*simp add*: *fps-compose-nth fps-inv-def*)
    next
      case (*Suc n1*)
       have *?i* \$ *n* = *sum* (λ*i*. (*fps-inv a* \$ *i*) ∗ (*a*^*i*)\$*n*) {*0* .. *n1*} + *fps-inv a* \$ *Suc n1* ∗ (*a* \$ *1*)^*Suc n1*
       by (*simp only*: *fps-compose-nth*) (*simp add*: *Suc startsby-zero-power-nth-same* [*OF a0*] *del*: *power-Suc*)
      also have . . . = *sum* (λ*i*. (*fps-inv a* \$ *i*) ∗ (*a*^*i*)\$*n*) {*0* .. *n1*} +
        (*fps-X*\$ *Suc n1* − *sum* (λ*i*. (*fps-inv a* \$ *i*) ∗ (*a*^*i*)\$*n*) {*0* .. *n1*})
        using *a0 a1 Suc* **by** (*simp add*: *fps-inv-def*)
      also have . . . = *fps-X*\$*n* **using** *Suc* **by** *simp*
      finally show *?thesis* **.**
    qed
  qed
  then show *?thesis*
    by (*simp add*: *fps-eq-iff*)
qed

fun *gcompinv* :: ′*a fps* ⇒ ′*a fps* ⇒ *nat* ⇒ ′*a*::*field*
where
  gcompinv b a 0 = *b*\$*0*
| *gcompinv b a* (*Suc n*) =
    (*b*\$ *Suc n* − *sum* (λ*i*. (*gcompinv b a i*) ∗ (*a*^*i*)\$*Suc n*) {*0* .. *n*}) / (*a*\$*1*) ^*Suc n*

definition *fps-ginv b a* = *Abs-fps* (*gcompinv b a*)

lemma *fps-ginv*:
  assumes *a0*: *a*\$*0* = *0*
    and *a1*: *a*\$*1* ≠ *0*
  shows *fps-ginv b a oo a* = *b*

```
proof –
  let ?i = fps-ginv b a oo a
  have ?i $n = b$n for n
  proof (induct n rule: nat-less-induct)
    fix n
    assume h: ∀ m<n. ?i$m = b$m
    show ?i $ n = b$n
    proof (cases n)
      case 0
      then show ?thesis using a0
        by (simp add: fps-compose-nth fps-ginv-def)
    next
      case (Suc n1)
      have ?i $ n = sum (λi. (fps-ginv b a $ i) * (a^i)$n) {0 .. n1} + fps-ginv b
a $ Suc n1 * (a $ 1)^ Suc n1
       by (simp only: fps-compose-nth) (simp add: Suc startsby-zero-power-nth-same
[OF a0] del: power-Suc)
      also have … = sum (λi. (fps-ginv b a $ i) * (a^i)$n) {0 .. n1} +
        (b$ Suc n1 – sum (λi. (fps-ginv b a $ i) * (a^i)$n) {0 .. n1})
        using a0 a1 Suc by (simp add: fps-ginv-def)
      also have … = b$n using Suc by simp
      finally show ?thesis .
    qed
  qed
  then show ?thesis
    by (simp add: fps-eq-iff)
qed

lemma fps-inv-ginv: fps-inv = fps-ginv fps-X
proof –
  have compinv x n = gcompinv fps-X x n for n and x :: 'a fps
  proof (induction n rule: nat-less-induct)
    case (1 n)
    then show ?case
      by (cases n) auto
  qed
  then show ?thesis
    by (auto simp add: fun-eq-iff fps-eq-iff fps-inv-def fps-ginv-def)
qed

lemma fps-compose-1[simp]: 1 oo a = 1
  by (simp add: fps-eq-iff fps-compose-nth mult-delta-left)

lemma fps-compose-0[simp]: 0 oo a = 0
  by (simp add: fps-eq-iff fps-compose-nth)

lemma fps-compose-0-right[simp]: a oo 0 = fps-const (a $ 0)
  by (simp add: fps-eq-iff fps-compose-nth power-0-left sum.neutral)
```

```
lemma fps-compose-add-distrib: (a + b) oo c = (a oo c) + (b oo c)
  by (simp add: fps-eq-iff fps-compose-nth field-simps sum.distrib)

lemma fps-compose-sum-distrib: (sum f S) oo a = sum (λi. f i oo a) S
proof (cases finite S)
  case True
  show ?thesis
  proof (rule finite-induct[OF True])
    show sum f {} oo a = (∑ i∈{}. f i oo a)
      by simp
  next
    fix x F
    assume fF: finite F
      and xF: x ∉ F
      and h: sum f F oo a = sum (λi. f i oo a) F
    show sum f (insert x F) oo a  = sum (λi. f i oo a) (insert x F)
      using fF xF h by (simp add: fps-compose-add-distrib)
  qed
next
  case False
  then show ?thesis by simp
qed

lemma convolution-eq:
  sum (λi. a (i :: nat) * b (n − i)) {0 .. n} =
    sum (λ(i,j). a i * b j) {(i,j). i ≤ n ∧ j ≤ n ∧ i + j = n}
  by (rule sum.reindex-bij-witness[where i=fst and j=λi. (i, n − i)]) auto

lemma product-composition-lemma:
  assumes c0: c$0 = (0::'a::idom)
    and d0: d$0 = 0
  shows ((a oo c) * (b oo d))$n =
    sum (λ(k,m). a$k * b$m * (c^k * d^m) $ n) {(k,m). k + m ≤ n}  (is ?l = ?r)
proof −
  let ?S = {(k::nat, m::nat). k + m ≤ n}
  have s: ?S ⊆ {0..n} × {0..n} by (simp add: subset-eq)
  have f: finite {(k::nat, m::nat). k + m ≤ n}
    by (auto intro: finite-subset[OF s])
  have ?r = (∑ (k, m) ∈ {(k, m). k + m ≤ n}. ∑ j = 0..n. a $ k * b $ m * (c ^
k $ j * d ^ m $ (n − j)))
    by (simp add: fps-mult-nth sum-distrib-left)
  also have . . . = (∑ i = 0..n. ∑ (k,m)∈{(k,m). k+m ≤ n}. a $ k * c ^ k $ i *
b $ m * d ^ m $ (n−i))
    unfolding sum.swap [where A = {0..n}] by (auto simp add: field-simps intro:
sum.cong)
  also have ... = (∑ i = 0..n.
                    ∑ q = 0..i. ∑ j = 0..n − i. a $ q * c ^ q $ i * (b $ j * d ^ j $ (n
− i)))
    apply (rule sum.cong [OF refl])
```

apply (*simp add*: *sum.cartesian-product mult.assoc*)
apply (*rule sum.mono-neutral-right*[*OF f*], *force*)
by *clarsimp* (*meson c0 d0 leI startsby-zero-power-prefix*)
also have ... = *?l*
by (*simp add*: *fps-mult-nth fps-compose-nth sum-product*)
finally show *?thesis* **by** *simp*
qed

lemma *sum-pair-less-iff*:
sum (λ((*k*::*nat*),*m*). *a k* ∗ *b m* ∗ *c* (*k* + *m*)) {(*k*,*m*). *k* + *m* ≤ *n*} =
sum (λ*s*. *sum* (λ*i*. *a i* ∗ *b* (*s* − *i*) ∗ *c s*) {*0*..*s*}) {*0*..*n*}
(**is** *?l* = *?r*)
proof −
have *th0*: {(*k*, *m*). *k* + *m* ≤ *n*} = (⋃*s*∈{*0*..*n*}. ⋃*i*∈{*0*..*s*}. {(*i*, *s* − *i*)})
by *auto*
show *?l* = *?r*
unfolding *th0*
by (*simp add*: *sum.UNION-disjoint eq-diff-iff disjoint-iff*)
qed

lemma *fps-compose-mult-distrib-lemma*:
assumes *c0*: *c*$*0* = (*0*::′*a*::*idom*)
shows ((*a oo c*) ∗ (*b oo c*))$*n* = *sum* (λ*s*. *sum* (λ*i*. *a*$*i* ∗ *b*$(*s* − *i*) ∗ (*c*^*s*) $ *n*) {*0*..*s*}) {*0*..*n*}
unfolding *product-composition-lemma*[*OF c0 c0*] *power-add*[*symmetric*]
unfolding *sum-pair-less-iff*[**where** *a* = λ*k*. *a*$*k* **and** *b*=λ*m*. *b*$*m* **and** *c*=λ*s*. (*c* ^ *s*)$*n* **and** *n* = *n*] **..**

lemma *fps-compose-mult-distrib*:
assumes *c0*: *c* $ *0* = (*0*::′*a*::*idom*)
shows (*a* ∗ *b*) *oo c* = (*a oo c*) ∗ (*b oo c*)
proof (*clarsimp simp add*: *fps-eq-iff fps-compose-mult-distrib-lemma* [*OF c0*])
show (*a* ∗ *b oo c*) $ *n* = (∑*s* = *0*..*n*. ∑*i* = *0*..*s*. *a* $ *i* ∗ *b* $ (*s* − *i*) ∗ *c* ^ *s* $ *n*) **for** *n*
by (*simp add*: *fps-compose-nth fps-mult-nth sum-distrib-right*)
qed

lemma *fps-compose-prod-distrib*:
assumes *c0*: *c*$*0* = (*0*::′*a*::*idom*)
shows *prod a S oo c* = *prod* (λ*k*. *a k oo c*) *S*
proof (*induct S rule*: *infinite-finite-induct*)
next
case (*insert*)
then show *?case*
by (*simp add*: *fps-compose-mult-distrib*[*OF c0*])
qed *auto*

lemma *fps-compose-divide*:

assumes [*simp*]: *g dvd f h* $ *0* = *0*
shows *fps-compose f h* = *fps-compose* (*f* / *g* :: ′*a* :: *field fps*) *h* ∗ *fps-compose g h*
proof −
have *f* = (*f* / *g*) ∗ *g* **by** *simp*
also have *fps-compose* . . . *h* = *fps-compose* (*f* / *g*) *h* ∗ *fps-compose g h*
by (*subst fps-compose-mult-distrib*) *simp-all*
finally show *?thesis* **.**
qed

lemma *fps-compose-divide-distrib*:
assumes *g dvd f h* $ *0* = *0 fps-compose g h* ≠ *0*
shows *fps-compose* (*f* / *g* :: ′*a* :: *field fps*) *h* = *fps-compose f h* / *fps-compose g h*
using *fps-compose-divide*[*OF assms*(*1*,*2*)] *assms*(*3*) **by** *simp*

lemma *fps-compose-power*:
assumes *c0*: *c*$*0* = (*0*::′*a*::*idom*)
shows (*a oo c*)^*n* = *a*^*n oo c*
proof (*cases n*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc m*)
have (∏ *n* = *0*..*m*. *a*) *oo c* = (∏ *n* = *0*..*m*. *a oo c*)
using *c0 fps-compose-prod-distrib* **by** *blast*
moreover have *th0*: *a*^*n* = *prod* (λ*k*. *a*) {*0*..*m*} (*a oo c*) ^ *n* = *prod* (λ*k*. *a oo c*) {*0*..*m*}
by (*simp-all add*: *prod-constant Suc*)
ultimately show *?thesis*
by *presburger*
qed

lemma *fps-compose-uminus*: − (*a*::′*a*::*ring-1 fps*) *oo c* = − (*a oo c*)
by (*simp add*: *fps-eq-iff fps-compose-nth field-simps sum-negf*[*symmetric*])

lemma *fps-compose-sub-distrib*: (*a* − *b*) *oo* (*c*::′*a*::*ring-1 fps*) = (*a oo c*) − (*b oo c*)
using *fps-compose-add-distrib* [*of a* − *b c*] **by** (*simp add*: *fps-compose-uminus*)

lemma *fps-X-fps-compose*: *fps-X oo a* = *Abs-fps* (λ*n*. *if n* = *0 then* (*0*::′*a*::*comm-ring-1*) *else a*$*n*)
by (*simp add*: *fps-eq-iff fps-compose-nth mult-delta-left*)

lemma *fps-inverse-compose*:
assumes *b0*: (*b*$*0* :: ′*a*::*field*) = *0*
and *a0*: *a*$*0* ≠ *0*
shows *inverse a oo b* = *inverse* (*a oo b*)
proof −

let *?ia* = *inverse a*
let *?ab* = *a oo b*
let *?iab* = *inverse ?ab*

from *a0* **have** *ia0*: *?ia* \$ *0* ≠ *0* **by** *simp*
from *a0* **have** *ab0*: *?ab* \$ *0* ≠ *0* **by** (*simp add*: *fps-compose-def*)
have (*?ia oo b*) ∗ (*a oo b*) = *1*
unfolding *fps-compose-mult-distrib*[*OF b0*, *symmetric*]
unfolding *inverse-mult-eq-1*[*OF a0*]
fps-compose-1 **..**

then have (*?ia oo b*) ∗ (*a oo b*) ∗ *?iab* = *1* ∗ *?iab* **by** *simp*
then have (*?ia oo b*) ∗ (*?iab* ∗ (*a oo b*)) = *?iab* **by** *simp*
then show *?thesis* **unfolding** *inverse-mult-eq-1*[*OF ab0*] **by** *simp*
qed

lemma *fps-divide-compose*:
assumes *c0*: (*c*\$*0* :: *'a*::*field*) = *0*
and *b0*: *b*\$*0* ≠ *0*
shows (*a*/*b*) *oo c* = (*a oo c*) / (*b oo c*)
using *b0 c0* **by** (*simp add*: *fps-divide-unit fps-inverse-compose fps-compose-mult-distrib*)

lemma *gp*:
assumes *a0*: *a*\$*0* = (*0*::*'a*::*field*)
shows (*Abs-fps* (*λn. 1*)) *oo a* = *1*/(*1* − *a*)
(**is** *?one oo a* = *-*)
proof −
have *o0*: *?one* \$ *0* ≠ *0* **by** *simp*
have *th0*: (*1* − *fps-X*) \$ *0* ≠ (*0*::*'a*) **by** *simp*
from *fps-inverse-gp*[**where** *?'a* = *'a*]
have *inverse ?one* = *1* − *fps-X* **by** (*simp add*: *fps-eq-iff*)
then have *inverse* (*inverse ?one*) = *inverse* (*1* − *fps-X*) **by** *simp*
then have *th*: *?one* = *1*/(*1* − *fps-X*) **unfolding** *fps-inverse-idempotent*[*OF o0*]
by (*simp add*: *fps-divide-def*)
show *?thesis*
unfolding *th*
unfolding *fps-divide-compose*[*OF a0 th0*]
fps-compose-1 fps-compose-sub-distrib fps-X-fps-compose-startby0[*OF a0*] **..**
qed

lemma *fps-compose-radical*:
assumes *b0*: *b*\$*0* = (*0*::*'a*::*field-char-0*)
and *ra0*: *r* (*Suc k*) (*a*\$*0*) ^ *Suc k* = *a*\$*0*
and *a0*: *a*\$*0* ≠ *0*
shows *fps-radical r* (*Suc k*) *a oo b* = *fps-radical r* (*Suc k*) (*a oo b*)
proof −
let *?r* = *fps-radical r* (*Suc k*)
let *?ab* = *a oo b*
have *ab0*: *?ab* \$ *0* = *a*\$*0*

by (*simp add*: *fps-compose-def*)
from *ab0 a0 ra0* **have** *rab0*: *?ab* $ *0* ≠ *0* *r* (*Suc k*) (*?ab* $ *0*) ^ *Suc k* = *?ab* $ *0*
by *simp-all*
have *th00*: *r* (*Suc k*) ((*a oo b*) $ *0*) = (*fps-radical r* (*Suc k*) *a oo b*) $ *0*
by (*simp add*: *ab0 fps-compose-def*)
have *th0*: (*?r a oo b*) ^ (*Suc k*) = *a oo b*
unfolding *fps-compose-power*[*OF b0*]
unfolding *iffD1*[*OF power-radical*[*of a r k*], *OF a0 ra0*] **..**
from *iffD1*[*OF radical-unique*[**where** *r*=*r* **and** *k*=*k* **and** *b*= *?ab* **and** *a* = *?r a oo b*, *OF rab0*(*2*) *th00 rab0*(*1*)], *OF th0*]
show *?thesis* **.**
qed

lemma *fps-const-mult-apply-left*: *fps-const c* ∗ (*a oo b*) = (*fps-const c* ∗ *a*) *oo b*
by (*simp add*: *fps-eq-iff fps-compose-nth sum-distrib-left mult.assoc*)

lemma *fps-const-mult-apply-right*:
(*a oo b*) ∗ *fps-const* (*c*::′*a*::*comm-semiring-1*) = (*fps-const c* ∗ *a*) *oo b*
by (*simp add*: *fps-const-mult-apply-left mult.commute*)

lemma *fps-compose-assoc*:
assumes *c0*: *c*$*0* = (*0*::′*a*::*idom*)
and *b0*: *b*$*0* = *0*
shows *a oo* (*b oo c*) = *a oo b oo c* (**is** *?l* = *?r*)
proof −
have *?l*$*n* = *?r*$*n* **for** *n*
proof −
have *?l*$*n* = (*sum* (λ*i*. (*fps-const* (*a*$*i*) ∗ *b*^*i*) *oo c*) {*0*..*n*})$*n*
by (*simp add*: *fps-compose-nth fps-compose-power*[*OF c0*] *fps-const-mult-apply-left sum-distrib-left mult.assoc fps-sum-nth*)
also have . . . = ((*sum* (λ*i*. *fps-const* (*a*$*i*) ∗ *b*^*i*) {*0*..*n*}) *oo c*)$*n*
by (*simp add*: *fps-compose-sum-distrib*)
also have ... = (∑ *i* = *0*..*n*. ∑ *j* = *0*..*n*. *a* $ *j* ∗ (*b* ^ *j* $ *i* ∗ *c* ^ *i* $ *n*))
by (*simp add*: *fps-compose-nth fps-sum-nth sum-distrib-right mult.assoc*)
also have ... = (∑ *i* = *0*..*n*. ∑ *j* = *0*..*i*. *a* $ *j* ∗ (*b* ^ *j* $ *i* ∗ *c* ^ *i* $ *n*))
by (*intro sum.cong* [*OF refl*] *sum.mono-neutral-right*; *simp add*: *b0 startsby-zero-power-prefix*)
also have . . . = *?r*$*n*
by (*simp add*: *fps-compose-nth sum-distrib-right mult.assoc*)
finally show *?thesis* **.**
qed
then show *?thesis*
by (*simp add*: *fps-eq-iff*)
qed

lemma *fps-X-power-compose*:
assumes *a0*: *a*$*0*=*0*
shows *fps-X*^*k oo a* = (*a*::′*a*::*idom fps*)^*k*
(**is** *?l* = *?r*)

```
proof (cases k)
  case 0
  then show ?thesis by simp
next
  case (Suc h)
  have ?l $ n = ?r $n for n
  proof -
    consider k > n | k ≤ n by arith
    then show ?thesis
    proof cases
      case 1
      then show ?thesis
        using a0 startsby-zero-power-prefix[OF a0] Suc
        by (simp add: fps-compose-nth del: power-Suc)
    next
      case 2
      then show ?thesis
        by (simp add: fps-compose-nth mult-delta-left)
    qed
  qed
  then show ?thesis
    unfolding fps-eq-iff by blast
qed

lemma fps-inv-right:
  assumes a0: a$0 = 0
    and a1: a$1 ≠ 0
  shows a oo fps-inv a = fps-X
proof -
  let ?ia = fps-inv a
  let ?iaa = a oo fps-inv a
  have th0: ?ia $ 0 = 0
    by (simp add: fps-inv-def)
  have th1: ?iaa $ 0 = 0
    using a0 a1 by (simp add: fps-inv-def fps-compose-nth)
  have th2: fps-X$0 = 0
    by simp
  from fps-inv[OF a0 a1] have a oo (fps-inv a oo a) = a oo fps-X
    by simp
  then have (a oo fps-inv a) oo a = fps-X oo a
     by (simp add: fps-compose-assoc[OF a0 th0] fps-X-fps-compose-startby0[OF
a0])
  with fps-compose-inj-right[OF a0 a1] show ?thesis
    by simp
qed

lemma fps-inv-deriv:
  assumes a0: a$0 = (0::'a::field)
    and a1: a$1 ≠ 0
```

shows *fps-deriv* (*fps-inv a*) = *inverse* (*fps-deriv a oo fps-inv a*)
proof −
let *?ia* = *fps-inv a*
let *?d* = *fps-deriv a oo ?ia*
let *?dia* = *fps-deriv ?ia*
have *ia0*: *?ia*$*0* = *0*
by (*simp add*: *fps-inv-def*)
have *th0*: *?d*$*0* ≠ *0*
using *a1* **by** (*simp add*: *fps-compose-nth*)
from *fps-inv-right*[*OF a0 a1*] **have** *?d* ∗ *?dia* = *1*
by (*simp add*: *fps-compose-deriv*[*OF ia0*, *of a*, *symmetric*])
then have *inverse ?d* ∗ *?d* ∗ *?dia* = *inverse ?d* ∗ *1*
by *simp*
with *inverse-mult-eq-1* [*OF th0*] **show** *?dia* = *inverse ?d*
by *simp*
qed

lemma *fps-inv-idempotent*:
assumes *a0*: *a*$*0* = *0*
and *a1*: *a*$*1* ≠ *0*
shows *fps-inv* (*fps-inv a*) = *a*
proof −
let *?r* = *fps-inv*
have *ra0*: *?r a* $ *0* = *0*
by (*simp add*: *fps-inv-def*)
from *a1* **have** *ra1*: *?r a* $ *1* ≠ *0*
by (*simp add*: *fps-inv-def field-simps*)
have *fps-X0*: *fps-X*$*0* = *0*
by *simp*
from *fps-inv*[*OF ra0 ra1*] **have** *?r* (*?r a*) *oo ?r a* = *fps-X* **.**
then have *?r* (*?r a*) *oo ?r a oo a* = *fps-X oo a*
by *simp*
then have *?r* (*?r a*) *oo* (*?r a oo a*) = *a*
unfolding *fps-X-fps-compose-startby0*[*OF a0*]
unfolding *fps-compose-assoc*[*OF a0 ra0*, *symmetric*] **.**
then show *?thesis*
unfolding *fps-inv*[*OF a0 a1*] **by** *simp*
qed

lemma *fps-ginv-ginv*:
assumes *a0*: *a*$*0* = *0*
and *a1*: *a*$*1* ≠ *0*
and *c0*: *c*$*0* = *0*
and *c1*: *c*$*1* ≠ *0*
shows *fps-ginv b* (*fps-ginv c a*) = *b oo a oo fps-inv c*
proof −
let *?r* = *fps-ginv*
from *c0* **have** *rca0*: *?r c a* $*0* = *0*
by (*simp add*: *fps-ginv-def*)

from *a1 c1* **have** *rca1*: *?r c a* $ *1* ≠ *0*
by (*simp add*: *fps-ginv-def field-simps*)
from *fps-ginv*[*OF rca0 rca1*]
have *?r b* (*?r c a*) *oo ?r c a* = *b* **.**
then have *?r b* (*?r c a*) *oo ?r c a oo a* = *b oo a*
by *simp*
then have *?r b* (*?r c a*) *oo* (*?r c a oo a*) = *b oo a*
by (*simp add*: *a0 fps-compose-assoc rca0*)
then have *?r b* (*?r c a*) *oo c* = *b oo a*
unfolding *fps-ginv*[*OF a0 a1*] **.**
then have *?r b* (*?r c a*) *oo c oo fps-inv c*= *b oo a oo fps-inv c*
by *simp*
then have *?r b* (*?r c a*) *oo* (*c oo fps-inv c*) = *b oo a oo fps-inv c*
by (*metis c0 c1 fps-compose-assoc fps-compose-nth-0 fps-inv fps-inv-right*)
then show *?thesis*
unfolding *fps-inv-right*[*OF c0 c1*] **by** *simp*
qed

lemma *fps-ginv-deriv*:
assumes *a0*:*a*$*0* = (*0*::′*a*::*field*)
and *a1*: *a*$*1* ≠ *0*
shows *fps-deriv* (*fps-ginv b a*) = (*fps-deriv b* / *fps-deriv a*) *oo fps-ginv fps-X a*
proof −
let *?ia* = *fps-ginv b a*
let *?ifps-Xa* = *fps-ginv fps-X a*
let *?d* = *fps-deriv*
let *?dia* = *?d ?ia*
have *ifps-Xa0*: *?ifps-Xa* $ *0* = *0*
by (*simp add*: *fps-ginv-def*)
have *da0*: *?d a* $ *0* ≠ *0*
using *a1* **by** *simp*
from *fps-ginv*[*OF a0 a1*, *of b*] **have** *?d* (*?ia oo a*) = *fps-deriv b*
by *simp*
then have (*?d ?ia oo a*) ∗ *?d a* = *?d b*
unfolding *fps-compose-deriv*[*OF a0*] **.**
then have (*?d ?ia oo a*) ∗ *?d a* ∗ *inverse* (*?d a*) = *?d b* ∗ *inverse* (*?d a*)
by *simp*
with *a1* **have** (*?d ?ia oo a*) ∗ (*inverse* (*?d a*) ∗ *?d a*) = *?d b* / *?d a*
by (*simp add*: *fps-divide-unit*)
then have (*?d ?ia oo a*) *oo ?ifps-Xa* = (*?d b* / *?d a*) *oo ?ifps-Xa*
unfolding *inverse-mult-eq-1*[*OF da0*] **by** *simp*
then have *?d ?ia oo* (*a oo ?ifps-Xa*) = (*?d b* / *?d a*) *oo ?ifps-Xa*
unfolding *fps-compose-assoc*[*OF ifps-Xa0 a0*] **.**
then show *?thesis* **unfolding** *fps-inv-ginv*[*symmetric*]
unfolding *fps-inv-right*[*OF a0 a1*] **by** *simp*
qed

lemma *fps-compose-linear*:
fps-compose (*f* :: ′*a* :: *comm-ring-1 fps*) (*fps-const c* ∗ *fps-X*) = *Abs-fps* (λ*n*. *c*^*n*

$* f \$ n$)
by (*simp add*: *fps-eq-iff fps-compose-def power-mult-distrib if-distrib cong*: *if-cong*)

lemma *fps-compose-uminus'*:
fps-compose f ($-$*fps-X* :: *'a* :: *comm-ring-1 fps*) = *Abs-fps* ($\lambda n.$ (-1)^$n * f \$ n$)
using *fps-compose-linear*[*of f* -1]
by (*simp only*: *fps-const-neg* [*symmetric*] *fps-const-1-eq-1*) *simp*

5.16 Elementary series

5.16.1 Exponential series

definition *fps-exp x* = *Abs-fps* ($\lambda n.$ *x*^*n* / *of-nat* (*fact n*))

lemma *fps-exp-deriv*[*simp*]: *fps-deriv* (*fps-exp a*) = *fps-const* (*a*::*'a*::*field-char-0*) $*$ *fps-exp a*
(**is** *?l* = *?r*)
proof $-$
have *?l*\$*n* = *?r* \$ *n* **for** *n*
using *of-nat-neq-0* **by** (*auto simp add*: *fps-exp-def divide-simps*)
then show *?thesis*
by (*simp add*: *fps-eq-iff*)
qed

lemma *fps-exp-unique-ODE*:
fps-deriv a = *fps-const c* $*$ *a* $\longleftrightarrow$ *a* = *fps-const* (*a*\$*0*) $*$ *fps-exp* (*c*::*'a*::*field-char-0*)
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
show *?rhs* **if** *?lhs*
proof $-$
from *that* **have** *th*: $\bigwedge n.$ *a* \$ *Suc n* = *c* $*$ *a*\$*n* / *of-nat* (*Suc n*)
by (*simp add*: *fps-deriv-def fps-eq-iff field-simps del*: *of-nat-Suc*)
have *th'*: *a*\$*n* = *a*\$*0* $*$ *c* ^ *n*/ (*fact n*) **for** *n*
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case *Suc*
then show *?case*
by (*simp add*: *th divide-simps*)
qed
show *?thesis*
by (*auto simp add*: *fps-eq-iff fps-const-mult-left fps-exp-def intro*: *th'*)
qed
show *?lhs* **if** *?rhs*
using *that* **by** (*metis fps-exp-deriv fps-deriv-mult-const-left mult.left-commute*)
qed

lemma *fps-exp-add-mult*: *fps-exp* (*a* + *b*) = *fps-exp* (*a*::*'a*::*field-char-0*) $*$ *fps-exp*

b (**is** *?l* = *?r*)
proof −
  have *fps-deriv ?r* = *fps-const* (*a* + *b*) ∗ *?r*
    by (*simp add*: *fps-const-add*[*symmetric*] *field-simps del*: *fps-const-add*)
  then have *?r* = *?l*
    by (*simp only*: *fps-exp-unique-ODE*) (*simp add*: *fps-mult-nth fps-exp-def*)
  then show *?thesis* **..**
qed

lemma *fps-exp-nth*[*simp*]: *fps-exp a* $ *n* = *a*^*n* / *of-nat* (*fact n*)
  by (*simp add*: *fps-exp-def*)

lemma *fps-exp-0*[*simp*]: *fps-exp* (*0*::*'a*::*field*) = *1*
  by (*simp add*: *fps-eq-iff power-0-left*)

lemma *fps-exp-neg*: *fps-exp* (− *a*) = *inverse* (*fps-exp* (*a*::*'a*::*field-char-0*))
proof −
  from *fps-exp-add-mult*[*of a* − *a*] **have** *th0*: *fps-exp a* ∗ *fps-exp* (− *a*) = *1* **by** *simp*
  from *fps-inverse-unique*[*OF th0*] **show** *?thesis* **by** *simp*
qed

lemma *fps-exp-nth-deriv*[*simp*]:
  fps-nth-deriv n (*fps-exp* (*a*::*'a*::*field-char-0*)) = (*fps-const a*)^*n* ∗ (*fps-exp a*)
  by (*induct n*) *auto*

lemma *fps-X-compose-fps-exp*[*simp*]: *fps-X oo fps-exp* (*a*::*'a*::*field*) = *fps-exp a* − *1*
  by (*simp add*: *fps-eq-iff fps-X-fps-compose*)

lemma *fps-inv-fps-exp-compose*:
  assumes *a*: *a* ≠ *0*
  shows *fps-inv* (*fps-exp a* − *1*) *oo* (*fps-exp a* − *1*) = *fps-X*
    and (*fps-exp a* − *1*) *oo fps-inv* (*fps-exp a* − *1*) = *fps-X*
proof −
  let *?b* = *fps-exp a* − *1*
  have *b0*: *?b* $ *0* = *0*
    by *simp*
  have *b1*: *?b* $ *1* ≠ *0*
    by (*simp add*: *a*)
  from *fps-inv*[*OF b0 b1*] **show** *fps-inv* (*fps-exp a* − *1*) *oo* (*fps-exp a* − *1*) = *fps-X* **.**
  from *fps-inv-right*[*OF b0 b1*] **show** (*fps-exp a* − *1*) *oo fps-inv* (*fps-exp a* − *1*) = *fps-X* **.**
qed

lemma *fps-exp-power-mult*: (*fps-exp* (*c*::*'a*::*field-char-0*))^*n* = *fps-exp* (*of-nat n* ∗ *c*)
  by (*induct n*) (*simp-all add*: *field-simps fps-exp-add-mult*)

lemma *radical-fps-exp*:
assumes *r*: *r* (*Suc k*) *1* = *1*
shows *fps-radical r* (*Suc k*) (*fps-exp* (*c*::′*a*::*field-char-0*)) = *fps-exp* (*c* / *of-nat* (*Suc k*))
proof −
let *?ck* = (*c* / *of-nat* (*Suc k*))
let *?r* = *fps-radical r* (*Suc k*)
have *eq0*[*simp*]: *?ck* ∗ *of-nat* (*Suc k*) = *c of-nat* (*Suc k*) ∗ *?ck* = *c*
by (*simp-all del*: *of-nat-Suc*)
have *th0*: *fps-exp ?ck* ^ (*Suc k*) = *fps-exp c* **unfolding** *fps-exp-power-mult eq0* ..
have *th*: *r* (*Suc k*) (*fps-exp c* $*0*) ^ *Suc k* = *fps-exp c* $ *0*
r (*Suc k*) (*fps-exp c* $ *0*) = *fps-exp ?ck* $ *0* *fps-exp c* $ *0* ≠ *0* **using** *r* **by** *simp-all*
from *th0 radical-unique*[**where** *r*=*r* **and** *k*=*k*, *OF th*] **show** *?thesis*
by *auto*
qed

lemma *fps-exp-compose-linear* [*simp*]:
fps-exp (*d*::′*a*::*field-char-0*) *oo* (*fps-const c* ∗ *fps-X*) = *fps-exp* (*c* ∗ *d*)
by (*simp add*: *fps-compose-linear fps-exp-def fps-eq-iff power-mult-distrib*)

lemma *fps-fps-exp-compose-minus* [*simp*]:
fps-compose (*fps-exp c*) (−*fps-X*) = *fps-exp* (−*c* :: ′*a* :: *field-char-0*)
using *fps-exp-compose-linear*[*of c* −*1* :: ′*a*]
unfolding *fps-const-neg* [*symmetric*] *fps-const-1-eq-1* **by** *simp*

lemma *fps-exp-eq-iff* [*simp*]: *fps-exp c* = *fps-exp d* ⟷ *c* = (*d* :: ′*a* :: *field-char-0*)
proof
assume *fps-exp c* = *fps-exp d*
from *arg-cong*[*of* - - λ*F*. *F* $ *1*, *OF this*] **show** *c* = *d* **by** *simp*
qed *simp-all*

lemma *fps-exp-eq-fps-const-iff* [*simp*]:
fps-exp (*c* :: ′*a* :: *field-char-0*) = *fps-const c*′ ⟷ *c* = *0* ∧ *c*′ = *1*
proof
assume *c* = *0* ∧ *c*′ = *1*
thus *fps-exp c* = *fps-const c*′ **by** (*simp add*: *fps-eq-iff*)
next
assume *fps-exp c* = *fps-const c*′
from *arg-cong*[*of* - - λ*F*. *F* $ *1*, *OF this*] *arg-cong*[*of* - - λ*F*. *F* $ *0*, *OF this*]
show *c* = *0* ∧ *c*′ = *1* **by** *simp-all*
qed

lemma *fps-exp-neq-0* [*simp*]: ¬*fps-exp* (*c* :: ′*a* :: *field-char-0*) = *0*
unfolding *fps-const-0-eq-0* [*symmetric*] *fps-exp-eq-fps-const-iff* **by** *simp*

lemma *fps-exp-eq-1-iff* [*simp*]: *fps-exp* (*c* :: ′*a* :: *field-char-0*) = *1* ⟷ *c* = *0*

unfolding *fps-const-1-eq-1* [*symmetric*] *fps-exp-eq-fps-const-iff* **by** *simp*

lemma *fps-exp-neq-numeral-iff* [*simp*]:
fps-exp (*c* :: *'a* :: *field-char-0*) = *numeral n* ⟷ *c* = *0* ∧ *n* = *Num.One*
unfolding *numeral-fps-const fps-exp-eq-fps-const-iff* **by** *simp*

5.16.2 Logarithmic series

lemma *Abs-fps-if-0*:
Abs-fps (λ*n*. *if n* = *0 then* (*v*::*'a*::*ring-1*) *else f n*) =
fps-const v + *fps-X* ∗ *Abs-fps* (λ*n*. *f* (*Suc n*))
by (*simp add*: *fps-eq-iff*)

definition *fps-ln* :: *'a*::*field-char-0* ⇒ *'a fps*
where *fps-ln c* = *fps-const* (*1*/*c*) ∗ *Abs-fps* (λ*n*. *if n* = *0 then 0 else* (− *1*) ^ (*n* − *1*) / *of-nat n*)

lemma *fps-ln-deriv*: *fps-deriv* (*fps-ln c*) = *fps-const* (*1*/*c*) ∗ *inverse* (*1* + *fps-X*)
unfolding *fps-inverse-fps-X-plus1*
by (*simp add*: *fps-ln-def fps-eq-iff del*: *of-nat-Suc*)

lemma *fps-ln-nth*: *fps-ln c* $ *n* = (*if n* = *0 then 0 else 1*/*c* ∗ ((− *1*) ^ (*n* − *1*) / *of-nat n*))
by (*simp add*: *fps-ln-def field-simps*)

lemma *fps-ln-0* [*simp*]: *fps-ln c* $ *0* = *0* **by** (*simp add*: *fps-ln-def*)

lemma *fps-ln-fps-exp-inv*:
fixes *a* :: *'a*::*field-char-0*
assumes *a*: *a* ≠ *0*
shows *fps-ln a* = *fps-inv* (*fps-exp a* − *1*) (**is** *?l* = *?r*)
proof −
let *?b* = *fps-exp a* − *1*
have *b0*: *?b* $ *0* = *0* **by** *simp*
have *b1*: *?b* $ *1* ≠ *0* **by** (*simp add*: *a*)
have *fps-deriv* (*fps-exp a* − *1*) *oo fps-inv* (*fps-exp a* − *1*) =
(*fps-const a* ∗ (*fps-exp a* − *1*) + *fps-const a*) *oo fps-inv* (*fps-exp a* − *1*)
by (*simp add*: *field-simps*)
also have . . . = *fps-const a* ∗ (*fps-X* + *1*)
by (*simp add*: *fps-compose-add-distrib fps-inv-right*[*OF b0 b1*] *distrib-left flip*: *fps-const-mult-apply-left*)
finally have *eq*: *fps-deriv* (*fps-exp a* − *1*) *oo fps-inv* (*fps-exp a* − *1*) = *fps-const a* ∗ (*fps-X* + *1*) **.**
from *fps-inv-deriv*[*OF b0 b1*, *unfolded eq*]
have *fps-deriv* (*fps-inv ?b*) = *fps-const* (*inverse a*) / (*fps-X* + *1*)
using *a* **by** (*simp add*: *fps-const-inverse eq fps-divide-def fps-inverse-mult*)
then have *fps-deriv ?l* = *fps-deriv ?r*
by (*simp add*: *fps-ln-deriv add.commute fps-divide-def divide-inverse*)
then show *?thesis* **unfolding** *fps-deriv-eq-iff*

by (*simp add*: *fps-ln-nth fps-inv-def*)
qed

lemma *fps-ln-mult-add*:
assumes *c0*: *c≠0*
and *d0*: *d≠0*
shows *fps-ln c* + *fps-ln d* = *fps-const* (*c*+*d*) ∗ *fps-ln* (*c*∗*d*)
(**is** *?r* = *?l*)
proof−
from *c0 d0* **have** *eq*: *1*/*c* + *1*/*d* = (*c*+*d*)/(*c*∗*d*) **by** (*simp add*: *field-simps*)
have *fps-deriv ?r* = *fps-const* (*1*/*c* + *1*/*d*) ∗ *inverse* (*1* + *fps-X*)
by (*simp add*: *fps-ln-deriv fps-const-add*[*symmetric*] *algebra-simps del*: *fps-const-add*)
also have . . . = *fps-deriv ?l*
by (*simp add*: *eq fps-ln-deriv*)
finally show *?thesis*
unfolding *fps-deriv-eq-iff* **by** *simp*
qed

lemma *fps-X-dvd-fps-ln* [*simp*]: *fps-X dvd fps-ln c*
proof −
have *fps-ln c* = *fps-X* ∗ *Abs-fps* (λ*n*. (−*1*) ^ *n* / (*of-nat* (*Suc n*) ∗ *c*))
by (*intro fps-ext*) (*simp add*: *fps-ln-def of-nat-diff*)
thus *?thesis* **by** *simp*
qed

5.16.3 Binomial series

definition *fps-binomial a* = *Abs-fps* (λ*n*. *a gchoose n*)

lemma *fps-binomial-nth*[*simp*]: *fps-binomial a* $ *n* = *a gchoose n*
by (*simp add*: *fps-binomial-def*)

lemma *fps-binomial-ODE-unique*:
fixes *c* :: ′*a*::*field-char-0*
shows *fps-deriv a* = (*fps-const c* ∗ *a*) / (*1* + *fps-X*) ⟷ *a* = *fps-const* (*a*$*0*) ∗ *fps-binomial c*
(**is** *?lhs* ⟷ *?rhs*)
proof
let *?da* = *fps-deriv a*
let *?x1* = (*1* + *fps-X*):: ′*a fps*
let *?l* = *?x1* ∗ *?da*
let *?r* = *fps-const c* ∗ *a*

have *eq*: *?l* = *?r* ⟷ *?lhs*
proof −
have *x10*: *?x1* $ *0* ≠ *0* **by** *simp*
have *?l* = *?r* ⟷ *inverse ?x1* ∗ *?l* = *inverse ?x1* ∗ *?r* **by** *simp*
also have . . . ⟷ *?da* = (*fps-const c* ∗ *a*) / *?x1*
unfolding *fps-divide-def mult.assoc*[*symmetric*] *inverse-mult-eq-1*[*OF x10*]

    by (*simp add*: *field-simps*)
  finally show *?thesis* **.**
qed

show *?rhs* **if** *?lhs*
proof −
  from *eq that* **have** *h*: *?l* = *?r* **..**
  have *th0*: *a$ Suc n* = ((*c* − *of-nat n*) / *of-nat* (*Suc n*)) ∗ *a* $*n* **for** *n*
  proof −
    from *h* **have** *?l* $ *n* = *?r* $ *n* **by** *simp*
    then show *?thesis*
      by (*simp add*: *field-simps del*: *of-nat-Suc split*: *if-split-asm*)
  qed
  have *th1*: *a* $ *n* = (*c gchoose n*) ∗ *a* $ *0* **for** *n*
  proof (*induct n*)
    case *0*
    then show *?case* **by** *simp*
  next
    case (*Suc m*)
    have (*c* − *of-nat m*) ∗ (*c gchoose m*) = (*c gchoose Suc m*) ∗ *of-nat* (*Suc m*)
      by (*metis gbinomial-absorb-comp gbinomial-absorption mult.commute*)
    with *Suc* **show** *?case*
      unfolding *th0*
      by (*simp add*: *divide-simps del*: *of-nat-Suc*)
  qed
  show *?thesis*
   by (*metis expand-fps-eq fps-binomial-nth fps-mult-right-const-nth mult.commute th1*)
qed

show *?lhs* **if** *?rhs*
proof −
  have *th00*: *x* ∗ (*a* $ *0* ∗ *y*) = *a* $ *0* ∗ (*x* ∗ *y*) **for** *x y*
    by (*simp add*: *mult.commute*)
  have *?l* = (*1* + *fps-X*) ∗ *fps-deriv* (*fps-const* (*a* $ *0*) ∗ *fps-binomial c*)
    using *that* **by** *auto*
  also have ... = *fps-const c* ∗ (*fps-const* (*a* $ *0*) ∗ *fps-binomial c*)
  proof (*clarsimp simp add*: *fps-eq-iff algebra-simps*)
     show *a* $ *0* ∗ (*c gchoose Suc n*) + (*of-nat n* ∗ ((*c gchoose n*) ∗ *a* $ *0*) + *of-nat n* ∗ (*a* $ *0* ∗ (*c gchoose Suc n*)))
        = *c* ∗ ((*c gchoose n*) ∗ *a* $ *0*) **for** *n*
    unfolding *mult.assoc*[*symmetric*]
    by (*simp add*: *field-simps gbinomial-mult-1*)
qed
  also have ... = *?r*
    using *that* **by** *auto*
  finally have *?l* = *?r* **.**
  with *eq* **show** *?thesis* **..**
qed

qed

lemma *fps-binomial-ODE-unique′*:
 (*fps-deriv a* = *fps-const c* ∗ *a* / (*1* + *fps-X*) ∧ *a* $ *0* = *1*) ⟷ (*a* = *fps-binomial c*)
 by (*subst fps-binomial-ODE-unique*) *auto*

lemma *fps-binomial-deriv*: *fps-deriv* (*fps-binomial c*) = *fps-const c* ∗ *fps-binomial c* / (*1* + *fps-X*)
proof −
 let *?a* = *fps-binomial c*
 have *th0*: *?a* = *fps-const* (*?a$0*) ∗ *?a* **by** (*simp*)
 from *iffD2*[*OF fps-binomial-ODE-unique*, *OF th0*] **show** *?thesis* **.**
qed

lemma *fps-binomial-add-mult*: *fps-binomial* (*c*+*d*) = *fps-binomial c* ∗ *fps-binomial d* (**is** *?l* = *?r*)
proof −
 let *?P* = *?r* − *?l*
 let *?b* = *fps-binomial*
 let *?db* = λ*x*. *fps-deriv* (*?b x*)
 have *fps-deriv ?P* = *?db c* ∗ *?b d* + *?b c* ∗ *?db d* − *?db* (*c* + *d*) **by** *simp*
 also have . . . = *inverse* (*1* + *fps-X*) ∗
  (*fps-const c* ∗ *?b c* ∗ *?b d* + *fps-const d* ∗ *?b c* ∗ *?b d* − *fps-const* (*c*+*d*) ∗ *?b* (*c* + *d*))
  unfolding *fps-binomial-deriv*
  by (*simp add*: *fps-divide-def field-simps*)
 also have . . . = (*fps-const* (*c* + *d*)/ (*1* + *fps-X*)) ∗ *?P*
 by (*simp add*: *field-simps fps-divide-unit fps-const-add*[*symmetric*] *del*: *fps-const-add*)
 finally have *th0*: *fps-deriv ?P* = *fps-const* (*c*+*d*) ∗ *?P* / (*1* + *fps-X*)
  by (*simp add*: *fps-divide-def*)
 have *?P* = *fps-const* (*?P$0*) ∗ *?b* (*c* + *d*)
  unfolding *fps-binomial-ODE-unique*[*symmetric*]
  using *th0* **by** *simp*
 then have *?P* = *0* **by** (*simp add*: *fps-mult-nth*)
 then show *?thesis* **by** *simp*
qed

lemma *fps-binomial-minus-one*: *fps-binomial* (− *1*) = *inverse* (*1* + *fps-X*)
 (**is** *?l* = *inverse ?r*)
proof−
 have *th*: *?r$0* ≠ *0* **by** *simp*
 have *th′*: *fps-deriv* (*inverse ?r*) = *fps-const* (− *1*) ∗ *inverse ?r* / (*1* + *fps-X*)
  by (*simp add*: *fps-inverse-deriv*[*OF th*] *fps-divide-def*
   power2-eq-square mult.commute fps-const-neg[*symmetric*] *del*: *fps-const-neg*)
 have *eq*: *inverse ?r* $ *0* = *1*
  by (*simp add*: *fps-inverse-def*)
 from *iffD1*[*OF fps-binomial-ODE-unique*[*of inverse* (*1* + *fps-X*) − *1*] *th′*] *eq*
 show *?thesis* **by** (*simp add*: *fps-inverse-def*)

qed

lemma *fps-binomial-of-nat*: *fps-binomial* (*of-nat n*) = (*1* + *fps-X* :: *'a* :: *field-char-0 fps*) ^ *n*
proof (*cases n* = *0*)
  case [*simp*]: *True*
  have *fps-deriv* ((*1* + *fps-X*) ^ *n* :: *'a fps*) = *0* **by** *simp*
  also have ... = *fps-const* (*of-nat n*) ∗ (*1* + *fps-X*) ^ *n* / (*1* + *fps-X*) **by** (*simp add*: *fps-binomial-def*)
  finally show *?thesis* **by** (*subst sym*, *subst fps-binomial-ODE-unique'* [*symmetric*]) *simp-all*
next
  case *False*
  have *fps-deriv* ((*1* + *fps-X*) ^ *n* :: *'a fps*) = *fps-const* (*of-nat n*) ∗ (*1* + *fps-X*) ^ (*n* − *1*)
    by (*simp add*: *fps-deriv-power*)
  also have (*1* + *fps-X* :: *'a fps*) $ *0* ≠ *0* **by** *simp*
  hence (*1* + *fps-X* :: *'a fps*) ≠ *0* **by** (*intro notI*) (*simp only*: , *simp*)
  with *False* **have** (*1* + *fps-X* :: *'a fps*) ^ (*n* − *1*) = (*1* + *fps-X*) ^ *n* / (*1* + *fps-X*)
    by (*cases n*) (*simp-all*)
  also have *fps-const* (*of-nat n* :: *'a*) ∗ ((*1* + *fps-X*) ^ *n* / (*1* + *fps-X*)) =
               fps-const (*of-nat n*) ∗ (*1* + *fps-X*) ^ *n* / (*1* + *fps-X*)
    by (*simp add*: *unit-div-mult-swap*)
  finally show *?thesis*
    by (*subst sym*, *subst fps-binomial-ODE-unique'* [*symmetric*]) (*simp-all add*: *fps-power-nth*)
qed

lemma *fps-binomial-0* [*simp*]: *fps-binomial 0* = *1*
  using *fps-binomial-of-nat*[*of 0*] **by** *simp*

lemma *fps-binomial-power*: *fps-binomial a* ^ *n* = *fps-binomial* (*of-nat n* ∗ *a*)
  by (*induction n*) (*simp-all add*: *fps-binomial-add-mult ring-distribs*)

lemma *fps-binomial-1*: *fps-binomial 1* = *1* + *fps-X*
  using *fps-binomial-of-nat*[*of 1*] **by** *simp*

lemma *fps-binomial-minus-of-nat*:
  fps-binomial (− *of-nat n*) = *inverse* ((*1* + *fps-X* :: *'a* :: *field-char-0 fps*) ^ *n*)
  by (*rule sym*, *rule fps-inverse-unique*)
    (*simp add*: *fps-binomial-of-nat* [*symmetric*] *fps-binomial-add-mult* [*symmetric*])

lemma *one-minus-const-fps-X-power*:
  c ≠ *0* ⟹ (*1* − *fps-const c* ∗ *fps-X*) ^ *n* =
    fps-compose (*fps-binomial* (*of-nat n*)) (−*fps-const c* ∗ *fps-X*)
  by (*subst fps-binomial-of-nat*)
    (*simp add*: *fps-compose-power* [*symmetric*] *fps-compose-add-distrib fps-const-neg* [*symmetric*]

del: *fps-const-neg*)

lemma *one-minus-fps-X-const-neg-power*:
inverse ((*1* − *fps-const c* ∗ *fps-X*) ^ *n*) =
fps-compose (*fps-binomial* (−*of-nat n*)) (−*fps-const c* ∗ *fps-X*)
proof (*cases c* = *0*)
case *False*
thus *?thesis*
by (*subst fps-binomial-minus-of-nat*)
(*simp add*: *fps-compose-power* [*symmetric*] *fps-inverse-compose fps-compose-add-distrib*
fps-const-neg [*symmetric*] *del*: *fps-const-neg*)
qed *simp*

lemma *fps-X-plus-const-power*:
c ≠ *0* ⟹ (*fps-X* + *fps-const c*) ^ *n* =
fps-const (*c*^*n*) ∗ *fps-compose* (*fps-binomial* (*of-nat n*)) (*fps-const* (*inverse c*)
∗ *fps-X*)
by (*subst fps-binomial-of-nat*)
(*simp add*: *fps-compose-power* [*symmetric*] *fps-binomial-of-nat fps-compose-add-distrib*
fps-const-power [*symmetric*] *power-mult-distrib* [*symmetric*]
algebra-simps inverse-mult-eq-1′ del: *fps-const-power*)

lemma *fps-X-plus-const-neg-power*:
c ≠ *0* ⟹ *inverse* ((*fps-X* + *fps-const c*) ^ *n*) =
fps-const (*inverse c*^*n*) ∗ *fps-compose* (*fps-binomial* (−*of-nat n*)) (*fps-const*
(*inverse c*) ∗ *fps-X*)
by (*subst fps-binomial-minus-of-nat*)
(*simp add*: *fps-compose-power* [*symmetric*] *fps-binomial-of-nat fps-compose-add-distrib*
fps-const-power [*symmetric*] *power-mult-distrib* [*symmetric*] *fps-inverse-compose*

algebra-simps fps-const-inverse [*symmetric*] *fps-inverse-mult* [*symmetric*]
fps-inverse-power [*symmetric*] *inverse-mult-eq-1′*
del: *fps-const-power*)

lemma *one-minus-const-fps-X-neg-power′*:
fixes *c* :: ′*a* :: *field-char-0*
assumes *n* > *0*
shows *inverse* ((*1* − *fps-const c* ∗ *fps-X*) ^ *n*) = *Abs-fps* (λ*k*. *of-nat* ((*n* + *k* −
1) *choose k*) ∗ *c*^*k*)
proof −
have §: ⋀*j*. *Abs-fps* (λ*na*. (− *c*) ^ *na* ∗ *fps-binomial* (− *of-nat n*) $ *na*) $ *j* =
Abs-fps (λ*k*. *of-nat* (*n* + *k* − *1 choose k*) ∗ *c* ^ *k*) $ *j*
using *assms*
by (*simp add*: *gbinomial-minus binomial-gbinomial of-nat-diff flip*: *power-mult-distrib*
mult.assoc)
show *?thesis*
apply (*rule fps-ext*)
using §

by (*metis* (*no-types*, *lifting*) *one-minus-fps-X-const-neg-power fps-const-neg fps-compose-linear fps-nth-Abs-fps*)
qed

Vandermonde's Identity as a consequence.

lemma *gbinomial-Vandermonde*:
sum (λ*k*. (*a gchoose k*) ∗ (*b gchoose* (*n* − *k*))) {*0*..*n*} = (*a* + *b*) *gchoose n*
proof −
let *?ba* = *fps-binomial a*
let *?bb* = *fps-binomial b*
let *?bab* = *fps-binomial* (*a* + *b*)
from *fps-binomial-add-mult*[*of a b*] **have** *?bab* $ *n* = (*?ba* ∗ *?bb*)$*n* **by** *simp*
then show *?thesis* **by** (*simp add*: *fps-mult-nth*)
qed

lemma *binomial-Vandermonde*:
sum (λ*k*. (*a choose k*) ∗ (*b choose* (*n* − *k*))) {*0*..*n*} = (*a* + *b*) *choose n*
using *gbinomial-Vandermonde*[*of* (*of-nat a*) *of-nat b n*]
by (*simp only*: *binomial-gbinomial*[*symmetric*] *of-nat-mult*[*symmetric*]
of-nat-sum[*symmetric*] *of-nat-add*[*symmetric*] *of-nat-eq-iff*)

lemma *binomial-Vandermonde-same*: *sum* (λ*k*. (*n choose k*)^2) {*0*..*n*} = (*2* ∗ *n*) *choose n*
using *binomial-Vandermonde*[*of n n n*, *symmetric*]
unfolding *mult-2*
by (*metis atMost-atLeast0 choose-square-sum mult-2*)

lemma *Vandermonde-pochhammer-lemma*:
fixes *a* :: ′*a*::*field-char-0*
assumes *b*: ⋀*j*. *j*<*n* ⟹ *b* ≠ *of-nat j*
shows *sum* (λ*k*. (*pochhammer* (− *a*) *k* ∗ *pochhammer* (− (*of-nat n*)) *k*) /
(*of-nat* (*fact k*) ∗ *pochhammer* (*b* − *of-nat n* + *1*) *k*)) {*0*..*n*} =
pochhammer (− (*a* + *b*)) *n* / *pochhammer* (− *b*) *n*
(**is** *?l* = *?r*)
proof −
let *?m1* = λ*m*. (− *1* :: ′*a*) ^ *m*
let *?f* = λ*m*. *of-nat* (*fact m*)
let *?p* = λ(*x*::′*a*). *pochhammer* (− *x*)
from *b* **have** *bn0*: *?p b n* ≠ *0*
unfolding *pochhammer-eq-0-iff* **by** *simp*
have *th00*:
b gchoose (*n* − *k*) =
(*?m1 n* ∗ *?p b n* ∗ *?m1 k* ∗ *?p* (*of-nat n*) *k*) / (*?f n* ∗ *pochhammer* (*b* − *of-nat n* + *1*) *k*)
(**is** *?gchoose*)
pochhammer (*1* + *b* − *of-nat n*) *k* ≠ *0*
(**is** *?pochhammer*)
if *kn*: *k* ∈ {*0*..*n*} **for** *k*
proof −

```
  from kn have k ≤ n by simp
  have nz: pochhammer (1 + b − of-nat n) n ≠ 0
  proof
    assume pochhammer (1 + b − of-nat n) n = 0
    then have c: pochhammer (b − of-nat n + 1) n = 0
      by (simp add: algebra-simps)
    then obtain j where j: j < n b − of-nat n + 1 = − of-nat j
      unfolding pochhammer-eq-0-iff by blast
    from j have b = of-nat n − of-nat j − of-nat 1
      by (simp add: algebra-simps)
    then show False
      using ‹j < n› j b
      by (metis bn0 c mult-cancel-right2 pochhammer-minus)
  qed

  from nz kn [simplified] have nz': pochhammer (1 + b − of-nat n) k ≠ 0
    by (rule pochhammer-neq-0-mono)

  consider k = 0 ∨ n = 0 | k ≠ 0 n ≠ 0
    by blast
  then have b gchoose (n − k) =
   (?m1 n ∗ ?p b n ∗ ?m1 k ∗ ?p (of-nat n) k) / (?f n ∗ pochhammer (b − of-nat
n + 1) k)
  proof cases
    case 1
    then show ?thesis
      using kn by (cases k = 0) (simp-all add: gbinomial-pochhammer)
  next
    case neq: 2
    then obtain m where m: n = Suc m
      by (cases n) auto
    from neq(1) obtain h where h: k = Suc h
      by (cases k) auto
    show ?thesis
    proof (cases k = n)
      case True
      with pochhammer-minus'[where k=k and b=b] bn0 show ?thesis
        by (simp add: pochhammer-same)
    next
      case False
      with kn have kn': k < n
        by simp
      have h ≤ m
        using ‹k ≤ n› h m by blast
      have m1nk: ?m1 n = prod (λi. − 1) {..m} ?m1 k = prod (λi. − 1) {0..h}
        by (simp-all add: m h)
      have bnz0: pochhammer (b − of-nat n + 1) k ≠ 0
        using bn0 kn
        unfolding pochhammer-eq-0-iff
```

by (*metis add.commute add-diff-eq nz′ pochhammer-eq-0-iff*)
have *eq1*: *prod* (λ*k*. (*1*::′*a*) + *of-nat m* − *of-nat k*) {..*h*} =
prod of-nat {*Suc* (*m* − *h*) .. *Suc m*}
using *kn′ h m*
by (*intro prod.reindex-bij-witness*[**where** *i*=λ*k*. *Suc m* − *k* **and** *j*=λ*k*. *Suc m* − *k*])
(*auto simp*: *of-nat-diff*)
have (∏ *i* = *0*..<*k*. *1* + *of-nat n* − *of-nat k* + *of-nat i*) = (∏ *x* = *n* − *k*..<*n*. (*1*::′*a*) + *of-nat x*)
using ‹*k* ≤ *n*›
using *prod.atLeastLessThan-shift-bounds* [**where** *?′a* = ′*a*, *of* λ*i*. *1* + *of-nat i 0 n* − *k k*]
by (*auto simp add*: *of-nat-diff field-simps*)
then have *fact* (*n* − *k*) ∗ *pochhammer* ((*1*::′*a*) + *of-nat n* − *of-nat k*) *k* = *fact n*
using ‹*k* ≤ *n*›
by (*auto simp add*: *fact-split* [*of k n*] *pochhammer-prod field-simps*)
then have *th1*: (*?m1 k* ∗ *?p* (*of-nat n*) *k*) / *?f n* = *1* / *of-nat*(*fact* (*n* − *k*))
by (*simp add*: *pochhammer-minus field-simps*)
have *?m1 n* ∗ *?p b n* = *pochhammer* (*b* − *of-nat m*) (*Suc m*)
by (*simp add*: *pochhammer-minus field-simps m*)
also have ... = (∏ *i* = *0*..*m*. *b* − *of-nat i*)
by (*auto simp add*: *pochhammer-prod-rev of-nat-diff prod.atLeast-Suc-atMost-Suc-shift simp del*: *prod.cl-ivl-Suc*)
finally have *th20*: *?m1 n* ∗ *?p b n* = *prod* (λ*i*. *b* − *of-nat i*) {*0*..*m*} **.**
have (∏ *x* = *0*..*h*. *b* − *of-nat m* + *of-nat* (*h* − *x*)) = (∏ *i* = *m* − *h*..*m*. *b* − *of-nat i*)
using ‹*h* ≤ *m*› *prod.atLeastAtMost-shift-0* [*of m* − *h m*, **where** *?′a* = ′*a*]
by (*auto simp add*: *of-nat-diff field-simps*)
then have *th21*:*pochhammer* (*b* − *of-nat n* + *1*) *k* = *prod* (λ*i*. *b* − *of-nat i*) {*n* − *k* .. *n* − *1*}
using *kn* **by** (*simp add*: *pochhammer-prod-rev m h prod.atLeast-Suc-atMost-Suc-shift del*: *prod.op-ivl-Suc del*: *prod.cl-ivl-Suc*)
have *?m1 n* ∗ *?p b n* =
prod (λ*i*. *b* − *of-nat i*) {*0*.. *n* − *k* − *1*} ∗ *pochhammer* (*b* − *of-nat n* + *1*) *k*
using *kn′ m h* **unfolding** *th20 th21*
by (*auto simp flip*: *prod.union-disjoint intro*: *prod.cong*)
then have *th2*: (*?m1 n* ∗ *?p b n*)/*pochhammer* (*b* − *of-nat n* + *1*) *k* =
prod (λ*i*. *b* − *of-nat i*) {*0*.. *n* − *k* − *1*}
using *nz′* **by** (*simp add*: *field-simps*)
have (*?m1 n* ∗ *?p b n* ∗ *?m1 k* ∗ *?p* (*of-nat n*) *k*) / (*?f n* ∗ *pochhammer* (*b* − *of-nat n* + *1*) *k*) =
((*?m1 k* ∗ *?p* (*of-nat n*) *k*) / *?f n*) ∗ ((*?m1 n* ∗ *?p b n*)/*pochhammer* (*b* − *of-nat n* + *1*) *k*)
using *bnz0*
by (*simp add*: *field-simps*)
also have ... = *b gchoose* (*n* − *k*)
unfolding *th1 th2*

```
          using kn′ m h
          by (auto simp: field-simps gbinomial-mult-fact intro: prod.cong)
        finally show ?thesis by simp
      qed
    qed
    then show ?gchoose and ?pochhammer
      using nz′ by force+
  qed
  have ?r = ((a + b) gchoose n) ∗ (of-nat (fact n) / (?m1 n ∗ pochhammer (− b)
n))
    unfolding gbinomial-pochhammer
    using bn0 by (auto simp add: field-simps)
  also have . . . = ?l
    using bn0
    unfolding gbinomial-Vandermonde[symmetric]
    apply (simp add: th00)
   by (simp add: gbinomial-pochhammer sum-distrib-right sum-distrib-left field-simps)
  finally show ?thesis by simp
qed

lemma Vandermonde-pochhammer:
  fixes a :: ′a::field-char-0
  assumes c: ∀ i ∈ {0..< n}. c ≠ − of-nat i
  shows sum (λk. (pochhammer a k ∗ pochhammer (− (of-nat n)) k) /
   (of-nat (fact k) ∗ pochhammer c k)) {0..n} = pochhammer (c − a) n / pochham-
mer c n
proof −
  let ?a = − a
  let ?b = c + of-nat n − 1
  have h: ?b ≠ of-nat j if j < n for j
  proof −
    have c ≠ − of-nat (n − j − 1)
      using c that by (auto simp: dest!: bspec [where x = n−j−1])
    with that show ?thesis
      by (auto simp add: algebra-simps of-nat-diff)
  qed
  have th0: pochhammer (− (?a + ?b)) n = (− 1)^n ∗ pochhammer (c − a) n
    unfolding pochhammer-minus
    by (simp add: algebra-simps)
  have th1: pochhammer (− ?b) n = (− 1)^n ∗ pochhammer c n
    unfolding pochhammer-minus
    by simp
  have nz: pochhammer c n ≠ 0 using c
    by (simp add: pochhammer-eq-0-iff)
  from Vandermonde-pochhammer-lemma[where a = ?a and b=?b and n=n, OF
h, unfolded th0 th1]
  show ?thesis
    using nz by (simp add: field-simps sum-distrib-left)
qed
```

5.16.4 Trigonometric functions

definition *fps-sin* (*c*::′*a*::*field-char-0*) =
Abs-fps (λ*n*. *if even n then 0 else* (− *1*) ^((*n* − *1*) *div 2*) ∗ *c*^*n* /(*of-nat* (*fact n*)))

definition *fps-cos* (*c*::′*a*::*field-char-0*) =
Abs-fps (λ*n*. *if even n then* (− *1*) ^(*n div 2*) ∗ *c*^*n* / (*of-nat* (*fact n*)) *else 0*)

lemma *fps-sin-0* [*simp*]: *fps-sin 0* = *0*
by (*intro fps-ext*) (*auto simp*: *fps-sin-def elim*!: *oddE*)

lemma *fps-cos-0* [*simp*]: *fps-cos 0* = *1*
by (*intro fps-ext*) (*simp add*: *fps-cos-def*)

lemma *fps-sin-deriv*:
fps-deriv (*fps-sin c*) = *fps-const c* ∗ *fps-cos c*
(**is** *?lhs* = *?rhs*)
proof (*rule fps-ext*)
fix *n* :: *nat*
show *?lhs* $ *n* = *?rhs* $ *n*
proof (*cases even n*)
case *True*
have *?lhs*$*n* = *of-nat* (*n*+*1*) ∗ (*fps-sin c* $ (*n*+*1*)) **by** *simp*
also have . . . = *of-nat* (*n*+*1*) ∗ ((− *1*)^(*n div 2*) ∗ *c*^*Suc n* / *of-nat* (*fact* (*Suc n*)))
using *True* **by** (*simp add*: *fps-sin-def*)
also have . . . = (− *1*)^(*n div 2*) ∗ *c*^*Suc n* ∗ (*of-nat* (*n*+*1*) / (*of-nat* (*Suc n*) ∗ *of-nat* (*fact n*)))
unfolding *fact-Suc of-nat-mult*
by (*simp add*: *field-simps del*: *of-nat-add of-nat-Suc*)
also have . . . = (− *1*)^(*n div 2*) ∗ *c*^*Suc n* / *of-nat* (*fact n*)
by (*simp add*: *field-simps del*: *of-nat-add of-nat-Suc*)
finally show *?thesis*
using *True* **by** (*simp add*: *fps-cos-def field-simps*)
next
case *False*
then show *?thesis*
by (*simp-all add*: *fps-deriv-def fps-sin-def fps-cos-def*)
qed
qed

lemma *fps-cos-deriv*: *fps-deriv* (*fps-cos c*) = *fps-const* (− *c*)∗ (*fps-sin c*)
(**is** *?lhs* = *?rhs*)
proof (*rule fps-ext*)
have *th0*: − ((− *1*::′*a*) ^ *n*) = (− *1*)^*Suc n* **for** *n*
by *simp*
show *?lhs* $ *n* = *?rhs* $ *n* **for** *n*
proof (*cases even n*)
case *False*

then have *n0*: $n \neq 0$ **by** *presburger*
from *False* **have** *th1*: *Suc* ((*n* − *1*) *div 2*) = *Suc n div 2*
by (*cases n*) *simp-all*
have *?lhs*\$*n* = *of-nat* (*n*+*1*) ∗ (*fps-cos c* \$ (*n*+*1*)) **by** *simp*
also have ... = *of-nat* (*n*+*1*) ∗ ((− *1*) ^(*n* + *1*) *div 2*) ∗ *c* ^*Suc n* / *of-nat* (*fact* (*Suc n*)))
using *False* **by** (*simp add*: *fps-cos-def*)
also have ... = (− *1*) ^((*n* + *1*) *div 2*) ∗ *c* ^*Suc n* ∗ (*of-nat* (*n*+*1*) / (*of-nat* (*Suc n*) ∗ *of-nat* (*fact n*)))
unfolding *fact-Suc of-nat-mult*
by (*simp add*: *field-simps del*: *of-nat-add of-nat-Suc*)
also have ... = (− *1*) ^((*n* + *1*) *div 2*) ∗ *c* ^*Suc n* / *of-nat* (*fact n*)
by (*simp add*: *field-simps del*: *of-nat-add of-nat-Suc*)
also have ... = (− ((− *1*) ^((*n* − *1*) *div 2*))) ∗ *c* ^*Suc n* / *of-nat* (*fact n*)
unfolding *th0* **unfolding** *th1* **by** *simp*
finally show *?thesis*
using *False* **by** (*simp add*: *fps-sin-def field-simps*)
next
case *True*
then show *?thesis*
by (*simp-all add*: *fps-deriv-def fps-sin-def fps-cos-def*)
qed
qed

lemma *fps-sin-cos-sum-of-squares*: $(fps\text{-}cos\ c)^2 + (fps\text{-}sin\ c)^2 = 1$
(**is** *?lhs* = -)
proof −
have *fps-deriv ?lhs* = *0*
by (*simp add*: *fps-deriv-power fps-sin-deriv fps-cos-deriv field-simps flip*: *fps-const-neg*)
then have *?lhs* = *fps-const* (*?lhs* \$ *0*)
unfolding *fps-deriv-eq-0-iff* **.**
also have ... = *1*
by (*simp add*: *fps-eq-iff numeral-2-eq-2 fps-mult-nth fps-cos-def fps-sin-def*)
finally show *?thesis* **.**
qed

lemma *fps-sin-nth-0* [*simp*]: *fps-sin c* \$ *0* = *0*
unfolding *fps-sin-def* **by** *simp*

lemma *fps-sin-nth-1* [*simp*]: *fps-sin c* \$ *Suc 0* = *c*
unfolding *fps-sin-def* **by** *simp*

lemma *fps-sin-nth-add-2*:
fps-sin c \$ (*n* + *2*) = − (*c* ∗ *c* ∗ *fps-sin c* \$ *n* / (*of-nat* (*n* + *1*) ∗ *of-nat* (*n* + *2*)))
proof (*cases n*)
case (*Suc n'*)
then show *?thesis*
unfolding *fps-sin-def* **by** (*simp add*: *field-simps*)

qed (*auto simp*: *fps-sin-def*)

lemma *fps-cos-nth-0* [*simp*]: *fps-cos c* $ *0* = *1*
  unfolding *fps-cos-def* **by** *simp*

lemma *fps-cos-nth-1* [*simp*]: *fps-cos c* $ *Suc 0* = *0*
  unfolding *fps-cos-def* **by** *simp*

lemma *fps-cos-nth-add-2*:
  fps-cos c $ (*n* + *2*) = − (*c* ∗ *c* ∗ *fps-cos c* $ *n* / (*of-nat* (*n* + *1*) ∗ *of-nat* (*n* +
2)))
proof (*cases n*)
  case (*Suc n′*)
  then show *?thesis*
    unfolding *fps-cos-def* **by** (*simp add*: *field-simps*)
qed (*auto simp*: *fps-cos-def*)

lemma *nat-add-1-add-1*: (*n*::*nat*) + *1* + *1* = *n* + *2*
  by *simp*

lemma *eq-fps-sin*:
  assumes *a0*: *a* $ *0* = *0*
    and *a1*: *a* $ *1* = *c*
    and *a2*: *fps-deriv* (*fps-deriv a*) = − (*fps-const c* ∗ *fps-const c* ∗ *a*)
  shows *fps-sin c* = *a*
proof (*rule fps-ext*)
  fix *n*
  show *fps-sin c* $ *n* = *a* $ *n*
  proof (*induction n rule*: *nat-induct2*)
    case (*step n*)
    then have *of-nat* (*n* + *1*) ∗ (*of-nat* (*n* + *2*) ∗ *a* $ (*n* + *2*)) =
     − (*c* ∗ *c* ∗ *fps-sin c* $ *n*)
      using *a2*
     by (*metis fps-const-mult fps-deriv-nth fps-mult-left-const-nth fps-neg-nth nat-add-1-add-1*)
    with *step* **show** *?case*
     by (*metis* (*no-types*, *lifting*) *a0 add.commute add.inverse-inverse fps-sin-nth-0*
fps-sin-nth-add-2 mult-divide-mult-cancel-left-if mult-minus-right nonzero-mult-div-cancel-left
not-less-zero of-nat-eq-0-iff plus-1-eq-Suc zero-less-Suc)
  qed (*use assms* **in** *auto*)
qed

lemma *eq-fps-cos*:
  assumes *a0*: *a* $ *0* = *1*
    and *a1*: *a* $ *1* = *0*
    and *a2*: *fps-deriv* (*fps-deriv a*) = − (*fps-const c* ∗ *fps-const c* ∗ *a*)
  shows *fps-cos c* = *a*
proof (*rule fps-ext*)
  fix *n*

show *fps-cos c $ n = a $ n*
proof (*induction n rule*: *nat-induct2*)
case (*step n*)
then have *of-nat* (*n* + *1*) ∗ (*of-nat* (*n* + *2*) ∗ *a* $ (*n* + *2*)) =
− (*c* ∗ *c* ∗ *fps-cos c* $ *n*)
using *a2*
by (*metis fps-const-mult fps-deriv-nth fps-mult-left-const-nth fps-neg-nth nat-add-1-add-1*)
with *step* **show** *?case*
by (*metis* (*no-types*, *lifting*) *a0 add.commute add.inverse-inverse fps-cos-nth-0 fps-cos-nth-add-2 mult-divide-mult-cancel-left-if mult-minus-right nonzero-mult-div-cancel-left not-less-zero of-nat-eq-0-iff plus-1-eq-Suc zero-less-Suc*)
qed (*use assms* **in** *auto*)
qed

lemma *fps-sin-add*: *fps-sin* (*a* + *b*) = *fps-sin a* ∗ *fps-cos b* + *fps-cos a* ∗ *fps-sin b*
proof −
have *fps-deriv* (*fps-deriv* (*fps-sin a* ∗ *fps-cos b* + *fps-cos a* ∗ *fps-sin b*)) =
− (*fps-const* (*a* + *b*) ∗ *fps-const* (*a* + *b*) ∗ (*fps-sin a* ∗ *fps-cos b* + *fps-cos a* ∗ *fps-sin b*))
by (*simp flip*: *fps-const-neg fps-const-add fps-const-mult*
add: *fps-sin-deriv fps-cos-deriv algebra-simps*)
then show *?thesis*
by (*auto intro*: *eq-fps-sin*)
qed

lemma *fps-cos-add*: *fps-cos* (*a* + *b*) = *fps-cos a* ∗ *fps-cos b* − *fps-sin a* ∗ *fps-sin b*
proof −
have *fps-deriv*
(*fps-deriv* (*fps-cos a* ∗ *fps-cos b* − *fps-sin a* ∗ *fps-sin b*)) =
− (*fps-const* (*a* + *b*) ∗ *fps-const* (*a* + *b*) ∗
(*fps-cos a* ∗ *fps-cos b* − *fps-sin a* ∗ *fps-sin b*))
by (*simp flip*: *fps-const-neg fps-const-add fps-const-mult*
add: *fps-sin-deriv fps-cos-deriv algebra-simps*)
then show *?thesis*
by (*auto intro*: *eq-fps-cos*)
qed

lemma *fps-sin-even*: *fps-sin* (− *c*) = − *fps-sin c*
by (*simp add*: *fps-eq-iff fps-sin-def*)

lemma *fps-cos-odd*: *fps-cos* (− *c*) = *fps-cos c*
by (*simp add*: *fps-eq-iff fps-cos-def*)

definition *fps-tan c* = *fps-sin c* / *fps-cos c*

lemma *fps-tan-0* [*simp*]: *fps-tan 0* = *0*
by (*simp add*: *fps-tan-def*)

lemma *fps-tan-deriv*: *fps-deriv* (*fps-tan c*) = *fps-const c* / (*fps-cos c*)^2
proof −
  have *th0*: *fps-cos c* \$ *0* ≠ *0* **by** (*simp add*: *fps-cos-def*)
  from *this* **have** *fps-cos c* ≠ *0* **by** (*intro notI*) *simp*
  hence *fps-deriv* (*fps-tan c*) =
        fps-const c ∗ (*fps-cos c*^*2* + *fps-sin c*^*2*) / (*fps-cos c*^*2*)
    by (*simp add*: *fps-tan-def fps-divide-deriv power2-eq-square algebra-simps*
          fps-sin-deriv fps-cos-deriv fps-const-neg[*symmetric*] *div-mult-swap*
        del: *fps-const-neg*)
  also note *fps-sin-cos-sum-of-squares*
  finally show *?thesis* **by** *simp*
qed

Connection to *fps-exp* over the complex numbers — Euler and de Moivre.

lemma *fps-exp-ii-sin-cos*: *fps-exp* (i ∗ *c*) = *fps-cos c* + *fps-const* i ∗ *fps-sin c*
  (**is** *?l* = *?r*)
proof −
  have *?l* \$ *n* = *?r* \$ *n* **for** *n*
  proof (*cases even n*)
    case *True*
    then obtain *m* **where** *m*: *n* = *2* ∗ *m* **..**
    show *?thesis*
     by (*simp add*: *m fps-sin-def fps-cos-def power-mult-distrib power-mult power-minus*
[*of c* ^ *2*])
  next
    case *False*
    then obtain *m* **where** *m*: *n* = *2* ∗ *m* + *1* **..**
    show *?thesis*
      by (*simp add*: *m fps-sin-def fps-cos-def power-mult-distrib*
        power-mult power-minus [*of c* ^ *2*])
  qed
  then show *?thesis*
    by (*simp add*: *fps-eq-iff*)
qed

lemma *fps-exp-minus-ii-sin-cos*: *fps-exp* (− (i ∗ *c*)) = *fps-cos c* − *fps-const* i ∗
fps-sin c
 unfolding *minus-mult-right fps-exp-ii-sin-cos* **by** (*simp add*: *fps-sin-even fps-cos-odd*)

lemma *fps-cos-fps-exp-ii*: *fps-cos c* = (*fps-exp* (i ∗ *c*) + *fps-exp* (− i ∗ *c*)) /
fps-const 2
proof −
  have *th*: *fps-cos c* + *fps-cos c* = *fps-cos c* ∗ *fps-const 2*
    by (*simp add*: *numeral-fps-const*)
  show *?thesis*
    unfolding *fps-exp-ii-sin-cos minus-mult-commute*
   by (*simp add*: *fps-sin-even fps-cos-odd numeral-fps-const fps-divide-unit fps-const-inverse*
th)
qed

lemma *fps-sin-fps-exp-ii*: *fps-sin c* = (*fps-exp* (i ∗ *c*) − *fps-exp* (− i ∗ *c*)) / *fps-const* (*2*∗i)
proof −
  have *th*: *fps-const* i ∗ *fps-sin c* + *fps-const* i ∗ *fps-sin c* = *fps-sin c* ∗ *fps-const* (*2* ∗ i)
    by (*simp add*: *fps-eq-iff numeral-fps-const*)
  show *?thesis*
    unfolding *fps-exp-ii-sin-cos minus-mult-commute*
    by (*simp add*: *fps-sin-even fps-cos-odd fps-divide-unit fps-const-inverse th*)
qed

lemma *fps-tan-fps-exp-ii*:
  fps-tan c = (*fps-exp* (i ∗ *c*) − *fps-exp* (− i ∗ *c*)) /
      (*fps-const* i ∗ (*fps-exp* (i ∗ *c*) + *fps-exp* (− i ∗ *c*)))
  unfolding *fps-tan-def fps-sin-fps-exp-ii fps-cos-fps-exp-ii*
  by (*simp add*: *fps-divide-unit fps-inverse-mult fps-const-inverse*)

lemma *fps-demoivre*:
  (*fps-cos a* + *fps-const* i ∗ *fps-sin a*)^*n* =
    fps-cos (*of-nat n* ∗ *a*) + *fps-const* i ∗ *fps-sin* (*of-nat n* ∗ *a*)
  unfolding *fps-exp-ii-sin-cos*[*symmetric*] *fps-exp-power-mult*
  by (*simp add*: *ac-simps*)

5.17 Hypergeometric series

definition *fps-hypergeo as bs* (*c*::*'a*::*field-char-0*) =
  Abs-fps (λ*n*. (*foldl* (λ*r a*. *r*∗ *pochhammer a n*) *1 as* ∗ *c*^*n*) /
    (*foldl* (λ*r b*. *r* ∗ *pochhammer b n*) *1 bs* ∗ *of-nat* (*fact n*)))

lemma *fps-hypergeo-nth*[*simp*]: *fps-hypergeo as bs c* $ *n* =
  (*foldl* (λ*r a*. *r*∗ *pochhammer a n*) *1 as* ∗ *c*^*n*) /
    (*foldl* (λ*r b*. *r* ∗ *pochhammer b n*) *1 bs* ∗ *of-nat* (*fact n*))
  by (*simp add*: *fps-hypergeo-def*)

lemma *foldl-mult-start*:
  fixes *v* :: *'a*::*comm-ring-1*
  shows *foldl* (λ*r x*. *r* ∗ *f x*) *v as* ∗ *x* = *foldl* (λ*r x*. *r* ∗ *f x*) (*v* ∗ *x*) *as*
  by (*induct as arbitrary*: *x v*) (*auto simp add*: *algebra-simps*)

lemma *foldr-mult-foldl*:
  fixes *v* :: *'a*::*comm-ring-1*
  shows *foldr* (λ*x r*. *r* ∗ *f x*) *as v* = *foldl* (λ*r x*. *r* ∗ *f x*) *v as*
  by (*induct as arbitrary*: *v*) (*simp-all add*: *foldl-mult-start*)

lemma *fps-hypergeo-nth-alt*:
  fps-hypergeo as bs c $ *n* = *foldr* (λ*a r*. *r* ∗ *pochhammer a n*) *as* (*c* ^ *n*) /
    foldr (λ*b r*. *r* ∗ *pochhammer b n*) *bs* (*of-nat* (*fact n*))
  by (*simp add*: *foldl-mult-start foldr-mult-foldl*)

lemma *fps-hypergeo-fps-exp*[*simp*]: *fps-hypergeo* [] [] *c* = *fps-exp c*
  by (*simp add*: *fps-eq-iff*)

lemma *fps-hypergeo-1-0*[*simp*]: *fps-hypergeo* [*1*] [] *c* = *1*/(*1* − *fps-const c* ∗ *fps-X*)
proof −
  let *?a* = (*Abs-fps* (λ*n*. *1*)) *oo* (*fps-const c* ∗ *fps-X*)
  have *th0*: (*fps-const c* ∗ *fps-X*) $ *0* = *0* **by** *simp*
  show *?thesis* **unfolding** *gp*[*OF th0*, *symmetric*]
    by (*simp add*: *fps-eq-iff pochhammer-fact*[*symmetric*]
      fps-compose-nth power-mult-distrib if-distrib cong del: *if-weak-cong*)
qed

lemma *fps-hypergeo-B*[*simp*]: *fps-hypergeo* [−*a*] [] (− *1*) = *fps-binomial a*
  by (*simp add*: *fps-eq-iff gbinomial-pochhammer algebra-simps*)

lemma *fps-hypergeo-0*[*simp*]: *fps-hypergeo as bs c* $ *0* = *1*
proof −
  have *foldl* (λ(*r*::′*a*) (*a*::′*a*). *r*) *1 as* = *1* **for** *as*
    by (*induction as*) *auto*
  then show *?thesis*
    by *auto*
qed

lemma *foldl-prod-prod*:
  foldl (λ(*r*::′*b*::*comm-ring-1*) (*x*::′*a*::*comm-ring-1*). *r* ∗ *f x*) *v as* ∗ *foldl* (λ*r x*. *r* ∗ *g x*) *w as* =
    foldl (λ*r x*. *r* ∗ *f x* ∗ *g x*) (*v* ∗ *w*) *as*
  by (*induct as arbitrary*: *v w*) (*simp-all add*: *algebra-simps*)

lemma *fps-hypergeo-rec*:
  fps-hypergeo as bs c $ *Suc n* = ((*foldl* (λ*r a*. *r*∗ (*a* + *of-nat n*)) *c as*) /
    (*foldl* (λ*r b*. *r* ∗ (*b* + *of-nat n*)) (*of-nat* (*Suc n*)) *bs*)) ∗ *fps-hypergeo as bs c* $ *n*
  apply (*simp add*: *foldl-mult-start del*: *of-nat-Suc of-nat-add fact-Suc*)
  unfolding *foldl-prod-prod*[*unfolded foldl-mult-start*] *pochhammer-Suc*
  by (*simp add*: *algebra-simps*)

lemma *fps-XD-nth*[*simp*]: *fps-XD a* $ *n* = *of-nat n* ∗ *a*$*n*
  by (*simp add*: *fps-XD-def*)

lemma *fps-XD-0th*[*simp*]: *fps-XD a* $ *0* = *0*
  by *simp*
lemma *fps-XD-Suc*[*simp*]: *fps-XD a* $ *Suc n* = *of-nat* (*Suc n*) ∗ *a* $ *Suc n*
  by *simp*

definition *fps-XDp c a* = *fps-XD a* + *fps-const c* ∗ *a*

lemma *fps-XDp-nth*[*simp*]: *fps-XDp c a* \$ *n* = (*c* + *of-nat n*) ∗ *a*\$*n*
  by (*simp add*: *fps-XDp-def algebra-simps*)

lemma *fps-XDp-commute*: *fps-XDp b* ∘ *fps-XDp* (*c*::′*a*::*comm-ring-1*) = *fps-XDp c* ∘ *fps-XDp b*
  by (*simp add*: *fps-XDp-def fun-eq-iff fps-eq-iff algebra-simps*)

lemma *fps-XDp0* [*simp*]: *fps-XDp 0* = *fps-XD*
  by (*simp add*: *fun-eq-iff fps-eq-iff*)

lemma *fps-XDp-fps-integral* [*simp*]:
  fixes *a* :: ′*a*::{*division-ring*,*ring-char-0*} *fps*
  shows *fps-XDp 0* (*fps-integral a c*) = *fps-X* ∗ *a*
  using *fps-deriv-fps-integral*[*of a c*]
  by (*simp add*: *fps-XD-def*)

lemma *fps-hypergeo-minus-nat*:
  fps-hypergeo [− *of-nat n*] [− *of-nat* (*n* + *m*)] (*c*::′*a*::*field-char-0*) \$ *k* =
    (*if k* ≤ *n then*
      pochhammer (− *of-nat n*) *k* ∗ *c* ^ *k* / (*pochhammer* (− *of-nat* (*n* + *m*)) *k* ∗ *of-nat* (*fact k*))
     else 0)
  fps-hypergeo [− *of-nat m*] [− *of-nat* (*m* + *n*)] (*c*::′*a*::*field-char-0*) \$ *k* =
    (*if k* ≤ *m then*
      pochhammer (− *of-nat m*) *k* ∗ *c* ^ *k* / (*pochhammer* (− *of-nat* (*m* + *n*)) *k* ∗ *of-nat* (*fact k*))
     else 0)
  by (*simp-all add*: *pochhammer-eq-0-iff*)

lemma *pochhammer-rec-if*: *pochhammer a n* = (*if n* = *0 then 1 else a* ∗ *pochhammer* (*a* + *1*) (*n* − *1*))
  by (*cases n*) (*simp-all add*: *pochhammer-rec*)

lemma *fps-XDp-foldr-nth* [*simp*]: *foldr* (λ*c r*. *fps-XDp c* ∘ *r*) *cs* (λ*c*. *fps-XDp c a*) *c0* \$ *n* =
    foldr (λ*c r*. (*c* + *of-nat n*) ∗ *r*) *cs* (*c0* + *of-nat n*) ∗ *a*\$*n*
  by (*induct cs arbitrary*: *c0*) (*simp-all add*: *algebra-simps*)

lemma *genric-fps-XDp-foldr-nth*:
  assumes *f*: ∀ *n c a*. *f c a* \$ *n* = (*of-nat n* + *k c*) ∗ *a*\$*n*
  shows *foldr* (λ*c r*. *f c* ∘ *r*) *cs* (λ*c*. *g c a*) *c0* \$ *n* =
    foldr (λ*c r*. (*k c* + *of-nat n*) ∗ *r*) *cs* (*g c0 a* \$ *n*)
  by (*induct cs arbitrary*: *c0*) (*simp-all add*: *algebra-simps f*)

lemma *dist-less-imp-nth-equal*:
  assumes *dist f g* < *inverse* (*2* ^ *i*)
    and*j* ≤ *i*
  shows *f* \$ *j* = *g* \$ *j*
proof (*rule ccontr*)

```
  assume f $ j ≠ g $ j
  hence f ≠ g by auto
  with assms have i < subdegree (f − g)
    by (simp add: if-split-asm dist-fps-def)
  also have ... ≤ j
    using ‹f $ j ≠ g $ j› by (intro subdegree-leI) simp-all
  finally show False using ‹j ≤ i› by simp
qed

lemma nth-equal-imp-dist-less:
  assumes ⋀j. j ≤ i ⟹ f $ j = g $ j
  shows dist f g < inverse (2 ^ i)
proof (cases f = g)
  case True
  then show ?thesis by simp
next
  case False
  with assms have dist f g = inverse (2 ^ subdegree (f − g))
    by (simp add: if-split-asm dist-fps-def)
  moreover
  from assms and False have i < subdegree (f − g)
    by (intro subdegree-greaterI) simp-all
  ultimately show ?thesis by simp
qed

lemma dist-less-eq-nth-equal: dist f g < inverse (2 ^ i) ⟷ (∀ j ≤ i. f $ j = g $
j)
  using dist-less-imp-nth-equal nth-equal-imp-dist-less by blast

instance fps :: (comm-ring-1) complete-space
proof
  fix fps-X :: nat ⇒ 'a fps
  assume Cauchy fps-X
  obtain M where M: ∀ i. ∀ m ≥ M i. ∀ j ≤ i. fps-X (M i) $ j = fps-X m $ j
  proof −
    have ∃ M. ∀ m ≥ M. ∀ j≤i. fps-X M $ j = fps-X m $ j for i
    proof −
      have 0 < inverse ((2::real)^i) by simp
      from metric-CauchyD[OF ‹Cauchy fps-X› this] dist-less-imp-nth-equal
      show ?thesis by blast
    qed
    then show ?thesis using that by metis
  qed

  show convergent fps-X
  proof (rule convergentI)
    show fps-X ⟶ Abs-fps (λi. fps-X (M i) $ i)
      unfolding tendsto-iff
    proof safe
```

fix *e*::*real* **assume** *e*: *0* < *e*
have (λ*n*. *inverse* (*2* ^ *n*) :: *real*) ⟶ *0* **by** (*rule LIMSEQ-inverse-realpow-zero*) *simp-all*
from *this* **and** *e* **have** *eventually* (λ*i*. *inverse* (*2* ^ *i*) < *e*) *sequentially*
by (*rule order-tendstoD*)
then obtain *i* **where** *inverse* (*2* ^ *i*) < *e*
by (*auto simp*: *eventually-sequentially*)
have *eventually* (λ*x*. *M i* ≤ *x*) *sequentially*
by (*auto simp*: *eventually-sequentially*)
then show *eventually* (λ*x*. *dist* (*fps-X x*) (*Abs-fps* (λ*i*. *fps-X* (*M i*) $ *i*)) < *e*) *sequentially*
proof *eventually-elim*
fix *x*
assume *x*: *M i* ≤ *x*
have *fps-X* (*M i*) $ *j* = *fps-X* (*M j*) $ *j* **if** *j* ≤ *i* **for** *j*
using *M that* **by** (*metis nat-le-linear*)
with *x* **have** *dist* (*fps-X x*) (*Abs-fps* (λ*j*. *fps-X* (*M j*) $ *j*)) < *inverse* (*2* ^ *i*)
using *M* **by** (*force simp*: *dist-less-eq-nth-equal*)
also note ‹*inverse* (*2* ^ *i*) < *e*›
finally show *dist* (*fps-X x*) (*Abs-fps* (λ*j*. *fps-X* (*M j*) $ *j*)) < *e* **.**
qed
qed
qed
qed

bundle *fps-syntax*
begin
notation *fps-nth* (**infixl** ‹$› *75*)
end

unbundle *no fps-syntax*

end

6 Converting polynomials to formal power series

theory *Polynomial-FPS*
imports *Polynomial Formal-Power-Series*
begin

context
includes *fps-syntax*
begin

definition *fps-of-poly* **where**
fps-of-poly p = *Abs-fps* (*coeff p*)

lemma *fps-of-poly-eq-iff*: *fps-of-poly p* = *fps-of-poly q* ⟷ *p* = *q*

by (*simp add*: *fps-of-poly-def poly-eq-iff fps-eq-iff*)

lemma *fps-of-poly-nth* [*simp*]: *fps-of-poly p* $ *n* = *coeff p n*
by (*simp add*: *fps-of-poly-def*)

lemma *fps-of-poly-const*: *fps-of-poly* [:*c*:] = *fps-const c*
proof (*subst fps-eq-iff* , *clarify*)
fix *n* :: *nat* **show** *fps-of-poly* [:*c*:] $ *n* = *fps-const c* $ *n*
by (*cases n*) (*auto simp*: *fps-of-poly-def*)
qed

lemma *fps-of-poly-0* [*simp*]: *fps-of-poly 0* = *0*
by (*subst fps-const-0-eq-0* [*symmetric*], *subst fps-of-poly-const* [*symmetric*]) *simp*

lemma *fps-of-poly-1* [*simp*]: *fps-of-poly 1* = *1*
by (*simp add*: *fps-eq-iff*)

lemma *fps-of-poly-1'* [*simp*]: *fps-of-poly* [:*1*:] = *1*
by (*subst fps-const-1-eq-1* [*symmetric*], *subst fps-of-poly-const* [*symmetric*])
(*simp add*: *one-poly-def*)

lemma *fps-of-poly-numeral* [*simp*]: *fps-of-poly* (*numeral n*) = *numeral n*
by (*simp add*: *numeral-fps-const fps-of-poly-const* [*symmetric*] *numeral-poly*)

lemma *fps-of-poly-numeral'* [*simp*]: *fps-of-poly* [:*numeral n*:] = *numeral n*
by (*simp add*: *numeral-fps-const fps-of-poly-const* [*symmetric*] *numeral-poly*)

lemma *fps-of-poly-fps-X* [*simp*]: *fps-of-poly* [:*0*, *1*:] = *fps-X*
by (*auto simp add*: *fps-of-poly-def fps-eq-iff coeff-pCons split*: *nat.split*)

lemma *fps-of-poly-add*: *fps-of-poly* (*p* + *q*) = *fps-of-poly p* + *fps-of-poly q*
by (*simp add*: *fps-of-poly-def plus-poly.rep-eq fps-plus-def*)

lemma *fps-of-poly-diff*: *fps-of-poly* (*p* − *q*) = *fps-of-poly p* − *fps-of-poly q*
by (*simp add*: *fps-of-poly-def minus-poly.rep-eq fps-minus-def*)

lemma *fps-of-poly-uminus*: *fps-of-poly* (−*p*) = −*fps-of-poly p*
by (*simp add*: *fps-of-poly-def uminus-poly.rep-eq fps-uminus-def*)

lemma *fps-of-poly-mult*: *fps-of-poly* (*p* ∗ *q*) = *fps-of-poly p* ∗ *fps-of-poly q*
by (*simp add*: *fps-of-poly-def fps-times-def fps-eq-iff coeff-mult atLeast0AtMost*)

lemma *fps-of-poly-smult*:
fps-of-poly (*smult c p*) = *fps-const c* ∗ *fps-of-poly p*
using *fps-of-poly-mult*[*of* [:*c*:] *p*] **by** (*simp add*: *fps-of-poly-mult fps-of-poly-const*)

lemma *fps-of-poly-sum*: *fps-of-poly* (*sum f A*) = *sum* (λ*x*. *fps-of-poly* (*f x*)) *A*
by (*cases finite A*, *induction rule*: *finite-induct*) (*simp-all add*: *fps-of-poly-add*)

lemma *fps-of-poly-sum-list*: *fps-of-poly* (*sum-list xs*) = *sum-list* (*map fps-of-poly xs*)
  by (*induction xs*) (*simp-all add*: *fps-of-poly-add*)

lemma *fps-of-poly-prod*: *fps-of-poly* (*prod f A*) = *prod* (λx. *fps-of-poly* (*f x*)) *A*
  by (*cases finite A*, *induction rule*: *finite-induct*) (*simp-all add*: *fps-of-poly-mult*)

lemma *fps-of-poly-prod-list*: *fps-of-poly* (*prod-list xs*) = *prod-list* (*map fps-of-poly xs*)
  by (*induction xs*) (*simp-all add*: *fps-of-poly-mult*)

lemma *fps-of-poly-pCons*:
  fps-of-poly (*pCons* (*c* :: *'a* :: *semiring-1*) *p*) = *fps-const c* + *fps-of-poly p* ∗ *fps-X*
  by (*subst fps-mult-fps-X-commute* [*symmetric*], *intro fps-ext*)
    (*auto simp*: *fps-of-poly-def coeff-pCons split*: *nat.split*)

lemma *fps-of-poly-pderiv*: *fps-of-poly* (*pderiv p*) = *fps-deriv* (*fps-of-poly p*)
  by (*intro fps-ext*) (*simp add*: *fps-of-poly-nth coeff-pderiv*)

lemma *fps-of-poly-power*: *fps-of-poly* (*p* ^ *n*) = *fps-of-poly p* ^ *n*
  by (*induction n*) (*simp-all add*: *fps-of-poly-mult*)

lemma *fps-of-poly-monom*: *fps-of-poly* (*monom* (*c* :: *'a* :: *comm-ring-1*) *n*) = *fps-const c* ∗ *fps-X* ^ *n*
  by (*intro fps-ext*) *simp-all*

lemma *fps-of-poly-monom'*: *fps-of-poly* (*monom* (*1* :: *'a* :: *comm-ring-1*) *n*) = *fps-X* ^ *n*
  by (*simp add*: *fps-of-poly-monom*)

lemma *fps-of-poly-div*:
  assumes (*q* :: *'a* :: *field poly*) *dvd p*
  shows *fps-of-poly* (*p div q*) = *fps-of-poly p* / *fps-of-poly q*
proof (*cases q* = *0*)
  case *False*
  from *False fps-of-poly-eq-iff*[*of q 0*] **have** *nz*: *fps-of-poly q* ≠ *0* **by** *simp*
  from *assms* **have** *p* = (*p div q*) ∗ *q* **by** *simp*
  also have *fps-of-poly* ... = *fps-of-poly* (*p div q*) ∗ *fps-of-poly q*
    by (*simp add*: *fps-of-poly-mult*)
  also from *nz* **have** ... / *fps-of-poly q* = *fps-of-poly* (*p div q*)
    by (*intro nonzero-mult-div-cancel-right*) (*auto simp*: *fps-of-poly-0*)
  finally show *?thesis* **..**
qed *simp*

lemma *fps-of-poly-divide-numeral*:
  fps-of-poly (*smult* (*inverse* (*numeral c* :: *'a* :: *field*)) *p*) = *fps-of-poly p* / *numeral c*
proof −
  have *smult* (*inverse* (*numeral c*)) *p* = [:*inverse* (*numeral c*):] ∗ *p* **by** *simp*

also have *fps-of-poly* ... = *fps-of-poly p* / *numeral c*
  by (*subst fps-of-poly-mult*) (*simp add*: *numeral-fps-const fps-of-poly-pCons*)
finally show *?thesis* **by** *simp*
qed

lemma *subdegree-fps-of-poly*:
  assumes $p \neq 0$
  defines $n \equiv$ *Polynomial.order 0 p*
  shows *subdegree* (*fps-of-poly p*) = *n*
proof (*rule subdegreeI*)
  from *assms* **have** *monom 1 n dvd p* **by** (*simp add*: *monom-1-dvd-iff*)
  thus *zero*: *fps-of-poly p* \$ *i* = *0* **if** $i < n$ **for** *i*
    using *that* **by** (*simp add*: *monom-1-dvd-iff'*)

  from *assms* **have** ¬*monom 1* (*Suc n*) *dvd p*
    by (*auto simp*: *monom-1-dvd-iff simp del*: *power-Suc*)
  then obtain *k* **where** *k*: $k \leq n$ *fps-of-poly p* \$ $k \neq 0$
    by (*auto simp*: *monom-1-dvd-iff' less-Suc-eq-le*)
  with *zero*[*of k*] **have** *k* = *n* **by** *linarith*
  with *k* **show** *fps-of-poly p* \$ $n \neq 0$ **by** *simp*
qed

lemma *fps-of-poly-dvd*:
  assumes *p dvd q*
  shows *fps-of-poly* (*p* :: *'a* :: *field poly*) *dvd fps-of-poly q*
proof (*cases p* = *0* ∨ *q* = *0*)
  case *False*
  with *assms fps-of-poly-eq-iff*[*of p 0*] *fps-of-poly-eq-iff*[*of q 0*] **show** *?thesis*
    by (*auto simp*: *fps-dvd-iff subdegree-fps-of-poly dvd-imp-order-le*)
qed (*insert assms*, *auto*)

lemmas *fps-of-poly-simps* =
  fps-of-poly-0 fps-of-poly-1 fps-of-poly-numeral fps-of-poly-const fps-of-poly-fps-X
  fps-of-poly-add fps-of-poly-diff fps-of-poly-uminus fps-of-poly-mult fps-of-poly-smult
  fps-of-poly-sum fps-of-poly-sum-list fps-of-poly-prod fps-of-poly-prod-list
  fps-of-poly-pCons fps-of-poly-pderiv fps-of-poly-power fps-of-poly-monom
  fps-of-poly-divide-numeral

lemma *fps-of-poly-pcompose*:
  assumes *coeff q 0* = (*0* :: *'a* :: *idom*)
  shows *fps-of-poly* (*pcompose p q*) = *fps-compose* (*fps-of-poly p*) (*fps-of-poly q*)
  using *assms* **by** (*induction p rule*: *pCons-induct*)
    (*auto simp*: *pcompose-pCons fps-of-poly-simps fps-of-poly-pCons*
      fps-compose-add-distrib fps-compose-mult-distrib)

lemmas *reify-fps-atom* =
  fps-of-poly-0 fps-of-poly-1' fps-of-poly-numeral' fps-of-poly-const fps-of-poly-fps-X

The following simproc can reduce the equality of two polynomial FPSs two equality of the respective polynomials. A polynomial FPS is one that only has finitely many non-zero coefficients and can therefore be written as *fps-of-poly p* for some polynomial *p*.

This may sound trivial, but it covers a number of annoying side conditions like $1 + fps\text{-}X \neq 0$ that would otherwise not be solved automatically.

ML ‹

```
(* TODO: Support for division *)
signature POLY-FPS = sig

val reify-conv : conv
val eq-conv : conv
val eq-simproc : cterm -> thm option

end


structure Poly-Fps = struct

fun const-binop-conv s conv ct =
  case Thm.term-of ct of
    (Const (s', -) $ - $ -) =>
      if s = s' then
        Conv.binop-conv conv ct
      else
        raise CTERM (const-binop-conv, [ct])
  | - => raise CTERM (const-binop-conv, [ct])

fun reify-conv ct =
  let
    val rewr = Conv.rewrs-conv o map (fn thm => thm RS @{thm eq-reflection})
    val un = Conv.arg-conv reify-conv
    val bin = Conv.binop-conv reify-conv
  in
    case Thm.term-of ct of
      (Const (const-name ‹fps-of-poly›, -) $ -) => ct |> Conv.all-conv
    | (Const (const-name ‹Groups.plus›, -) $ - $ -) => ct |> (
        bin then-conv rewr @{thms fps-of-poly-add [symmetric]})
    | (Const (const-name ‹Groups.uminus›, -) $ -) => ct |> (
        un then-conv rewr @{thms fps-of-poly-uminus [symmetric]})
    | (Const (const-name ‹Groups.minus›, -) $ - $ -) => ct |> (
        bin then-conv rewr @{thms fps-of-poly-diff [symmetric]})
    | (Const (const-name ‹Groups.times›, -) $ - $ -) => ct |> (
        bin then-conv rewr @{thms fps-of-poly-mult [symmetric]})
   | (Const (const-name ‹Rings.divide›, -) $ - $ (Const (const-name ‹Num.numeral›,
-) $ -))
        => ct |> (Conv.fun-conv (Conv.arg-conv reify-conv)
```

then-conv rewr @{*thms fps-of-poly-divide-numeral* [*symmetric*]})
| (*Const* (***const-name*** ‹*Power.power*›, -) $ *Const* (***const-name*** ‹*fps-X*›,-) $ -)
=> *ct* |> (
rewr @{*thms fps-of-poly-monom′* [*symmetric*]})
| (*Const* (***const-name*** ‹*Power.power*›, -) $ - $ -) => *ct* |> (
Conv.fun-conv (*Conv.arg-conv reify-conv*)
then-conv rewr @{*thms fps-of-poly-power* [*symmetric*]})
| - => *ct* |> (
rewr @{*thms reify-fps-atom* [*symmetric*]})
end

fun eq-conv ct =
case Thm.term-of ct of
(*Const* (***const-name*** ‹*HOL.eq*›, -) $ - $ -) => *ct* |> (
Conv.binop-conv reify-conv
then-conv Conv.rewr-conv @{*thm fps-of-poly-eq-iff*[*THEN eq-reflection*]})
| - => *raise CTERM* (*poly-fps-eq-conv*, [*ct*])

val eq-simproc = *try eq-conv*

end
›

simproc-setup *poly-fps-eq* ((*f* :: ′*a fps*) = *g*) = ‹*K* (*K Poly-Fps.eq-simproc*)›

lemma *fps-of-poly-linear*: *fps-of-poly* [:*a*,*1* :: ′*a* :: *field*:] = *fps-X* + *fps-const a*
by *simp*

lemma *fps-of-poly-linear′*: *fps-of-poly* [:*1*,*a* :: ′*a* :: *field*:] = *1* + *fps-const a* ∗ *fps-X*
by *simp*

lemma *fps-of-poly-cutoff* [*simp*]:
fps-of-poly (*poly-cutoff n p*) = *fps-cutoff n* (*fps-of-poly p*)
by (*simp add*: *fps-eq-iff coeff-poly-cutoff*)

lemma *fps-of-poly-shift* [*simp*]: *fps-of-poly* (*poly-shift n p*) = *fps-shift n* (*fps-of-poly p*)
by (*simp add*: *fps-eq-iff coeff-poly-shift*)

definition *poly-subdegree* :: ′*a*::*zero poly* ⇒ *nat* **where**
poly-subdegree p = *subdegree* (*fps-of-poly p*)

lemma *coeff-less-poly-subdegree*:
k < *poly-subdegree p* ⟹ *coeff p k* = *0*
unfolding *poly-subdegree-def* **using** *nth-less-subdegree-zero*[*of k fps-of-poly p*] **by** *simp*

definition *prefix-length* :: ('*a* ⇒ *bool*) ⇒ '*a list* ⇒ *nat* **where**
prefix-length P xs = *length* (*takeWhile P xs*)

primrec *prefix-length-aux* :: ('*a* ⇒ *bool*) ⇒ *nat* ⇒ '*a list* ⇒ *nat* **where**
prefix-length-aux P acc [] = *acc*
| *prefix-length-aux P acc* (*x*#*xs*) = (*if P x then prefix-length-aux P* (*Suc acc*) *xs else acc*)

lemma *prefix-length-aux-correct*: *prefix-length-aux P acc xs* = *prefix-length P xs* + *acc*
by (*induction xs arbitrary*: *acc*) (*simp-all add*: *prefix-length-def*)

lemma *prefix-length-code* [*code*]: *prefix-length P xs* = *prefix-length-aux P 0 xs*
by (*simp add*: *prefix-length-aux-correct*)

lemma *prefix-length-le-length*: *prefix-length P xs* ≤ *length xs*
by (*induction xs*) (*simp-all add*: *prefix-length-def*)

lemma *prefix-length-less-length*: (∃ *x*∈*set xs*. ¬*P x*) ⟹ *prefix-length P xs* < *length xs*
by (*induction xs*) (*simp-all add*: *prefix-length-def*)

lemma *nth-prefix-length*:
(∃ *x*∈*set xs*. ¬*P x*) ⟹ ¬*P* (*xs* ! *prefix-length P xs*)
by (*induction xs*) (*simp-all add*: *prefix-length-def*)

lemma *nth-less-prefix-length*:
n < *prefix-length P xs* ⟹ *P* (*xs* ! *n*)
by (*induction xs arbitrary*: *n*)
(*auto simp*: *prefix-length-def nth-Cons split*: *if-splits nat.splits*)

lemma *poly-subdegree-code* [*code*]: *poly-subdegree p* = *prefix-length* ((=) *0*) (*coeffs p*)
proof (*cases p* = *0*)
case *False*
note [*simp*] = *this*
define *n* **where** *n* = *prefix-length* ((=) *0*) (*coeffs p*)
from *False* **have** ∃ *k*. *coeff p k* ≠ *0* **by** (*auto simp*: *poly-eq-iff*)
hence *ex*: ∃ *x*∈*set* (*coeffs p*). *x* ≠ *0* **by** (*auto simp*: *coeffs-def*)
hence *n-less*: *n* < *length* (*coeffs p*) **and** *nonzero*: *coeffs p* ! *n* ≠ *0*
unfolding *n-def* **by** (*auto intro*!: *prefix-length-less-length nth-prefix-length*)
show *?thesis* **unfolding** *poly-subdegree-def*
proof (*intro subdegreeI*)
from *n-less* **have** *fps-of-poly p* $ *n* = *coeffs p* ! *n*
by (*subst coeffs-nth*) (*simp-all add*: *degree-eq-length-coeffs*)
with *nonzero* **show** *fps-of-poly p* $ *prefix-length* ((=) *0*) (*coeffs p*) ≠ *0*
unfolding *n-def* **by** *simp*

next
fix *k* **assume** *A*: *k* < *prefix-length* ((=) *0*) (*coeffs p*)
also have ... ≤ *length* (*coeffs p*) **by** (*rule prefix-length-le-length*)
finally show *fps-of-poly p* $ *k* = *0*
using *nth-less-prefix-length*[*OF A*]
by (*simp add*: *coeffs-nth degree-eq-length-coeffs*)
qed
qed (*simp-all add*: *poly-subdegree-def prefix-length-def*)

end

end

7 A formalization of formal Laurent series

theory *Formal-Laurent-Series*
imports
Polynomial-FPS
begin

7.1 The type of formal Laurent series

7.1.1 Type definition

typedef (**overloaded**) *'a fls* = {*f*::*int* ⇒ *'a*::*zero*. $\forall_\infty$ *n*::*nat*. *f* (− *int n*) = *0*}
morphisms *fls-nth Abs-fls*
proof
show (λ*x*. *0*) ∈ {*f*::*int* ⇒ *'a*::*zero*. $\forall_\infty$ *n*::*nat*. *f* (− *int n*) = *0*}
by *simp*
qed

setup-lifting *type-definition-fls*

unbundle *fps-syntax*
notation *fls-nth* (**infixl** ‹$$› *75*)

lemmas *fls-eqI* = *iffD1*[*OF fls-nth-inject*, *OF iffD2*, *OF fun-eq-iff*, *OF allI*]

lemma *fls-eq-iff*: *f* = *g* ⟷ (∀ *n*. *f* $$ *n* = *g* $$ *n*)
by (*simp add*: *fls-nth-inject*[*symmetric*] *fun-eq-iff*)

lemma *nth-Abs-fls* [*simp*]: $\forall_\infty$*n*. *f* (− *int n*) = *0* ⟹ *Abs-fls f* $$ *n* = *f n*
by (*simp add*: *Abs-fls-inverse*[*OF CollectI*])

lemmas *nth-Abs-fls-finite-nonzero-neg-nth* = *nth-Abs-fls*[*OF iffD2*, *OF eventually-cofinite*]
lemmas *nth-Abs-fls-ex-nat-lower-bound* = *nth-Abs-fls*[*OF iffD2*, *OF MOST-nat*]
lemmas *nth-Abs-fls-nat-lower-bound* = *nth-Abs-fls-ex-nat-lower-bound*[*OF exI*]

lemma *nth-Abs-fls-ex-lower-bound*:
 assumes $\exists N. \forall n<N. f\, n = 0$
 shows *Abs-fls f* $$ $n = f\, n$
proof (*intro nth-Abs-fls-ex-nat-lower-bound*)
 from *assms* **obtain** $N::int$ **where** $\forall n<N. f\, n = 0$ **by** *fast*
 hence $\forall n > (\textit{if } N < 0 \textit{ then nat } (-N) \textit{ else } 0). f\, (-\textit{int } n) = 0$ **by** *auto*
 thus $\exists M. \forall n>M. f\, (- \textit{int } n) = 0$ **by** *fast*
qed

lemmas *nth-Abs-fls-lower-bound* = *nth-Abs-fls-ex-lower-bound*[*OF exI*]

lemmas *MOST-fls-neg-nth-eq-0* [*simp*] = *CollectD*[*OF fls-nth*]
lemmas *fls-finite-nonzero-neg-nth* = *iffD1*[*OF eventually-cofinite MOST-fls-neg-nth-eq-0*]

lemma *fls-nth-vanishes-below-natE*:
 fixes f :: *'a::zero fls*
 obtains N :: *nat*
 where $\forall n>N. f$$$(-\textit{int } n) = 0$
 using *iffD1*[*OF MOST-nat MOST-fls-neg-nth-eq-0*]
 by *blast*

lemma *fls-nth-vanishes-belowE*:
 fixes f :: *'a::zero fls*
 obtains N :: *int*
 where $\forall n<N. f$$$n = 0$
proof−
 obtain K :: *nat* **where** K: $\forall n>K. f$$$(-\textit{int } n) = 0$ **by** (*elim fls-nth-vanishes-below-natE*)
 have $\forall n < -\textit{int } K. f$$$n = 0$
 proof *clarify*
  fix n **assume** n: $n < -\textit{int } K$
  define m **where** $m \equiv \textit{nat } (-n)$
  with n **have** $m > K$ **by** *simp*
  moreover from *n m-def* **have** $f$$$n = f$ $$ $(-\textit{int } m)$ **by** *simp*
  ultimately show f $$ $n = 0$ **using** K **by** *simp*
 qed
 thus $(\bigwedge N. \forall n<N. f$ $$ $n = 0 \Longrightarrow \textit{thesis}) \Longrightarrow \textit{thesis}$ **by** *fast*
qed

7.1.2 Definition of basic Laurent series

instantiation *fls* :: (*zero*) *zero*
begin
 lift-definition *zero-fls* :: *'a fls* **is** λ-. *0* **by** *simp*
 instance ..
end

lemma *fls-zero-nth* [*simp*]: *0* $$ $n = 0$
 by (*simp add*: *zero-fls-def*)

lemma *fls-zero-eqI*: $(\bigwedge n.\ f\$\$n = 0) \Longrightarrow f = 0$
  by (*fastforce intro*: *fls-eqI*)

lemma *fls-nonzeroI*: $f\$\$n \neq 0 \Longrightarrow f \neq 0$
  by *auto*

lemma *fls-nonzero-nth*: $f \neq 0 \longleftrightarrow (\exists\ n.\ f\ \$\$\ n \neq 0)$
  using *fls-zero-eqI* **by** *fastforce*

lemma *fls-trivial-delta-eq-zero* [*simp*]: $b = 0 \Longrightarrow$ *Abs-fls* ($\lambda n.$ *if n=a then b else 0*) = *0*
  by (*intro fls-zero-eqI*) *simp*

lemma *fls-delta-nth* [*simp*]:
  Abs-fls ($\lambda n.$ *if n=a then b else 0*) \$\$ *n* = (*if n=a then b else 0*)
  using *nth-Abs-fls-lower-bound*[*of a* $\lambda n.$ *if n=a then b else 0*] **by** *simp*

instantiation *fls* :: ({*zero*,*one*}) *one*
begin
  lift-definition *one-fls* :: *'a fls* **is** $\lambda k.$ *if k = 0 then 1 else 0*
    by (*simp add*: *eventually-cofinite*)
  instance ..
end

lemma *fls-one-nth* [*simp*]:
  1 \$\$ *n* = (*if n = 0 then 1 else 0*)
  by (*simp add*: *one-fls-def eventually-cofinite*)

instance *fls* :: (*zero-neq-one*) *zero-neq-one*
proof (*standard*, *standard*)
  assume (*0*::*'a fls*) = (*1*::*'a fls*)
  hence (*0*::*'a fls*) \$\$ *0* = (*1*::*'a fls*) \$\$ *0* **by** *simp*
  thus *False* **by** *simp*
qed

definition *fls-const* :: *'a*::*zero* ⇒ *'a fls*
  where *fls-const c* ≡ *Abs-fls* ($\lambda n.$ *if n = 0 then c else 0*)

lemma *fls-const-nth* [*simp*]: *fls-const c* \$\$ *n* = (*if n = 0 then c else 0*)
  by (*simp add*: *fls-const-def eventually-cofinite*)

lemma *fls-const-0* [*simp*]: *fls-const 0* = *0*
  unfolding *fls-const-def* **using** *fls-trivial-delta-eq-zero* **by** *fast*

lemma *fls-const-nonzero*: $c \neq 0 \Longrightarrow$ *fls-const c* $\neq 0$
  using *fls-nonzeroI*[*of fls-const c 0*] **by** *simp*

lemma *fls-const-eq-0-iff* [*simp*]: *fls-const c* = *0* ⟷ *c* = *0*
  by (*auto simp*: *fls-eq-iff*)

lemma *fls-const-1* [*simp*]: *fls-const 1* = *1*
  unfolding *fls-const-def one-fls-def* ..

lemma *fls-const-eq-1-iff* [*simp*]: *fls-const c* = *1* ⟷ *c* = *1*
  by (*auto simp*: *fls-eq-iff*)

lift-definition *fls-X* :: *'a*::{*zero*,*one*} *fls*
  is λ*n*. *if n* = *1 then 1 else 0*
  by *simp*

lemma *fls-X-nth* [*simp*]:
  fls-X $$ *n* = (*if n* = *1 then 1 else 0*)
  by (*simp add*: *fls-X-def*)

lemma *fls-X-nonzero* [*simp*]: (*fls-X* :: *'a* :: *zero-neq-one fls*) ≠ *0*
  by (*intro fls-nonzeroI*) *simp*

lift-definition *fls-X-inv* :: *'a*::{*zero*,*one*} *fls*
  is λ*n*. *if n* = −*1 then 1 else 0*
  by (*simp add*: *eventually-cofinite*)

lemma *fls-X-inv-nth* [*simp*]:
  fls-X-inv $$ *n* = (*if n* = −*1 then 1 else 0*)
  by (*simp add*: *fls-X-inv-def eventually-cofinite*)

lemma *fls-X-inv-nonzero* [*simp*]: (*fls-X-inv* :: *'a* :: *zero-neq-one fls*) ≠ *0*
  by (*intro fls-nonzeroI*) *simp*

7.2 Subdegrees

lemma *unique-fls-subdegree*:
  assumes *f* ≠ *0*
  shows ∃!*n*. *f*$$*n* ≠ *0* ∧ (∀ *m*. *f*$$*m* ≠ *0* ⟶ *n* ≤ *m*)
proof−
  obtain *N*::*nat* **where** *N*: ∀ *n*>*N*. *f*$$(−*int n*) = *0* **by** (*elim fls-nth-vanishes-below-natE*)
  define *M* **where** *M* ≡ −*int N*
  have *M*: ⋀*m*. *f*$$*m* ≠ *0* ⟹ *M* ≤ *m*
  proof−
    fix *m* **assume** *m*: *f*$$*m* ≠ *0*
    show *M* ≤ *m*
    proof (*cases m*<*0*)
      case *True* **with** *m N M-def* **show** *?thesis*
        using *allE*[*OF N*, *of nat* (−*m*) *False*] **by** *force*
    qed (*simp add*: *M-def*)
  qed
  have ¬ (∀ *k*::*nat*. *f*$$(*M* + *int k*) = *0*)
  proof
    assume *above0*: ∀ *k*::*nat*. *f*$$(*M* + *int k*) = *0*

have *f=0*
proof (*rule fls-zero-eqI*)
fix *n* **show** *f$$n = 0*
proof (*cases M ≤ n*)
case *True*
define *k* **where** *k = nat (n − M)*
from *True* **have** *n = M + int k* **by** (*simp add*: *k-def*)
with *above0* **show** *?thesis* **by** *simp*
next
case *False* **with** *M* **show** *?thesis* **by** *auto*
qed
qed
with *assms* **show** *False* **by** *fast*
qed
hence *ex-k*: ∃ *k*::*nat*. *f$$(M + int k) ≠ 0* **by** *fast*
define *k* **where** *k ≡ (LEAST k::nat. f$$(M + int k) ≠ 0)*
define *n* **where** *n ≡ M + int k*
from *k-def n-def* **have** *fn*: *f$$n ≠ 0* **using** *LeastI-ex*[*OF ex-k*] **by** *simp*
moreover have ∀ *m*. *f$$m ≠ 0 ⟶ n ≤ m*
proof (*clarify*)
fix *m* **assume** *m*: *f$$m ≠ 0*
with *M* **have** *M ≤ m* **by** *fast*
define *l* **where** *l = nat (m − M)*
from ‹*M ≤ m*› **have** *l*: *m = M + int l* **by** (*simp add*: *l-def*)
with *n-def m k-def l* **show** *n ≤ m*
using *Least-le*[*of λk. f$$(M + int k) ≠ 0 l*] **by** *auto*
qed
moreover have ⋀*n′*. *f$$n′ ≠ 0 ⟹ (∀ m. f$$m ≠ 0 ⟶ n′ ≤ m) ⟹ n′ = n*
proof−
fix *n′* :: *int*
assume *n′*: *f$$n′ ≠ 0* ∀ *m*. *f$$m ≠ 0 ⟶ n′ ≤ m*
from *n′(1) M* **have** *M ≤ n′* **by** *fast*
define *l* **where** *l = nat (n′ − M)*
from ‹*M ≤ n′*› **have** *l*: *n′ = M + int l* **by** (*simp add*: *l-def*)
with *n-def k-def n′ fn* **show** *n′ = n*
using *Least-le*[*of λk. f$$(M + int k) ≠ 0 l*] **by** *force*
qed
ultimately show *?thesis*
using *ex1I*[*of λn. f$$n ≠ 0 ∧ (∀ m. f$$m ≠ 0 ⟶ n ≤ m) n*] **by** *blast*
qed

definition *fls-subdegree* :: (*′a*::*zero*) *fls* ⇒ *int*
where *fls-subdegree f ≡ (if f = 0 then 0 else LEAST n::int. f$$n ≠ 0)*

lemma *fls-zero-subdegree* [*simp*]: *fls-subdegree 0 = 0*
by (*simp add*: *fls-subdegree-def*)

lemma *nth-fls-subdegree-nonzero* [*simp*]: *f ≠ 0 ⟹ f $$ fls-subdegree f ≠ 0*
using *Least1I*[*OF unique-fls-subdegree*] **by** (*simp add*: *fls-subdegree-def*)

lemma *nth-fls-subdegree-zero-iff*: $(f\ \$\$\ \textit{fls-subdegree}\ f = 0) \longleftrightarrow (f = 0)$
  using *nth-fls-subdegree-nonzero* **by** *auto*

lemma *fls-subdegree-leI*: $f\ \$\$\ n \neq 0 \Longrightarrow \textit{fls-subdegree}\ f \leq n$
  using *Least1-le*[*OF unique-fls-subdegree*]
  by (*auto simp*: *fls-subdegree-def*)

lemma *fls-subdegree-leI'*: $f\ \$\$\ n \neq 0 \Longrightarrow n \leq m \Longrightarrow \textit{fls-subdegree}\ f \leq m$
  using *fls-subdegree-leI* **by** *fastforce*

lemma *fls-eq0-below-subdegree* [*simp*]: $n < \textit{fls-subdegree}\ f \Longrightarrow f\ \$\$\ n = 0$
  using *fls-subdegree-leI* **by** *fastforce*

lemma *fls-subdegree-geI*: $f \neq 0 \Longrightarrow (\bigwedge k.\ k < n \Longrightarrow f\ \$\$\ k = 0) \Longrightarrow n \leq$ *fls-subdegree f*
  using *nth-fls-subdegree-nonzero* **by** *force*

lemma *fls-subdegree-ge0I*: $(\bigwedge k.\ k < 0 \Longrightarrow f\ \$\$\ k = 0) \Longrightarrow 0 \leq \textit{fls-subdegree}\ f$
  using *fls-subdegree-geI*[*of f 0*] **by** (*cases f=0*) *auto*

lemma *fls-subdegree-greaterI*:
  assumes $f \neq 0$ $\bigwedge k.\ k \leq n \Longrightarrow f\ \$\$\ k = 0$
  shows $n < \textit{fls-subdegree}\ f$
  using *assms*(*1*) *assms*(*2*)[*of fls-subdegree f*] *nth-fls-subdegree-nonzero*[*of f*]
  by *force*

lemma *fls-subdegree-eqI*: $f\ \$\$\ n \neq 0 \Longrightarrow (\bigwedge k.\ k < n \Longrightarrow f\ \$\$\ k = 0) \Longrightarrow$ *fls-subdegree f* $= n$
  using *fls-subdegree-leI fls-subdegree-geI*[*of f*]
  by *fastforce*

lemma *fls-delta-subdegree* [*simp*]:
  $b \neq 0 \Longrightarrow \textit{fls-subdegree}\ (\textit{Abs-fls}\ (\lambda n.\ \textit{if}\ n{=}a\ \textit{then}\ b\ \textit{else}\ 0)) = a$
  by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-delta0-subdegree*: $\textit{fls-subdegree}\ (\textit{Abs-fls}\ (\lambda n.\ \textit{if}\ n{=}0\ \textit{then}\ a\ \textit{else}\ 0)) = 0$
  by (*cases a=0*) *simp-all*

lemma *fls-one-subdegree* [*simp*]: *fls-subdegree* $1 = 0$
  by (*auto intro*: *fls-delta0-subdegree simp*: *one-fls-def*)

lemma *fls-const-subdegree* [*simp*]: *fls-subdegree* (*fls-const c*) $= 0$
  by (*cases c=0*) (*auto intro*: *fls-subdegree-eqI*)

lemma *fls-X-subdegree* [*simp*]: *fls-subdegree* (*fls-X*::'*a*::{*zero-neq-one*} *fls*) $= 1$
  by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-X-inv-subdegree* [*simp*]: *fls-subdegree* (*fls-X-inv*::'*a*::{*zero-neq-one*} *fls*)

= −*1*
by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-eq-above-subdegreeI*:
assumes $N \leq$ *fls-subdegree f* $N \leq$ *fls-subdegree g* $\forall k \geq N.$ *f* $$ *k* = *g* $$ *k*
shows *f* = *g*
proof (*rule fls-eqI*)
fix *n* **from** *assms* **show** *f* $$ *n* = *g* $$ *n* **by** (*cases n* < *N*) *auto*
qed

7.3 Shifting

7.3.1 Shift definition

definition *fls-shift* :: *int* ⇒ (*'a*::*zero*) *fls* ⇒ *'a fls*
where *fls-shift n f* ≡ *Abs-fls* (λ*k*. *f* $$ (*k*+*n*))
— Since the index set is unbounded in both directions, we can shift in either direction.

lemma *fls-shift-nth* [*simp*]: *fls-shift m f* $$ *n* = *f* $$ (*n*+*m*)
unfolding *fls-shift-def*
proof (*rule nth-Abs-fls-ex-lower-bound*)
obtain *K*::*int* **where** *K*: ∀ *n*<*K*. *f*$$*n* = *0* **by** (*elim fls-nth-vanishes-belowE*)
hence ∀ *n*<*K*−*m*. *f*$$(*n*+*m*) = *0* **by** *auto*
thus ∃ *N*. ∀ *n*<*N*. *f* $$ (*n* + *m*) = *0* **by** *fast*
qed

lemma *fls-shift-eq-iff*: (*fls-shift m f* = *fls-shift m g*) ⟷ (*f* = *g*)
proof (*rule iffI*, *rule fls-eqI*)
fix *k*
assume *1*: *fls-shift m f* = *fls-shift m g*
have *f* $$ *k* = *fls-shift m g* $$ (*k* − *m*) **by** (*simp add*: *1*[*symmetric*])
thus *f* $$ *k* = *g* $$ *k* **by** *simp*
qed (*intro fls-eqI*, *simp*)

lemma *fls-shift-0* [*simp*]: *fls-shift 0 f* = *f*
by (*intro fls-eqI*) *simp*

lemma *fls-shift-subdegree* [*simp*]:
f ≠ *0* ⟹ *fls-subdegree* (*fls-shift n f*) = *fls-subdegree f* − *n*
by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-shift-fls-shift* [*simp*]: *fls-shift m* (*fls-shift k f*) = *fls-shift* (*k*+*m*) *f*
by (*intro fls-eqI*) (*simp add*: *algebra-simps*)

lemma *fls-shift-fls-shift-reorder*:
fls-shift m (*fls-shift k f*) = *fls-shift k* (*fls-shift m f*)
using *fls-shift-fls-shift*[*of m k f*] *fls-shift-fls-shift*[*of k m f*] **by** (*simp add*: *add.commute*)

lemma *fls-shift-zero* [*simp*]: *fls-shift m 0* = *0*

by (*intro fls-zero-eqI*) *simp*

lemma *fls-shift-eq0-iff*: *fls-shift m f = 0 ⟷ f = 0*
using *fls-shift-eq-iff*[*of m f 0*] **by** *simp*

lemma *fls-shift-eq-1-iff*: *fls-shift n f = 1 ⟷ f = fls-shift* (*−n*) *1*
by (*metis add-minus-cancel fls-shift-eq-iff fls-shift-fls-shift*)

lemma *fls-shift-nonneg-subdegree*: *m ≤ fls-subdegree f ⟹ fls-subdegree* (*fls-shift m f*) *≥ 0*
by (*cases f=0*) (*auto intro*: *fls-subdegree-geI*)

lemma *fls-shift-delta*:
fls-shift m (*Abs-fls* (*λn. if n=a then b else 0*)) = *Abs-fls* (*λn. if n=a−m then b else 0*)
by (*intro fls-eqI*) *simp*

lemma *fls-shift-const*:
fls-shift m (*fls-const c*) = *Abs-fls* (*λn. if n=−m then c else 0*)
by (*intro fls-eqI*) *simp*

lemma *fls-shift-const-nth*:
fls-shift m (*fls-const c*) $$ *n* = (*if n=−m then c else 0*)
by (*simp add*: *fls-shift-const*)

lemma *fls-X-conv-shift-1*: *fls-X = fls-shift* (*−1*) *1*
by (*intro fls-eqI*) *simp*

lemma *fls-X-shift-to-one* [*simp*]: *fls-shift 1 fls-X = 1*
using *fls-shift-fls-shift*[*of −1 1 1*] **by** (*simp add*: *fls-X-conv-shift-1*)

lemma *fls-X-inv-conv-shift-1*: *fls-X-inv = fls-shift 1 1*
by (*intro fls-eqI*) *simp*

lemma *fls-X-inv-shift-to-one* [*simp*]: *fls-shift* (*−1*) *fls-X-inv = 1*
using *fls-shift-fls-shift*[*of 1 −1 1*] **by** (*simp add*: *fls-X-inv-conv-shift-1*)

lemma *fls-X-fls-X-inv-conv*:
fls-X = fls-shift (*−2*) *fls-X-inv fls-X-inv = fls-shift 2 fls-X*
by (*simp-all add*: *fls-X-conv-shift-1 fls-X-inv-conv-shift-1*)

7.3.2 Base factor

Similarly to the *unit-factor* for formal power series, we can decompose a formal Laurent series as a power of the implied variable times a series of subdegree 0. (See lemma *fls-base-factor-X-power-decompose*.) But we will call this something other *unit-factor* because it will not satisfy assumption *is-unit-unit-factor* of *semidom-divide-unit-factor*.

definition *fls-base-factor* :: (*'a*::*zero*) *fls* ⇒ *'a fls*
where *fls-base-factor-def*[*simp*]: *fls-base-factor f* = *fls-shift* (*fls-subdegree f*) *f*

lemma *fls-base-factor-nth*: *fls-base-factor f* $$ *n* = *f* $$ (*n* + *fls-subdegree f*)
by *simp*

lemma *fls-base-factor-nonzero* [*simp*]: *f* ≠ *0* ⟹ *fls-base-factor f* ≠ *0*
using *fls-nonzeroI*[*of fls-base-factor f 0*] **by** *simp*

lemma *fls-base-factor-subdegree* [*simp*]: *fls-subdegree* (*fls-base-factor f*) = *0*
by (*cases f*=*0*) *auto*

lemma *fls-base-factor-base* [*simp*]:
fls-base-factor f $$ *fls-subdegree* (*fls-base-factor f*) = *f* $$ *fls-subdegree f*
using *fls-base-factor-subdegree*[*of f*] **by** *simp*

lemma *fls-conv-base-factor-shift-subdegree*:
f = *fls-shift* (−*fls-subdegree f*) (*fls-base-factor f*)
by *simp*

lemma *fls-base-factor-idem*:
fls-base-factor (*fls-base-factor* (*f*::*'a*::*zero fls*)) = *fls-base-factor f*
using *fls-base-factor-subdegree*[*of f*] **by** *simp*

lemma *fls-base-factor-zero*: *fls-base-factor* (*0*::*'a*::*zero fls*) = *0*
by *simp*

lemma *fls-base-factor-zero-iff*: *fls-base-factor* (*f*::*'a*::*zero fls*) = *0* ⟷ *f* = *0*
proof
have *fls-shift* (−*fls-subdegree f*) (*fls-shift* (*fls-subdegree f*) *f*) = *f* **by** *simp*
thus *fls-base-factor f* = *0* ⟹ *f*=*0* **by** *simp*
qed *simp*

lemma *fls-base-factor-nth-0*: *f* ≠ *0* ⟹ *fls-base-factor f* $$ *0* ≠ *0*
by *simp*

lemma *fls-base-factor-one*: *fls-base-factor* (*1*::*'a*::{*zero*,*one*} *fls*) = *1*
by *simp*

lemma *fls-base-factor-const*: *fls-base-factor* (*fls-const c*) = *fls-const c*
by *simp*

lemma *fls-base-factor-delta*:
fls-base-factor (*Abs-fls* (λ*n*. *if n*=*a then c else 0*)) = *fls-const c*
by (*cases c*=*0*) (*auto intro*: *fls-eqI*)

lemma *fls-base-factor-X*: *fls-base-factor* (*fls-X*::*'a*::{*zero-neq-one*} *fls*) = *1*
by *simp*

lemma *fls-base-factor-X-inv*: *fls-base-factor* (*fls-X-inv*::'*a*::{*zero-neq-one*} *fls*) = *1*
by *simp*

lemma *fls-base-factor-shift* [*simp*]: *fls-base-factor* (*fls-shift n f*) = *fls-base-factor f*
by (*cases f=0*) *simp-all*

7.4 Conversion between formal power and Laurent series

7.4.1 Converting Laurent to power series

We can truncate a Laurent series at index 0 to create a power series, called the regular part.

lift-definition *fls-regpart* :: ('*a*::*zero*) *fls* ⇒ '*a fps*
is λf. *Abs-fps* (λn. *f* (*int n*))
.

lemma *fls-regpart-nth* [*simp*]: *fls-regpart f* \$ *n* = *f* \$\$ (*int n*)
by (*simp add*: *fls-regpart-def*)

lemma *fls-regpart-zero* [*simp*]: *fls-regpart 0* = *0*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-one* [*simp*]: *fls-regpart 1* = *1*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-Abs-fls*:
$\forall_\infty n$. *F* (− *int n*) = *0* ⟹ *fls-regpart* (*Abs-fls F*) = *Abs-fps* (λn. *F* (*int n*))
by (*intro fps-ext*) *auto*

lemma *fls-regpart-delta*:
fls-regpart (*Abs-fls* (λn. *if n=a then b else 0*)) =
(*if a* < *0 then 0 else Abs-fps* (λn. *if n=nat a then b else 0*))
by (*rule fps-ext*, *auto*)

lemma *fls-regpart-const* [*simp*]: *fls-regpart* (*fls-const c*) = *fps-const c*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-fls-X* [*simp*]: *fls-regpart fls-X* = *fps-X*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-fls-X-inv* [*simp*]: *fls-regpart fls-X-inv* = *0*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-eq0-imp-nonpos-subdegree*:
assumes *fls-regpart f* = *0*
shows *fls-subdegree f* ≤ *0*
proof (*cases f=0*)
case *False*
have *fls-subdegree f* ≥ *0* ⟹ *f* \$\$ *fls-subdegree f* = *0*

proof−
assume *fls-subdegree f* ≥ *0*
hence *f* $$ (*fls-subdegree f*) = (*fls-regpart f*) $ (*nat* (*fls-subdegree f*)) **by** *simp*
with *assms* **show** *f* $$ (*fls-subdegree f*) = *0* **by** *simp*
qed
with *False* **show** *?thesis* **by** *fastforce*
qed *simp*

lemma *fls-subdegree-lt-fls-regpart-subdegree*:
fls-subdegree f ≤ *int* (*subdegree* (*fls-regpart f*))
using *fls-subdegree-leI nth-subdegree-nonzero*[*of fls-regpart f*]
by (*cases* (*fls-regpart f*) = *0*)
(*simp-all add*: *fls-regpart-eq0-imp-nonpos-subdegree*)

lemma *fls-regpart-subdegree-conv*:
assumes *fls-subdegree f* ≥ *0*
shows *subdegree* (*fls-regpart f*) = *nat* (*fls-subdegree f*)
— This is the best we can do since if the subdegree is negative, we might still have the bad luck that the term at index 0 is equal to 0.
proof (*cases f*=*0*)
case *False* **with** *assms* **show** *?thesis* **by** (*intro subdegreeI*) *simp-all*
qed *simp*

lemma *fls-eq-conv-fps-eqI*:
assumes *0* ≤ *fls-subdegree f 0* ≤ *fls-subdegree g fls-regpart f* = *fls-regpart g*
shows *f* = *g*
proof (*rule fls-eq-above-subdegreeI*, *rule assms*(*1*), *rule assms*(*2*), *clarify*)
fix *k*::*int* **assume** *0* ≤ *k*
with *assms*(*3*) **show** *f* $$ *k* = *g* $$ *k*
using *fls-regpart-nth*[*of f nat k*] *fls-regpart-nth*[*of g*] **by** *simp*
qed

lemma *fls-regpart-shift-conv-fps-shift*:
m ≥ *0* ⟹ *fls-regpart* (*fls-shift m f*) = *fps-shift* (*nat m*) (*fls-regpart f*)
by (*intro fps-ext*) *simp-all*

lemma *fps-shift-fls-regpart-conv-fls-shift*:
fps-shift m (*fls-regpart f*) = *fls-regpart* (*fls-shift m f*)
by (*intro fps-ext*) *simp-all*

lemma *fps-unit-factor-fls-regpart*:
fls-subdegree f ≥ *0* ⟹ *unit-factor* (*fls-regpart f*) = *fls-regpart* (*fls-base-factor f*)
by (*auto intro*: *fps-ext simp*: *fls-regpart-subdegree-conv*)

The terms below the zeroth form a polynomial in the inverse of the implied variable, called the principle part.

lift-definition *fls-prpart* :: (*'a*::*zero*) *fls* ⇒ *'a poly*
is *λf*. *Abs-poly* (*λn*. *if n* = *0 then 0 else f* (− *int n*))
.

lemma *fls-prpart-coeff* [*simp*]: *coeff* (*fls-prpart f*) *n* = (*if n* = *0 then 0 else f* $$ (− *int n*))
proof−
  have {*x*. (*if x* = *0 then 0 else f* $$ − *int x*) ≠ *0*} ⊆ {*x*. *f* $$ − *int x* ≠ *0*}
    by *auto*
  hence *finite* {*x*. (*if x* = *0 then 0 else f* $$ − *int x*) ≠ *0*}
    using *fls-finite-nonzero-neg-nth*[*of f*] **by** (*simp add*: *rev-finite-subset*)
  hence *coeff* (*fls-prpart f*) = (λ*n*. *if n* = *0 then 0 else f* $$ (− *int n*))
    using *Abs-poly-inverse*[*OF CollectI*, *OF iffD2*, *OF eventually-cofinite*]
    by (*simp add*: *fls-prpart-def*)
  thus *?thesis* **by** *simp*
qed

lemma *fls-prpart-eq0-iff*: (*fls-prpart f* = *0*) ⟷ (*fls-subdegree f* ≥ *0*)
proof
  assume *1*: *fls-prpart f* = *0*
  show *fls-subdegree f* ≥ *0*
  proof (*intro fls-subdegree-ge0I*)
    fix *k*::*int* **assume** *k* < *0*
    with *1* **show** *f* $$ *k* = *0* **using** *fls-prpart-coeff*[*of f nat* (−*k*)] **by** *simp*
  qed
qed (*intro poly-eqI*, *simp*)

lemma *fls-prpart0* [*simp*]: *fls-prpart 0* = *0*
  by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-prpart-one* [*simp*]: *fls-prpart 1* = *0*
  by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-prpart-delta*:
  fls-prpart (*Abs-fls* (λ*n*. *if n*=*a then b else 0*)) =
    (*if a*<*0 then Poly* (*replicate* (*nat* (−*a*)) *0* @ [*b*]) *else 0*)
  by (*intro poly-eqI*) (*auto simp*: *nth-default-def nth-append*)

lemma *fls-prpart-const* [*simp*]: *fls-prpart* (*fls-const c*) = *0*
  by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-prpart-X* [*simp*]: *fls-prpart fls-X* = *0*
  by (*intro poly-eqI*) *simp*

lemma *fls-prpart-X-inv*: *fls-prpart fls-X-inv* = [:*0*,*1*:]
proof (*intro poly-eqI*)
  fix *n* **show** *coeff* (*fls-prpart fls-X-inv*) *n* = *coeff* [:*0*,*1*:] *n*
  proof (*cases n*)
    case (*Suc i*) **thus** *?thesis* **by** (*cases i*) *simp-all*
  qed *simp*
qed

lemma *degree-fls-prpart* [*simp*]:
degree (*fls-prpart f*) = *nat* (−*fls-subdegree f*)
proof (*cases f=0*)
case *False* **show** *?thesis* **unfolding** *degree-def*
proof (*intro Least-equality*)
fix *N* **assume** *N*: ∀ *i*>*N*. *coeff* (*fls-prpart f*) *i* = *0*
have ∀ *i* < −*int N*. *f* $$ *i* = *0*
proof *clarify*
fix *i* **assume** *i*: *i* < −*int N*
hence *nat* (−*i*) > *N* **by** *simp*
with *N i* **show** *f* $$ *i* = *0* **using** *fls-prpart-coeff*[*of f nat* (−*i*)] **by** *auto*
qed
with *False* **have** *fls-subdegree f* ≥ −*int N* **using** *fls-subdegree-geI* **by** *auto*
thus *nat* (− *fls-subdegree f*) ≤ *N* **by** *simp*
qed *auto*
qed *simp*

lemma *fls-prpart-shift*:
assumes *m* ≤ *0*
shows *fls-prpart* (*fls-shift m f*) = *pCons 0* (*poly-shift* (*Suc* (*nat* (−*m*))) (*fls-prpart f*))
proof (*intro poly-eqI*)
fix *n*
define *LHS RHS*
where *LHS* ≡ *fls-prpart* (*fls-shift m f*)
and *RHS* ≡ *pCons 0* (*poly-shift* (*Suc* (*nat* (−*m*))) (*fls-prpart f*))
show *coeff LHS n* = *coeff RHS n*
proof (*cases n*)
case (*Suc k*)
from *assms* **have** *1*: −*int* (*Suc k* + *nat* (−*m*)) = −*int* (*Suc k*) + *m* **by** *simp*
have *coeff RHS n* = *f* $$ (−*int* (*Suc k*) + *m*)
using *arg-cong*[*OF 1*, *of* ($$) *f*] **by** (*simp add*: *Suc RHS-def coeff-poly-shift*)
with *Suc* **show** *?thesis* **by** (*simp add*: *LHS-def*)
qed (*simp add*: *LHS-def RHS-def*)
qed

lemma *fls-prpart-base-factor*: *fls-prpart* (*fls-base-factor f*) = *0*
using *fls-base-factor-subdegree*[*of f*] **by** (*simp add*: *fls-prpart-eq0-iff*)

The essential data of a formal Laurant series resides from the subdegree up.

abbreviation *fls-base-factor-to-fps* :: (′*a*::*zero*) *fls* ⇒ ′*a fps*
where *fls-base-factor-to-fps f* ≡ *fls-regpart* (*fls-base-factor f*)

lemma *fls-base-factor-to-fps-conv-fps-shift*:
assumes *fls-subdegree f* ≥ *0*
shows *fls-base-factor-to-fps f* = *fps-shift* (*nat* (*fls-subdegree f*)) (*fls-regpart f*)
by (*simp add*: *assms fls-regpart-shift-conv-fps-shift*)

lemma *fls-base-factor-to-fps-nth*:

fls-base-factor-to-fps f $ *n* = *f* $$ (*fls-subdegree f* + *int n*)
by (*simp add*: *algebra-simps*)

lemma *fls-base-factor-to-fps-base*: *f* ≠ *0* ⟹ *fls-base-factor-to-fps f* $ *0* ≠ *0*
by *simp*

lemma *fls-base-factor-to-fps-nonzero*: *f* ≠ *0* ⟹ *fls-base-factor-to-fps f* ≠ *0*
using *fps-nonzeroI*[*of fls-base-factor-to-fps f 0*] *fls-base-factor-to-fps-base* **by** *simp*

lemma *fls-base-factor-to-fps-subdegree* [*simp*]: *subdegree* (*fls-base-factor-to-fps f*) = *0*
by (*cases f=0*) *auto*

lemma *fls-base-factor-to-fps-trivial*:
fls-subdegree f = *0* ⟹ *fls-base-factor-to-fps f* = *fls-regpart f*
by *simp*

lemma *fls-base-factor-to-fps-zero*: *fls-base-factor-to-fps 0* = *0*
by *simp*

lemma *fls-base-factor-to-fps-one*: *fls-base-factor-to-fps 1* = *1*
by *simp*

lemma *fls-base-factor-to-fps-delta*:
fls-base-factor-to-fps (*Abs-fls* (λ*n. if n=a then c else 0*)) = *fps-const c*
using *fls-base-factor-delta*[*of a c*] **by** *simp*

lemma *fls-base-factor-to-fps-const*:
fls-base-factor-to-fps (*fls-const c*) = *fps-const c*
by *simp*

lemma *fls-base-factor-to-fps-X*:
fls-base-factor-to-fps (*fls-X*::′*a*::{*zero-neq-one*} *fls*) = *1*
by *simp*

lemma *fls-base-factor-to-fps-X-inv*:
fls-base-factor-to-fps (*fls-X-inv*::′*a*::{*zero-neq-one*} *fls*) = *1*
by *simp*

lemma *fls-base-factor-to-fps-shift*:
fls-base-factor-to-fps (*fls-shift m f*) = *fls-base-factor-to-fps f*
using *fls-base-factor-shift*[*of m f*] **by** *simp*

lemma *fls-base-factor-to-fps-base-factor*:
fls-base-factor-to-fps (*fls-base-factor f*) = *fls-base-factor-to-fps f*
using *fls-base-factor-to-fps-shift* **by** *simp*

lemma *fps-unit-factor-fls-base-factor*:
unit-factor (*fls-base-factor-to-fps f*) = *fls-base-factor-to-fps f*

using *fls-base-factor-to-fps-subdegree*[*of f*] **by** *simp*

7.4.2 Converting power to Laurent series

We can extend a power series by 0s below to create a Laurent series.

definition *fps-to-fls* :: (*'a*::*zero*) *fps* ⇒ *'a fls*
where *fps-to-fls f* ≡ *Abs-fls* (λ*k*::*int*. *if k<0 then 0 else f* $ (*nat k*))

lemma *fps-to-fls-nth* [*simp*]:
(*fps-to-fls f*) $$ *n* = (*if n* < *0 then 0 else f*$(*nat n*))
using *nth-Abs-fls-lower-bound*[*of 0* (λ*k*::*int*. *if k<0 then 0 else f* $ (*nat k*))]
unfolding *fps-to-fls-def*
by *simp*

lemma *fps-to-fls-eq-imp-fps-eq*:
assumes *fps-to-fls f* = *fps-to-fls g*
shows *f* = *g*
proof (*intro fps-ext*)
fix *n*
have *f* $ *n* = *fps-to-fls g* $$ *int n* **by** (*simp add*: *assms*[*symmetric*])
thus *f* $ *n* = *g* $ *n* **by** *simp*
qed

lemma *fps-to-fls-eq-iff* [*simp*]: *fps-to-fls f* = *fps-to-fls g* ⟷ *f* = *g*
using *fps-to-fls-eq-imp-fps-eq* **by** *blast*

lemma *fps-zero-to-fls* [*simp*]: *fps-to-fls 0* = *0*
by (*intro fls-zero-eqI*) *simp*

lemma *fps-to-fls-nonzeroI*: *f* ≠ *0* ⟹ *fps-to-fls f* ≠ *0*
using *fps-to-fls-eq-imp-fps-eq*[*of f 0*] **by** *auto*

lemma *fps-one-to-fls* [*simp*]: *fps-to-fls 1* = *1*
by (*intro fls-eqI*) *simp*

lemma *fps-to-fls-Abs-fps*:
fps-to-fls (*Abs-fps F*) = *Abs-fls* (λ*n*. *if n<0 then 0 else F* (*nat n*))
using *nth-Abs-fls-lower-bound*[*of 0* (λ*n*::*int*. *if n<0 then 0 else F* (*nat n*))]
by (*intro fls-eqI*) *simp*

lemma *fps-delta-to-fls*:
fps-to-fls (*Abs-fps* (λ*n*. *if n=a then b else 0*)) = *Abs-fls* (λ*n*. *if n=int a then b else 0*)
using *fls-eqI*[*of - Abs-fls* (λ*n*. *if n=int a then b else 0*)] **by** *force*

lemma *fps-const-to-fls* [*simp*]: *fps-to-fls* (*fps-const c*) = *fls-const c*
by (*intro fls-eqI*) *simp*

lemma *fps-X-to-fls* [*simp*]: *fps-to-fls fps-X* = *fls-X* by (*fastforce intro*: *fls-eqI*)

lemma *fps-to-fls-eq-0-iff* [*simp*]: (*fps-to-fls f* = *0*) ⟷ (*f*=*0*)
using *fps-to-fls-nonzeroI* **by** *auto*

lemma *fps-to-fls-eq-1-iff* [*simp*]: *fps-to-fls f* = *1* ⟷ *f* = *1*
using *fps-to-fls-eq-iff* **by** *fastforce*

lemma *fls-subdegree-fls-to-fps-gt0*: *fls-subdegree* (*fps-to-fls f*) ≥ *0*
proof (*cases f*=*0*)
case *False* **show** *?thesis*
proof (*rule fls-subdegree-geI*, *rule fls-nonzeroI*)
from *False* **show** *fps-to-fls f* $$ *int* (*subdegree f*) ≠ *0*
by *simp*
qed *simp*
qed *simp*

lemma *fls-subdegree-fls-to-fps*: *fls-subdegree* (*fps-to-fls f*) = *int* (*subdegree f*)
proof (*cases f*=*0*)
case *False*
have *subdegree f* = *nat* (*fls-subdegree* (*fps-to-fls f*))
proof (*rule subdegreeI*)
from *False* **show** *f* $ (*nat* (*fls-subdegree* (*fps-to-fls f*))) ≠ *0*
using *fls-subdegree-fls-to-fps-gt0*[*of f*] *nth-fls-subdegree-nonzero*[*of fps-to-fls f*]
fps-to-fls-nonzeroI[*of f*]
by *simp*
next
fix *k* **assume** *k*: *k* < *nat* (*fls-subdegree* (*fps-to-fls f*))
thus *f* $ *k* = *0*
using *fls-eq0-below-subdegree*[*of int k fps-to-fls f*] **by** *simp*
qed
thus *?thesis* **by** (*simp add*: *fls-subdegree-fls-to-fps-gt0*)
qed *simp*

lemma *fps-shift-to-fls* [*simp*]:
n ≤ *subdegree f* ⟹ *fps-to-fls* (*fps-shift n f*) = *fls-shift* (*int n*) (*fps-to-fls f*)
by (*auto intro*: *fls-eqI simp*: *nat-add-distrib nth-less-subdegree-zero*)

lemma *fls-base-factor-fps-to-fls*: *fls-base-factor* (*fps-to-fls f*) = *fps-to-fls* (*unit-factor f*)
using *nth-less-subdegree-zero*[*of - f*]
by (*auto intro*: *fls-eqI simp*: *fls-subdegree-fls-to-fps nat-add-distrib*)

lemma *fls-regpart-to-fls-trivial* [*simp*]:
fls-subdegree f ≥ *0* ⟹ *fps-to-fls* (*fls-regpart f*) = *f*
by (*intro fls-eqI*) *simp*

lemma *fls-regpart-fps-trivial* [*simp*]: *fls-regpart* (*fps-to-fls f*) = *f*
by (*intro fps-ext*) *simp*

lemma *fps-to-fls-base-factor-to-fps*:
fps-to-fls (*fls-base-factor-to-fps f*) = *fls-base-factor f*
by (*intro fls-eqI*) *simp*

lemma *fls-conv-base-factor-to-fps-shift-subdegree*:
f = *fls-shift* (−*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*))
using *fps-to-fls-base-factor-to-fps*[*of f*] *fps-to-fls-base-factor-to-fps*[*of f*] **by** *simp*

lemma *fls-base-factor-to-fps-to-fls*:
fls-base-factor-to-fps (*fps-to-fls f*) = *unit-factor f*
using *fls-base-factor-fps-to-fls*[*of f*] *fls-regpart-fps-trivial*[*of unit-factor f*]
by *simp*

lemma *fls-as-fps*:
fixes *f* :: ′*a* :: *zero fls* **and** *n* :: *int*
assumes *n*: *n* ≥ −*fls-subdegree f*
obtains *f*′ **where** *f* = *fls-shift n* (*fps-to-fls f*′)
proof −
have *fls-subdegree* (*fls-shift* (− *n*) *f*) ≥ *0*
by (*rule fls-shift-nonneg-subdegree*) (*use n* **in** *simp*)
hence *f* = *fls-shift n* (*fps-to-fls* (*fls-regpart* (*fls-shift* (−*n*) *f*)))
by (*subst fls-regpart-to-fls-trivial*) *simp-all*
thus *?thesis*
by (*rule that*)
qed

lemma *fls-as-fps*′:
fixes *f* :: ′*a* :: *zero fls* **and** *n* :: *int*
assumes *n*: *n* ≥ −*fls-subdegree f*
shows ∃*f*′. *f* = *fls-shift n* (*fps-to-fls f*′)
using *fls-as-fps*[*OF assms*] **by** *metis*

abbreviation
fls-regpart-as-fls f ≡ *fps-to-fls* (*fls-regpart f*)
abbreviation
fls-prpart-as-fls f ≡
fls-shift (−*fls-subdegree f*) (*fps-to-fls* (*fps-of-poly* (*reflect-poly* (*fls-prpart f*))))

lemma *fls-regpart-as-fls-nth*:
fls-regpart-as-fls f $$ *n* = (*if n* < *0 then 0 else f* $$ *n*)
by *simp*

lemma *fls-regpart-idem*:
fls-regpart (*fls-regpart-as-fls f*) = *fls-regpart f*
by *simp*

lemma *fls-prpart-as-fls-nth*:
fls-prpart-as-fls f $$ *n* = (*if n* < *0 then f* $$ *n else 0*)

proof (*cases n* < *fls-subdegree f n* < *0 rule*: *case-split*[*case-product case-split*])
  case *False-True*
    hence *nat* (−*fls-subdegree f*) − *nat* (*n* − *fls-subdegree f*) = *nat* (−*n*) **by** *auto*
    with *False-True* **show** *?thesis*
      using *coeff-reflect-poly*[*of fls-prpart f nat* (*n* − *fls-subdegree f*)] **by** *auto*
  next
    case *False-False* **thus** *?thesis*
      using *coeff-reflect-poly*[*of fls-prpart f nat* (*n* − *fls-subdegree f*)] **by** *auto*
qed *simp-all*

lemma *fls-prpart-idem* [*simp*]: *fls-prpart* (*fls-prpart-as-fls f*) = *fls-prpart f*
  using *fls-prpart-as-fls-nth*[*of f*] **by** (*intro poly-eqI*) *simp*

lemma *fls-regpart-prpart*: *fls-regpart* (*fls-prpart-as-fls f*) = *0*
  using *fls-prpart-as-fls-nth*[*of f*] **by** (*intro fps-ext*) *simp*

lemma *fls-prpart-regpart*: *fls-prpart* (*fls-regpart-as-fls f*) = *0*
  by (*intro poly-eqI*) *simp*

7.5 Algebraic structures

7.5.1 Addition

instantiation *fls* :: (*monoid-add*) *plus*
begin
  lift-definition *plus-fls* :: *'a fls* ⇒ *'a fls* ⇒ *'a fls* **is** λ*f g n*. *f n* + *g n*
  proof−
    fix *f f'* :: *int* ⇒ *'a*
    assume $\forall_\infty$*n*. *f* (− *int n*) = *0* $\forall_\infty$*n*. *f'* (− *int n*) = *0*
    from *this* **obtain** *N N'* **where** ∀ *n*>*N*. *f* (−*int n*) = *0* ∀ *n*>*N'*. *f'* (−*int n*) =
0
      by (*auto simp*: *MOST-nat*)
    hence ∀ *n* > *max N N'*. *f* (−*int n*) + *f'* (−*int n*) = *0* **by** *auto*
    hence ∃ *K*. ∀ *n*>*K*. *f* (−*int n*) + *f'* (−*int n*) = *0* **by** *fast*
    thus $\forall_\infty$*n*. *f* (− *int n*) + *f'* (−*int n*) = *0* **by** (*simp add*: *MOST-nat*)
  qed
  instance ..
end

lemma *fls-plus-nth* [*simp*]: (*f* + *g*) $$ *n* = *f* $$ *n* + *g* $$ *n*
  by *transfer simp*

lemma *fls-plus-const*: *fls-const x* + *fls-const y* = *fls-const* (*x*+*y*)
  by (*intro fls-eqI*) *simp*

lemma *fls-plus-subdegree*:
  f + *g* ≠ *0* ⟹ *fls-subdegree* (*f* + *g*) ≥ *min* (*fls-subdegree f*) (*fls-subdegree g*)
  by (*auto intro*: *fls-subdegree-geI*)

lemma *fls-shift-plus* [*simp*]:

fls-shift m (*f* + *g*) = (*fls-shift m f*) + (*fls-shift m g*)
by (*intro fls-eqI*) *simp*

lemma *fls-regpart-plus* [*simp*]: *fls-regpart* (*f* + *g*) = *fls-regpart f* + *fls-regpart g*
by (*intro fps-ext*) *simp*

lemma *fls-prpart-plus* [*simp*] : *fls-prpart* (*f* + *g*) = *fls-prpart f* + *fls-prpart g*
by (*intro poly-eqI*) *simp*

lemma *fls-decompose-reg-pr-parts*:
fixes *f* :: *'a* :: *monoid-add fls*
defines $R \equiv$ *fls-regpart-as-fls f*
and $P \equiv$ *fls-prpart-as-fls f*
shows $f = P + R$
and $f = R + P$
using *fls-prpart-as-fls-nth*[*of f*]
by (*auto intro*: *fls-eqI simp add*: *assms*)

lemma *fps-to-fls-plus* [*simp*]: *fps-to-fls* (*f* + *g*) = *fps-to-fls f* + *fps-to-fls g*
by (*intro fls-eqI*) *simp*

instance *fls* :: (*monoid-add*) *monoid-add*
proof
fix *a b c* :: *'a fls*
show $a + b + c = a + (b + c)$ **by** *transfer* (*simp add*: *add.assoc*)
show $0 + a = a$ **by** *transfer simp*
show $a + 0 = a$ **by** *transfer simp*
qed

instance *fls* :: (*comm-monoid-add*) *comm-monoid-add*
by (*standard*, *transfer*, *auto simp*: *add.commute*)

7.5.2 Subtraction and negatives

instantiation *fls* :: (*group-add*) *minus*
begin
lift-definition *minus-fls* :: *'a fls* ⇒ *'a fls* ⇒ *'a fls* **is** $\lambda f\ g\ n.\ f\ n - g\ n$
proof−
fix $f\ f'$ :: *int* ⇒ *'a*
assume $\forall_\infty n.\ f\ (-\ int\ n) = 0$ $\forall_\infty n.\ f'\ (-\ int\ n) = 0$
from *this* **obtain** $N\ N'$ **where** $\forall\, n{>}N.\ f\ (-int\ n) = 0$ $\forall\, n{>}N'.\ f'\ (-int\ n) =$
0
by (*auto simp*: *MOST-nat*)
hence $\forall\, n > max\ N\ N'.\ f\ (-int\ n) - f'\ (-int\ n) = 0$ **by** *auto*
hence $\exists K.\ \forall\, n{>}K.\ f\ (-int\ n) - f'\ (-int\ n) = 0$ **by** *fast*
thus $\forall_\infty n.\ f\ (-\ int\ n) - f'\ (-int\ n) = 0$ **by** (*simp add*: *MOST-nat*)
qed
instance ..
end

lemma *fls-minus-nth* [*simp*]: (*f* − *g*) $$ *n* = *f* $$ *n* − *g* $$ *n*
  by *transfer simp*

lemma *fls-minus-const*: *fls-const x* − *fls-const y* = *fls-const* (*x*−*y*)
  by (*intro fls-eqI*) *simp*

lemma *fls-subdegree-minus*:
  f − *g* ≠ *0* ⟹ *fls-subdegree* (*f* − *g*) ≥ *min* (*fls-subdegree f*) (*fls-subdegree g*)
  by (*intro fls-subdegree-geI*) *simp-all*

lemma *fls-shift-minus* [*simp*]: *fls-shift m* (*f* − *g*) = (*fls-shift m f*) − (*fls-shift m g*)
  by (*auto intro*: *fls-eqI*)

lemma *fls-regpart-minus* [*simp*]: *fls-regpart* (*f* − *g*) = *fls-regpart f* − *fls-regpart g*
  by (*intro fps-ext*) *simp*

lemma *fls-prpart-minus* [*simp*] : *fls-prpart* (*f* − *g*) = *fls-prpart f* − *fls-prpart g*
  by (*intro poly-eqI*) *simp*

lemma *fps-to-fls-minus* [*simp*]: *fps-to-fls* (*f* − *g*) = *fps-to-fls f* − *fps-to-fls g*
  by (*intro fls-eqI*) *simp*

instantiation *fls* :: (*group-add*) *uminus*
begin
  lift-definition *uminus-fls* :: *'a fls* ⇒ *'a fls* **is** *λf n.* − *f n*
  proof−
    fix *f* :: *int* ⇒ *'a* **assume** $\forall_\infty$*n. f* (− *int n*) = *0*
    from *this* **obtain** *N* **where** ∀ *n*>*N*. *f* (−*int n*) = *0*
      by (*auto simp*: *MOST-nat*)
    hence ∀ *n*>*N*. − *f* (−*int n*) = *0* **by** *auto*
    hence ∃ *K*. ∀ *n*>*K*. − *f* (−*int n*) = *0* **by** *fast*
    thus $\forall_\infty$*n.* − *f* (− *int n*) = *0* **by** (*simp add*: *MOST-nat*)
  qed
  instance ..
end

lemma *fls-uminus-nth* [*simp*]: (−*f*) $$ *n* = − (*f* $$ *n*)
  by *transfer simp*

lemma *fls-const-uminus*[*simp*]: *fls-const* (−*x*) = −*fls-const x*
  by (*intro fls-eqI*) *simp*

lemma *fls-shift-uminus* [*simp*]: *fls-shift m* (− *f*) = − (*fls-shift m f*)
  by (*auto intro*: *fls-eqI*)

lemma *fls-regpart-uminus* [*simp*]: *fls-regpart* (− *f*) = − *fls-regpart f*
  by (*intro fps-ext*) *simp*

lemma *fls-prpart-uminus* [*simp*] : *fls-prpart* (− *f*) = − *fls-prpart f*
  by (*intro poly-eqI*) *simp*

lemma *fps-to-fls-uminus* [*simp*]: *fps-to-fls* (− *f*) = − *fps-to-fls f*
  by (*intro fls-eqI*) *simp*

instance *fls* :: (*group-add*) *group-add*
proof
  fix *a b* :: ′*a fls*
  show − *a* + *a* = *0* **by** *transfer simp*
  show *a* + − *b* = *a* − *b* **by** *transfer simp*
qed

instance *fls* :: (*ab-group-add*) *ab-group-add*
proof
  fix *a b* :: ′*a fls*
  show − *a* + *a* = *0* **by** *transfer simp*
  show *a* − *b* = *a* + − *b* **by** *transfer simp*
qed

lemma *fls-uminus-subdegree* [*simp*]: *fls-subdegree* (−*f*) = *fls-subdegree f*
  by (*cases f*=*0*) (*auto intro*: *fls-subdegree-eqI*)

lemma *fls-subdegree-minus-sym*: *fls-subdegree* (*g* − *f*) = *fls-subdegree* (*f* − *g*)
  using *fls-uminus-subdegree*[*of g*−*f*] **by** (*simp add*: *algebra-simps*)

lemma *fls-regpart-sub-prpart*: *fls-regpart* (*f* − *fls-prpart-as-fls f*) = *fls-regpart f*
  using *fls-decompose-reg-pr-parts*(*2*)[*of f*]
        add-diff-cancel[*of fls-regpart-as-fls f fls-prpart-as-fls f*]
  by *simp*

lemma *fls-prpart-sub-regpart*: *fls-prpart* (*f* − *fls-regpart-as-fls f*) = *fls-prpart f*
  using *fls-decompose-reg-pr-parts*(*1*)[*of f*]
        add-diff-cancel[*of fls-prpart-as-fls f fls-regpart-as-fls f*]
  by *simp*

7.5.3 Multiplication

instantiation *fls* :: ({*comm-monoid-add*, *times*}) *times*
begin
  definition *fls-times-def*:
    (∗) = (λ*f g*.
      fls-shift
        (− (*fls-subdegree f* + *fls-subdegree g*))
        (*fps-to-fls* (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*))
)
  instance ..
end

lemma *fls-times-nth-eq0*: $n <$ *fls-subdegree* f + *fls-subdegree* $g \Longrightarrow (f * g)$ $$ $n = 0$
  by (*simp add*: *fls-times-def*)

lemma *fls-times-nth*:
  fixes *f df g dg*
  defines $df \equiv$ *fls-subdegree* f **and** $dg \equiv$ *fls-subdegree* g
  shows $(f * g)$ $$ $n = (\sum i{=}df + dg..n.\ f$ $$ $(i - dg) * g$ $$ $(dg + n - i))$
  and $(f * g)$ $$ $n = (\sum i{=}df..n - dg.\ f$ $$ $i * g$ $$ $(n - i))$
  and $(f * g)$ $$ $n = (\sum i{=}dg..n - df.\ f$ $$ $(df + i - dg) * g$ $$ $(dg + n - df - i))$
  and $(f * g)$ $$ $n = (\sum i{=}0..n - (df + dg).\ f$ $$ $(df + i) * g$ $$ $(n - df - i))$
proof−

  define *dfg* **where** $dfg \equiv df + dg$

  show *4*: $(f * g)$ $$ $n = (\sum i{=}0..n - dfg.\ f$ $$ $(df + i) * g$ $$ $(n - df - i))$
  proof (*cases* $n < dfg$)
    case *False*
    from *False assms* **have**
      $(f * g)$ $$ $n =$
        $(\sum i = 0..nat\ (n - dfg).\ f$ $$ $(df + int\ i) * g$ $$ $(dg + int\ (nat\ (n - dfg) - i)))$
      using *fps-mult-nth*[*of fls-base-factor-to-fps f fls-base-factor-to-fps g*]
        fls-base-factor-to-fps-nth[*of f*]
        fls-base-factor-to-fps-nth[*of g*]
      by (*simp add*: *dfg-def fls-times-def algebra-simps*)
    moreover from *False* **have** *index*:
      $\bigwedge i.\ i \in \{0..nat\ (n - dfg)\} \Longrightarrow dg + int\ (nat\ (n - dfg) - i) = n - df - int\ i$
      by (*auto simp*: *dfg-def*)
    ultimately have
      $(f * g)$ $$ $n = (\sum i{=}0..nat\ (n - dfg).\ f$ $$ $(df + int\ i) * g$ $$ $(n - df - int\ i))$
      by (*simp del*: *of-nat-diff*)
    moreover have
      $(\sum i{=}0..nat\ (n - dfg).\ f$ $$ $(df + int\ i) * g$ $$ $(n - df - int\ i)) =$
        $(\sum i{=}0..n - dfg.\ f$ $$ $(df + i) * g$ $$ $(n - df - i))$
    proof (*intro sum.reindex-cong*)
      show *inj-on nat* $\{0..n - dfg\}$ **by** *standard auto*
      show $\{0..nat\ (n - dfg)\} = nat$ ‘ $\{0..n - dfg\}$
      proof
        show $\{0..nat\ (n - dfg)\} \subseteq nat$ ‘ $\{0..n - dfg\}$
        proof
          fix i **assume** $i \in \{0..nat\ (n - dfg)\}$
          hence i: $i \geq 0$ $i \leq nat\ (n - dfg)$ **by** *auto*
          with *False* **have** $int\ i \geq 0$ $int\ i \leq n - dfg$ **by** *auto*

hence *int i* ∈ {*0*..*n* − *dfg*} **by** *simp*
moreover from *i*(*1*) **have** *i* = *nat* (*int i*) **by** *simp*
ultimately show *i* ∈ *nat* ‘ {*0*..*n* − *dfg*} **by** *fast*
qed
qed (*auto simp*: *False*)
qed (*simp add*: *False*)
ultimately show (*f* ∗ *g*) $$ *n* = (∑ *i*=*0*..*n* − *dfg*. *f* $$ (*df* + *i*) ∗ *g* $$ (*n* − *df* − *i*))
by *simp*
qed (*simp add*: *fls-times-nth-eq0 assms dfg-def*)

have
(∑ *i*=*dfg*..*n*. *f* $$ (*i* − *dg*) ∗ *g* $$ (*dg* + *n* − *i*)) =
(∑ *i*=*0*..*n* − *dfg*. *f* $$ (*df* + *i*) ∗ *g* $$ (*n* − *df* − *i*))
proof (*intro sum.reindex-cong*)
define *T* **where** *T* ≡ λ*i*. *i* + *dfg*
show *inj-on T* {*0*..*n* − *dfg*} **by** *standard* (*simp add*: *T-def*)
qed (*simp-all add*: *dfg-def algebra-simps*)
with *4* **show** *1*: (*f* ∗ *g*) $$ *n* = (∑ *i*=*dfg*..*n*. *f* $$ (*i* − *dg*) ∗ *g* $$ (*dg* + *n* − *i*))
by *simp*

have
(∑ *i*=*dfg*..*n*. *f* $$ (*i* − *dg*) ∗ *g* $$ (*dg* + *n* − *i*)) = (∑ *i*=*df*..*n* − *dg*. *f* $$ *i* ∗ *g* $$ (*n* − *i*))
proof (*intro sum.reindex-cong*)
define *T* **where** *T* ≡ λ*i*. *i* + *dg*
show *inj-on T* {*df*..*n* − *dg*} **by** *standard* (*simp add*: *T-def*)
qed (*auto simp*: *dfg-def*)
with *1* **show** (*f* ∗ *g*) $$ *n* = (∑ *i*=*df*..*n* − *dg*. *f* $$ *i* ∗ *g* $$ (*n* − *i*))
by *simp*

have
(∑ *i*=*dfg*..*n*. *f* $$ (*i* − *dg*) ∗ *g* $$ (*dg* + *n* − *i*)) =
(∑ *i*=*dg*..*n* − *df*. *f* $$ (*df* + *i* − *dg*) ∗ *g* $$ (*dg* + *n* − *df* − *i*))
proof (*intro sum.reindex-cong*)
define *T* **where** *T* ≡ λ*i*. *i* + *df*
show *inj-on T* {*dg*..*n* − *df*} **by** *standard* (*simp add*: *T-def*)
qed (*simp-all add*: *dfg-def algebra-simps*)
with *1* **show** (*f* ∗ *g*) $$ *n* = (∑ *i*=*dg*..*n* − *df*. *f* $$ (*df* + *i* − *dg*) ∗ *g* $$ (*dg* + *n* − *df* − *i*))
by *simp*

qed

lemma *fls-times-base* [*simp*]:
(*f* ∗ *g*) $$ (*fls-subdegree f* + *fls-subdegree g*) =
(*f* $$ *fls-subdegree f*) ∗ (*g* $$ *fls-subdegree g*)
by (*simp add*: *fls-times-nth*(*1*))

instance *fls* :: ({*comm-monoid-add*, *mult-zero*}) *mult-zero*
proof
  fix *a* :: *'a fls*
  have
    (*0*::*'a fls*) ∗ *a* =
      fls-shift (*fls-subdegree a*) (*fps-to-fls* ((*0*::*'a fps*)∗(*fls-base-factor-to-fps a*)))
    by (*simp add*: *fls-times-def*)
  moreover have
    a ∗ (*0*::*'a fls*) =
      fls-shift (*fls-subdegree a*) (*fps-to-fls* ((*fls-base-factor-to-fps a*)∗(*0*::*'a fps*)))
    by (*simp add*: *fls-times-def*)
  ultimately show *0* ∗ *a* = (*0*::*'a fls*) *a* ∗ *0* = (*0*::*'a fls*)
    by *auto*
qed

lemma *fls-mult-one*:
  fixes *f* :: *'a*::{*comm-monoid-add*, *mult-zero*, *monoid-mult*} *fls*
  shows *1* ∗ *f* = *f*
  and *f* ∗ *1* = *f*
  using *fls-conv-base-factor-to-fps-shift-subdegree*[*of f*]
  by (*simp-all add*: *fls-times-def fps-one-mult*)

lemma *fls-mult-const-nth* [*simp*]:
  fixes *f* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
  shows (*fls-const x* ∗ *f*) $$ *n* = *x* ∗ *f*$$*n*
  and (*f* ∗ *fls-const x*) $$ *n* = *f*$$*n* ∗ *x*
proof−
  show (*fls-const x* ∗ *f*) $$ *n* = *x* ∗ *f*$$*n*
  proof (*cases n*<*fls-subdegree f*)
    case *False*
    hence {*fls-subdegree f*..*n*} = *insert* (*fls-subdegree f*) {*fls-subdegree f*+*1*..*n*} **by**
auto
    thus *?thesis* **by** (*simp add*: *fls-times-nth*(*1*))
  qed (*simp add*: *fls-times-nth-eq0*)
  show (*f* ∗ *fls-const x*) $$ *n* = *f*$$*n* ∗ *x*
  proof (*cases n*<*fls-subdegree f*)
    case *False*
    hence {*fls-subdegree f*..*n*} = *insert n* {*fls-subdegree f*..*n*−*1*} **by** *auto*
    thus *?thesis* **by** (*simp add*: *fls-times-nth*(*1*))
  qed (*simp add*: *fls-times-nth-eq0*)
qed

lemma *fls-const-mult-const*[*simp*]:
  fixes *x y* :: *'a*::{*comm-monoid-add*, *mult-zero*}
  shows *fls-const x* ∗ *fls-const y* = *fls-const* (*x*∗*y*)
  by (*intro fls-eqI*) *simp*

lemma *fls-mult-subdegree-ge*:

fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes *f*∗*g* ≠ *0*
shows *fls-subdegree* (*f*∗*g*) ≥ *fls-subdegree f* + *fls-subdegree g*
by (*auto intro*: *fls-subdegree-geI simp*: *assms fls-times-nth-eq0*)

lemma *fls-mult-subdegree-ge-0*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *fls-subdegree* (*f*∗*g*) ≥ *0*
using *assms fls-mult-subdegree-ge*[*of f g*]
by *fastforce*

lemma *fls-mult-nonzero-base-subdegree-eq*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes *f* $$ (*fls-subdegree f*) ∗ *g* $$ (*fls-subdegree g*) ≠ *0*
shows *fls-subdegree* (*f*∗*g*) = *fls-subdegree f* + *fls-subdegree g*
proof−
from *assms* **have** *fls-subdegree* (*f*∗*g*) ≥ *fls-subdegree f* + *fls-subdegree g*
using *fls-nonzeroI*[*of f*∗*g fls-subdegree f* + *fls-subdegree g*]
fls-mult-subdegree-ge[*of f g*]
by *simp*
moreover from *assms* **have** *fls-subdegree* (*f*∗*g*) ≤ *fls-subdegree f* + *fls-subdegree g*
by (*intro fls-subdegree-leI*) *simp*
ultimately show *?thesis* **by** *simp*
qed

lemma *fls-subdegree-mult* [*simp*]:
fixes *f g* :: *'a*::*semiring-no-zero-divisors fls*
assumes *f* ≠ *0 g* ≠ *0*
shows *fls-subdegree* (*f* ∗ *g*) = *fls-subdegree f* + *fls-subdegree g*
using *assms*
by (*auto intro*: *fls-subdegree-eqI simp*: *fls-times-nth-eq0*)

lemma *fls-shifted-times-simps*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows *f* ∗ (*fls-shift n g*) = *fls-shift n* (*f*∗*g*) (*fls-shift n f*) ∗ *g* = *fls-shift n* (*f*∗*g*)
proof−

show *f* ∗ (*fls-shift n g*) = *fls-shift n* (*f*∗*g*)
proof (*cases g*=*0*)
case *False*
hence
f ∗ (*fls-shift n g*) =
fls-shift (− (*fls-subdegree f* + (*fls-subdegree g* − *n*)))
(*fps-to-fls* (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*))
unfolding *fls-times-def* **by** (*simp add*: *fls-base-factor-to-fps-shift*)
thus *f* ∗ (*fls-shift n g*) = *fls-shift n* (*f*∗*g*)
by (*simp add*: *algebra-simps fls-times-def*)

qed *auto*

show (*fls-shift n f*)∗*g* = *fls-shift n* (*f*∗*g*)
proof (*cases f*=*0*)
case *False*
hence
(*fls-shift n f*)∗*g* =
fls-shift (− ((*fls-subdegree f* − *n*) + *fls-subdegree g*))
(*fps-to-fls* (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*))
unfolding *fls-times-def* **by** (*simp add*: *fls-base-factor-to-fps-shift*)
thus (*fls-shift n f*) ∗ *g* = *fls-shift n* (*f*∗*g*)
by (*simp add*: *algebra-simps fls-times-def*)
qed *auto*

qed

lemma *fls-shifted-times-transfer*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows *fls-shift n f* ∗ *g* = *f* ∗ *fls-shift n g*
using *fls-shifted-times-simps*(*1*)[*of f n g*] *fls-shifted-times-simps*(*2*)[*of n f g*]
by *simp*

lemma *fls-times-both-shifted-simp*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows (*fls-shift m f*) ∗ (*fls-shift n g*) = *fls-shift* (*m*+*n*) (*f*∗*g*)
by (*simp add*: *fls-shifted-times-simps*)

lemma *fls-base-factor-mult-base-factor*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows *fls-base-factor* (*f* ∗ *fls-base-factor g*) = *fls-base-factor* (*f* ∗ *g*)
and *fls-base-factor* (*fls-base-factor f* ∗ *g*) = *fls-base-factor* (*f* ∗ *g*)
using *fls-base-factor-shift*[*of fls-subdegree g f*∗*g*]
fls-base-factor-shift[*of fls-subdegree f f*∗*g*]
by (*simp-all add*: *fls-shifted-times-simps*)

lemma *fls-base-factor-mult-both-base-factor*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
shows *fls-base-factor* (*fls-base-factor f* ∗ *fls-base-factor g*) = *fls-base-factor* (*f* ∗ *g*)
using *fls-base-factor-mult-base-factor*(*1*)[*of fls-base-factor f g*]
fls-base-factor-mult-base-factor(*2*)[*of f g*]
by *simp*

lemma *fls-base-factor-mult*:
fixes *f g* :: *'a*::*semiring-no-zero-divisors fls*
shows *fls-base-factor* (*f* ∗ *g*) = *fls-base-factor f* ∗ *fls-base-factor g*
by (*cases f*≠*0* ∧ *g*≠*0*)
(*auto simp*: *fls-times-both-shifted-simp*)

lemma *fls-times-conv-base-factor-times*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows
f ∗ *g* =
fls-shift (−(*fls-subdegree f* + *fls-subdegree g*)) (*fls-base-factor f* ∗ *fls-base-factor g*)
by (*simp add*: *fls-times-both-shifted-simp*)

lemma *fls-times-base-factor-conv-shifted-times*:
— Convenience form of lemma *fls-times-both-shifted-simp*.
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows
fls-base-factor f ∗ *fls-base-factor g* = *fls-shift* (*fls-subdegree f* + *fls-subdegree g*) (*f* ∗ *g*)
by (*simp add*: *fls-times-both-shifted-simp*)

lemma *fls-times-conv-regpart*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0* *fls-subdegree g* ≥ *0*
shows *fls-regpart* (*f* ∗ *g*) = *fls-regpart f* ∗ *fls-regpart g*
proof−
from *assms* **have** *1*:
f ∗ *g* =
fls-shift (− (*fls-subdegree f* + *fls-subdegree g*)) (
fps-to-fls (
fps-shift (*nat* (*fls-subdegree f*) + *nat* (*fls-subdegree g*)) (
fls-regpart f ∗ *fls-regpart g*
)
)
)
by (*simp add*:
fls-times-def fls-base-factor-to-fps-conv-fps-shift[*symmetric*]
fls-regpart-subdegree-conv fps-shift-mult-both[*symmetric*]
)
show *?thesis*
proof (*cases fls-regpart f* ∗ *fls-regpart g* = *0*)
case *False*
with *assms* **have**
subdegree (*fls-regpart f* ∗ *fls-regpart g*) ≥
nat (*fls-subdegree f*) + *nat* (*fls-subdegree g*)
by (*simp add*: *fps-mult-subdegree-ge fls-regpart-subdegree-conv*[*symmetric*])
with *1 assms* **show** *?thesis* **by** *simp*
qed (*simp add*: *1*)
qed

lemma *fls-base-factor-to-fps-mult-conv-unit-factor*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
shows
fls-base-factor-to-fps (*f* ∗ *g*) =

unit-factor (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*)
using *fls-base-factor-mult-both-base-factor*[*of f g*]
fps-unit-factor-fls-regpart[*of fls-base-factor f* ∗ *fls-base-factor g*]
fls-base-factor-subdegree[*of f*] *fls-base-factor-subdegree*[*of g*]
fls-mult-subdegree-ge-0[*of fls-base-factor f fls-base-factor g*]
fls-times-conv-regpart[*of fls-base-factor f fls-base-factor g*]
by *simp*

lemma *fls-base-factor-to-fps-mult'*:
fixes *f g* :: *'a*::{*comm-monoid-add,mult-zero*} *fls*
assumes (*f* $$ *fls-subdegree f*) ∗ (*g* $$ *fls-subdegree g*) ≠ *0*
shows *fls-base-factor-to-fps* (*f* ∗ *g*) = *fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*
using *assms fls-mult-nonzero-base-subdegree-eq*[*of f g*]
fls-times-base-factor-conv-shifted-times[*of f g*]
fls-times-conv-regpart[*of fls-base-factor f fls-base-factor g*]
fls-base-factor-subdegree[*of f*] *fls-base-factor-subdegree*[*of g*]
by *fastforce*

lemma *fls-base-factor-to-fps-mult*:
fixes *f g* :: *'a*::*semiring-no-zero-divisors fls*
shows *fls-base-factor-to-fps* (*f* ∗ *g*) = *fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*
using *fls-base-factor-to-fps-mult'*[*of f g*]
by (*cases f=0* ∨ *g=0*) *auto*

lemma *fls-times-conv-fps-times*:
fixes *f g* :: *'a*::{*comm-monoid-add,mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *f* ∗ *g* = *fps-to-fls* (*fls-regpart f* ∗ *fls-regpart g*)
using *assms fls-mult-subdegree-ge*[*of f g*]
by (*cases f* ∗ *g* = *0*) (*simp-all add*: *fls-times-conv-regpart*[*symmetric*])

lemma *fps-times-conv-fls-times*:
fixes *f g* :: *'a*::{*comm-monoid-add,mult-zero*} *fps*
shows *f* ∗ *g* = *fls-regpart* (*fps-to-fls f* ∗ *fps-to-fls g*)
using *fls-subdegree-fls-to-fps-gt0 fls-times-conv-regpart*[*symmetric*]
by *fastforce*

lemma *fls-times-fps-to-fls*:
fixes *f g* :: *'a*::{*comm-monoid-add,mult-zero*} *fps*
shows *fps-to-fls* (*f* ∗ *g*) = *fps-to-fls f* ∗ *fps-to-fls g*
proof (*intro fls-eq-conv-fps-eqI*, *rule fls-subdegree-fls-to-fps-gt0*)
show *fls-subdegree* (*fps-to-fls f* ∗ *fps-to-fls g*) ≥ *0*
proof (*cases fps-to-fls f* ∗ *fps-to-fls g* = *0*)
case *False* **thus** *?thesis*
using *fls-mult-subdegree-ge fls-subdegree-fls-to-fps-gt0*[*of f*]
fls-subdegree-fls-to-fps-gt0[*of g*]
by *fastforce*

qed *simp*
qed (*simp add*: *fps-times-conv-fls-times*)

lemma *fls-X-times-conv-shift*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
shows *fls-X* ∗ *f* = *fls-shift* (−*1*) *f* *f* ∗ *fls-X* = *fls-shift* (−*1*) *f*
by (*simp-all add*: *fls-X-conv-shift-1 fls-mult-one fls-shifted-times-simps*)

lemmas *fls-X-times-comm* = *trans-sym*[*OF fls-X-times-conv-shift*]

lemma *fls-subdegree-mult-fls-X*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
assumes *f* ≠ *0*
shows *fls-subdegree* (*fls-X* ∗ *f*) = *fls-subdegree f* + *1*
and *fls-subdegree* (*f* ∗ *fls-X*) = *fls-subdegree f* + *1*
by (*auto simp*: *fls-X-times-conv-shift assms*)

lemma *fls-mult-fls-X-nonzero*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
assumes *f* ≠ *0*
shows *fls-X* ∗ *f* ≠ *0*
and *f* ∗ *fls-X* ≠ *0*
by (*auto simp*: *fls-X-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-base-factor-mult-fls-X*:
fixes *f* :: *'a*::{*comm-monoid-add*,*monoid-mult*,*mult-zero*} *fls*
shows *fls-base-factor* (*fls-X* ∗ *f*) = *fls-base-factor f*
and *fls-base-factor* (*f* ∗ *fls-X*) = *fls-base-factor f*
using *fls-base-factor-shift*[*of* −*1 f*]
by (*auto simp*: *fls-X-times-conv-shift*)

lemma *fls-X-inv-times-conv-shift*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
shows *fls-X-inv* ∗ *f* = *fls-shift 1 f* *f* ∗ *fls-X-inv* = *fls-shift 1 f*
by (*simp-all add*: *fls-X-inv-conv-shift-1 fls-mult-one fls-shifted-times-simps*)

lemmas *fls-X-inv-times-comm* = *trans-sym*[*OF fls-X-inv-times-conv-shift*]

lemma *fls-subdegree-mult-fls-X-inv*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
assumes *f* ≠ *0*
shows *fls-subdegree* (*fls-X-inv* ∗ *f*) = *fls-subdegree f* − *1*
and *fls-subdegree* (*f* ∗ *fls-X-inv*) = *fls-subdegree f* − *1*
by (*auto simp*: *fls-X-inv-times-conv-shift assms*)

lemma *fls-mult-fls-X-inv-nonzero*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
assumes *f* ≠ *0*
shows *fls-X-inv* ∗ *f* ≠ *0*

and *f* ∗ *fls-X-inv* ≠ *0*
by (*auto simp*: *fls-X-inv-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-base-factor-mult-fls-X-inv*:
fixes *f* :: *'a*::{*comm-monoid-add*,*monoid-mult*,*mult-zero*} *fls*
shows *fls-base-factor* (*fls-X-inv* ∗ *f*) = *fls-base-factor f*
and *fls-base-factor* (*f* ∗ *fls-X-inv*) = *fls-base-factor f*
using *fls-base-factor-shift*[*of 1 f*]
by (*auto simp*: *fls-X-inv-times-conv-shift*)

lemma *fls-mult-assoc-subdegree-ge-0*:
fixes *f g h* :: *'a*::*semiring-0 fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0 fls-subdegree h* ≥ *0*
shows *f* ∗ *g* ∗ *h* = *f* ∗ (*g* ∗ *h*)
using *assms*
by (*simp add*: *fls-times-conv-fps-times fls-subdegree-fls-to-fps-gt0 mult.assoc*)

lemma *fls-mult-assoc-base-factor*:
fixes *a b c* :: *'a*::*semiring-0 fls*
shows
fls-base-factor a ∗ *fls-base-factor b* ∗ *fls-base-factor c* =
fls-base-factor a ∗ (*fls-base-factor b* ∗ *fls-base-factor c*)
by (*simp add*: *fls-mult-assoc-subdegree-ge-0 del*: *fls-base-factor-def*)

lemma *fls-mult-distrib-subdegree-ge-0*:
fixes *f g h* :: *'a*::*semiring-0 fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0 fls-subdegree h* ≥ *0*
shows (*f* + *g*) ∗ *h* = *f* ∗ *h* + *g* ∗ *h*
and *h* ∗ (*f* + *g*) = *h* ∗ *f* + *h* ∗ *g*
proof−
have *fls-subdegree* (*f*+*g*) ≥ *0*
proof (*cases f*+*g* = *0*)
case *False*
with *assms*(*1*,*2*) **show** *?thesis*
using *fls-plus-subdegree* **by** *fastforce*
qed *simp*
with *assms* **show** (*f* + *g*) ∗ *h* = *f* ∗ *h* + *g* ∗ *h h* ∗ (*f* + *g*) = *h* ∗ *f* + *h* ∗ *g*
using *distrib-right*[*of fls-regpart f*] *distrib-left*[*of fls-regpart h*]
by (*simp-all add*: *fls-times-conv-fps-times*)
qed

lemma *fls-mult-distrib-base-factor*:
fixes *a b c* :: *'a*::*semiring-0 fls*
shows
fls-base-factor a ∗ (*fls-base-factor b* + *fls-base-factor c*) =
fls-base-factor a ∗ *fls-base-factor b* + *fls-base-factor a* ∗ *fls-base-factor c*
by (*simp add*: *fls-mult-distrib-subdegree-ge-0 del*: *fls-base-factor-def*)

instance *fls* :: (*semiring-0*) *semiring-0*

proof

fix *a b c* :: *'a fls*
have
a ∗ *b* ∗ *c* =
fls-shift (− (*fls-subdegree a* + *fls-subdegree b* + *fls-subdegree c*))
(*fls-base-factor a* ∗ *fls-base-factor b* ∗ *fls-base-factor c*)
by (*simp add*: *fls-times-both-shifted-simp*)
moreover have
a ∗ (*b* ∗ *c*) =
fls-shift (− (*fls-subdegree a* + *fls-subdegree b* + *fls-subdegree c*))
(*fls-base-factor a* ∗ *fls-base-factor b* ∗ *fls-base-factor c*)
using *fls-mult-assoc-base-factor*[*of a b c*] **by** (*simp add*: *fls-times-both-shifted-simp*)
ultimately show *a* ∗ *b* ∗ *c* = *a* ∗ (*b* ∗ *c*) **by** *simp*

have *ab*:
fls-subdegree (*fls-shift* (*min* (*fls-subdegree a*) (*fls-subdegree b*)) *a*) ≥ *0*
fls-subdegree (*fls-shift* (*min* (*fls-subdegree a*) (*fls-subdegree b*)) *b*) ≥ *0*
by (*simp-all add*: *fls-shift-nonneg-subdegree*)
have
(*a* + *b*) ∗ *c* =
fls-shift (− (*min* (*fls-subdegree a*) (*fls-subdegree b*) + *fls-subdegree c*)) (
(
fls-shift (*min* (*fls-subdegree a*) (*fls-subdegree b*)) *a* +
fls-shift (*min* (*fls-subdegree a*) (*fls-subdegree b*)) *b*
) ∗ *fls-base-factor c*)
using *fls-times-both-shifted-simp*[*of*
−*min* (*fls-subdegree a*) (*fls-subdegree b*)
fls-shift (*min* (*fls-subdegree a*) (*fls-subdegree b*)) *a* +
fls-shift (*min* (*fls-subdegree a*) (*fls-subdegree b*)) *b*
−*fls-subdegree c fls-base-factor c*
]
by *simp*
also have
. . . =
fls-shift (−(*min* (*fls-subdegree a*) (*fls-subdegree b*) + *fls-subdegree c*))
(*fls-shift* (*min* (*fls-subdegree a*) (*fls-subdegree b*)) *a* ∗ *fls-base-factor c*)
+
fls-shift (−(*min* (*fls-subdegree a*) (*fls-subdegree b*) + *fls-subdegree c*))
(*fls-shift* (*min* (*fls-subdegree a*) (*fls-subdegree b*)) *b* ∗ *fls-base-factor c*)
using *ab*
by (*simp add*: *fls-mult-distrib-subdegree-ge-0*(*1*) *del*: *fls-base-factor-def*)
finally show (*a* + *b*) ∗ *c* = *a* ∗ *c* + *b* ∗ *c* **by** (*simp add*: *fls-times-both-shifted-simp*)

have *bc*:
fls-subdegree (*fls-shift* (*min* (*fls-subdegree b*) (*fls-subdegree c*)) *b*) ≥ *0*
fls-subdegree (*fls-shift* (*min* (*fls-subdegree b*) (*fls-subdegree c*)) *c*) ≥ *0*
by (*simp-all add*: *fls-shift-nonneg-subdegree*)
have

a ∗ (*b* + *c*) =
fls-shift (− (*fls-subdegree a* + *min* (*fls-subdegree b*) (*fls-subdegree c*))) (
fls-base-factor a ∗ (
fls-shift (*min* (*fls-subdegree b*) (*fls-subdegree c*)) *b* +
fls-shift (*min* (*fls-subdegree b*) (*fls-subdegree c*)) *c*
)
)

using *fls-times-both-shifted-simp*[*of*
−*fls-subdegree a fls-base-factor a*
−*min* (*fls-subdegree b*) (*fls-subdegree c*)
fls-shift (*min* (*fls-subdegree b*) (*fls-subdegree c*)) *b* +
fls-shift (*min* (*fls-subdegree b*) (*fls-subdegree c*)) *c*
]
by *simp*
also have
. . . =
fls-shift (−(*fls-subdegree a* + *min* (*fls-subdegree b*) (*fls-subdegree c*)))
(*fls-base-factor a* ∗ *fls-shift* (*min* (*fls-subdegree b*) (*fls-subdegree c*)) *b*)
+
fls-shift (−(*fls-subdegree a* + *min* (*fls-subdegree b*) (*fls-subdegree c*)))
(*fls-base-factor a* ∗ *fls-shift* (*min* (*fls-subdegree b*) (*fls-subdegree c*)) *c*)

using *bc*
by (*simp add*: *fls-mult-distrib-subdegree-ge-0*(*2*) *del*: *fls-base-factor-def*)
finally show *a* ∗ (*b* + *c*) = *a* ∗ *b* + *a* ∗ *c* **by** (*simp add*: *fls-times-both-shifted-simp*)

qed

lemma *fls-mult-commute-subdegree-ge-0*:
fixes *f g* :: *'a*::*comm-semiring-0 fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *f* ∗ *g* = *g* ∗ *f*
using *assms*
by (*simp add*: *fls-times-conv-fps-times mult.commute*)

lemma *fls-mult-commute-base-factor*:
fixes *a b c* :: *'a*::*comm-semiring-0 fls*
shows *fls-base-factor a* ∗ *fls-base-factor b* = *fls-base-factor b* ∗ *fls-base-factor a*
by (*simp add*: *fls-mult-commute-subdegree-ge-0 del*: *fls-base-factor-def*)

instance *fls* :: (*comm-semiring-0*) *comm-semiring-0*
proof
fix *a b c* :: *'a fls*
show *a* ∗ *b* = *b* ∗ *a*
using *fls-times-conv-base-factor-times*[*of a b*] *fls-times-conv-base-factor-times*[*of b a*]
fls-mult-commute-base-factor[*of a b*]
by (*simp add*: *add.commute*)

qed (*simp add*: *distrib-right*)

instance *fls* :: (*semiring-1*) *semiring-1*
  by (*standard*, *simp-all add*: *fls-mult-one*)

lemma *fls-of-nat*: (*of-nat n* :: *'a*::*semiring-1 fls*) = *fls-const* (*of-nat n*)
  by (*induct n*) (*auto intro*: *fls-eqI*)

lemma *fls-of-nat-nth*: *of-nat n* $$ *k* = (*if k=0 then of-nat n else 0*)
  by (*simp add*: *fls-of-nat*)

lemma *fls-mult-of-nat-nth* [*simp*]:
  shows (*of-nat k* ∗ *f*) $$ *n* = *of-nat k* ∗ *f*$$*n*
  and (*f* ∗ *of-nat k*) $$ *n* = *f*$$*n* ∗ *of-nat k*
  by (*simp-all add*: *fls-of-nat*)

lemma *fls-subdegree-of-nat* [*simp*]: *fls-subdegree* (*of-nat n*) = *0*
  by (*simp add*: *fls-of-nat*)

lemma *fls-shift-of-nat-nth*:
  fls-shift k (*of-nat a*) $$ *n* = (*if n*=−*k then of-nat a else 0*)
  by (*simp add*: *fls-of-nat fls-shift-const-nth*)

lemma *fls-base-factor-of-nat* [*simp*]:
  fls-base-factor (*of-nat n* :: *'a*::*semiring-1 fls*) = (*of-nat n* :: *'a fls*)
  by (*simp add*: *fls-of-nat*)

lemma *fls-regpart-of-nat* [*simp*]: *fls-regpart* (*of-nat n*) = (*of-nat n* :: *'a*::*semiring-1 fps*)
  by (*simp add*: *fls-of-nat fps-of-nat*)

lemma *fls-prpart-of-nat* [*simp*]: *fls-prpart* (*of-nat n*) = *0*
  by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-base-factor-to-fps-of-nat*:
  fls-base-factor-to-fps (*of-nat n*) = (*of-nat n* :: *'a*::*semiring-1 fps*)
  by *simp*

lemma *fps-to-fls-of-nat*:
  fps-to-fls (*of-nat n*) = (*of-nat n* :: *'a*::*semiring-1 fls*)
proof −
  have *fps-to-fls* (*of-nat n*) = *fps-to-fls* (*fps-const* (*of-nat n*))
    by (*simp add*: *fps-of-nat*)
  thus *?thesis* **by** (*simp add*: *fls-of-nat*)
qed

lemma *fps-to-fls-numeral* [*simp*]: *fps-to-fls* (*numeral n*) = *numeral n*
  by (*metis fps-to-fls-of-nat of-nat-numeral*)

lemma *fls-const-power*: *fls-const* (*a* ^ *b*) = *fls-const a* ^ *b*
  by (*induction b*) (*auto simp flip*: *fls-const-mult-const*)

lemma *fls-const-numeral* [*simp*]: *fls-const* (*numeral n*) = *numeral n*
  by (*metis fls-of-nat of-nat-numeral*)

lemma *fls-mult-of-numeral-nth* [*simp*]:
  shows (*numeral k* ∗ *f*) $$ *n* = *numeral k* ∗ *f* $$ *n*
  and (*f* ∗ *numeral k*) $$ *n* = *f* $$ *n* ∗ *numeral k*
  by (*metis fls-const-numeral fls-mult-const-nth*)+

lemma *fls-nth-numeral′* [*simp*]:
  numeral n $$ *0* = *numeral n k* ≠ *0* ⟹ *numeral n* $$ *k* = *0*
  by (*metis fls-const-nth fls-const-numeral*)+

instance *fls* :: (*comm-semiring-1*) *comm-semiring-1*
  by *standard simp*

instance *fls* :: (*ring*) *ring* ..

instance *fls* :: (*comm-ring*) *comm-ring* ..

instance *fls* :: (*ring-1*) *ring-1* ..

lemma *fls-of-int-nonneg*: (*of-int* (*int n*) :: ′*a*::*ring-1 fls*) = *fls-const* (*of-int* (*int n*))
  by (*induct n*) (*auto intro*: *fls-eqI*)

lemma *fls-of-int*: (*of-int i* :: ′*a*::*ring-1 fls*) = *fls-const* (*of-int i*)
proof (*induct i*)
  case (*neg i*)
  have *of-int* (*int* (*Suc i*)) = *fls-const* (*of-int* (*int* (*Suc i*)) :: ′*a*)
    using *fls-of-int-nonneg*[*of Suc i*] **by** *simp*
  hence − *of-int* (*int* (*Suc i*)) = − *fls-const* (*of-int* (*int* (*Suc i*)) :: ′*a*)
    by *simp*
  thus *?case* **by** (*simp add*: *fls-const-uminus*[*symmetric*])
qed (*rule fls-of-int-nonneg*)

lemma *fls-of-int-nth*: *of-int n* $$ *k* = (*if k*=*0 then of-int n else 0*)
  by (*simp add*: *fls-of-int*)

lemma *fls-mult-of-int-nth* [*simp*]:
  shows (*of-int k* ∗ *f*) $$ *n* = *of-int k* ∗ *f*$$*n*
  and (*f* ∗ *of-int k*) $$ *n* = *f*$$*n* ∗ *of-int k*
  by (*simp-all add*: *fls-of-int*)

lemma *fls-subdegree-of-int* [*simp*]: *fls-subdegree* (*of-int i*) = *0*
  by (*simp add*: *fls-of-int*)

lemma *fls-shift-of-int-nth*:
fls-shift k (*of-int i*) $$ *n* = (*if n*=−*k then of-int i else 0*)
by (*simp add*: *fls-of-int-nth*)

lemma *fls-base-factor-of-int* [*simp*]:
fls-base-factor (*of-int i* :: *'a*::*ring-1 fls*) = (*of-int i* :: *'a fls*)
by (*simp add*: *fls-of-int*)

lemma *fls-regpart-of-int* [*simp*]:
fls-regpart (*of-int i*) = (*of-int i* :: *'a*::*ring-1 fps*)
by (*simp add*: *fls-of-int fps-of-int*)

lemma *fls-prpart-of-int* [*simp*]: *fls-prpart* (*of-int n*) = *0*
by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-base-factor-to-fps-of-int*:
fls-base-factor-to-fps (*of-int i*) = (*of-int i* :: *'a*::*ring-1 fps*)
by *simp*

lemma *fps-to-fls-of-int*:
fps-to-fls (*of-int i*) = (*of-int i* :: *'a*::*ring-1 fls*)
proof −
have *fps-to-fls* (*of-int i*) = *fps-to-fls* (*fps-const* (*of-int i*))
by (*simp add*: *fps-of-int*)
thus *?thesis* **by** (*simp add*: *fls-of-int*)
qed

instance *fls* :: (*comm-ring-1*) *comm-ring-1* **..**

instance *fls* :: (*semiring-no-zero-divisors*) *semiring-no-zero-divisors*
proof
fix *a b* :: *'a fls*
assume *a* ≠ *0* **and** *b* ≠ *0*
hence (*a* ∗ *b*) $$ (*fls-subdegree a* + *fls-subdegree b*) ≠ *0* **by** *simp*
thus *a* ∗ *b* ≠ *0* **using** *fls-nonzeroI* **by** *fast*
qed

instance *fls* :: (*semiring-1-no-zero-divisors*) *semiring-1-no-zero-divisors* **..**

instance *fls* :: (*ring-no-zero-divisors*) *ring-no-zero-divisors* **..**

instance *fls* :: (*ring-1-no-zero-divisors*) *ring-1-no-zero-divisors* **..**

instance *fls* :: (*idom*) *idom* **..**

lemma *semiring-char-fls* [*simp*]: *CHAR*(*'a* :: *comm-semiring-1 fls*) = *CHAR*(*'a*)
by (*rule CHAR-eqI*) (*auto simp*: *fls-of-nat of-nat-eq-0-iff-char-dvd fls-const-nonzero*)

instance *fls* :: ({*semiring-prime-char*,*comm-semiring-1*}) *semiring-prime-char*

by (*rule semiring-prime-charI*) *auto*
instance *fls* :: ({*comm-semiring-prime-char*,*comm-semiring-1*}) *comm-semiring-prime-char*
by *standard*
instance *fls* :: ({*comm-ring-prime-char*,*comm-semiring-1*}) *comm-ring-prime-char*
by *standard*
instance *fls* :: ({*idom-prime-char*,*comm-semiring-1*}) *idom-prime-char*
by *standard*

7.5.4 Powers

lemma *fls-subdegree-prod*:
fixes F :: $'a \Rightarrow 'b$ :: *field-char-0 fls*
assumes $\bigwedge x.\ x \in I \Longrightarrow F\ x \neq 0$
shows *fls-subdegree* ($\prod x{\in}I.\ F\ x$) = ($\sum x{\in}I.$ *fls-subdegree* ($F\ x$))
using *assms* **by** (*induction I rule*: *infinite-finite-induct*) *auto*

lemma *fls-subdegree-prod'*:
fixes F :: $'a \Rightarrow 'b$ :: *field-char-0 fls*
assumes $\bigwedge x.\ x \in I \Longrightarrow$ *fls-subdegree* ($F\ x$) $\neq 0$
shows *fls-subdegree* ($\prod x{\in}I.\ F\ x$) = ($\sum x{\in}I.$ *fls-subdegree* ($F\ x$))
proof (*intro fls-subdegree-prod*)
show $F\ x \neq 0$ **if** $x \in I$ **for** x
using *assms*[*OF that*] **by** *auto*
qed

lemma *fls-pow-subdegree-ge*:
f^$n \neq 0 \Longrightarrow$ *fls-subdegree* (f^n) $\geq n *$ *fls-subdegree f*
proof (*induct n*)
case (*Suc n*) **thus** *?case*
using *fls-mult-subdegree-ge*[*of f f*^*n*] **by** (*fastforce simp*: *algebra-simps*)
qed *simp*

lemma *fls-pow-nth-below-subdegree*:
$k < n *$ *fls-subdegree f* $\Longrightarrow$ (f^n) \$\$ $k = 0$
using *fls-pow-subdegree-ge*[*of f n*] **by** (*cases f*^*n* = *0*) *auto*

lemma *fls-pow-base* [*simp*]:
(f ^ n) \$\$ ($n *$ *fls-subdegree f*) = (f \$\$ *fls-subdegree f*) ^ n
proof (*induct n*)
case (*Suc n*)
show *?case*
proof (*cases Suc n* $*$ *fls-subdegree f* < *fls-subdegree f* + *fls-subdegree* (f^n))
case *True* **with** *Suc* **show** *?thesis*
by (*simp-all add*: *fls-times-nth-eq0 distrib-right*)
next
case *False*
from *False* **have**
{*0*..*int n* $*$ *fls-subdegree f* − *fls-subdegree* (f ^ n)} =
insert 0 {*1*..*int n* $*$ *fls-subdegree f* − *fls-subdegree* (f ^ n)}

by (*auto simp*: *algebra-simps*)
with *False Suc* **show** *?thesis*
by (*simp add*: *algebra-simps fls-times-nth*(*4*) *fls-pow-nth-below-subdegree*)
qed
qed *simp*

lemma *fls-pow-subdegree-eqI*:
(*f* $$ *fls-subdegree f*) ^ *n* ≠ *0* ⟹ *fls-subdegree* (*f^n*) = *n* ∗ *fls-subdegree f*
using *fls-pow-nth-below-subdegree* **by** (*fastforce intro*: *fls-subdegree-eqI*)

lemma *fls-unit-base-subdegree-power*:
x ∗ *f* $$ *fls-subdegree f* = *1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* ∗ *fls-subdegree f*
f $$ *fls-subdegree f* ∗ *y* = *1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* ∗ *fls-subdegree f*
proof−
show *x* ∗ *f* $$ *fls-subdegree f* = *1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* ∗ *fls-subdegree f*
using *left-right-inverse-power*[*of x f* $$ *fls-subdegree f n*]
by (*auto intro*: *fls-pow-subdegree-eqI*)
show *f* $$ *fls-subdegree f* ∗ *y* = *1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* ∗ *fls-subdegree f*
using *left-right-inverse-power*[*of f* $$ *fls-subdegree f y n*]
by (*auto intro*: *fls-pow-subdegree-eqI*)
qed

lemma *fls-base-dvd1-subdegree-power*:
f $$ *fls-subdegree f dvd 1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* ∗ *fls-subdegree f*
using *fls-unit-base-subdegree-power* **unfolding** *dvd-def* **by** *auto*

lemma *fls-pow-subdegree-ge0*:
assumes *fls-subdegree f* ≥ *0*
shows *fls-subdegree* (*f^n*) ≥ *0*
proof (*cases f^n* = *0*)
case *False*
moreover from *assms* **have** *int n* ∗ *fls-subdegree f* ≥ *0* **by** *simp*
ultimately show *?thesis* **using** *fls-pow-subdegree-ge* **by** *fastforce*
qed *simp*

lemma *fls-subdegree-pow*:
fixes *f* :: *'a*::*semiring-1-no-zero-divisors fls*
shows *fls-subdegree* (*f* ^ *n*) = *n* ∗ *fls-subdegree f*
proof (*cases f*=*0*)
case *False* **thus** *?thesis* **by** (*induct n*) (*simp-all add*: *algebra-simps*)
qed (*cases n*=*0*, *auto simp*: *zero-power*)

lemma *fls-shifted-pow*:
(*fls-shift m f*) ^ *n* = *fls-shift* (*n*∗*m*) (*f* ^ *n*)
by (*induct n*) (*simp-all add*: *fls-times-both-shifted-simp algebra-simps*)

lemma *fls-pow-conv-fps-pow*:
assumes *fls-subdegree f* ≥ *0*
shows *f* ^ *n* = *fps-to-fls* ((*fls-regpart f*) ^ *n*)

proof (*induct n*)
  case (*Suc n*) **with** *assms* **show** *?case*
    using *fls-pow-subdegree-ge0*[*of f n*]
    by (*simp add*: *fls-times-conv-fps-times*)
qed *simp*

lemma *fps-to-fls-power*: *fps-to-fls* (*f* ^ *n*) = *fps-to-fls f* ^ *n*
  by (*simp add*: *fls-pow-conv-fps-pow fls-subdegree-fls-to-fps-gt0*)

lemma *fls-pow-conv-regpart*:
  fls-subdegree f ≥ *0* ⟹ *fls-regpart* (*f* ^ *n*) = (*fls-regpart f*) ^ *n*
  by (*simp add*: *fls-pow-conv-fps-pow*)

These two lemmas show that shifting 1 is equivalent to powers of the implied variable.

lemma *fls-X-power-conv-shift-1*: *fls-X* ^ *n* = *fls-shift* (−*n*) *1*
  by (*simp add*: *fls-X-conv-shift-1 fls-shifted-pow*)

lemma *fls-X-inv-power-conv-shift-1*: *fls-X-inv* ^ *n* = *fls-shift n 1*
  by (*simp add*: *fls-X-inv-conv-shift-1 fls-shifted-pow*)

abbreviation *fls-X-intpow* ≡ (λ*i*. *fls-shift* (−*i*) *1*)
— Unifies *fls-X* and *fls-X-inv* so that *fls-X-intpow* returns the equivalent of the implied variable raised to the supplied integer argument of *fls-X-intpow*, whether positive or negative.

lemma *fls-X-intpow-nonzero*[*simp*]: (*fls-X-intpow i* :: *'a*::*zero-neq-one fls*) ≠ *0*
  by (*simp add*: *fls-shift-eq0-iff*)

lemma *fls-X-intpow-power*: (*fls-X-intpow i*) ^ *n* = *fls-X-intpow* (*n* ∗ *i*)
  by (*simp add*: *fls-shifted-pow*)

lemma *fls-X-power-nth* [*simp*]: *fls-X* ^ *n* $$ *k* = (*if k*=*n then 1 else 0*)
  by (*simp add*: *fls-X-power-conv-shift-1*)

lemma *fls-X-inv-power-nth* [*simp*]: *fls-X-inv* ^ *n* $$ *k* = (*if k*=−*n then 1 else 0*)
  by (*simp add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-X-pow-nonzero*[*simp*]: (*fls-X* ^ *n* :: *'a* :: *semiring-1 fls*) ≠ *0*
proof
  assume (*fls-X* ^ *n* :: *'a fls*) = *0*
  hence (*fls-X* ^ *n* :: *'a fls*) $$ *n* = *0* **by** *simp*
  thus *False* **by** *simp*
qed

lemma *fls-X-inv-pow-nonzero*[*simp*]: (*fls-X-inv* ^ *n* :: *'a* :: *semiring-1 fls*) ≠ *0*
proof
  assume (*fls-X-inv* ^ *n* :: *'a fls*) = *0*
  hence (*fls-X-inv* ^ *n* :: *'a fls*) $$ −*n* = *0* **by** *simp*

thus *False* **by** *simp*
qed

lemma *fls-subdegree-fls-X-pow* [*simp*]: *fls-subdegree* (*fls-X* ^ *n*) = *n*
by (*intro fls-subdegree-eqI*) (*simp-all add*: *fls-X-power-conv-shift-1*)

lemma *fls-subdegree-fls-X-inv-pow* [*simp*]: *fls-subdegree* (*fls-X-inv* ^ *n*) = −*n*
by (*intro fls-subdegree-eqI*) (*simp-all add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-subdegree-fls-X-intpow* [*simp*]:
fls-subdegree ((*fls-X-intpow i*) :: ′*a*::*zero-neq-one fls*) = *i*
by *simp*

lemma *fls-X-pow-conv-fps-X-pow*: *fls-regpart* (*fls-X* ^ *n*) = *fps-X* ^ *n*
by (*simp add*: *fls-pow-conv-regpart*)

lemma *fls-X-inv-pow-regpart*: *n* > *0* ⟹ *fls-regpart* (*fls-X-inv* ^ *n*) = *0*
by (*auto intro*: *fps-ext simp*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-X-intpow-regpart*:
fls-regpart (*fls-X-intpow i*) = (*if i*≥*0 then fps-X* ^ *nat i else 0*)
using *fls-X-pow-conv-fps-X-pow*[*of nat i*]
fls-regpart-shift-conv-fps-shift[*of* −*i 1*]
by (*auto simp*: *fls-X-power-conv-shift-1 fps-shift-one*)

lemma *fls-X-power-times-conv-shift*:
fls-X ^ *n* ∗ *f* = *fls-shift* (−*int n*) *f f* ∗ *fls-X* ^ *n* = *fls-shift* (−*int n*) *f*
using *fls-times-both-shifted-simp*[*of* −*int n 1 0 f*]
fls-times-both-shifted-simp[*of 0 f* −*int n 1*]
by (*simp-all add*: *fls-X-power-conv-shift-1*)

lemma *fls-X-inv-power-times-conv-shift*:
fls-X-inv ^ *n* ∗ *f* = *fls-shift* (*int n*) *f f* ∗ *fls-X-inv* ^ *n* = *fls-shift* (*int n*) *f*
using *fls-times-both-shifted-simp*[*of int n 1 0 f*]
fls-times-both-shifted-simp[*of 0 f int n 1*]
by (*simp-all add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-X-intpow-times-conv-shift*:
fixes *f* :: ′*a*::*semiring-1 fls*
shows *fls-X-intpow i* ∗ *f* = *fls-shift* (−*i*) *f f* ∗ *fls-X-intpow i* = *fls-shift* (−*i*) *f*
by (*simp-all add*: *fls-shifted-times-simps*)

lemmas *fls-X-power-times-comm* = *trans-sym*[*OF fls-X-power-times-conv-shift*]
lemmas *fls-X-inv-power-times-comm* = *trans-sym*[*OF fls-X-inv-power-times-conv-shift*]

lemma *fls-X-intpow-times-comm*:
fixes *f* :: ′*a*::*semiring-1 fls*
shows *fls-X-intpow i* ∗ *f* = *f* ∗ *fls-X-intpow i*
by (*simp add*: *fls-X-intpow-times-conv-shift*)

lemma *fls-X-intpow-times-fls-X-intpow*:
(*fls-X-intpow i* :: *'a*::*semiring-1 fls*) ∗ *fls-X-intpow j* = *fls-X-intpow* (*i*+*j*)
by (*simp add*: *fls-times-both-shifted-simp*)

lemma *fls-X-intpow-diff-conv-times*:
fls-X-intpow (*i*−*j*) = (*fls-X-intpow i* :: *'a*::*semiring-1 fls*) ∗ *fls-X-intpow* (−*j*)
using *fls-X-intpow-times-fls-X-intpow*[*of i* −*j*,*symmetric*] **by** *simp*

lemma *fls-mult-fls-X-power-nonzero*:
assumes *f* ≠ *0*
shows *fls-X* ^ *n* ∗ *f* ≠ *0 f* ∗ *fls-X* ^ *n* ≠ *0*
by (*auto simp*: *fls-X-power-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-mult-fls-X-inv-power-nonzero*:
assumes *f* ≠ *0*
shows *fls-X-inv* ^ *n* ∗ *f* ≠ *0 f* ∗ *fls-X-inv* ^ *n* ≠ *0*
by (*auto simp*: *fls-X-inv-power-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-mult-fls-X-intpow-nonzero*:
fixes *f* :: *'a*::*semiring-1 fls*
assumes *f* ≠ *0*
shows *fls-X-intpow i* ∗ *f* ≠ *0 f* ∗ *fls-X-intpow i* ≠ *0*
by (*auto simp*: *fls-X-intpow-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-subdegree-mult-fls-X-power*:
assumes *f* ≠ *0*
shows *fls-subdegree* (*fls-X* ^ *n* ∗ *f*) = *fls-subdegree f* + *n*
and *fls-subdegree* (*f* ∗ *fls-X* ^ *n*) = *fls-subdegree f* + *n*
by (*auto simp*: *fls-X-power-times-conv-shift assms*)

lemma *fls-subdegree-mult-fls-X-inv-power*:
assumes *f* ≠ *0*
shows *fls-subdegree* (*fls-X-inv* ^ *n* ∗ *f*) = *fls-subdegree f* − *n*
and *fls-subdegree* (*f* ∗ *fls-X-inv* ^ *n*) = *fls-subdegree f* − *n*
by (*auto simp*: *fls-X-inv-power-times-conv-shift assms*)

lemma *fls-subdegree-mult-fls-X-intpow*:
fixes *f* :: *'a*::*semiring-1 fls*
assumes *f* ≠ *0*
shows *fls-subdegree* (*fls-X-intpow i* ∗ *f*) = *fls-subdegree f* + *i*
and *fls-subdegree* (*f* ∗ *fls-X-intpow i*) = *fls-subdegree f* + *i*
by (*auto simp*: *fls-X-intpow-times-conv-shift assms*)

lemma *fls-X-shift*:
fls-shift (−*int n*) *fls-X* = *fls-X* ^ *Suc n*
fls-shift (*int* (*Suc n*)) *fls-X* = *fls-X-inv* ^ *n*
using *fls-X-power-conv-shift-1*[*of Suc n*, *symmetric*]
by (*simp-all add*: *fls-X-conv-shift-1 fls-X-inv-power-conv-shift-1*)

lemma *fls-X-inv-shift*:
fls-shift (*int n*) *fls-X-inv* = *fls-X-inv* ^ *Suc n*
fls-shift (− *int* (*Suc n*)) *fls-X-inv* = *fls-X* ^ *n*
using *fls-X-inv-power-conv-shift-1*[*of Suc n*, *symmetric*]
by (*simp-all add*: *fls-X-inv-conv-shift-1 fls-X-power-conv-shift-1*)

lemma *fls-X-power-base-factor*: *fls-base-factor* (*fls-X* ^ *n*) = *1*
by (*simp add*: *fls-X-power-conv-shift-1*)

lemma *fls-X-inv-power-base-factor*: *fls-base-factor* (*fls-X-inv* ^ *n*) = *1*
by (*simp add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-X-intpow-base-factor*: *fls-base-factor* (*fls-X-intpow i*) = *1*
using *fls-base-factor-shift*[*of* −*i 1*] **by** *simp*

lemma *fls-base-factor-mult-fls-X-power*:
shows *fls-base-factor* (*fls-X* ^ *n* ∗ *f*) = *fls-base-factor f*
and *fls-base-factor* (*f* ∗ *fls-X* ^ *n*) = *fls-base-factor f*
using *fls-base-factor-shift*[*of* −*int n f*]
by (*auto simp*: *fls-X-power-times-conv-shift*)

lemma *fls-base-factor-mult-fls-X-inv-power*:
shows *fls-base-factor* (*fls-X-inv* ^ *n* ∗ *f*) = *fls-base-factor f*
and *fls-base-factor* (*f* ∗ *fls-X-inv* ^ *n*) = *fls-base-factor f*
using *fls-base-factor-shift*[*of int n f*]
by (*auto simp*: *fls-X-inv-power-times-conv-shift*)

lemma *fls-base-factor-mult-fls-X-intpow*:
fixes *f* :: ′*a*::*semiring-1 fls*
shows *fls-base-factor* (*fls-X-intpow i* ∗ *f*) = *fls-base-factor f*
and *fls-base-factor* (*f* ∗ *fls-X-intpow i*) = *fls-base-factor f*
using *fls-base-factor-shift*[*of* −*i f*]
by (*auto simp*: *fls-X-intpow-times-conv-shift*)

lemma *fls-X-power-base-factor-to-fps*: *fls-base-factor-to-fps* (*fls-X* ^ *n*) = *1*
proof−
define *X* **where** *X* ≡ *fls-X* :: ′*a*::*semiring-1 fls*
hence *fls-base-factor* (*X* ^ *n*) = *1* **using** *fls-X-power-base-factor* **by** *simp*
thus *fls-base-factor-to-fps* (*X*^*n*) = *1* **by** *simp*
qed

lemma *fls-X-inv-power-base-factor-to-fps*: *fls-base-factor-to-fps* (*fls-X-inv* ^ *n*) = *1*
proof−
define *iX* **where** *iX* ≡ *fls-X-inv* :: ′*a*::*semiring-1 fls*
hence *fls-base-factor* (*iX* ^ *n*) = *1* **using** *fls-X-inv-power-base-factor* **by** *simp*
thus *fls-base-factor-to-fps* (*iX*^*n*) = *1* **by** *simp*
qed

lemma *fls-X-intpow-base-factor-to-fps*: *fls-base-factor-to-fps* (*fls-X-intpow i*) = *1*
proof−
  define *f* :: *'a fls* **where** *f* ≡ *fls-X-intpow i*
  moreover have *fls-base-factor* (*fls-X-intpow i*) = *1* **by** (*rule fls-X-intpow-base-factor*)
  ultimately have *fls-base-factor f* = *1* **by** *simp*
  thus *fls-base-factor-to-fps f* = *1* **by** *simp*
qed

lemma *fls-base-factor-X-power-decompose*:
  fixes *f* :: *'a::semiring-1 fls*
  shows *f* = *fls-base-factor f* ∗ *fls-X-intpow* (*fls-subdegree f*)
  and *f* = *fls-X-intpow* (*fls-subdegree f*) ∗ *fls-base-factor f*
  by (*simp-all add*: *fls-times-both-shifted-simp*)

lemma *fls-normalized-product-of-inverses*:
  assumes *f* ∗ *g* = *1*
  shows *fls-base-factor f* ∗ *fls-base-factor g* =
    fls-X ^ (*nat* (−(*fls-subdegree f*+*fls-subdegree g*)))
  and *fls-base-factor f* ∗ *fls-base-factor g* =
    fls-X-intpow (−(*fls-subdegree f*+*fls-subdegree g*))
  using *fls-mult-subdegree-ge*[*of f g*]
    fls-times-base-factor-conv-shifted-times[*of f g*]
  by (*simp-all add*: *assms fls-X-power-conv-shift-1 algebra-simps*)

lemma *fls-fps-normalized-product-of-inverses*:
  assumes *f* ∗ *g* = *1*
  shows *fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g* =
    fps-X ^ (*nat* (−(*fls-subdegree f*+*fls-subdegree g*)))
  using *fls-times-conv-regpart*[*of fls-base-factor f fls-base-factor g*]
    fls-base-factor-subdegree[*of f*] *fls-base-factor-subdegree*[*of g*]
    fls-normalized-product-of-inverses(*1*)[*OF assms*]
  by (*force simp*: *fls-X-pow-conv-fps-X-pow*)

7.5.5 Inverses

abbreviation *fls-left-inverse* ::
  'a::{*comm-monoid-add*,*uminus*,*times*} *fls* ⇒ *'a* ⇒ *'a fls*
  where
  fls-left-inverse f x ≡
    fls-shift (*fls-subdegree f*) (*fps-to-fls* (*fps-left-inverse* (*fls-base-factor-to-fps f*) *x*))

abbreviation *fls-right-inverse* ::
  'a::{*comm-monoid-add*,*uminus*,*times*} *fls* ⇒ *'a* ⇒ *'a fls*
  where
  fls-right-inverse f y ≡
    fls-shift (*fls-subdegree f*) (*fps-to-fls* (*fps-right-inverse* (*fls-base-factor-to-fps f*)
y))

instantiation *fls* :: ({*comm-monoid-add*,*uminus*,*times*,*inverse*}) *inverse*
begin
  definition *fls-divide-def*:
    f div g =
      fls-shift (*fls-subdegree g* − *fls-subdegree f*) (
        fps-to-fls ((*fls-base-factor-to-fps f*) *div* (*fls-base-factor-to-fps g*))
)

  definition *fls-inverse-def*:
    inverse f = *fls-shift* (*fls-subdegree f*) (*fps-to-fls* (*inverse* (*fls-base-factor-to-fps f*)))
  instance ..
end

lemma *fls-inverse-def'*:
  inverse f = *fls-right-inverse f* (*inverse* (*f* $$ *fls-subdegree f*))
  by (*simp add*: *fls-inverse-def fps-inverse-def*)

lemma *fls-lr-inverse-base*:
  fls-left-inverse f x $$ (−*fls-subdegree f*) = *x*
  fls-right-inverse f y $$ (−*fls-subdegree f*) = *y*
  by *auto*

lemma *fls-inverse-base*:
  f ≠ *0* ⟹ *inverse f* $$ (−*fls-subdegree f*) = *inverse* (*f* $$ *fls-subdegree f*)
  by (*simp add*: *fls-inverse-def'*)

lemma *fls-lr-inverse-starting0*:
  fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*} *fls*
  and *g* :: *'b*::{*ab-group-add*,*mult-zero*} *fls*
  shows *fls-left-inverse f 0* = *0*
  and *fls-right-inverse g 0* = *0*
  by (*simp-all add*: *fps-lr-inverse-starting0*)

lemma *fls-lr-inverse-eq0-imp-starting0*:
  fls-left-inverse f x = *0* ⟹ *x* = *0*
  fls-right-inverse f x = *0* ⟹ *x* = *0*
  by (*metis fls-lr-inverse-base fls-nonzeroI*)+

lemma *fls-lr-inverse-eq-0-iff*:
  fixes *x* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
  and *y* :: *'b*::{*ab-group-add*,*mult-zero*}
  shows *fls-left-inverse f x* = *0* ⟷ *x* = *0*
  and *fls-right-inverse g y* = *0* ⟷ *y* = *0*
  using *fls-lr-inverse-starting0 fls-lr-inverse-eq0-imp-starting0*
  by *auto*

lemma *fls-inverse-eq-0-iff'*:
  fixes *f* :: *'a*::{*ab-group-add*,*inverse*,*mult-zero*} *fls*

shows *inverse f* = *0* ⟷ (*inverse* (*f* $$ *fls-subdegree f*) = *0*)
using *fls-lr-inverse-eq-0-iff* (*2*)[*of f inverse* (*f* $$ *fls-subdegree f*)]
by (*simp add*: *fls-inverse-def′*)

lemma *fls-inverse-eq-0-iff* [*simp*]:
inverse f = (*0*:: (*′a*::*division-ring*) *fls*) ⟷ *f* $$ *fls-subdegree f* = *0*
using *fls-inverse-eq-0-iff′*[*of f*] **by** (*cases f*=*0*) *auto*

lemmas *fls-inverse-eq-0′* = *iffD2*[*OF fls-inverse-eq-0-iff′*]
lemmas *fls-inverse-eq-0* = *iffD2*[*OF fls-inverse-eq-0-iff*]

lemma *fls-lr-inverse-const*:
fixes *a* :: *′a*::{*ab-group-add*,*mult-zero*}
and *b* :: *′b*::{*comm-monoid-add*,*mult-zero*,*uminus*}
shows *fls-left-inverse* (*fls-const a*) *x* = *fls-const x*
and *fls-right-inverse* (*fls-const b*) *y* = *fls-const y*
by (*simp-all add*: *fps-const-lr-inverse*)

lemma *fls-inverse-const*:
fixes *a* :: *′a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}
shows *inverse* (*fls-const a*) = *fls-const* (*inverse a*)
using *fls-lr-inverse-const*(*2*)
by (*auto simp*: *fls-inverse-def′*)

lemma *fls-lr-inverse-of-nat*:
fixes *x* :: *′a*::{*ring-1*,*mult-zero*}
and *y* :: *′b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*of-nat n*) *x* = *fls-const x*
and *fls-right-inverse* (*of-nat n*) *y* = *fls-const y*
using *fls-lr-inverse-const*
by (*auto simp*: *fls-of-nat*)

lemma *fls-inverse-of-nat*:
inverse (*of-nat n* :: *′a* :: {*semiring-1*,*inverse*,*uminus*} *fls*) = *fls-const* (*inverse* (*of-nat n*))
by (*simp add*: *fls-inverse-const fls-of-nat*)

lemma *fls-lr-inverse-of-int*:
fixes *x* :: *′a*::{*ring-1*,*mult-zero*}
shows *fls-left-inverse* (*of-int n*) *x* = *fls-const x*
and *fls-right-inverse* (*of-int n*) *x* = *fls-const x*
using *fls-lr-inverse-const*
by (*auto simp*: *fls-of-int*)

lemma *fls-inverse-of-int*:
inverse (*of-int n* :: *′a* :: {*ring-1*,*inverse*,*uminus*} *fls*) = *fls-const* (*inverse* (*of-int n*))
by (*simp add*: *fls-inverse-const fls-of-int*)

lemma *fls-lr-inverse-zero*:
fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*}
and *y* :: *'b*::{*comm-monoid-add*,*mult-zero*,*uminus*}
shows *fls-left-inverse 0 x* = *fls-const x*
and *fls-right-inverse 0 y* = *fls-const y*
using *fls-lr-inverse-const*[*of 0*]
by *auto*

lemma *fls-inverse-zero-conv-fls-const*:
inverse (*0*::*'a*::{*comm-monoid-add*,*mult-zero*,*uminus*,*inverse*} *fls*) = *fls-const* (*inverse 0*)
using *fls-lr-inverse-zero*(*2*)[*of inverse* (*0*::*'a*)] **by** (*simp add*: *fls-inverse-def'*)

lemma *fls-inverse-zero'*:
assumes *inverse* (*0*::*'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}) = *0*
shows *inverse* (*0*::*'a fls*) = *0*
by (*simp add*: *fls-inverse-zero-conv-fls-const assms*)

lemma *fls-inverse-zero* [*simp*]: *inverse* (*0*::*'a*::*division-ring fls*) = *0*
by (*rule fls-inverse-zero'*[*OF inverse-zero*])

lemma *fls-inverse-base2*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*,*inverse*} *fls*
shows *inverse f* $$ (−*fls-subdegree f*) = *inverse* (*f* $$ *fls-subdegree f*)
by (*cases f*=*0*) (*simp-all add*: *fls-inverse-zero-conv-fls-const fls-inverse-def'*)

lemma *fls-lr-inverse-one*:
fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*,*one*}
and *y* :: *'b*::{*comm-monoid-add*,*mult-zero*,*uminus*,*one*}
shows *fls-left-inverse 1 x* = *fls-const x*
and *fls-right-inverse 1 y* = *fls-const y*
using *fls-lr-inverse-const*[*of 1*]
by *auto*

lemma *fls-lr-inverse-one-one*:
fls-left-inverse 1 1 =
(*1*::*'a*::{*ab-group-add*,*mult-zero*,*one*} *fls*)
fls-right-inverse 1 1 =
(*1*::*'b*::{*comm-monoid-add*,*mult-zero*,*uminus*,*one*} *fls*)
using *fls-lr-inverse-one*[*of 1*] **by** *auto*

lemma *fls-inverse-one*:
assumes *inverse* (*1*::*'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*one*}) = *1*
shows *inverse* (*1*::*'a fls*) = *1*
using *assms fls-lr-inverse-one-one*(*2*)
by (*simp add*: *fls-inverse-def'*)

lemma *fls-left-inverse-delta*:
fixes *b* :: *'a*::{*ab-group-add*,*mult-zero*}

assumes $b \neq 0$
shows *fls-left-inverse* (*Abs-fls* (λn. *if* $n=a$ *then* b *else* 0)) x =
Abs-fls (λn. *if* $n=-a$ *then* x *else* 0)
proof (*intro fls-eqI*)
fix n **from** *assms* **show**
fls-left-inverse (*Abs-fls* (λn. *if* $n=a$ *then* b *else* 0)) x $$ n
= *Abs-fls* (λn. *if* $n = - a$ *then* x *else* 0) $$ n
using *fls-base-factor-to-fps-delta*[*of a b*]
fls-lr-inverse-const(*1*)[*of b*]
fls-shift-const
by *simp*
qed

lemma *fls-right-inverse-delta*:
fixes b :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
assumes $b \neq 0$
shows *fls-right-inverse* (*Abs-fls* (λn. *if* $n=a$ *then* b *else* 0)) x =
Abs-fls (λn. *if* $n=-a$ *then* x *else* 0)
proof (*intro fls-eqI*)
fix n **from** *assms* **show**
fls-right-inverse (*Abs-fls* (λn. *if* $n=a$ *then* b *else* 0)) x $$ n
= *Abs-fls* (λn. *if* $n = - a$ *then* x *else* 0) $$ n
using *fls-base-factor-to-fps-delta*[*of a b*]
fls-lr-inverse-const(*2*)[*of b*]
fls-shift-const
by *simp*
qed

lemma *fls-inverse-delta-nonzero*:
fixes b :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}
assumes $b \neq 0$
shows *inverse* (*Abs-fls* (λn. *if* $n=a$ *then* b *else* 0)) =
Abs-fls (λn. *if* $n=-a$ *then inverse* b *else* 0)
using *assms fls-nonzeroI*[*of Abs-fls* (λn. *if* $n=a$ *then* b *else* 0) *a*]
by (*simp add*: *fls-inverse-def' fls-right-inverse-delta*[*symmetric*])

lemma *fls-inverse-delta*:
fixes b :: *'a*::*division-ring*
shows *inverse* (*Abs-fls* (λn. *if* $n=a$ *then* b *else* 0)) =
Abs-fls (λn. *if* $n=-a$ *then inverse* b *else* 0)
by (*cases* $b=0$) (*simp-all add*: *fls-inverse-delta-nonzero*)

lemma *fls-lr-inverse-X*:
fixes x :: *'a*::{*ab-group-add*,*mult-zero*,*zero-neq-one*}
and y :: *'b*::{*comm-monoid-add*,*uminus*,*mult-zero*,*zero-neq-one*}
shows *fls-left-inverse fls-X* x = *fls-shift* 1 (*fls-const* x)
and *fls-right-inverse fls-X* y = *fls-shift* 1 (*fls-const* y)
using *fls-lr-inverse-one*(*1*)[*of x*] *fls-lr-inverse-one*(*2*)[*of y*]
by *auto*

lemma *fls-lr-inverse-X'*:
fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*,*zero-neq-one*,*monoid-mult*}
and *y* :: *'b*::{*comm-monoid-add*,*uminus*,*mult-zero*,*zero-neq-one*,*monoid-mult*}
shows *fls-left-inverse fls-X x* = *fls-const x* ∗ *fls-X-inv*
and *fls-right-inverse fls-X y* = *fls-const y* ∗ *fls-X-inv*
using *fls-lr-inverse-X*(*1*)[*of x*] *fls-lr-inverse-X*(*2*)[*of y*]
by (*simp-all add*: *fls-X-inv-times-conv-shift*(*2*))

lemma *fls-inverse-X'*:
assumes *inverse 1* = (*1*::*'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*})
shows *inverse* (*fls-X*::*'a fls*) = *fls-X-inv*
using *assms fls-lr-inverse-X*(*2*)[*of 1*::*'a*]
by (*simp add*: *fls-inverse-def' fls-X-inv-conv-shift-1*)

lemma *fls-inverse-X*: *inverse* (*fls-X*::*'a*::*division-ring fls*) = *fls-X-inv*
by (*simp add*: *fls-inverse-X'*)

lemma *fls-lr-inverse-X-inv*:
fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*,*zero-neq-one*}
and *y* :: *'b*::{*comm-monoid-add*,*uminus*,*mult-zero*,*zero-neq-one*}
shows *fls-left-inverse fls-X-inv x* = *fls-shift* (−*1*) (*fls-const x*)
and *fls-right-inverse fls-X-inv y* = *fls-shift* (−*1*) (*fls-const y*)
using *fls-lr-inverse-one*(*1*)[*of x*] *fls-lr-inverse-one*(*2*)[*of y*]
by *auto*

lemma *fls-lr-inverse-X-inv'*:
fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*,*zero-neq-one*,*monoid-mult*}
and *y* :: *'b*::{*comm-monoid-add*,*uminus*,*mult-zero*,*zero-neq-one*,*monoid-mult*}
shows *fls-left-inverse fls-X-inv x* = *fls-const x* ∗ *fls-X*
and *fls-right-inverse fls-X-inv y* = *fls-const y* ∗ *fls-X*
using *fls-lr-inverse-X-inv*(*1*)[*of x*] *fls-lr-inverse-X-inv*(*2*)[*of y*]
by (*simp-all add*: *fls-X-times-conv-shift*(*2*))

lemma *fls-inverse-X-inv'*:
assumes *inverse 1* = (*1*::*'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*})
shows *inverse* (*fls-X-inv*::*'a fls*) = *fls-X*
using *assms fls-lr-inverse-X-inv*(*2*)[*of 1*::*'a*]
by (*simp add*: *fls-inverse-def' fls-X-conv-shift-1*)

lemma *fls-inverse-X-inv*: *inverse* (*fls-X-inv*::*'a*::*division-ring fls*) = *fls-X*
by (*simp add*: *fls-inverse-X-inv'*)

lemma *fls-lr-inverse-subdegree*:
assumes *x* ≠ *0*
shows *fls-subdegree* (*fls-left-inverse f x*) = − *fls-subdegree f*
and *fls-subdegree* (*fls-right-inverse f x*) = − *fls-subdegree f*
by (*auto intro*: *fls-subdegree-eqI simp*: *assms*)

lemma *fls-inverse-subdegree′*:
inverse (*f* $$ *fls-subdegree f*) ≠ *0* ⟹ *fls-subdegree* (*inverse f*) = − *fls-subdegree f*
using *fls-lr-inverse-subdegree*(*2*)[*of inverse* (*f* $$ *fls-subdegree f*)]
by (*simp add*: *fls-inverse-def′*)

lemma *fls-inverse-subdegree* [*simp*]:
fixes *f* :: ′*a*::*division-ring fls*
shows *fls-subdegree* (*inverse f*) = − *fls-subdegree f*
by (*cases f=0*)
(*auto intro*: *fls-inverse-subdegree′ simp*: *nonzero-imp-inverse-nonzero*)

lemma *fls-inverse-subdegree-base-nonzero*:
assumes *f* ≠ *0 inverse* (*f* $$ *fls-subdegree f*) ≠ *0*
shows *inverse f* $$ (*fls-subdegree* (*inverse f*)) = *inverse* (*f* $$ *fls-subdegree f*)
using *assms fls-inverse-subdegree′*[*of f*] *fls-inverse-base*[*of f*]
by *simp*

lemma *fls-inverse-subdegree-base*:
fixes *f* :: ′*a*::{*ab-group-add*,*inverse*,*mult-zero*} *fls*
shows *inverse f* $$ (*fls-subdegree* (*inverse f*)) = *inverse* (*f* $$ *fls-subdegree f*)
using *fls-inverse-eq-0-iff′*[*of f*] *fls-inverse-subdegree-base-nonzero*[*of f*]
by (*cases f=0* ∨ *inverse* (*f* $$ *fls-subdegree f*) = *0*)
(*auto simp*: *fls-inverse-zero-conv-fls-const*)

lemma *fls-lr-inverse-subdegree-0*:
assumes *fls-subdegree f* = *0*
shows *fls-subdegree* (*fls-left-inverse f x*) ≥ *0*
and *fls-subdegree* (*fls-right-inverse f x*) ≥ *0*
using *fls-subdegree-ge0I*[*of fls-left-inverse f x*]
fls-subdegree-ge0I[*of fls-right-inverse f x*]
by (*auto simp*: *assms*)

lemma *fls-inverse-subdegree-0*:
fls-subdegree f = *0* ⟹ *fls-subdegree* (*inverse f*) ≥ *0*
using *fls-lr-inverse-subdegree-0*(*2*)[*of f*] **by** (*simp add*: *fls-inverse-def′*)

lemma *fls-lr-inverse-shift-nonzero*:
fixes *f* :: ′*a*::{*comm-monoid-add*,*mult-zero*,*uminus*} *fls*
assumes *f* ≠ *0*
shows *fls-left-inverse* (*fls-shift m f*) *x* = *fls-shift* (−*m*) (*fls-left-inverse f x*)
and *fls-right-inverse* (*fls-shift m f*) *x* = *fls-shift* (−*m*) (*fls-right-inverse f x*)
using *assms fls-base-factor-to-fps-shift*[*of m f*] *fls-shift-subdegree*
by *auto*

lemma *fls-inverse-shift-nonzero*:
fixes *f* :: ′*a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
assumes *f* ≠ *0*
shows *inverse* (*fls-shift m f*) = *fls-shift* (−*m*) (*inverse f*)

using *assms fls-lr-inverse-shift-nonzero(2)[of f m inverse (f $$ fls-subdegree f)]*
by *(simp add: fls-inverse-def′)*

lemma *fls-inverse-shift*:
fixes *f* :: *′a::division-ring fls*
shows *inverse (fls-shift m f) = fls-shift (−m) (inverse f)*
using *fls-inverse-shift-nonzero*
by *(cases f=0) simp-all*

lemma *fls-left-inverse-base-factor*:
fixes *x* :: *′a::{ab-group-add,mult-zero}*
assumes *x ≠ 0*
shows *fls-left-inverse (fls-base-factor f) x = fls-base-factor (fls-left-inverse f x)*
using *assms fls-lr-inverse-zero(1)[of x] fls-lr-inverse-subdegree(1)[of x]*
by *(cases f=0) auto*

lemma *fls-right-inverse-base-factor*:
fixes *y* :: *′a::{comm-monoid-add,mult-zero,uminus}*
assumes *y ≠ 0*
shows *fls-right-inverse (fls-base-factor f) y = fls-base-factor (fls-right-inverse f y)*
using *assms fls-lr-inverse-zero(2)[of y] fls-lr-inverse-subdegree(2)[of y]*
by *(cases f=0) auto*

lemma *fls-inverse-base-factor′*:
fixes *f* :: *′a::{comm-monoid-add,inverse,mult-zero,uminus} fls*
assumes *inverse (f $$ fls-subdegree f) ≠ 0*
shows *inverse (fls-base-factor f) = fls-base-factor (inverse f)*
by *(cases f=0)*
(simp-all add:
assms fls-inverse-shift-nonzero fls-inverse-subdegree′
fls-inverse-zero-conv-fls-const
)

lemma *fls-inverse-base-factor*:
fixes *f* :: *′a::{ab-group-add,inverse,mult-zero} fls*
shows *inverse (fls-base-factor f) = fls-base-factor (inverse f)*
using *fls-base-factor-base[of f] fls-inverse-eq-0-iff′[of f]*
fls-inverse-eq-0-iff′[of fls-base-factor f] fls-inverse-base-factor′[of f]
by *(cases inverse (f $$ fls-subdegree f) = 0) simp-all*

lemma *fls-lr-inverse-regpart*:
assumes *fls-subdegree f = 0*
shows *fls-regpart (fls-left-inverse f x) = fps-left-inverse (fls-regpart f) x*
and *fls-regpart (fls-right-inverse f y) = fps-right-inverse (fls-regpart f) y*
using *assms*
by *auto*

lemma *fls-inverse-regpart*:

assumes *fls-subdegree f* = *0*
shows *fls-regpart* (*inverse f*) = *inverse* (*fls-regpart f*)
by (*simp add*: *assms fls-inverse-def*)

lemma *fls-base-factor-to-fps-left-inverse*:
fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*}
shows *fls-base-factor-to-fps* (*fls-left-inverse f x*) =
fps-left-inverse (*fls-base-factor-to-fps f*) *x*
using *fls-left-inverse-base-factor*[*of x f*] *fls-base-factor-subdegree*[*of f*]
by (*cases x*=*0*) (*simp-all add*: *fls-lr-inverse-starting0*(*1*) *fps-lr-inverse-starting0*(*1*))

lemma *fls-base-factor-to-fps-right-inverse-nonzero*:
fixes *y* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
assumes *y* ≠ *0*
shows *fls-base-factor-to-fps* (*fls-right-inverse f y*) =
fps-right-inverse (*fls-base-factor-to-fps f*) *y*
using *assms fls-right-inverse-base-factor*[*of y f*]
fls-base-factor-subdegree[*of f*]
by *simp*

lemma *fls-base-factor-to-fps-right-inverse*:
fixes *y* :: *'a*::{*ab-group-add*,*mult-zero*}
shows *fls-base-factor-to-fps* (*fls-right-inverse f y*) =
fps-right-inverse (*fls-base-factor-to-fps f*) *y*
using *fls-base-factor-to-fps-right-inverse-nonzero*[*of y f*]
by (*cases y*=*0*) (*simp-all add*: *fls-lr-inverse-starting0*(*2*) *fps-lr-inverse-starting0*(*2*))

lemma *fls-base-factor-to-fps-inverse-nonzero*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
assumes *inverse* (*f* $$ *fls-subdegree f*) ≠ *0*
shows *fls-base-factor-to-fps* (*inverse f*) = *inverse* (*fls-base-factor-to-fps f*)
using *assms fls-base-factor-to-fps-right-inverse-nonzero*
by (*simp add*: *fls-inverse-def' fps-inverse-def*)

lemma *fls-base-factor-to-fps-inverse*:
fixes *f* :: *'a*::{*ab-group-add*,*inverse*,*mult-zero*} *fls*
shows *fls-base-factor-to-fps* (*inverse f*) = *inverse* (*fls-base-factor-to-fps f*)
using *fls-base-factor-to-fps-right-inverse*
by (*simp add*: *fls-inverse-def' fps-inverse-def*)

lemma *fls-lr-inverse-fps-to-fls*:
assumes *subdegree f* = *0*
shows *fls-left-inverse* (*fps-to-fls f*) *x* = *fps-to-fls* (*fps-left-inverse f x*)
and *fls-right-inverse* (*fps-to-fls f*) *x* = *fps-to-fls* (*fps-right-inverse f x*)
using *assms fls-base-factor-to-fps-to-fls*[*of f*]
by (*simp-all add*: *fls-subdegree-fls-to-fps*)

lemma *fls-inverse-fps-to-fls*:
subdegree f = *0* ⟹ *inverse* (*fps-to-fls f*) = *fps-to-fls* (*inverse f*)

using *nth-subdegree-nonzero*[*of f*]
by (*cases f=0*)
(*auto simp add*:
fps-to-fls-nonzeroI fls-inverse-def′ fls-subdegree-fls-to-fps fps-inverse-def
fls-lr-inverse-fps-to-fls(*2*)
)

lemma *fls-lr-inverse-X-power*:
fixes *x* :: ′*a*::*ring-1*
and *y* :: ′*b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X* ^ *n*) *x* = *fls-shift n* (*fls-const x*)
and *fls-right-inverse* (*fls-X* ^ *n*) *y* = *fls-shift n* (*fls-const y*)
using *fls-lr-inverse-one*(*1*)[*of x*] *fls-lr-inverse-one*(*2*)[*of y*]
by (*simp-all add*: *fls-X-power-conv-shift-1*)

lemma *fls-lr-inverse-X-power′*:
fixes *x* :: ′*a*::*ring-1*
and *y* :: ′*b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X* ^ *n*) *x* = *fls-const x* ∗ *fls-X-inv* ^ *n*
and *fls-right-inverse* (*fls-X* ^ *n*) *y* = *fls-const y* ∗ *fls-X-inv* ^ *n*
using *fls-lr-inverse-X-power*(*1*)[*of n x*] *fls-lr-inverse-X-power*(*2*)[*of n y*]
by (*simp-all add*: *fls-X-inv-power-times-conv-shift*(*2*))

lemma *fls-inverse-X-power′*:
assumes *inverse 1* = (*1*::′*a*::{*semiring-1*,*uminus*,*inverse*})
shows *inverse* ((*fls-X* ^ *n*)::′*a fls*) = *fls-X-inv* ^ *n*
using *fls-lr-inverse-X-power′*(*2*)[*of n 1*]
by (*simp add*: *fls-inverse-def′ assms*)

lemma *fls-inverse-X-power*:
inverse ((*fls-X*::′*a*::*division-ring fls*) ^ *n*) = *fls-X-inv* ^ *n*
by (*simp add*: *fls-inverse-X-power′*)

lemma *fls-lr-inverse-X-inv-power*:
fixes *x* :: ′*a*::*ring-1*
and *y* :: ′*b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-inv* ^ *n*) *x* = *fls-shift* (−*n*) (*fls-const x*)
and *fls-right-inverse* (*fls-X-inv* ^ *n*) *y* = *fls-shift* (−*n*) (*fls-const y*)
using *fls-lr-inverse-one*(*1*)[*of x*] *fls-lr-inverse-one*(*2*)[*of y*]
by (*simp-all add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-lr-inverse-X-inv-power′*:
fixes *x* :: ′*a*::*ring-1*
and *y* :: ′*b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-inv* ^ *n*) *x* = *fls-const x* ∗ *fls-X* ^ *n*
and *fls-right-inverse* (*fls-X-inv* ^ *n*) *y* = *fls-const y* ∗ *fls-X* ^ *n*
using *fls-lr-inverse-X-inv-power*(*1*)[*of n x*] *fls-lr-inverse-X-inv-power*(*2*)[*of n y*]
by (*simp-all add*: *fls-X-power-times-conv-shift*(*2*))

lemma *fls-inverse-X-inv-power′*:
assumes *inverse 1* = (*1*::′*a*::{*semiring-1*,*uminus*,*inverse*})
shows *inverse* ((*fls-X-inv* ^ *n*)::′*a fls*) = *fls-X* ^ *n*
using *fls-lr-inverse-X-inv-power′*(*2*)[*of n 1*]
by (*simp add*: *fls-inverse-def′ assms*)

lemma *fls-inverse-X-inv-power*:
inverse ((*fls-X-inv*::′*a*::*division-ring fls*) ^ *n*) = *fls-X* ^ *n*
by (*simp add*: *fls-inverse-X-inv-power′*)

lemma *fls-lr-inverse-X-intpow*:
fixes *x* :: ′*a*::*ring-1*
and *y* :: ′*b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-intpow i*) *x* = *fls-shift i* (*fls-const x*)
and *fls-right-inverse* (*fls-X-intpow i*) *y* = *fls-shift i* (*fls-const y*)
using *fls-lr-inverse-one*(*1*)[*of x*] *fls-lr-inverse-one*(*2*)[*of y*]
by *auto*

lemma *fls-lr-inverse-X-intpow′*:
fixes *x* :: ′*a*::*ring-1*
and *y* :: ′*b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-intpow i*) *x* = *fls-const x* ∗ *fls-X-intpow* (−*i*)
and *fls-right-inverse* (*fls-X-intpow i*) *y* = *fls-const y* ∗ *fls-X-intpow* (−*i*)
using *fls-lr-inverse-X-intpow*(*1*)[*of i x*] *fls-lr-inverse-X-intpow*(*2*)[*of i y*]
by (*simp-all add*: *fls-shifted-times-simps*(*1*))

lemma *fls-inverse-X-intpow′*:
assumes *inverse 1* = (*1*::′*a*::{*semiring-1*,*uminus*,*inverse*})
shows *inverse* (*fls-X-intpow i* :: ′*a fls*) = *fls-X-intpow* (−*i*)
using *fls-lr-inverse-X-intpow′*(*2*)[*of i 1*]
by (*simp add*: *fls-inverse-def′ assms*)

lemma *fls-inverse-X-intpow*:
inverse (*fls-X-intpow i* :: ′*a*::*division-ring fls*) = *fls-X-intpow* (−*i*)
by (*simp add*: *fls-inverse-X-intpow′*)

lemma *fls-left-inverse*:
fixes *f* :: ′*a*::*ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1*
shows *fls-left-inverse f x* ∗ *f* = *1*
proof−
from *assms* **have** *x* ≠ *0 x* ∗ (*fls-base-factor-to-fps f*$*0*) = *1* **by** *auto*
thus *?thesis*
using *fls-base-factor-to-fps-left-inverse*[*of f x*]
fls-lr-inverse-subdegree(*1*)[*of x*] *fps-left-inverse*
by (*fastforce simp*: *fls-times-def*)
qed

lemma *fls-right-inverse*:

fixes *f* :: *'a*::*ring-1 fls*
assumes *f* $$ *fls-subdegree f* ∗ *y* = *1*
shows *f* ∗ *fls-right-inverse f y* = *1*
proof−
from *assms* **have** *y* ≠ *0* (*fls-base-factor-to-fps f*$*0*) ∗ *y* = *1* **by** *auto*
thus *?thesis*
using *fls-base-factor-to-fps-right-inverse*[*of f y*]
fls-lr-inverse-subdegree(*2*)[*of y*] *fps-right-inverse*
by (*fastforce simp*: *fls-times-def*)
qed

— It is possible in a ring for an element to have a left inverse but not a right inverse, or vice versa. But when an element has both, they must be the same.
lemma *fls-left-inverse-eq-fls-right-inverse*:
fixes *f* :: *'a*::*ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *fls-left-inverse f x* = *fls-right-inverse f y*
using *assms*
by (*simp add*: *fps-left-inverse-eq-fps-right-inverse*)

lemma *fls-left-inverse-eq-inverse*:
fixes *f* :: *'a*::*division-ring fls*
shows *fls-left-inverse f* (*inverse* (*f* $$ *fls-subdegree f*)) = *inverse f*
proof (*cases f*=*0*)
case *True*
hence *fls-left-inverse f* (*inverse* (*f* $$ *fls-subdegree f*)) = *fls-const* (*0*::*'a*)
by (*simp add*: *fls-lr-inverse-zero*(*1*)[*symmetric*])
with *True* **show** *?thesis* **by** *simp*
next
case *False* **thus** *?thesis*
using *fls-left-inverse-eq-fls-right-inverse*[*of inverse* (*f* $$ *fls-subdegree f*)]
by (*auto simp add*: *fls-inverse-def'*)
qed

lemma *fls-right-inverse-eq-inverse*:
fixes *f* :: *'a*::*division-ring fls*
shows *fls-right-inverse f* (*inverse* (*f* $$ *fls-subdegree f*)) = *inverse f*
proof (*cases f*=*0*)
case *True*
hence *fls-right-inverse f* (*inverse* (*f* $$ *fls-subdegree f*)) = *fls-const* (*0*::*'a*)
by (*simp add*: *fls-lr-inverse-zero*(*2*)[*symmetric*])
with *True* **show** *?thesis* **by** *simp*
qed (*simp add*: *fls-inverse-def'*)

lemma *fls-left-inverse-eq-fls-right-inverse-comm*:
fixes *f* :: *'a*::*comm-ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1*
shows *fls-left-inverse f x* = *fls-right-inverse f x*

using *assms fls-left-inverse-eq-fls-right-inverse*[*of x f x*]
by (*simp add*: *mult.commute*)

lemma *fls-left-inverse*′:
fixes *f* :: ′*a*::*ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *fls-right-inverse f y* ∗ *f* = *1*
using *assms fls-left-inverse-eq-fls-right-inverse*[*of x f y*] *fls-left-inverse*[*of x f*]
by *simp*

lemma *fls-right-inverse*′:
fixes *f* :: ′*a*::*ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *f* ∗ *fls-left-inverse f x* = *1*
using *assms fls-left-inverse-eq-fls-right-inverse*[*of x f y*] *fls-right-inverse*[*of f y*]
by *simp*

lemma *fls-mult-left-inverse-base-factor*:
fixes *f* :: ′*a*::*ring-1 fls*
assumes *x* ∗ (*f* $$ *fls-subdegree f*) = *1*
shows *fls-left-inverse* (*fls-base-factor f*) *x* ∗ *f* = *fls-X-intpow* (*fls-subdegree f*)
using *assms fls-base-factor-to-fps-base-factor*[*of f*] *fls-base-factor-subdegree*[*of f*]
fls-shifted-times-simps(*2*)[*of* −*fls-subdegree f fls-left-inverse f x f*]
fls-left-inverse[*of x f*]
by *simp*

lemma *fls-mult-right-inverse-base-factor*:
fixes *f* :: ′*a*::*ring-1 fls*
assumes (*f* $$ *fls-subdegree f*) ∗ *y* = *1*
shows *f* ∗ *fls-right-inverse* (*fls-base-factor f*) *y* = *fls-X-intpow* (*fls-subdegree f*)
using *assms fls-base-factor-to-fps-base-factor*[*of f*] *fls-base-factor-subdegree*[*of f*]
fls-shifted-times-simps(*1*)[*of f* −*fls-subdegree f fls-right-inverse f y*]
fls-right-inverse[*of f y*]
by *simp*

lemma *fls-mult-inverse-base-factor*:
fixes *f* :: ′*a*::*division-ring fls*
assumes *f* ≠ *0*
shows *f* ∗ *inverse* (*fls-base-factor f*) = *fls-X-intpow* (*fls-subdegree f*)
using *fls-mult-right-inverse-base-factor*[*of f inverse* (*f* $$ *fls-subdegree f*)]
fls-base-factor-base[*of f*]
by (*simp add*: *assms fls-right-inverse-eq-inverse*[*symmetric*])

lemma *fls-left-inverse-idempotent-ring1*:
fixes *f* :: ′*a*::*ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 y* ∗ *x* = *1*
— These assumptions imply y equals *f* $$ *fls-subdegree f*, but no need to assume

that.
shows *fls-left-inverse* (*fls-left-inverse f x*) *y* = *f*
proof−
from *assms*(*1*) **have**
fls-left-inverse (*fls-left-inverse f x*) *y* ∗ *fls-left-inverse f x* ∗ *f* =
fls-left-inverse (*fls-left-inverse f x*) *y*
using *fls-left-inverse*[*of x f*]
by (*simp add*: *mult.assoc*)
moreover have
fls-left-inverse (*fls-left-inverse f x*) *y* ∗ *fls-left-inverse f x* = *1*
using *assms fls-lr-inverse-subdegree*(*1*)[*of x f*] *fls-lr-inverse-base*(*1*)[*of f x*]
by (*fastforce intro*: *fls-left-inverse*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fls-left-inverse-idempotent-comm-ring1*:
fixes *f* :: ′*a*::*comm-ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1*
shows *fls-left-inverse* (*fls-left-inverse f x*) (*f* $$ *fls-subdegree f*) = *f*
using *assms fls-left-inverse-idempotent-ring1*[*of x f f* $$ *fls-subdegree f*]
by (*simp add*: *mult.commute*)

lemma *fls-right-inverse-idempotent-ring1*:
fixes *f* :: ′*a*::*ring-1 fls*
assumes *f* $$ *fls-subdegree f* ∗ *x* = *1 x* ∗ *y* = *1*
— These assumptions imply y equals *f* $$ *fls-subdegree f*, but no need to assume that.
shows *fls-right-inverse* (*fls-right-inverse f x*) *y* = *f*
proof−
from *assms*(*1*) **have**
f ∗ (*fls-right-inverse f x* ∗ *fls-right-inverse* (*fls-right-inverse f x*) *y*) =
fls-right-inverse (*fls-right-inverse f x*) *y*
using *fls-right-inverse* [*of f*]
by (*simp add*: *mult.assoc*[*symmetric*])
moreover have
fls-right-inverse f x ∗ *fls-right-inverse* (*fls-right-inverse f x*) *y* = *1*
using *assms fls-lr-inverse-subdegree*(*2*)[*of x f*] *fls-lr-inverse-base*(*2*)[*of f x*]
by (*fastforce intro*: *fls-right-inverse*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fls-right-inverse-idempotent-comm-ring1*:
fixes *f* :: ′*a*::*comm-ring-1 fls*
assumes *f* $$ *fls-subdegree f* ∗ *x* = *1*
shows *fls-right-inverse* (*fls-right-inverse f x*) (*f* $$ *fls-subdegree f*) = *f*
using *assms fls-right-inverse-idempotent-ring1*[*of f x f* $$ *fls-subdegree f*]
by (*simp add*: *mult.commute*)

lemma *fls-lr-inverse-unique-ring1*:

```
  fixes   f g :: 'a :: ring-1 fls
  assumes fg: f ∗ g = 1 g $$ fls-subdegree g ∗ f $$ fls-subdegree f = 1
  shows   fls-left-inverse g (f $$ fls-subdegree f) = f
  and     fls-right-inverse f (g $$ fls-subdegree g) = g
proof−

  have f $$ fls-subdegree f ∗ g $$ fls-subdegree g ≠ 0
  proof
    assume f $$ fls-subdegree f ∗ g $$ fls-subdegree g = 0
    hence f $$ fls-subdegree f ∗ (g $$ fls-subdegree g ∗ f $$ fls-subdegree f) = 0
      by (simp add: mult.assoc[symmetric])
    with fg(2) show False by simp
  qed
  with fg(1) have subdeg-sum: fls-subdegree f + fls-subdegree g = 0
    using fls-mult-nonzero-base-subdegree-eq[of f g] by simp
  hence subdeg-sum':
    fls-subdegree f = −fls-subdegree g fls-subdegree g = −fls-subdegree f
    by auto

  from fg(1) have f-ne-0: f≠0 by auto
  moreover have
    fps-left-inverse (fls-base-factor-to-fps g) (fls-regpart (fls-shift (−fls-subdegree g)
f)$0)
      = fls-regpart (fls-shift (−fls-subdegree g) f)
  proof (intro fps-lr-inverse-unique-ring1(1))
    from fg(1) show
      fls-regpart (fls-shift (−fls-subdegree g) f) ∗ fls-base-factor-to-fps g = 1
     using f-ne-0 fls-times-conv-regpart[of fls-shift (−fls-subdegree g) f fls-base-factor
g]
           fls-base-factor-subdegree[of g]
      by    (simp add: fls-times-both-shifted-simp subdeg-sum)
    from fg(2) show
     fls-base-factor-to-fps g $ 0 ∗ fls-regpart (fls-shift (−fls-subdegree g) f) $ 0 = 1
      by (simp add: subdeg-sum'(2))
  qed
  ultimately show fls-left-inverse g (f $$ fls-subdegree f) = f
    by (simp add: subdeg-sum'(2))

  from fg(1) have g-ne-0: g≠0 by auto
  moreover have
     fps-right-inverse (fls-base-factor-to-fps f) (fls-regpart (fls-shift (−fls-subdegree
f) g)$0)
      = fls-regpart (fls-shift (−fls-subdegree f) g)
  proof (intro fps-lr-inverse-unique-ring1(2))
    from fg(1) show
      fls-base-factor-to-fps f ∗ fls-regpart (fls-shift (−fls-subdegree f) g) = 1
      using g-ne-0 fls-times-conv-regpart[of fls-base-factor f fls-shift (−fls-subdegree
f) g]
           fls-base-factor-subdegree[of f]
```

by (*simp add*: *fls-times-both-shifted-simp subdeg-sum add.commute*)
from *fg*(*2*) **show**
fls-regpart (*fls-shift* (−*fls-subdegree f*) *g*) $ *0* ∗ *fls-base-factor-to-fps f* $ *0* = *1*
by (*simp add*: *subdeg-sum′*(*1*))
qed
ultimately show *fls-right-inverse f* (*g* $$ *fls-subdegree g*) = *g*
by (*simp add*: *subdeg-sum′*(*2*))

qed

lemma *fls-lr-inverse-unique-divring*:
fixes *f g* :: ′*a* ::*division-ring fls*
assumes *fg*: *f* ∗ *g* = *1*
shows *fls-left-inverse g* (*f* $$ *fls-subdegree f*) = *f*
and *fls-right-inverse f* (*g* $$ *fls-subdegree g*) = *g*
proof−
from *fg* **have** *f* ≠*0 g* ≠ *0* **by** *auto*
with *fg* **have** *fls-subdegree f* + *fls-subdegree g* = *0* **using** *fls-subdegree-mult* **by** *force*
with *fg* **have** *f* $$ *fls-subdegree f* ∗ *g* $$ *fls-subdegree g* = *1*
using *fls-times-base*[*of f g*] **by** *simp*
hence *g* $$ *fls-subdegree g* ∗ *f* $$ *fls-subdegree f* = *1*
using *inverse-unique*[*of f* $$ *fls-subdegree f*] *left-inverse*[*of f* $$ *fls-subdegree f*]
by *force*
thus
fls-left-inverse g (*f* $$ *fls-subdegree f*) = *f*
fls-right-inverse f (*g* $$ *fls-subdegree g*) = *g*
using *fg fls-lr-inverse-unique-ring1*
by *auto*
qed

lemma *fls-lr-inverse-minus*:
fixes *f* :: ′*a*::*ring-1 fls*
shows *fls-left-inverse* (−*f*) (−*x*) = − *fls-left-inverse f x*
and *fls-right-inverse* (−*f*) (−*x*) = − *fls-right-inverse f x*
by (*simp-all add*: *fps-lr-inverse-minus*)

lemma *fls-inverse-minus* [*simp*]: *inverse* (−*f*) = −*inverse* (*f* :: ′*a* :: *division-ring fls*)
using *fls-lr-inverse-minus*(*2*)[*of f*] **by** (*simp add*: *fls-inverse-def′*)

lemma *fls-lr-inverse-mult-ring1*:
fixes *f g* :: ′*a*::*ring-1 fls*
assumes *x*: *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *x* = *1*
and *y*: *y* ∗ *g* $$ *fls-subdegree g* = *1 g* $$ *fls-subdegree g* ∗ *y* = *1*
shows *fls-left-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fls-left-inverse g y* ∗ *fls-left-inverse f x*
and *fls-right-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fls-right-inverse g y* ∗ *fls-right-inverse f x*
proof−

from *x(1) y(2)* **have** *x ∗ (f $$ fls-subdegree f ∗ g $$ fls-subdegree g) ∗ y = 1*
  by (*simp add*: *mult.assoc*)
hence *base-prod*: *f $$ fls-subdegree f ∗ g $$ fls-subdegree g ≠ 0* **by** *auto*
hence *subdegrees*: *fls-subdegree (f∗g) = fls-subdegree f + fls-subdegree g*
  using *fls-mult-nonzero-base-subdegree-eq[of f g]* **by** *simp*

have *norm*:
  fls-base-factor-to-fps (f ∗ g) = fls-base-factor-to-fps f ∗ fls-base-factor-to-fps g
  using *base-prod fls-base-factor-to-fps-mult'[of f g]* **by** *simp*

have
  fls-left-inverse (f ∗ g) (y∗x) =
    fls-shift (fls-subdegree (f ∗ g)) (
      fps-to-fls (
        fps-left-inverse (fls-base-factor-to-fps f ∗ fls-base-factor-to-fps g) (y∗x)
)
)

  using *norm*
  by *simp*
thus *fls-left-inverse (f ∗ g) (y∗x) = fls-left-inverse g y ∗ fls-left-inverse f x*
  using *x y*
    fps-lr-inverse-mult-ring1(1)[of
      x fls-base-factor-to-fps f y fls-base-factor-to-fps g
    ]
  by (*simp add*:
      fls-times-both-shifted-simp fls-times-fps-to-fls subdegrees algebra-simps
)

have
  fls-right-inverse (f ∗ g) (y∗x) =
    fls-shift (fls-subdegree (f ∗ g)) (
      fps-to-fls (
        fps-right-inverse (fls-base-factor-to-fps f ∗ fls-base-factor-to-fps g) (y∗x)
)
)

  using *norm*
  by *simp*
thus *fls-right-inverse (f ∗ g) (y∗x) = fls-right-inverse g y ∗ fls-right-inverse f x*
  using *x y*
    fps-lr-inverse-mult-ring1(2)[of
      x fls-base-factor-to-fps f y fls-base-factor-to-fps g
    ]
  by (*simp add*:
      fls-times-both-shifted-simp fls-times-fps-to-fls subdegrees algebra-simps
)

qed

lemma *fls-lr-inverse-power-ring1*:
fixes *f* :: *'a::ring-1 fls*
assumes *x*: *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *x* = *1*
shows *fls-left-inverse* (*f* ^ *n*) (*x* ^ *n*) = (*fls-left-inverse f x*) ^ *n*
fls-right-inverse (*f* ^ *n*) (*x* ^ *n*) = (*fls-right-inverse f x*) ^ *n*
proof−

show *fls-left-inverse* (*f* ^ *n*) (*x* ^ *n*) = (*fls-left-inverse f x*) ^ *n*
proof (*induct n*)
case *0* **show** *?case* **using** *fls-lr-inverse-one*(*1*)[*of 1*] **by** *simp*
next
case (*Suc n*) **with** *assms* **show** *?case*
using *fls-lr-inverse-mult-ring1*(*1*)[*of x f x^n f^n*]
by (*simp add*:
power-Suc2[*symmetric*] *fls-unit-base-subdegree-power*(*1*) *left-right-inverse-power*
)
qed

show *fls-right-inverse* (*f* ^ *n*) (*x* ^ *n*) = (*fls-right-inverse f x*) ^ *n*
proof (*induct n*)
case *0* **show** *?case* **using** *fls-lr-inverse-one*(*2*)[*of 1*] **by** *simp*
next
case (*Suc n*) **with** *assms* **show** *?case*
using *fls-lr-inverse-mult-ring1*(*2*)[*of x f x^n f^n*]
by (*simp add*:
power-Suc2[*symmetric*] *fls-unit-base-subdegree-power*(*1*) *left-right-inverse-power*
)
qed

qed

lemma *fls-divide-convert-times-inverse*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
shows *f* / *g* = *f* ∗ *inverse g*
using *fls-base-factor-to-fps-subdegree*[*of g*] *fps-to-fls-base-factor-to-fps*[*of f*]
fls-times-both-shifted-simp[*of* −*fls-subdegree f fls-base-factor f*]
by (*simp add*:
fls-divide-def fps-divide-unit' fls-times-fps-to-fls
fls-conv-base-factor-shift-subdegree fls-inverse-def
)

instance *fls* :: (*division-ring*) *division-ring*
proof
fix *a b* :: *'a fls*
show *a* ≠ *0* ⟹ *inverse a* ∗ *a* = *1*
using *fls-left-inverse'*[*of inverse* (*a* $$ *fls-subdegree a*) *a*]
by (*simp add*: *fls-inverse-def'*)
show *a* ≠ *0* ⟹ *a* ∗ *inverse a* = *1*

using *fls-right-inverse*[*of a*]
by (*simp add*: *fls-inverse-def′*)
show *a* / *b* = *a* ∗ *inverse b* **using** *fls-divide-convert-times-inverse* **by** *fast*
show *inverse* (*0*::′*a fls*) = *0* **by** *simp*
qed

lemma *fls-lr-inverse-mult-divring*:
fixes *f g* :: ′*a*::*division-ring fls*
and *df dg* :: *int*
defines *df* ≡ *fls-subdegree f*
and *dg* ≡ *fls-subdegree g*
shows *fls-left-inverse* (*f*∗*g*) (*inverse* ((*f*∗*g*)$$(*df*+*dg*))) =
fls-left-inverse g (*inverse* (*g*$$*dg*)) ∗ *fls-left-inverse f* (*inverse* (*f*$$*df*))
and *fls-right-inverse* (*f*∗*g*) (*inverse* ((*f*∗*g*)$$(*df*+*dg*))) =
fls-right-inverse g (*inverse* (*g*$$*dg*)) ∗ *fls-right-inverse f* (*inverse* (*f*$$*df*))
proof −
show
fls-left-inverse (*f*∗*g*) (*inverse* ((*f*∗*g*)$$(*df*+*dg*))) =
fls-left-inverse g (*inverse* (*g*$$*dg*)) ∗ *fls-left-inverse f* (*inverse* (*f*$$*df*))
proof (*cases f*=*0* ∨ *g*=*0*)
case *True* **thus** *?thesis*
using *fls-lr-inverse-zero*(*1*)[*of inverse* (*0*::′*a*)] **by** (*auto simp add*: *assms*)
next
case *False* **thus** *?thesis*
using *fls-left-inverse-eq-inverse*[*of f*∗*g*] *nonzero-inverse-mult-distrib*[*of f g*]
fls-left-inverse-eq-inverse[*of g*] *fls-left-inverse-eq-inverse*[*of f*]
by (*simp add*: *assms*)
qed
show
fls-right-inverse (*f*∗*g*) (*inverse* ((*f*∗*g*)$$(*df*+*dg*))) =
fls-right-inverse g (*inverse* (*g*$$*dg*)) ∗ *fls-right-inverse f* (*inverse* (*f*$$*df*))
proof (*cases f*=*0* ∨ *g*=*0*)
case *True* **thus** *?thesis*
using *fls-lr-inverse-zero*(*2*)[*of inverse* (*0*::′*a*)] **by** (*auto simp add*: *assms*)
next
case *False* **thus** *?thesis*
using *fls-inverse-def′*[*of f*∗*g*] *nonzero-inverse-mult-distrib*[*of f g*]
fls-inverse-def′[*of g*] *fls-inverse-def′*[*of f*]
by (*simp add*: *assms*)
qed
qed

lemma *fls-lr-inverse-power-divring*:
fls-left-inverse (*f* ^ *n*) ((*inverse* (*f* $$ *fls-subdegree f*)) ^ *n*) =
(*fls-left-inverse f* (*inverse* (*f* $$ *fls-subdegree f*))) ^ *n* (**is** *?P*)
and *fls-right-inverse* (*f* ^ *n*) ((*inverse* (*f* $$ *fls-subdegree f*)) ^ *n*) =
(*fls-right-inverse f* (*inverse* (*f* $$ *fls-subdegree f*))) ^ *n* (**is** *?Q*)
for *f* :: ′*a*::*division-ring fls*
proof −

note *fls-left-inverse-eq-inverse* [*of f*] *fls-right-inverse-eq-inverse*[*of f*]
moreover have
fls-right-inverse (*f* ^ *n*) ((*inverse* (*f* $$ *fls-subdegree f*)) ^ *n*) =
inverse f ^ *n*
using *fls-right-inverse-eq-inverse* [*of f* ^ *n*]
by (*simp add*: *fls-subdegree-pow power-inverse*)
moreover have
fls-left-inverse (*f* ^ *n*) ((*inverse* (*f* $$ *fls-subdegree f*)) ^ *n*) =
inverse f ^ *n*
using *fls-left-inverse-eq-inverse* [*of f* ^ *n*]
by (*simp add*: *fls-subdegree-pow power-inverse*)
ultimately show *?P* **and** *?Q*
by *simp-all*
qed

instance *fls* :: (*field*) *field*
by (*standard*, *simp-all add*: *field-simps*)

instance *fls* :: ({*field-prime-char*,*comm-semiring-1*}) *field-prime-char*
by (*rule field-prime-charI'*) *auto*

7.5.6 Division

lemma *fls-divide-nth-below*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*uminus*,*times*,*inverse*} *fls*
shows *n* < *fls-subdegree f* − *fls-subdegree g* ⟹ (*f div g*) $$ *n* = *0*
by (*simp add*: *fls-divide-def*)

lemma *fls-divide-nth-base*:
fixes *f g* :: *'a*::*division-ring fls*
shows
(*f div g*) $$ (*fls-subdegree f* − *fls-subdegree g*) =
f $$ *fls-subdegree f* / *g* $$ *fls-subdegree g*
using *fps-divide-nth-0'*[*of fls-base-factor-to-fps g fls-base-factor-to-fps f*]
fls-base-factor-to-fps-subdegree[*of g*]
by (*simp add*: *fls-divide-def*)

lemma *fls-div-zero* [*simp*]:
0 div (*g* :: *'a* :: {*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*) = *0*
by (*simp add*: *fls-divide-def*)

lemma *fls-div-by-zero*:
fixes *g* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
assumes *inverse* (*0*::*'a*) = *0*
shows *g div 0* = *0*
by (*simp add*: *fls-divide-def assms fps-div-by-zero'*)

lemma *fls-divide-times*:
fixes *f g* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fls*

shows (*f* ∗ *g*) / *h* = *f* ∗ (*g* / *h*)
by (*simp add*: *fls-divide-convert-times-inverse mult.assoc*)

lemma *fls-divide-times2*:
fixes *f g* :: *'a*::{*comm-semiring-0*,*inverse*,*uminus*} *fls*
shows (*f* ∗ *g*) / *h* = (*f* / *h*) ∗ *g*
using *fls-divide-times*[*of g f h*]
by (*simp add*: *mult.commute*)

lemma *fls-divide-subdegree-ge*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*uminus*,*times*,*inverse*} *fls*
assumes *f* / *g* ≠ *0*
shows *fls-subdegree* (*f* / *g*) ≥ *fls-subdegree f* − *fls-subdegree g*
using *assms fls-divide-nth-below*
by (*intro fls-subdegree-geI*) *simp*

lemma *fls-divide-subdegree*:
fixes *f g* :: *'a*::*division-ring fls*
assumes *f* ≠ *0 g* ≠ *0*
shows *fls-subdegree* (*f* / *g*) = *fls-subdegree f* − *fls-subdegree g*
proof (*intro antisym*)
from *assms* **have** *f* $$ *fls-subdegree f* / *g* $$ *fls-subdegree g* ≠ *0* **by** (*simp add*: *field-simps*)
thus *fls-subdegree* (*f*/*g*) ≤ *fls-subdegree f* − *fls-subdegree g*
using *fls-divide-nth-base*[*of f g*] **by** (*intro fls-subdegree-leI*) *simp*
from *assms* **have** *f* / *g* ≠ *0* **by** (*simp add*: *field-simps*)
thus *fls-subdegree* (*f*/*g*) ≥ *fls-subdegree f* − *fls-subdegree g*
using *fls-divide-subdegree-ge* **by** *fast*
qed

lemma *fls-divide-shift-numer-nonzero*:
fixes *f g* :: *'a* :: {*comm-monoid-add*,*inverse*,*times*,*uminus*} *fls*
assumes *f* ≠ *0*
shows *fls-shift m f* / *g* = *fls-shift m* (*f*/*g*)
using *assms fls-base-factor-to-fps-shift*[*of m f*]
by (*simp add*: *fls-divide-def algebra-simps*)

lemma *fls-divide-shift-numer*:
fixes *f g* :: *'a* :: {*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
shows *fls-shift m f* / *g* = *fls-shift m* (*f*/*g*)
using *fls-divide-shift-numer-nonzero*
by (*cases f*=*0*) *auto*

lemma *fls-divide-shift-denom-nonzero*:
fixes *f g* :: *'a* :: {*comm-monoid-add*,*inverse*,*times*,*uminus*} *fls*
assumes *g* ≠ *0*
shows *f* / *fls-shift m g* = *fls-shift* (−*m*) (*f*/*g*)
using *assms fls-base-factor-to-fps-shift*[*of m g*]
by (*simp add*: *fls-divide-def algebra-simps*)

lemma *fls-divide-shift-denom*:
fixes *f g* :: *'a* :: *division-ring fls*
shows *f* / *fls-shift m g* = *fls-shift* (−*m*) (*f*/*g*)
using *fls-divide-shift-denom-nonzero*
by (*cases g=0*) *auto*

lemma *fls-divide-shift-both-nonzero*:
fixes *f g* :: *'a* :: {*comm-monoid-add*,*inverse*,*times*,*uminus*} *fls*
assumes *f* ≠ *0 g* ≠ *0*
shows *fls-shift n f* / *fls-shift m g* = *fls-shift* (*n*−*m*) (*f*/*g*)
by (*simp add*: *assms fls-divide-shift-numer-nonzero fls-divide-shift-denom-nonzero*)

lemma *fls-divide-shift-both* [*simp*]:
fixes *f g* :: *'a* :: *division-ring fls*
shows *fls-shift n f* / *fls-shift m g* = *fls-shift* (*n*−*m*) (*f*/*g*)
using *fls-divide-shift-both-nonzero*
by (*cases f=0* ∨ *g=0*) *auto*

lemma *fls-divide-base-factor-numer*:
fls-base-factor f / *g* = *fls-shift* (*fls-subdegree f*) (*f*/*g*)
using *fls-base-factor-to-fps-base-factor*[*of f*]
fls-base-factor-subdegree[*of f*]
by (*simp add*: *fls-divide-def algebra-simps*)

lemma *fls-divide-base-factor-denom*:
f / *fls-base-factor g* = *fls-shift* (−*fls-subdegree g*) (*f*/*g*)
using *fls-base-factor-to-fps-base-factor*[*of g*]
fls-base-factor-subdegree[*of g*]
by (*simp add*: *fls-divide-def*)

lemma *fls-divide-base-factor'*:
fls-base-factor f / *fls-base-factor g* = *fls-shift* (*fls-subdegree f* − *fls-subdegree g*) (*f*/*g*)
using *fls-divide-base-factor-numer*[*of f fls-base-factor g*]
fls-divide-base-factor-denom[*of f g*]
by *simp*

lemma *fls-divide-base-factor*:
fixes *f g* :: *'a* :: *division-ring fls*
shows *fls-base-factor f* / *fls-base-factor g* = *fls-base-factor* (*f*/*g*)
using *fls-divide-subdegree*[*of f g*] *fls-divide-base-factor'*
by *fastforce*

lemma *fls-divide-regpart*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *fls-regpart* (*f* / *g*) = *fls-regpart f* / *fls-regpart g*
proof −

have *deg0*:
⋀*g*. *fls-subdegree g* = *0* ⟹
fls-regpart (*f* / *g*) = *fls-regpart f* / *fls-regpart g*
by (*simp add*:
assms(*1*) *fls-divide-convert-times-inverse fls-inverse-subdegree-0*
fls-times-conv-regpart fls-inverse-regpart fls-regpart-subdegree-conv fps-divide-unit′
)
show *?thesis*
proof (*cases fls-subdegree g* = *0*)
case *False*
hence *fls-base-factor g* ≠ *0* **using** *fls-base-factor-nonzero*[*of g*] **by** *force*
with *assms*(*2*) **show** *?thesis*
using *fls-divide-shift-denom-nonzero*[*of fls-base-factor g f* −*fls-subdegree g*]
fps-shift-fls-regpart-conv-fls-shift[*of*
nat (*fls-subdegree g*) *f* / *fls-base-factor g*
]
fls-base-factor-subdegree[*of g*] *deg0*
fls-regpart-subdegree-conv[*of g*] *fps-unit-factor-fls-regpart*[*of g*]
by (*simp add*:
fls-conv-base-factor-shift-subdegree fls-regpart-subdegree-conv fps-divide-def
)
qed (*rule deg0*)
qed

lemma *fls-divide-fls-base-factor-to-fps′*:
fixes *f g* :: ′*a*::{*comm-monoid-add*,*uminus*,*inverse*,*mult-zero*} *fls*
shows
fls-base-factor-to-fps f / *fls-base-factor-to-fps g* =
fls-regpart (*fls-shift* (*fls-subdegree f* − *fls-subdegree g*) (*f* / *g*))
using *fls-base-factor-subdegree*[*of f*] *fls-base-factor-subdegree*[*of g*]
fls-divide-regpart[*of fls-base-factor f fls-base-factor g*]
fls-divide-base-factor′[*of f g*]
by *simp*

lemma *fls-divide-fls-base-factor-to-fps*:
fixes *f g* :: ′*a*::*division-ring fls*
shows *fls-base-factor-to-fps f* / *fls-base-factor-to-fps g* = *fls-base-factor-to-fps* (*f* / *g*)
using *fls-divide-fls-base-factor-to-fps′ fls-divide-subdegree*[*of f g*]
by *fastforce*

lemma *fls-divide-fps-to-fls*:
fixes *f g* :: ′*a*::{*inverse*,*ab-group-add*,*mult-zero*} *fps*
assumes *subdegree f* ≥ *subdegree g*
shows *fps-to-fls f* / *fps-to-fls g* = *fps-to-fls* (*f*/*g*)
proof−
have *1*:
fps-to-fls f / *fps-to-fls g* =
fls-shift (*int* (*subdegree g*)) (*fps-to-fls* (*f* ∗ *inverse* (*unit-factor g*)))

using *fls-base-factor-to-fps-to-fls*[*of f*] *fls-base-factor-to-fps-to-fls*[*of g*]
fls-subdegree-fls-to-fps[*of f*] *fls-subdegree-fls-to-fps*[*of g*]
fps-divide-def[*of unit-factor f unit-factor g*]
fls-times-fps-to-fls[*of unit-factor f inverse* (*unit-factor g*)]
fls-shifted-times-simps(*2*)[*of* −*int* (*subdegree f*) *fps-to-fls* (*unit-factor f*)]
fls-times-fps-to-fls[*of f inverse* (*unit-factor g*)]
by (*simp add*: *fls-divide-def*)
with *assms* **show** *?thesis*
using *fps-mult-subdegree-ge*[*of f inverse* (*unit-factor g*)]
fps-shift-to-fls[*of subdegree g f* ∗ *inverse* (*unit-factor g*)]
by (*cases f* ∗ *inverse* (*unit-factor g*) = *0*) (*simp-all add*: *fps-divide-def*)
qed

lemma *fls-divide-1'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / *1* = *f*
using *assms fls-conv-base-factor-to-fps-shift-subdegree*[*of f*]
by (*simp add*: *fls-divide-def fps-divide-1'*)

lemma *fls-divide-1* [*simp*]: *a* / *1* = (*a*::*'a*::*division-ring fls*)
by (*rule fls-divide-1'*[*OF inverse-1*])

lemma *fls-const-divide-const*:
fixes *x y* :: *'a*::*division-ring*
shows *fls-const x* / *fls-const y* = *fls-const* (*x*/*y*)
by (*simp add*: *fls-divide-def fls-base-factor-to-fps-const fps-const-divide*)

lemma *fls-divide-X'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / *fls-X* = *fls-shift 1 f*
proof−
from *assms* **have**
f / *fls-X* =
fls-shift 1 (*fls-shift* (−*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*)))
by (*simp add*: *fls-divide-def fps-divide-1'*)
also have ... = *fls-shift 1 f*
using *fls-conv-base-factor-to-fps-shift-subdegree*[*of f*]
by *simp*
finally show *?thesis* **by** *simp*
qed

lemma *fls-divide-X* [*simp*]:
fixes *f* :: *'a*::*division-ring fls*
shows *f* / *fls-X* = *fls-shift 1 f*
by (*rule fls-divide-X'*[*OF inverse-1*])

lemma *fls-divide-X-power'*:
  fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fls*
  assumes *inverse* (*1*::*'a*) = *1*
  shows *f* / (*fls-X* ^ *n*) = *fls-shift n f*
proof−
  have *fls-base-factor-to-fps* ((*fls-X*::*'a fls*) ^ *n*) = *1* **by** (*rule fls-X-power-base-factor-to-fps*)
  with *assms* **have**
    f / (*fls-X* ^ *n*) =
      fls-shift n (*fls-shift* (−*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*)))
    by (*simp add*: *fls-divide-def fps-divide-1'*)
  also have ... = *fls-shift n f*
    using *fls-conv-base-factor-to-fps-shift-subdegree*[*of f*] **by** *simp*
  finally show *?thesis* **by** *simp*
qed

lemma *fls-divide-X-power* [*simp*]:
  fixes *f* :: *'a*::*division-ring fls*
  shows *f* / (*fls-X* ^ *n*) = *fls-shift n f*
  by (*rule fls-divide-X-power'*[*OF inverse-1*])

lemma *fls-divide-X-inv'*:
  fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fls*
  assumes *inverse* (*1*::*'a*) = *1*
  shows *f* / *fls-X-inv* = *fls-shift* (−*1*) *f*
proof−
  from *assms* **have**
    f / *fls-X-inv* =
      fls-shift (−*1*) (*fls-shift* (−*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*)))
    by (*simp add*: *fls-divide-def fps-divide-1' algebra-simps*)
  also have ... = *fls-shift* (−*1*) *f*
    using *fls-conv-base-factor-to-fps-shift-subdegree*[*of f*]
    by *simp*
  finally show *?thesis* **by** *simp*
qed

lemma *fls-divide-X-inv* [*simp*]:
  fixes *f* :: *'a*::*division-ring fls*
  shows *f* / *fls-X-inv* = *fls-shift* (−*1*) *f*
  by (*rule fls-divide-X-inv'*[*OF inverse-1*])

lemma *fls-divide-X-inv-power'*:
  fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fls*
  assumes *inverse* (*1*::*'a*) = *1*
  shows *f* / (*fls-X-inv* ^ *n*) = *fls-shift* (−*int n*) *f*
proof−
  have *fls-base-factor-to-fps* ((*fls-X-inv*::*'a fls*) ^ *n*) = *1*
    by (*rule fls-X-inv-power-base-factor-to-fps*)

with *assms* **have**
f / (*fls-X-inv* ^ *n*) =
fls-shift (−*int n* + −*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*))
by (*simp add*: *fls-divide-def fps-divide-1'*)
also have
... = *fls-shift* (−*int n*) (*fls-shift* (−*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*)))
by (*simp add*: *add.commute*)
also have ... = *fls-shift* (−*int n*) *f*
using *fls-conv-base-factor-to-fps-shift-subdegree*[*of f*] **by** *simp*
finally show *?thesis* **by** *simp*
qed

lemma *fls-divide-X-inv-power* [*simp*]:
fixes *f* :: *'a*::*division-ring fls*
shows *f* / (*fls-X-inv* ^ *n*) = *fls-shift* (−*int n*) *f*
by (*rule fls-divide-X-inv-power'*[*OF inverse-1*])

lemma *fls-divide-X-intpow'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / (*fls-X-intpow i*) = *fls-shift i f*
using *assms*
by (*simp add*: *fls-divide-shift-denom-nonzero fls-divide-1'*)

lemma *fls-divide-X-intpow-conv-times'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / (*fls-X-intpow i*) = *f* ∗ *fls-X-intpow* (−*i*)
using *assms fls-X-intpow-times-conv-shift*(*2*)[*of f* −*i*]
by (*simp add*: *fls-divide-X-intpow'*)

lemma *fls-divide-X-intpow*:
fixes *f* :: *'a*::*division-ring fls*
shows *f* / (*fls-X-intpow i*) = *fls-shift i f*
by (*rule fls-divide-X-intpow'*[*OF inverse-1*])

lemma *fls-divide-X-intpow-conv-times*:
fixes *f* :: *'a*::*division-ring fls*
shows *f* / (*fls-X-intpow i*) = *f* ∗ *fls-X-intpow* (−*i*)
by (*rule fls-divide-X-intpow-conv-times'*[*OF inverse-1*])

lemma *fls-X-intpow-div-fls-X-intpow-semiring1*:
assumes *inverse* (*1*::*'a*::{*semiring-1*,*inverse*,*uminus*}) = *1*
shows (*fls-X-intpow i* :: *'a fls*) / *fls-X-intpow j* = *fls-X-intpow* (*i*−*j*)
by (*simp add*: *assms fls-divide-shift-both-nonzero fls-divide-1'*)

lemma *fls-X-intpow-div-fls-X-intpow*:
(*fls-X-intpow i* :: *'a*::*division-ring fls*) / *fls-X-intpow j* = *fls-X-intpow* (*i*−*j*)

by (*rule fls-X-intpow-div-fls-X-intpow-semiring1*[*OF inverse-1*])

lemma *fls-divide-add*:
fixes *f g h* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fls*
shows (*f* + *g*) / *h* = *f* / *h* + *g* / *h*
by (*simp add*: *fls-divide-convert-times-inverse algebra-simps*)

lemma *fls-divide-diff*:
fixes *f g h* :: *'a*::{*ring*,*inverse*} *fls*
shows (*f* − *g*) / *h* = *f* / *h* − *g* / *h*
by (*simp add*: *fls-divide-convert-times-inverse algebra-simps*)

lemma *fls-divide-uminus*:
fixes *f g h* :: *'a*::{*ring*,*inverse*} *fls*
shows (− *f*) / *g* = − (*f* / *g*)
by (*simp add*: *fls-divide-convert-times-inverse*)

lemma *fls-divide-uminus'*:
fixes *f g h* :: *'a*::*division-ring fls*
shows *f* / (− *g*) = − (*f* / *g*)
by (*simp add*: *fls-divide-convert-times-inverse*)

7.5.7 Units

lemma *fls-is-left-unit-iff-base-is-left-unit*:
fixes *f* :: *'a* :: *ring-1-no-zero-divisors fls*
shows (∃ *g*. *1* = *f* ∗ *g*) ⟷ (∃ *k*. *1* = *f* $$ *fls-subdegree f* ∗ *k*)
proof
assume ∃ *g*. *1* = *f* ∗ *g*
then obtain *g* **where** *1* = *f* ∗ *g* **by** *fast*
hence *1* = (*f* $$ *fls-subdegree f*) ∗ (*g* $$ *fls-subdegree g*)
using *fls-subdegree-mult*[*of f g*] *fls-times-base*[*of f g*] **by** *fastforce*
thus ∃ *k*. *1* = *f* $$ *fls-subdegree f* ∗ *k* **by** *fast*
next
assume ∃ *k*. *1* = *f* $$ *fls-subdegree f* ∗ *k*
then obtain *k* **where** *1* = *f* $$ *fls-subdegree f* ∗ *k* **by** *fast*
hence *1* = *f* ∗ *fls-right-inverse f k*
using *fls-right-inverse* **by** *simp*
thus ∃ *g*. *1* = *f* ∗ *g* **by** *fast*
qed

lemma *fls-is-right-unit-iff-base-is-right-unit*:
fixes *f* :: *'a* :: *ring-1-no-zero-divisors fls*
shows (∃ *g*. *1* = *g* ∗ *f*) ⟷ (∃ *k*. *1* = *k* ∗ *f* $$ *fls-subdegree f*)
proof
assume ∃ *g*. *1* = *g* ∗ *f*
then obtain *g* **where** *1* = *g* ∗ *f* **by** *fast*
hence *1* = (*g* $$ *fls-subdegree g*) ∗ (*f* $$ *fls-subdegree f*)
using *fls-subdegree-mult*[*of g f*] *fls-times-base*[*of g f*] **by** *fastforce*

thus ∃ *k*. *1* = *k* ∗ *f* $$ *fls-subdegree f* **by** *fast*
next
assume ∃ *k*. *1* = *k* ∗ *f* $$ *fls-subdegree f*
then obtain *k* **where** *1* = *k* ∗ *f* $$ *fls-subdegree f* **by** *fast*
hence *1* = *fls-left-inverse f k* ∗ *f*
using *fls-left-inverse* **by** *simp*
thus ∃ *g*. *1* = *g* ∗ *f* **by** *fast*
qed

7.6 Composition

definition *fls-compose-fps* :: *'a* :: *field fls* ⇒ *'a fps* ⇒ *'a fls* **where**
fls-compose-fps F G =
fps-to-fls (*fps-compose* (*fls-base-factor-to-fps F*) *G*) ∗ *fps-to-fls G powi fls-subdegree F*

lemma *fps-compose-of-nat* [*simp*]: *fps-compose* (*of-nat n* :: *'a* :: *comm-ring-1 fps*) *H* = *of-nat n*
and *fps-compose-of-int* [*simp*]: *fps-compose* (*of-int i*) *H* = *of-int i*
unfolding *fps-of-nat* [*symmetric*] *fps-of-int* [*symmetric*] *numeral-fps-const*
by (*rule fps-const-compose*)+

lemmas [*simp*] = *fps-to-fls-of-nat fps-to-fls-of-int*

lemma *fls-compose-fps-0* [*simp*]: *fls-compose-fps 0 H* = *0*
and *fls-compose-fps-1* [*simp*]: *fls-compose-fps 1 H* = *1*
and *fls-compose-fps-const* [*simp*]: *fls-compose-fps* (*fls-const c*) *H* = *fls-const c*
and *fls-compose-fps-of-nat* [*simp*]: *fls-compose-fps* (*of-nat n*) *H* = *of-nat n*
and *fls-compose-fps-of-int* [*simp*]: *fls-compose-fps* (*of-int i*) *H* = *of-int i*
and *fls-compose-fps-X* [*simp*]: *fls-compose-fps fls-X F* = *fps-to-fls F*
by (*simp-all add*: *fls-compose-fps-def*)

lemma *fls-compose-fps-0-right*:
fls-compose-fps F 0 = (*if 0* ≤ *fls-subdegree F then fls-const* (*F* $$ *0*) *else 0*)
by (*cases fls-subdegree F* = *0*) (*simp-all add*: *fls-compose-fps-def*)

lemma *fls-compose-fps-shift*:
assumes *H* ≠ *0*
shows *fls-compose-fps* (*fls-shift n F*) *H* = *fls-compose-fps F H* ∗ *fps-to-fls H powi* (−*n*)
proof (*cases F* = *0*)
case *False*
thus *?thesis*
using *assms* **by** (*simp add*: *fls-compose-fps-def power-int-diff power-int-minus field-simps*)
qed *auto*

lemma *fls-compose-fps-to-fls* [*simp*]:
assumes [*simp*]: *G* ≠ *0 fps-nth G 0* = *0*

shows *fls-compose-fps* (*fps-to-fls F*) *G* = *fps-to-fls* (*fps-compose F G*)
proof (*cases F* = *0*)
case *False*
define *n* **where** *n* = *subdegree F*
define *F′* **where** *F′* = *fps-shift n F*
have [*simp*]: *F′* ≠ *0 subdegree F′* = *0*
using *False* **by** (*auto simp*: *F′-def n-def*)
have *F-eq*: *F* = *F′* ∗ *fps-X* ^ *n*
unfolding *F′-def n-def* **using** *subdegree-decompose* **by** *blast*
have *fls-compose-fps* (*fps-to-fls F*) *G* =
fps-to-fls (*fps-shift n* (*fls-regpart* (*fps-to-fls F′* ∗ *fls-X-intpow* (*int n*))) *oo G*) ∗ *fps-to-fls* (*G* ^ *n*)
unfolding *F-eq fls-compose-fps-def*
by (*simp add*: *fls-times-fps-to-fls fls-X-power-conv-shift-1 power-int-add fls-subdegree-fls-to-fps fps-to-fls-power fls-regpart-shift-conv-fps-shift flip*: *fls-times-both-shifted-simp*)
also have *fps-to-fls F′* ∗ *fls-X-intpow* (*int n*) = *fps-to-fls F*
by (*simp add*: *F-eq fls-times-fps-to-fls fps-to-fls-power fls-X-power-conv-shift-1*)
also have *fps-to-fls* (*fps-shift n* (*fls-regpart* (*fps-to-fls F*)) *oo G*) ∗ *fps-to-fls* (*G* ^ *n*) =
fps-to-fls ((*fps-shift n* (*fls-regpart* (*fps-to-fls F*)) ∗ *fps-X* ^ *n*) *oo G*)
by (*simp add*: *fls-times-fps-to-fls flip*: *fps-compose-power add*: *fps-compose-mult-distrib*)
also have *fps-shift n* (*fls-regpart* (*fps-to-fls F*)) ∗ *fps-X* ^ *n* = *F*
by (*simp add*: *F-eq*)
finally show *?thesis* **.**
qed (*auto simp*: *fls-compose-fps-def*)

lemma *fls-compose-fps-mult*:
assumes [*simp*]: *H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*F* ∗ *G*) *H* = *fls-compose-fps F H* ∗ *fls-compose-fps G H*
using *assms*
proof (*cases F* ∗ *G* = *0*)
case *False*
hence [*simp*]: *F* ≠ *0 G* ≠ *0*
by *auto*
define *n m* **where** *n* = *fls-subdegree F m* = *fls-subdegree G*
define *F′* **where** *F′* = *fls-regpart* (*fls-shift n F*)
define *G′* **where** *G′* = *fls-regpart* (*fls-shift m G*)
have *F-eq*: *F* = *fls-shift* (−*n*) (*fps-to-fls F′*) **and** *G-eq*: *G* = *fls-shift* (−*m*) (*fps-to-fls G′*)
by (*simp-all add*: *F′-def G′-def n-m-def*)
have *fls-compose-fps* (*F* ∗ *G*) *H* = *fls-compose-fps* (*fls-shift* (−(*n* + *m*)) (*fps-to-fls* (*F′* ∗ *G′*))) *H*
by (*simp add*: *fls-times-fps-to-fls F-eq G-eq fls-shifted-times-simps*)
also have . . . = *fps-to-fls* ((*F′ oo H*) ∗ (*G′ oo H*)) ∗ *fps-to-fls H powi* (*m* + *n*)
by (*simp add*: *fls-compose-fps-shift fps-compose-mult-distrib*)
also have . . . = *fls-compose-fps F H* ∗ *fls-compose-fps G H*
by (*simp add*: *F-eq G-eq fls-compose-fps-shift fls-times-fps-to-fls power-int-add*)

finally show *?thesis* **.**
qed *auto*

lemma *fls-compose-fps-power*:
assumes [*simp*]: $G \neq 0$ *fps-nth G 0* = *0*
shows *fls-compose-fps* (F ^ n) G = *fls-compose-fps F G* ^ n
by (*induction n*) (*auto simp*: *fls-compose-fps-mult*)

lemma *fls-compose-fps-add*:
assumes [*simp*]: $H \neq 0$ *fps-nth H 0* = *0*
shows *fls-compose-fps* ($F + G$) H = *fls-compose-fps F H* + *fls-compose-fps G H*
proof (*cases* $F = 0 \vee G = 0$)
case *False*
hence [*simp*]: $F \neq 0$ $G \neq 0$
by *auto*
define n **where** n = *min* (*fls-subdegree F*) (*fls-subdegree G*)
define F' **where** F' = *fls-regpart* (*fls-shift n F*)
define G' **where** G' = *fls-regpart* (*fls-shift n G*)
have *F-eq*: F = *fls-shift* ($-n$) (*fps-to-fls* F') **and** *G-eq*: G = *fls-shift* ($-n$) (*fps-to-fls* G')
unfolding *n-def* **by** (*simp-all add*: *F′-def G′-def n-def*)
have $F + G$ = *fls-shift* ($-n$) (*fps-to-fls* ($F' + G'$))
by (*simp add*: *F-eq G-eq*)
also have *fls-compose-fps* ... H = *fls-compose-fps* (*fps-to-fls* ($F' + G'$)) H ∗ *fps-to-fls H powi n*
by (*subst fls-compose-fps-shift*) *auto*
also have ... = *fps-to-fls* (*fps-compose* ($F' + G'$) H) ∗ *fps-to-fls H powi n*
by (*subst fls-compose-fps-to-fls*) *auto*
also have ... = *fls-compose-fps F H* + *fls-compose-fps G H*
by (*simp add*: *F-eq G-eq fls-compose-fps-shift fps-compose-add-distrib algebra-simps*)
finally show *?thesis* **.**
qed *auto*

lemma *fls-compose-fps-uminus* [*simp*]: *fls-compose-fps* ($-F$) H = $-$*fls-compose-fps F H*
by (*simp add*: *fls-compose-fps-def fps-compose-uminus*)

lemma *fls-compose-fps-diff*:
assumes [*simp*]: $H \neq 0$ *fps-nth H 0* = *0*
shows *fls-compose-fps* ($F - G$) H = *fls-compose-fps F H* $-$ *fls-compose-fps G H*
using *fls-compose-fps-add*[*of H F* $-G$] **by** *simp*

lemma *fps-compose-eq-0-iff*:
fixes F G :: *'a* :: *idom fps*
assumes *fps-nth G 0* = *0*
shows *fps-compose F G* = *0* $\longleftrightarrow$ F = *0* $\vee$ (G = *0* $\wedge$ *fps-nth F 0* = *0*)

proof *safe*
  assume ∗: *fps-compose F G = 0 F ≠ 0*
  have *fps-nth* (*fps-compose F G*) *0* = *fps-nth F 0*
    by *simp*
  also have *fps-compose F G = 0*
    by (*simp add*: ∗)
  finally show *fps-nth F 0 = 0*
    by *simp*
  show *G = 0*
  proof (*rule ccontr*)
    assume *G ≠ 0*
    hence *subdegree G > 0* **using** *assms*
      using *subdegree-eq-0-iff* **by** *blast*
    define *N* **where** *N = subdegree F ∗ subdegree G*
    have *fps-nth* (*fps-compose F G*) *N* = (∑ *i = 0..N*. *fps-nth F i* ∗ *fps-nth* (*G* ^ *i*) *N*)
      unfolding *fps-compose-def* **by** (*simp add*: *N-def*)
    also have ... = (∑ *i*∈{*subdegree F*}. *fps-nth F i* ∗ *fps-nth* (*G* ^ *i*) *N*)
    proof (*intro sum.mono-neutral-right ballI*)
      fix *i* **assume** *i*: *i* ∈ {*0..N*} − {*subdegree F*}
      show *fps-nth F i* ∗ *fps-nth* (*G* ^ *i*) *N = 0*
      proof (*cases i subdegree F rule*: *linorder-cases*)
        assume *i > subdegree F*
        hence *fps-nth* (*G* ^ *i*) *N = 0*
          using *i* ‹*subdegree G > 0*› **by** (*intro fps-pow-nth-below-subdegree*) (*auto simp*: *N-def*)
        thus *?thesis* **by** *simp*
      qed (*use i* **in** ‹*auto simp*: *N-def*›)
    qed (*use* ‹*subdegree G > 0*› **in** ‹*auto simp*: *N-def*›)
    also have ... = *fps-nth F* (*subdegree F*) ∗ *fps-nth* (*G* ^ *subdegree F*) *N*
      by *simp*
    also have ... ≠ *0*
      using ‹*G ≠ 0*› ‹*F ≠ 0*› **by** (*auto simp*: *N-def*)
    finally show *False* **using** ∗ **by** *auto*
  qed
qed *auto*

lemma *fls-compose-fps-eq-0-iff*:
  assumes *H ≠ 0 fps-nth H 0 = 0*
  shows *fls-compose-fps F H = 0* ⟷ *F = 0*
  using *assms fls-base-factor-to-fps-nonzero*[*of F*]
  by (*cases F = 0*) (*auto simp*: *fls-compose-fps-def fps-compose-eq-0-iff*)

lemma *fls-compose-fps-inverse*:
  assumes [*simp*]: *H ≠ 0 fps-nth H 0 = 0*
  shows *fls-compose-fps* (*inverse F*) *H* = *inverse* (*fls-compose-fps F H*)
proof (*cases F = 0*)
  case *False*
  have *fls-compose-fps* (*inverse F*) *H* ∗ *fls-compose-fps F H* =

fls-compose-fps (*inverse F* ∗ *F*) *H*
by (*subst fls-compose-fps-mult*) *auto*
also have *inverse F* ∗ *F* = *1*
using *False* **by** *simp*
finally show *?thesis*
using *False* **by** (*simp add*: *field-simps fls-compose-fps-eq-0-iff*)
qed *auto*

lemma *fls-compose-fps-divide*:
assumes [*simp*]: *H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*F* / *G*) *H* = *fls-compose-fps F H* / *fls-compose-fps G H*
using *fls-compose-fps-mult*[*of H F inverse G*] *fls-compose-fps-inverse*[*of H G*]
by (*simp add*: *field-simps*)

lemma *fls-compose-fps-powi*:
assumes [*simp*]: *H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*F powi n*) *H* = *fls-compose-fps F H powi n*
by (*simp add*: *power-int-def fls-compose-fps-power fls-compose-fps-inverse*)

lemma *fls-compose-fps-assoc*:
assumes [*simp*]: *G* ≠ *0 fps-nth G 0* = *0 H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*fls-compose-fps F G*) *H* = *fls-compose-fps F* (*fps-compose G H*)
proof (*cases F* = *0*)
case [*simp*]: *False*
define *n* **where** *n* = *fls-subdegree F*
define *F′* **where** *F′* = *fls-regpart* (*fls-shift n F*)
have *F-eq*: *F* = *fls-shift* (−*n*) (*fps-to-fls F′*)
by (*simp add*: *F′-def n-def*)
show *?thesis*
by (*simp add*: *F-eq fls-compose-fps-shift fls-compose-fps-mult fls-compose-fps-powi fps-compose-eq-0-iff fps-compose-assoc*)
qed *auto*

lemma *subdegree-pos-iff*: *subdegree F* > *0* ⟷ *F* ≠ *0* ∧ *fps-nth F 0* = *0*
using *subdegree-eq-0-iff*[*of F*] **by** *auto*

lemma *fls-X-power-int* [*simp*]: *fls-X powi n* = (*fls-X-intpow n* :: ′*a* :: *division-ring fls*)
by (*auto simp*: *power-int-def fls-X-power-conv-shift-1 fls-inverse-X fls-inverse-shift simp flip*: *fls-inverse-X-power*)

lemma *fls-const-power-int*: *fls-const* (*c powi n*) = *fls-const* (*c* :: ′*a* :: *division-ring*) *powi n*
by (*auto simp*: *power-int-def fls-const-power fls-inverse-const*)

lemma *fls-nth-fls-compose-fps-linear*:
fixes *c* :: ′*a* :: *field*

assumes [*simp*]: *c* ≠ *0*
shows *fls-compose-fps F* (*fps-const c* ∗ *fps-X*) $$ *n* = *F* $$ *n* ∗ *c powi n*
proof −
{
assume ∗: *n* ≥ *fls-subdegree F*
hence *c* ^ *nat* (*n* − *fls-subdegree F*) = *c powi int* (*nat* (*n* − *fls-subdegree F*))
by (*simp add*: *power-int-def*)
also have ... ∗ *c powi fls-subdegree F* = *c powi* (*int* (*nat* (*n* − *fls-subdegree F*)) + *fls-subdegree F*)
using ∗ **by** (*subst power-int-add*) *auto*
also have ... = *c powi n*
using ∗ **by** *simp*
finally have *c* ^ *nat* (*n* − *fls-subdegree F*) ∗ *c powi fls-subdegree F* = *c powi n*
.
}
thus *?thesis*
by (*simp add*: *fls-compose-fps-def fps-compose-linear fls-times-fps-to-fls power-int-mult-distrib fls-shifted-times-simps*
flip: *fls-const-power-int*)
qed

lemma *fls-const-transfer* [*transfer-rule*]:
rel-fun (=) (*pcr-fls* (=))
(λ*c n. if n* = *0 then c else 0*) *fls-const*
by (*auto simp*: *fls-const-def rel-fun-def pcr-fls-def OO-def cr-fls-def*)

lemma *fls-shift-transfer* [*transfer-rule*]:
rel-fun (=) (*rel-fun* (*pcr-fls* (=)) (*pcr-fls* (=)))
(λ*n f k. f* (*k*+*n*)) *fls-shift*
by (*auto simp*: *fls-const-def rel-fun-def pcr-fls-def OO-def cr-fls-def*)

lift-definition *fls-compose-power* :: ′*a* :: *zero fls* ⇒ *nat* ⇒ ′*a fls* **is**
λ*f d n. if d* > *0* ∧ *int d dvd n then f* (*n div int d*) *else 0*
proof −
fix *f* :: *int* ⇒ ′*a* **and** *d* :: *nat*
assume ∗: *eventually* (λ*n. f* (−*int n*) = *0*) *cofinite*
show *eventually* (λ*n.* (*if d* > *0* ∧ *int d dvd* −*int n then f* (−*int n div int d*) *else 0*) = *0*) *cofinite*
proof (*cases d* = *0*)
case *False*
from ∗ **have** *eventually* (λ*n. f* (−*int n*) = *0*) *at-top*
by (*simp add*: *cofinite-eq-sequentially*)
hence *eventually* (λ*n. f* (−*int* (*n div d*)) = *0*) *at-top*
by (*rule eventually-compose-filterlim*[*OF* - *filterlim-at-top-div-const-nat*]) (*use False* **in** *auto*)
hence *eventually* (λ*n.* (*if d* > *0* ∧ *int d dvd* −*int n then f* (−*int n div int d*) *else 0*) = *0*) *at-top*
by *eventually-elim* (*auto simp*: *zdiv-int dvd-neg-div*)
thus *?thesis*

by (*simp add*: *cofinite-eq-sequentially*)
qed *auto*
qed

lemma *fls-nth-compose-power*:
assumes *d* > *0*
shows *fls-compose-power f d* $$ *n* = (*if int d dvd n then f* $$ (*n div int d*) *else* *0*)
by (*simp add*: *assms fls-compose-power.rep-eq*)

lemma *fls-compose-power-0-left* [*simp*]: *fls-compose-power 0 d* = *0*
by *transfer auto*

lemma *fls-compose-power-1-left* [*simp*]: *d* > *0* ⟹ *fls-compose-power 1 d* = *1*
by *transfer* (*auto simp*: *fun-eq-iff*)

lemma *fls-compose-power-const-left* [*simp*]:
d > *0* ⟹ *fls-compose-power* (*fls-const c*) *d* = *fls-const c*
by *transfer* (*auto simp*: *fun-eq-iff*)

lemma *fls-compose-power-shift* [*simp*]:
d > *0* ⟹ *fls-compose-power* (*fls-shift n f*) *d* = *fls-shift* (*d* ∗ *n*) (*fls-compose-power f d*)
by *transfer* (*auto simp*: *fun-eq-iff add-ac mult-ac*)

lemma *fls-compose-power-X-intpow* [*simp*]:
d > *0* ⟹ *fls-compose-power* (*fls-X-intpow n*) *d* = *fls-X-intpow* (*int d* ∗ *n*)
by *simp*

lemma *fls-compose-power-X* [*simp*]:
d > *0* ⟹ *fls-compose-power fls-X d* = *fls-X-intpow* (*int d*)
by *transfer* (*auto simp*: *fun-eq-iff*)

lemma *fls-compose-power-X-inv* [*simp*]:
d > *0* ⟹ *fls-compose-power fls-X-inv d* = *fls-X-intpow* (−*int d*)
by (*simp add*: *fls-X-inv-conv-shift-1*)

lemma *fls-compose-power-0-right* [*simp*]: *fls-compose-power f 0* = *0*
by *transfer auto*

lemma *fls-compose-power-add* [*simp*]:
fls-compose-power (*f* + *g*) *d* = *fls-compose-power f d* + *fls-compose-power g d*
by *transfer auto*

lemma *fls-compose-power-diff* [*simp*]:
fls-compose-power (*f* − *g*) *d* = *fls-compose-power f d* − *fls-compose-power g d*
by *transfer auto*

lemma *fls-compose-power-uminus* [*simp*]:
fls-compose-power (−*f*) *d* = −*fls-compose-power f d*
by *transfer auto*

lemma *fps-nth-compose-X-power*:
fps-nth (*f oo* (*fps-X* ^ *d*)) *n* = (*if d dvd n then fps-nth f* (*n div d*) *else 0*)
proof −
have *fps-nth* (*f oo* (*fps-X* ^ *d*)) *n* = ($\sum$ *i* = *0..n*. *f* $ *i* ∗ (*fps-X* ^(*d* ∗ *i*)) $ *n*)
unfolding *fps-compose-def* **by** (*simp add*: *power-mult*)
also have ... = ($\sum$ *i*∈(*if d dvd n then* {*n div d*} *else* {}). *f* $ *i* ∗ (*fps-X* ^(*d* ∗ *i*)) $ *n*)
by (*intro sum.mono-neutral-right*) *auto*
also have ... = (*if d dvd n then fps-nth f* (*n div d*) *else 0*)
by *auto*
finally show *?thesis* **.**
qed

lemma *fls-compose-power-fps-to-fls*:
assumes *d* > *0*
shows *fls-compose-power* (*fps-to-fls f*) *d* = *fps-to-fls* (*fps-compose f* (*fps-X* ^ *d*))
using *assms*
by (*intro fls-eqI*) (*auto simp*: *fls-nth-compose-power fps-nth-compose-X-power*
pos-imp-zdiv-neg-iff div-neg-pos-less0 nat-div-distrib
simp flip: *int-dvd-int-iff*)

lemma *fls-compose-power-mult* [*simp*]:
fls-compose-power (*f* ∗ *g* :: ′*a* :: *idom fls*) *d* = *fls-compose-power f d* ∗ *fls-compose-power g d*
proof (*cases d* > *0*)
case *True*
define *n* **where** *n* = *nat* (*max 0* (*max* (− *fls-subdegree f*) (− *fls-subdegree g*)))
have *n-ge*: −*fls-subdegree f* ≤ *int n* −*fls-subdegree g* ≤ *int n*
unfolding *n-def* **by** *auto*
obtain *f*′ **where** *f*′: *f* = *fls-shift n* (*fps-to-fls f*′)
using *fls-as-fps*[*OF n-ge*(*1*)] **by** (*auto simp*: *n-def*)
obtain *g*′ **where** *g*′: *g* = *fls-shift n* (*fps-to-fls g*′)
using *fls-as-fps*[*OF n-ge*(*2*)] **by** (*auto simp*: *n-def*)
show *?thesis* **using** ‹*d* > *0*›
by (*simp add*: *f*′ *g*′ *fls-shifted-times-simps mult-ac fls-compose-power-fps-to-fls*
fps-compose-mult-distrib flip: *fls-times-fps-to-fls*)
qed *auto*

lemma *fls-compose-power-power* [*simp*]:
assumes *d* > *0* ∨ *n* > *0*
shows *fls-compose-power* (*f* ^ *n* :: ′*a* :: *idom fls*) *d* = *fls-compose-power f d* ^ *n*
proof (*cases d* > *0*)
case *True*
thus *?thesis* **by** (*induction n*) *auto*

qed (*use assms* **in** *auto*)

lemma *fls-nth-compose-power′* [*simp*]:
d = *0* ∨ ¬*d dvd n* ⟹ *fls-compose-power f d* $$ *int n* = *0*
d dvd n ⟹ *d* > *0* ⟹ *fls-compose-power f d* $$ *int n* = *f* $$ *int* (*n div d*)
by (*transfer*; *force*; *fail*)+

7.7 Formal differentiation and integration

7.7.1 Derivative

definition *fls-deriv f* = *Abs-fls* (*λn. of-int* (*n+1*) ∗ *f*$$(*n+1*))

lemma *fls-deriv-nth*[*simp*]: *fls-deriv f* $$ *n* = *of-int* (*n+1*) ∗ *f*$$(*n+1*)
proof−
obtain *N* **where** ∀ *n<N. f*$$*n* = *0* **by** (*elim fls-nth-vanishes-belowE*)
hence ∀ *n<N−1. of-int* (*n+1*) ∗ *f*$$(*n+1*) = *0* **by** *auto*
thus *?thesis* **using** *nth-Abs-fls-lower-bound* **unfolding** *fls-deriv-def* **by** *simp*
qed

lemma *fls-deriv-residue*: *fls-deriv f* $$ −*1* = *0*
by *simp*

lemma *fls-deriv-const*[*simp*]: *fls-deriv* (*fls-const x*) = *0*
proof (*intro fls-eqI*)
fix *n* **show** *fls-deriv* (*fls-const x*) $$ *n* = *0*$$*n*
by (*cases n+1=0*) *auto*
qed

lemma *fls-deriv-of-nat*[*simp*]: *fls-deriv* (*of-nat n*) = *0*
by (*simp add*: *fls-of-nat*)

lemma *fls-deriv-of-int*[*simp*]: *fls-deriv* (*of-int i*) = *0*
by (*simp add*: *fls-of-int*)

lemma *fls-deriv-zero*[*simp*]: *fls-deriv 0* = *0*
using *fls-deriv-const*[*of 0*] **by** *simp*

lemma *fls-deriv-one*[*simp*]: *fls-deriv 1* = *0*
using *fls-deriv-const*[*of 1*] **by** *simp*

lemma *fls-deriv-numeral* [*simp*]: *fls-deriv* (*numeral n*) = *0*
by (*metis fls-deriv-of-int of-int-numeral*)

lemma *fls-deriv-subdegree′*:
assumes *of-int* (*fls-subdegree f*) ∗ *f* $$ *fls-subdegree f* ≠ *0*
shows *fls-subdegree* (*fls-deriv f*) = *fls-subdegree f* − *1*
by (*auto intro*: *fls-subdegree-eqI simp*: *assms*)

lemma *fls-deriv-subdegree0*:

assumes *fls-subdegree f* = *0*
shows *fls-subdegree* (*fls-deriv f*) ≥ *0*
proof (*cases fls-deriv f* = *0*)
case *False*
show *?thesis*
proof (*intro fls-subdegree-geI*, *rule False*)
fix *k* :: *int* **assume** *k* < *0*
with *assms* **show** *fls-deriv f* $$ *k* = *0* **by** (*cases k*=−*1*) *auto*
qed
qed *simp*

lemma *fls-subdegree-deriv′*:
fixes *f* :: *′a*::*ring-1-no-zero-divisors fls*
assumes (*of-int* (*fls-subdegree f*) :: *′a*) ≠ *0*
shows *fls-subdegree* (*fls-deriv f*) = *fls-subdegree f* − *1*
using *assms nth-fls-subdegree-zero-iff* [*of f*]
by (*auto intro*: *fls-deriv-subdegree′*)

lemma *fls-subdegree-deriv*:
fixes *f* :: *′a*::{*ring-1-no-zero-divisors*,*ring-char-0*} *fls*
assumes *fls-subdegree f* ≠ *0*
shows *fls-subdegree* (*fls-deriv f*) = *fls-subdegree f* − *1*
by (*auto intro*: *fls-subdegree-deriv′ simp*: *assms*)

Shifting is like multiplying by a power of the implied variable, and so satisfies a product-like rule.

lemma *fls-deriv-shift*:
fls-deriv (*fls-shift n f*) = *of-int* (−*n*) ∗ *fls-shift* (*n*+*1*) *f* + *fls-shift n* (*fls-deriv f*)
by (*intro fls-eqI*) (*simp flip*: *fls-shift-fls-shift add*: *algebra-simps*)

lemma *fls-deriv-X* [*simp*]: *fls-deriv fls-X* = *1*
by (*intro fls-eqI*) *simp*

lemma *fls-deriv-X-inv* [*simp*]: *fls-deriv fls-X-inv* = − (*fls-X-inv*^{2})
proof−
have *fls-deriv fls-X-inv* = − (*fls-shift 2 1*)
by (*simp add*: *fls-X-inv-conv-shift-1 fls-deriv-shift*)
thus *?thesis* **by** (*simp add*: *fls-X-inv-power-conv-shift-1*)
qed

lemma *fls-deriv-delta*:
fls-deriv (*Abs-fls* (λ*n*. *if n*=*m then c else 0*)) =
Abs-fls (λ*n*. *if n*=*m*−*1 then of-int m* ∗ *c else 0*)
proof−
have
fls-deriv (*Abs-fls* (λ*n*. *if n*=*m then c else 0*)) = *fls-shift* (*1*−*m*) (*fls-const* (*of-int m* ∗ *c*))
using *fls-deriv-shift*[*of* −*m fls-const c*]
by (*simp*

add: *fls-shift-const fls-of-int fls-shifted-times-simps*(*1*)[*symmetric*]
fls-const-mult-const[*symmetric*]
del: *fls-const-mult-const*
)
thus *?thesis* **by** (*simp add*: *fls-shift-const*)
qed

lemma *fls-deriv-base-factor*:
fls-deriv (*fls-base-factor f*) =
of-int (−*fls-subdegree f*) ∗ *fls-shift* (*fls-subdegree f* + *1*) *f* +
fls-shift (*fls-subdegree f*) (*fls-deriv f*)
by (*simp add*: *fls-deriv-shift*)

lemma *fls-regpart-deriv*: *fls-regpart* (*fls-deriv f*) = *fps-deriv* (*fls-regpart f*)
proof (*intro fps-ext*)
fix *n*
have *1*: (*of-nat n* :: ′*a*) + *1* = *of-nat* (*n*+*1*)
and *2*: *int n* + *1* = *int* (*n* + *1*)
by *auto*
show *fls-regpart* (*fls-deriv f*) $ *n* = *fps-deriv* (*fls-regpart f*) $ *n* **by** (*simp add*: *1*
2)
qed

lemma *fls-prpart-deriv*:
fixes *f* :: ′*a* :: {*comm-ring-1*,*ring-no-zero-divisors*} *fls*
— Commutivity and no zero divisors are required by the definition of *pderiv*.
shows *fls-prpart* (*fls-deriv f*) = − *pCons 0* (*pCons 0* (*pderiv* (*fls-prpart f*)))
proof (*intro poly-eqI*)
fix *n*
show
coeff (*fls-prpart* (*fls-deriv f*)) *n* =
coeff (− *pCons 0* (*pCons 0* (*pderiv* (*fls-prpart f*)))) *n*
proof (*cases n*)
case (*Suc m*)
hence *n*: *n* = *Suc m* **by** *fast*
show *?thesis*
proof (*cases m*)
case (*Suc k*)
with *n* **have**
coeff (− *pCons 0* (*pCons 0* (*pderiv* (*fls-prpart f*)))) *n* =
− *coeff* (*pderiv* (*fls-prpart f*)) *k*
by (*simp flip*: *coeff-minus*)
with *Suc n* **show** *?thesis* **by** (*simp add*: *coeff-pderiv algebra-simps*)
qed (*simp add*: *n*)
qed *simp*
qed

lemma *pderiv-fls-prpart*:
pderiv (*fls-prpart f*) = − *poly-shift 2* (*fls-prpart* (*fls-deriv f*))

by (*intro poly-eqI*) (*simp add*: *coeff-pderiv coeff-poly-shift algebra-simps*)

lemma *fls-deriv-fps-to-fls*: *fls-deriv* (*fps-to-fls f*) = *fps-to-fls* (*fps-deriv f*)
proof (*intro fls-eqI*)
  fix *n*
  show *fls-deriv* (*fps-to-fls f*) $$ *n* = *fps-to-fls* (*fps-deriv f*) $$ *n*
  proof (*cases n≥0*)
    case *True*
    from *True* **have** *1*: *nat* (*n* + *1*) = *nat n* + *1* **by** *simp*
    from *True* **have** *2*: (*of-int* (*n* + *1*) :: *'a*) = *of-nat* (*nat* (*n*+*1*)) **by** *simp*
    from *True* **show** *?thesis* **using** *arg-cong*[*OF 2*, *of λx. x ∗ f* $ (*nat n*+*1*)] **by** (*simp add*: *1*)
  next
    case *False* **thus** *?thesis* **by** (*cases n*=−*1*) *auto*
  qed
qed

7.7.2 Algebraic rules of the derivative

lemma *fls-deriv-add* [*simp*]: *fls-deriv* (*f*+*g*) = *fls-deriv f* + *fls-deriv g*
  by (*auto intro*: *fls-eqI simp*: *algebra-simps*)

lemma *fls-deriv-sub* [*simp*]: *fls-deriv* (*f*−*g*) = *fls-deriv f* − *fls-deriv g*
  by (*auto intro*: *fls-eqI simp*: *algebra-simps*)

lemma *fls-deriv-neg* [*simp*]: *fls-deriv* (−*f*) = − *fls-deriv f*
  using *fls-deriv-sub*[*of 0 f*] **by** *simp*

lemma *fls-deriv-mult* [*simp*]:
  fls-deriv (*f*∗*g*) = *f* ∗ *fls-deriv g* + *fls-deriv f* ∗ *g*
proof−
  define *df dg* :: *int*
    where *df* ≡ *fls-subdegree f*
    and *dg* ≡ *fls-subdegree g*
  define *uf ug* :: *'a fls*
    where *uf* ≡ *fls-base-factor f*
    and *ug* ≡ *fls-base-factor g*
  have
    f ∗ *fls-deriv g* =
      of-int dg ∗ *fls-shift* (*1* − *dg*) (*f* ∗ *ug*) + *fls-shift* (−*dg*) (*f* ∗ *fls-deriv ug*)
    fls-deriv f ∗ *g* =
      of-int df ∗ *fls-shift* (*1* − *df*) (*uf* ∗ *g*) + *fls-shift* (−*df*) (*fls-deriv uf* ∗ *g*)
    using *fls-deriv-shift*[*of* −*df uf*] *fls-deriv-shift*[*of* −*dg ug*]
          mult-of-int-commute[*of dg f*]
          mult.assoc[*of of-int dg f*]
          fls-shifted-times-simps(*1*)[*of f 1* − *dg ug*]
          fls-shifted-times-simps(*1*)[*of f* −*dg fls-deriv ug*]
          fls-shifted-times-simps(*2*)[*of 1* − *df uf g*]
          fls-shifted-times-simps(*2*)[*of* −*df fls-deriv uf g*]

by (*auto simp add*: *algebra-simps df-def dg-def uf-def ug-def*)
moreover have
fls-deriv (*f*∗*g*) =
(*of-int dg* ∗ *fls-shift* (*1* − *dg*) (*f* ∗ *ug*) + *fls-shift* (−*dg*) (*f* ∗ *fls-deriv ug*)) +
(*of-int df* ∗ *fls-shift* (*1* − *df*) (*uf* ∗ *g*) + *fls-shift* (−*df*) (*fls-deriv uf* ∗ *g*))

using *fls-deriv-shift*[*of*
− (*df* + *dg*) *fps-to-fls* (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*)
]
fls-deriv-fps-to-fls[*of fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*]
fps-deriv-mult[*of fls-base-factor-to-fps f fls-base-factor-to-fps g*]
distrib-right[*of*
of-int df of-int dg
fls-shift (*1* − (*df* + *dg*)) (
fps-to-fls (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*)
)
]
fls-times-conv-fps-times[*of uf ug*]
fls-base-factor-subdegree[*of f*] *fls-base-factor-subdegree*[*of g*]
fls-regpart-deriv[*of ug*]
fls-times-conv-fps-times[*of uf fls-deriv ug*]
fls-deriv-subdegree0[*of ug*]
fls-regpart-deriv[*of uf*]
fls-times-conv-fps-times[*of fls-deriv uf ug*]
fls-deriv-subdegree0[*of uf*]
fls-shifted-times-simps(*1*)[*of uf* −*dg ug*]
fls-shifted-times-simps(*1*)[*of fls-deriv uf* −*dg ug*]
fls-shifted-times-simps(*2*)[*of* −*df uf ug*]
fls-shifted-times-simps(*2*)[*of* −*df uf fls-deriv ug*]
by (*simp add*: *fls-times-def algebra-simps df-def dg-def uf-def ug-def*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fls-deriv-mult-const-left*:
fls-deriv (*fls-const c* ∗ *f*) = *fls-const c* ∗ *fls-deriv f*
by *simp*

lemma *fls-deriv-linear*:
fls-deriv (*fls-const a* ∗ *f* + *fls-const b* ∗ *g*) =
fls-const a ∗ *fls-deriv f* + *fls-const b* ∗ *fls-deriv g*
by *simp*

lemma *fls-deriv-mult-const-right*:
fls-deriv (*f* ∗ *fls-const c*) = *fls-deriv f* ∗ *fls-const c*
by *simp*

lemma *fls-deriv-linear2*:
fls-deriv (*f* ∗ *fls-const a* + *g* ∗ *fls-const b*) =
fls-deriv f ∗ *fls-const a* + *fls-deriv g* ∗ *fls-const b*

by *simp*

lemma *fls-deriv-sum*:
fls-deriv (*sum f S*) = *sum* (λ*i*. *fls-deriv* (*f i*)) *S*
proof (*cases finite S*)
case *True* **show** *?thesis*
by (*induct rule*: *finite-induct* [*OF True*]) *simp-all*
qed *simp*

lemma *fls-deriv-power*:
fixes *f* :: *'a*::*comm-ring-1 fls*
shows *fls-deriv* (*f*^*n*) = *of-nat n* ∗ *f*^(*n*−*1*) ∗ *fls-deriv f*
proof (*cases n*)
case (*Suc m*)
have *fls-deriv* (*f*^*Suc m*) = *of-nat* (*Suc m*) ∗ *f*^*m* ∗ *fls-deriv f*
by (*induct m*) (*simp-all add*: *algebra-simps*)
with *Suc* **show** *?thesis* **by** *simp*
qed *simp*

lemma *fls-deriv-X-power*:
fls-deriv (*fls-X* ^ *n*) = *of-nat n* ∗ *fls-X* ^ (*n*−*1*)
proof (*cases n*)
case (*Suc m*)
have *fls-deriv* (*fls-X*^*Suc m*) = *of-nat* (*Suc m*) ∗ *fls-X*^*m*
by (*induct m*) (*simp-all add*: *mult-of-nat-commute algebra-simps*)
with *Suc* **show** *?thesis* **by** *simp*
qed *simp*

lemma *fls-deriv-X-inv-power*:
fls-deriv (*fls-X-inv* ^ *n*) = − *of-nat n* ∗ *fls-X-inv* ^ (*Suc n*)
proof (*cases n*)
case (*Suc m*)
define *iX* :: *'a fls* **where** *iX* ≡ *fls-X-inv*
have *fls-deriv* (*iX* ^ *Suc m*) = − *of-nat* (*Suc m*) ∗ *iX* ^ (*Suc* (*Suc m*))
proof (*induct m*)
case (*Suc m*)
have − *of-nat* (*Suc m* + *1*) ∗ *iX* ^ *Suc* (*Suc* (*Suc m*)) =
iX ∗ (−*of-nat* (*Suc m*) ∗ *iX* ^ *Suc* (*Suc m*)) +
− (*iX* ^ *2* ∗ *iX* ^ *Suc m*)
using *distrib-right*[*of* −*of-nat* (*Suc m*) −(*1*::*'a fls*) *fls-X-inv* ^ *Suc* (*Suc* (*Suc m*))]
by (*simp add*: *algebra-simps mult-of-nat-commute power2-eq-square Suc iX-def*)
thus *?case* **using** *Suc* **by** (*simp add*: *iX-def*)
qed (*simp add*: *numeral-2-eq-2 iX-def*)
with *Suc* **show** *?thesis* **by** (*simp add*: *iX-def*)
qed *simp*

lemma *fls-deriv-X-intpow*:
fls-deriv (*fls-X-intpow i*) = *of-int i* ∗ *fls-X-intpow* (*i*−*1*)

by (*simp add*: *fls-deriv-shift*)

lemma *fls-deriv-lr-inverse*:
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *fls-deriv* (*fls-left-inverse f x*) =
− *fls-left-inverse f x* ∗ *fls-deriv f* ∗ *fls-left-inverse f x*
and *fls-deriv* (*fls-right-inverse f y*) =
− *fls-right-inverse f y* ∗ *fls-deriv f* ∗ *fls-right-inverse f y*
proof−

define *L* **where** *L* ≡ *fls-left-inverse f x*
hence *fls-deriv* (*L* ∗ *f*) = *0* **using** *fls-left-inverse*[*OF assms*(*1*)] **by** *simp*
with *assms* **show** *fls-deriv L* = − *L* ∗ *fls-deriv f* ∗ *L*
using *fls-right-inverse′*[*OF assms*]
by (*simp add*: *minus-unique mult.assoc L-def*)

define *R* **where** *R* ≡ *fls-right-inverse f y*
hence *fls-deriv* (*f* ∗ *R*) = *0* **using** *fls-right-inverse*[*OF assms*(*2*)] **by** *simp*
hence *1*: *f* ∗ *fls-deriv R* + *fls-deriv f* ∗ *R* = *0* **by** *simp*
have *R* ∗ *f* ∗ *fls-deriv R* = − *R* ∗ *fls-deriv f* ∗ *R*
using *iffD2*[*OF eq-neg-iff-add-eq-0*, *OF 1*] **by** (*simp add*: *mult.assoc*)
thus *fls-deriv R* = − *R* ∗ *fls-deriv f* ∗ *R*
using *fls-left-inverse′*[*OF assms*] **by** (*simp add*: *R-def*)

qed

lemma *fls-deriv-lr-inverse-comm*:
fixes *x y* :: ′*a*::*comm-ring-1*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1*
shows *fls-deriv* (*fls-left-inverse f x*) = − *fls-deriv f* ∗ (*fls-left-inverse f x*)^2
and *fls-deriv* (*fls-right-inverse f x*) = − *fls-deriv f* ∗ (*fls-right-inverse f x*)^2
using *assms fls-deriv-lr-inverse*[*of x f x*]
by (*simp-all add*: *mult.commute power2-eq-square*)

lemma *fls-inverse-deriv-divring*:
fixes *a* :: ′*a*::*division-ring fls*
shows *fls-deriv* (*inverse a*) = − *inverse a* ∗ *fls-deriv a* ∗ *inverse a*
proof (*cases a*=*0*)
case *False* **thus** *?thesis*
using *fls-deriv-lr-inverse*(*2*)[*of*
inverse (*a* $$ *fls-subdegree a*) *a inverse* (*a* $$ *fls-subdegree a*)
]
by (*auto simp add*: *fls-inverse-def′*)
qed *simp*

lemma *fls-inverse-deriv*:
fixes *a* :: ′*a*::*field fls*
shows *fls-deriv* (*inverse a*) = − *fls-deriv a* ∗ (*inverse a*)^2

by (*simp add*: *fls-inverse-deriv-divring power2-eq-square*)

lemma *fls-inverse-deriv′*:
fixes *a* :: *′a*::*field fls*
shows *fls-deriv* (*inverse a*) = − *fls-deriv a* / a^2
using *fls-inverse-deriv*[*of a*]
by (*simp add*: *field-simps*)

7.7.3 Equality of derivatives

lemma *fls-deriv-eq-0-iff*:
fls-deriv f = *0* ⟷ *f* = *fls-const* (*f*$$*0* :: *′a*::{*ring-1-no-zero-divisors*,*ring-char-0*})
proof
assume *f*: *fls-deriv f* = *0*
show *f* = *fls-const* (*f*$$*0*)
proof (*intro fls-eqI*)
fix *n*
from *f* **have** *of-int n* ∗ *f*$$ *n* = *0* **using** *fls-deriv-nth*[*of f n*−*1*] **by** *simp*
thus *f*$$*n* = *fls-const* (*f*$$*0*) $$ *n* **by** (*cases n*=*0*) *auto*
qed
next
show *f* = *fls-const* (*f*$$*0*) ⟹ *fls-deriv f* = *0* **using** *fls-deriv-const*[*of f*$$*0*] **by** *simp*
qed

lemma *fls-deriv-eq-iff*:
fixes *f g* :: *′a*::{*ring-1-no-zero-divisors*,*ring-char-0*} *fls*
shows *fls-deriv f* = *fls-deriv g* ⟷ (*f* = *fls-const*(*f*$$*0* − *g*$$*0*) + *g*)
proof −
have *fls-deriv f* = *fls-deriv g* ⟷ *fls-deriv* (*f* − *g*) = *0*
by *simp*
also have . . . ⟷ *f* − *g* = *fls-const* ((*f* − *g*) $$ *0*)
unfolding *fls-deriv-eq-0-iff* **..**
finally show *?thesis*
by (*simp add*: *field-simps*)
qed

lemma *fls-deriv-eq-iff-ex*:
fixes *f g* :: *′a*::{*ring-1-no-zero-divisors*,*ring-char-0*} *fls*
shows (*fls-deriv f* = *fls-deriv g*) ⟷ (∃ *c*. *f* = *fls-const c* + *g*)
by (*auto simp*: *fls-deriv-eq-iff*)

7.7.4 Residues

definition *fls-residue-def*[*simp*]: *fls-residue f* ≡ *f* $$ −*1*

lemma *fls-residue-deriv*: *fls-residue* (*fls-deriv f*) = *0*
by *simp*

lemma *fls-residue-add*: *fls-residue* (*f*+*g*) = *fls-residue f* + *fls-residue g*

by *simp*

lemma *fls-residue-times-deriv*:
fls-residue (*fls-deriv f* ∗ *g*) = − *fls-residue* (*f* ∗ *fls-deriv g*)
using *fls-residue-deriv*[*of f*∗*g*] *minus-unique*[*of fls-residue* (*f* ∗ *fls-deriv g*)]
by *simp*

lemma *fls-residue-power-series*: *fls-subdegree f* ≥ *0* ⟹ *fls-residue f* = *0*
by *simp*

lemma *fls-residue-fls-X-intpow*:
fls-residue (*fls-X-intpow i*) = (*if i*=−*1 then 1 else 0*)
by *simp*

lemma *fls-residue-shift-nth*:
fixes *f* :: ′*a*::*semiring-1 fls*
shows *f*$$*n* = *fls-residue* (*fls-X-intpow* (−*n*−*1*) ∗ *f*)
by (*simp add*: *fls-shifted-times-transfer*)

lemma *fls-residue-fls-const-times*:
fixes *f* :: ′*a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows *fls-residue* (*fls-const c* ∗ *f*) = *c* ∗ *fls-residue f*
and *fls-residue* (*f* ∗ *fls-const c*) = *fls-residue f* ∗ *c*
by *simp-all*

lemma *fls-residue-of-int-times*:
fixes *f* :: ′*a*::*ring-1 fls*
shows *fls-residue* (*of-int i* ∗ *f*) = *of-int i* ∗ *fls-residue f*
and *fls-residue* (*f* ∗ *of-int i*) = *fls-residue f* ∗ *of-int i*
by (*simp-all add*: *fls-residue-fls-const-times fls-of-int*)

lemma *fls-residue-deriv-times-lr-inverse-eq-subdegree*:
fixes *f g* :: ′*a*::*ring-1 fls*
assumes *y* ∗ (*f* $$ *fls-subdegree f*) = *1* (*f* $$ *fls-subdegree f*) ∗ *y* = *1*
shows *fls-residue* (*fls-deriv f* ∗ *fls-right-inverse f y*) = *of-int* (*fls-subdegree f*)
and *fls-residue* (*fls-deriv f* ∗ *fls-left-inverse f y*) = *of-int* (*fls-subdegree f*)
and *fls-residue* (*fls-left-inverse f y* ∗ *fls-deriv f*) = *of-int* (*fls-subdegree f*)
and *fls-residue* (*fls-right-inverse f y* ∗ *fls-deriv f*) = *of-int* (*fls-subdegree f*)
proof−
define *df* :: *int* **where** *df* ≡ *fls-subdegree f*
define *B X* :: ′*a fls*
where *B* ≡ *fls-base-factor f*
and *X* ≡ (*fls-X-intpow df* :: ′*a fls*)
define *D L R* :: ′*a fls*
where *D* ≡ *fls-deriv B*
and *L* ≡ *fls-left-inverse B y*
and *R* ≡ *fls-right-inverse B y*
have *intpow-diff*: *fls-X-intpow* (*df* − *1*) = *X* ∗ *fls-X-inv*
using *fls-X-intpow-diff-conv-times*[*of df 1*] **by** (*simp add*: *X-def fls-X-inv-conv-shift-1*)

show *fls-residue* (*fls-deriv f* ∗ *fls-right-inverse f y*) = *of-int df*
proof−
have *subdegree-DR*: *fls-subdegree* (*D* ∗ *R*) ≥ *0*
using *fls-base-factor-subdegree*[*of f*] *fls-base-factor-subdegree*[*of fls-right-inverse f y*]
assms(*1*) *fls-right-inverse-base-factor*[*of y f*] *fls-mult-subdegree-ge-0*[*of D R*]
by (*force simp*: *fls-deriv-subdegree0 D-def R-def B-def*)
have *decomp*: *f* = *X* ∗ *B*
unfolding *X-def B-def df-def* **by** (*rule fls-base-factor-X-power-decompose*(*2*)[*of f*])
hence *fls-deriv f* = *X* ∗ *D* + *of-int df* ∗ *X* ∗ *fls-X-inv* ∗ *B*
using *intpow-diff fls-deriv-mult*[*of X B*]
by (*simp add*: *fls-deriv-X-intpow X-def B-def D-def mult.assoc*)
moreover from *assms* **have** *fls-right-inverse* (*X* ∗ *B*) *y* = *R* ∗ *fls-right-inverse X 1*
using *fls-base-factor-base*[*of f*] *fls-lr-inverse-mult-ring1*(*2*)[*of 1 X*]
by (*simp add*: *X-def B-def R-def*)
ultimately have
fls-deriv f ∗ *fls-right-inverse f y* =
(*D* + *of-int df* ∗ *fls-X-inv* ∗ *B*) ∗ *R* ∗ (*X* ∗ *fls-right-inverse X 1*)
by (*simp add*: *decomp algebra-simps X-def fls-X-intpow-times-comm*)
also have ... = *D* ∗ *R* + *of-int df* ∗ *fls-X-inv*
using *fls-right-inverse*[*of X 1*]
assms fls-base-factor-base[*of f*] *fls-right-inverse*[*of B y*]
by (*simp add*: *X-def distrib-right mult.assoc B-def R-def*)
finally show *?thesis* **using** *subdegree-DR* **by** *simp*
qed

with *assms* **show** *fls-residue* (*fls-deriv f* ∗ *fls-left-inverse f y*) = *of-int df*
using *fls-left-inverse-eq-fls-right-inverse*[*of y f*] **by** *simp*

show *fls-residue* (*fls-left-inverse f y* ∗ *fls-deriv f*) = *of-int df*
proof−
have *subdegree-LD*: *fls-subdegree* (*L* ∗ *D*) ≥ *0*
using *fls-base-factor-subdegree*[*of f*] *fls-base-factor-subdegree*[*of fls-left-inverse f y*]
assms(*1*) *fls-left-inverse-base-factor*[*of y f*] *fls-mult-subdegree-ge-0*[*of L D*]
by (*force simp*: *fls-deriv-subdegree0 D-def L-def B-def*)
have *decomp*: *f* = *B* ∗ *X*
unfolding *X-def B-def df-def* **by** (*rule fls-base-factor-X-power-decompose*(*1*)[*of f*])
hence *fls-deriv f* = *D* ∗ *X* + *B* ∗ *of-int df* ∗ *X* ∗ *fls-X-inv*
using *intpow-diff fls-deriv-mult*[*of B X*]
by (*simp add*: *fls-deriv-X-intpow X-def D-def B-def mult.assoc*)
moreover from *assms* **have** *fls-left-inverse* (*B* ∗ *X*) *y* = *fls-left-inverse X 1* ∗

L
  using *fls-base-factor-base*[*of f*] *fls-lr-inverse-mult-ring1*(*1*)[*of - - 1 X*]
  by (*simp add*: *X-def B-def L-def*)
 ultimately have
  fls-left-inverse f y ∗ *fls-deriv f* =
   fls-left-inverse X 1 ∗ *X* ∗ *L* ∗ (*D* + *B* ∗ (*of-int df* ∗ *fls-X-inv*))
  by (*simp add*: *decomp algebra-simps X-def fls-X-intpow-times-comm*)
 also have ... = *L* ∗ *D* + *of-int df* ∗ *fls-X-inv*
  using *assms fls-left-inverse*[*of 1 X*] *fls-base-factor-base*[*of f*] *fls-left-inverse*[*of y B*]
  by (*simp add*: *X-def distrib-left mult.assoc*[*symmetric*] *L-def B-def*)
 finally show *?thesis* **using** *subdegree-LD* **by** *simp*
qed

with *assms* **show** *fls-residue* (*fls-right-inverse f y* ∗ *fls-deriv f*) = *of-int df*
 using *fls-left-inverse-eq-fls-right-inverse*[*of y f*] **by** *simp*

qed

lemma *fls-residue-deriv-times-inverse-eq-subdegree*:
 fixes *f g* :: *'a*::*division-ring fls*
 shows *fls-residue* (*fls-deriv f* ∗ *inverse f*) = *of-int* (*fls-subdegree f*)
 and *fls-residue* (*inverse f* ∗ *fls-deriv f*) = *of-int* (*fls-subdegree f*)
proof−
 show *fls-residue* (*fls-deriv f* ∗ *inverse f*) = *of-int* (*fls-subdegree f*)
  using *fls-residue-deriv-times-lr-inverse-eq-subdegree*(*1*)[*of - f*]
  by (*cases f=0*) (*auto simp*: *fls-inverse-def'*)
 show *fls-residue* (*inverse f* ∗ *fls-deriv f*) = *of-int* (*fls-subdegree f*)
  using *fls-residue-deriv-times-lr-inverse-eq-subdegree*(*4*)[*of - f*]
  by (*cases f=0*) (*auto simp*: *fls-inverse-def'*)
qed

7.7.5 Integral definition and basic properties

definition *fls-integral* :: *'a*::{*ring-1*,*inverse*} *fls* ⇒ *'a fls*
 where *fls-integral a* = *Abs-fls* (λ*n. if n=0 then 0 else inverse* (*of-int n*) ∗ *a*$$(*n* − *1*))

lemma *fls-integral-nth* [*simp*]:
 fls-integral a $$ *n* = (*if n=0 then 0 else inverse* (*of-int n*) ∗ *a*$$(*n*−*1*))
proof−
 define *F* **where** *F* ≡ (λ*n. if n=0 then 0 else inverse* (*of-int n*) ∗ *a*$$(*n* − *1*))
 obtain *N* **where** ∀ *n*<*N*. *a*$$*n* = *0* **by** (*elim fls-nth-vanishes-belowE*)
 hence ∀ *n*<*N*. *F n* = *0* **by** (*auto simp add*: *F-def*)
 thus *?thesis* **using** *nth-Abs-fls-lower-bound*[*of N F*] **unfolding** *fls-integral-def F-def* **by** *simp*
qed

lemma *fls-integral-conv-fps-zeroth-integral*:

assumes *fls-subdegree a* ≥ *0*
shows *fls-integral a* = *fps-to-fls* (*fps-integral0* (*fls-regpart a*))
proof (*rule fls-eqI*)
fix *n*
show *fls-integral a* $$ *n* = *fps-to-fls* (*fps-integral0* (*fls-regpart a*)) $$ *n*
proof (*cases n>0*)
case *False* **with** *assms* **show** *?thesis* **by** *simp*
next
case *True*
hence *int* ((*nat n*) − *1*) = *n* − *1* **by** *simp*
with *True* **show** *?thesis* **by** (*simp add*: *fps-integral-def*)
qed
qed

lemma *fls-integral-zero* [*simp*]: *fls-integral 0* = *0*
by (*intro fls-eqI*) *simp*

lemma *fls-integral-const′*:
fixes *x* :: *′a*::{*ring-1*,*inverse*}
assumes *inverse* (*1*::*′a*) = *1*
shows *fls-integral* (*fls-const x*) = *fls-const x* ∗ *fls-X*
by (*intro fls-eqI*) (*simp add*: *assms*)

lemma *fls-integral-const*:
fixes *x* :: *′a*::*division-ring*
shows *fls-integral* (*fls-const x*) = *fls-const x* ∗ *fls-X*
by (*rule fls-integral-const′*[*OF inverse-1*])

lemma *fls-integral-of-nat′*:
assumes *inverse* (*1*::*′a*::{*ring-1*,*inverse*}) = *1*
shows *fls-integral* (*of-nat n* :: *′a fls*) = *of-nat n* ∗ *fls-X*
by (*simp add*: *assms fls-integral-const′ fls-of-nat*)

lemma *fls-integral-of-nat*:
fls-integral (*of-nat n* :: *′a*::*division-ring fls*) = *of-nat n* ∗ *fls-X*
by (*rule fls-integral-of-nat′*[*OF inverse-1*])

lemma *fls-integral-of-int′*:
assumes *inverse* (*1*::*′a*::{*ring-1*,*inverse*}) = *1*
shows *fls-integral* (*of-int i* :: *′a fls*) = *of-int i* ∗ *fls-X*
by (*simp add*: *assms fls-integral-const′ fls-of-int*)

lemma *fls-integral-of-int*:
fls-integral (*of-int i* :: *′a*::*division-ring fls*) = *of-int i* ∗ *fls-X*
by (*rule fls-integral-of-int′*[*OF inverse-1*])

lemma *fls-integral-one′*:
assumes *inverse* (*1*::*′a*::{*ring-1*,*inverse*}) = *1*
shows *fls-integral* (*1*::*′a fls*) = *fls-X*

using *fls-integral-const′*[*of 1*]
by (*force simp*: *assms*)

lemma *fls-integral-one*: *fls-integral* (*1*::*′a*::*division-ring fls*) = *fls-X*
by (*rule fls-integral-one′*[*OF inverse-1*])

lemma *fls-subdegree-integral-ge*:
fls-integral f ≠ *0* ⟹ *fls-subdegree* (*fls-integral f*) ≥ *fls-subdegree f* + *1*
by (*intro fls-subdegree-geI*) *simp-all*

lemma *fls-subdegree-integral*:
fixes *f* :: *′a*::{*division-ring*,*ring-char-0*} *fls*
assumes *f* ≠ *0 fls-subdegree f* ≠ −*1*
shows *fls-subdegree* (*fls-integral f*) = *fls-subdegree f* + *1*
using *assms of-int-0-eq-iff*[*of fls-subdegree f* + *1*] *fls-subdegree-integral-ge*
by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-integral-X* [*simp*]:
fls-integral (*fls-X*::*′a*::{*ring-1*,*inverse*} *fls*) =
fls-const (*inverse* (*of-int 2*)) ∗ *fls-X*^{2}
proof (*intro fls-eqI*)
fix *n*
show *fls-integral* (*fls-X*::*′a fls*) $$ *n* = (*fls-const* (*inverse* (*of-int 2*)) ∗ *fls-X*^{2}) $$ *n*
using *arg-cong*[*OF fls-X-power-nth*, *of λx. inverse* (*of-int 2*) ∗ *x*, *of 2 n*, *symmetric*]
by (*auto simp add*:)
qed

lemma *fls-integral-X-power*:
fls-integral (*fls-X* ^ *n* ::*′a* :: {*ring-1*,*inverse*} *fls*) =
fls-const (*inverse* (*of-nat* (*Suc n*))) ∗ *fls-X* ^ *Suc n*
proof (*intro fls-eqI*)
fix *k*
have (*fls-X* :: *′a fls*) ^ *Suc n* $$ *k* = (*if k*=*Suc n then 1 else 0*)
by (*rule fls-X-power-nth*)
thus
fls-integral ((*fls-X*::*′a fls*) ^ *n*) $$ *k* =
(*fls-const* (*inverse* (*of-nat* (*Suc n*))) ∗ (*fls-X*::*′a fls*) ^ *Suc n*) $$ *k*
by *simp*
qed

lemma *fls-integral-X-power-char0*:
fls-integral (*fls-X* ^ *n* :: *′a* :: {*ring-char-0*,*inverse*} *fls*) =
inverse (*of-nat* (*Suc n*)) ∗ *fls-X* ^ *Suc n*
proof −
have (*of-nat* (*Suc n*) :: *′a*) ≠ *0* **by** (*rule of-nat-neq-0*)
hence *fls-const* (*inverse* (*of-nat* (*Suc n*) :: *′a*)) = *inverse* (*fls-const* (*of-nat* (*Suc n*)))

by (*simp add*: *fls-inverse-const*)
moreover have
fls-integral ((*fls-X*::*'a fls*) ^ *n*) = *fls-const* (*inverse* (*of-nat* (*Suc n*))) ∗ *fls-X* ^ *Suc n*
by (*rule fls-integral-X-power*)
ultimately show *?thesis* **by** (*simp add*: *fls-of-nat*)
qed

lemma *fls-integral-X-inv* [*simp*]: *fls-integral* (*fls-X-inv*::*'a*::{*ring-1*,*inverse*} *fls*) = *0*
by (*intro fls-eqI*) *simp*

lemma *fls-integral-X-inv-power*:
assumes *n* ≥ *2*
shows
fls-integral (*fls-X-inv* ^ *n* :: *'a* :: {*ring-1*,*inverse*} *fls*) =
fls-const (*inverse* (*of-int* (*1* − *int n*))) ∗ *fls-X-inv* ^(*n*−*1*)
proof (*rule fls-eqI*)
fix *k* **show**
fls-integral (*fls-X-inv* ^ *n* :: *'a fls*) $$ *k*=
(*fls-const* (*inverse* (*of-int* (*1* − *int n*))) ∗ *fls-X-inv* ^(*n*−*1*)) $$ *k*
proof (*cases k*=*0*)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False*
from *assms* **have** *int* (*n*−*1*) = *int n* − *1* **by** *simp*
hence
(*fls-const* (*inverse* (*of-int* (*1* − *int n*))) ∗ (*fls-X-inv*:: *'a fls*) ^(*n*−*1*)) $$ *k* =
(*if k* = *1* − *int n then inverse* (*of-int k*) *else 0*)
by (*simp add*: *fls-X-inv-power-times-conv-shift*(*2*))
with *False* **show** *?thesis* **by** (*simp add*: *algebra-simps*)
qed
qed

lemma *fls-integral-X-inv-power-char0*:
assumes *n* ≥ *2*
shows
fls-integral (*fls-X-inv* ^ *n* :: *'a* :: {*ring-char-0*,*inverse*} *fls*) =
inverse (*of-int* (*1* − *int n*)) ∗ *fls-X-inv* ^(*n*−*1*)
proof−
from *assms* **have** (*of-int* (*1* − *int n*) :: *'a*) ≠ *0* **by** *simp*
hence
fls-const (*inverse* (*of-int* (*1* − *int n*) :: *'a*)) = *inverse* (*fls-const* (*of-int* (*1* − *int n*)))
by (*simp add*: *fls-inverse-const*)
moreover have
fls-integral (*fls-X-inv* ^ *n* :: *'a fls*) =
fls-const (*inverse* (*of-int* (*1* − *int n*))) ∗ *fls-X-inv* ^(*n*−*1*)
using *assms* **by** (*rule fls-integral-X-inv-power*)

ultimately show *?thesis* **by** (*simp add*: *fls-of-int*)
qed

lemma *fls-integral-X-inv-power′*:
assumes $n \geq 1$
shows
fls-integral (*fls-X-inv* ^ *n* :: ′*a* :: *division-ring fls*) =
− *fls-const* (*inverse* (*of-nat* (*n*−*1*))) ∗ *fls-X-inv* ^(*n*−*1*)
proof (*cases n* = *1*)
case *False*
with *assms* **have** *n*: $n \geq 2$ **by** *simp*
hence
fls-integral (*fls-X-inv* ^ *n* :: ′*a fls*) =
fls-const (*inverse* (− *of-nat* (*nat* (*int n* − *1*)))) ∗ *fls-X-inv* ^(*n*−*1*)
by (*simp add*: *fls-integral-X-inv-power*)
moreover from *n* **have** *nat* (*int n* − *1*) = *n* − *1* **by** *simp*
ultimately show *?thesis*
using *inverse-minus-eq*[*of of-nat* (*n*−*1*) :: ′*a*] **by** *simp*
qed *simp*

lemma *fls-integral-X-inv-power-char0′*:
assumes $n \geq 1$
shows
fls-integral (*fls-X-inv* ^ *n* :: ′*a* :: {*division-ring*,*ring-char-0*} *fls*) =
− *inverse* (*of-nat* (*n*−*1*)) ∗ *fls-X-inv* ^(*n*−*1*)
proof (*cases n*=*1*)
case *False* **with** *assms* **show** *?thesis*
by (*simp add*: *fls-integral-X-inv-power′ fls-inverse-const fls-of-nat*)
qed *simp*

lemma *fls-integral-delta*:
assumes $m \neq -1$
shows
fls-integral (*Abs-fls* (λ*n*. *if n*=*m then c else 0*)) =
Abs-fls (λ*n*. *if n*=*m*+*1 then inverse* (*of-int* (*m*+*1*)) ∗ *c else 0*)
using *assms*
by (*intro fls-eqI*) *auto*

lemma *fls-regpart-integral*:
fls-regpart (*fls-integral f*) = *fps-integral0* (*fls-regpart f*)
proof (*rule fps-ext*)
fix *n*
show *fls-regpart* (*fls-integral f*) $ *n* = *fps-integral0* (*fls-regpart f*) $ *n*
by (*cases n*) (*simp-all add*: *fps-integral-def*)
qed

lemma *fls-integral-fps-to-fls*:
fls-integral (*fps-to-fls f*) = *fps-to-fls* (*fps-integral0 f*)
proof (*intro fls-eqI*)

fix *n* :: *int*
show *fls-integral* (*fps-to-fls f*) $$ *n* = *fps-to-fls* (*fps-integral0 f*) $$ *n*
proof (*cases n<1*)
case *True* **thus** *?thesis* **by** *simp*
next
case *False*
hence *nat* (*n−1*) = *nat n* − *1* **by** *simp*
with *False* **show** *?thesis* **by** (*cases nat n*) *auto*
qed
qed

7.7.6 Algebraic rules of the integral

lemma *fls-integral-add* [*simp*]: *fls-integral* (*f*+*g*) = *fls-integral f* + *fls-integral g*
by (*intro fls-eqI*) (*simp add*: *algebra-simps*)

lemma *fls-integral-sub* [*simp*]: *fls-integral* (*f*−*g*) = *fls-integral f* − *fls-integral g*
by (*intro fls-eqI*) (*simp add*: *algebra-simps*)

lemma *fls-integral-neg* [*simp*]: *fls-integral* (−*f*) = − *fls-integral f*
using *fls-integral-sub*[*of 0 f*] **by** *simp*

lemma *fls-integral-mult-const-left*:
fls-integral (*fls-const c* ∗ *f*) = *fls-const c* ∗ *fls-integral* (*f* :: ′*a*::*division-ring fls*)
by (*intro fls-eqI*) (*simp add*: *mult.assoc mult-inverse-of-int-commute*)

lemma *fls-integral-mult-const-left-comm*:
fixes *f* :: ′*a*::{*comm-ring-1*,*inverse*} *fls*
shows *fls-integral* (*fls-const c* ∗ *f*) = *fls-const c* ∗ *fls-integral f*
by (*intro fls-eqI*) (*simp add*: *mult.assoc mult.commute*)

lemma *fls-integral-linear*:
fixes *f g* :: ′*a*::*division-ring fls*
shows
fls-integral (*fls-const a* ∗ *f* + *fls-const b* ∗ *g*) =
fls-const a ∗ *fls-integral f* + *fls-const b* ∗ *fls-integral g*
by (*simp add*: *fls-integral-mult-const-left*)

lemma *fls-integral-linear-comm*:
fixes *f g* :: ′*a*::{*comm-ring-1*,*inverse*} *fls*
shows
fls-integral (*fls-const a* ∗ *f* + *fls-const b* ∗ *g*) =
fls-const a ∗ *fls-integral f* + *fls-const b* ∗ *fls-integral g*
by (*simp add*: *fls-integral-mult-const-left-comm*)

lemma *fls-integral-mult-const-right*:
fls-integral (*f* ∗ *fls-const c*) = *fls-integral f* ∗ *fls-const c*
by (*intro fls-eqI*) (*simp add*: *mult.assoc*)

lemma *fls-integral-linear2*:
  fls-integral (*f* ∗ *fls-const a* + *g* ∗ *fls-const b*) =
    fls-integral f ∗ *fls-const a* + *fls-integral g* ∗ *fls-const b*
  by (*simp add*: *fls-integral-mult-const-right*)

lemma *fls-integral-sum*:
  fls-integral (*sum f S*) = *sum* (λ*i*. *fls-integral* (*f i*)) *S*
proof (*cases finite S*)
  case *True* **show** *?thesis*
    by (*induct rule*: *finite-induct* [*OF True*]) *simp-all*
qed *simp*

7.7.7 Derivatives of integrals and vice versa

lemma *fls-integral-fls-deriv*:
  fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
  shows *fls-integral* (*fls-deriv a*) + *fls-const* (*a*$$*0*) = *a*
  by (*intro fls-eqI*) (*simp add*: *mult.assoc*[*symmetric*])

lemma *fls-deriv-fls-integral*:
  fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
  assumes *fls-residue a* = *0*
  shows *fls-deriv* (*fls-integral a*) = *a*
proof (*intro fls-eqI*)
  fix *n* :: *int*
  show *fls-deriv* (*fls-integral a*) $$ *n* = *a* $$ *n*
  proof (*cases n*=−*1*)
    case *True* **with** *assms* **show** *?thesis* **by** *simp*
  next
    case *False*
    hence (*of-int* (*n*+*1*) :: *'a*) ≠ *0* **using** *of-int-eq-0-iff*[*of n*+*1*] **by** *simp*
    hence (*of-int* (*n*+*1*) :: *'a*) ∗ *inverse* (*of-int* (*n*+*1*) :: *'a*) = (*1*::*'a*)
      using *of-int-eq-0-iff*[*of n*+*1*] **by** *simp*
    moreover have
      fls-deriv (*fls-integral a*) $$ *n* =
        (*if n*=−*1 then 0 else of-int* (*n*+*1*) ∗ *inverse* (*of-int* (*n*+*1*)) ∗ *a*$$*n*)
      by (*simp add*: *mult.assoc*)
    ultimately show *?thesis*
      by (*simp add*: *False*)
  qed
qed

Series with zero residue are precisely the derivatives.

lemma *fls-residue-nonzero-ex-antiderivative*:
  fixes *f* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
  assumes *fls-residue f* = *0*
  shows ∃*F*. *fls-deriv F* = *f*
  using *assms fls-deriv-fls-integral*
  by *auto*

lemma *fls-ex-antiderivative-residue-nonzero*:
assumes ∃ *F*. *fls-deriv F* = *f*
shows *fls-residue f* = *0*
using *assms fls-residue-deriv*
by *auto*

lemma *fls-residue-nonzero-ex-anitderivative-iff*:
fixes *f* :: ′*a*::{*division-ring*,*ring-char-0*} *fls*
shows *fls-residue f* = *0* ⟷ (∃ *F*. *fls-deriv F* = *f*)
using *fls-residue-nonzero-ex-antiderivative fls-ex-antiderivative-residue-nonzero*
by *fast*

7.8 Topology

instantiation *fls* :: (*group-add*) *metric-space*
begin

definition
dist-fls-def:
dist (*a* :: ′*a fls*) *b* =
(*if a* = *b*
then 0
else if fls-subdegree (*a*−*b*) ≥ *0*
then inverse (*2* ^ *nat* (*fls-subdegree* (*a*−*b*)))
else 2 ^ *nat* (−*fls-subdegree* (*a*−*b*))
)

lemma *dist-fls-ge0*: *dist* (*a* :: ′*a fls*) *b* ≥ *0*
by (*simp add*: *dist-fls-def*)

definition *uniformity-fls-def* [*code del*]:
(*uniformity* :: (′*a fls* × ′*a fls*) *filter*) = (*INF e* ∈ {*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-fls-def* ′ [*code del*]:
open (*U* :: ′*a fls set*) ⟷ (∀ *x*∈*U*. *eventually* (λ(*x*′, *y*). *x*′ = *x* ⟶ *y* ∈ *U*) *uniformity*)

lemma *dist-fls-sym*: *dist* (*a* :: ′*a fls*) *b* = *dist b a*
by (*cases a*≠*b*, *cases fls-subdegree* (*a*−*b*) ≥ *0*)
(*simp-all add*: *fls-subdegree-minus-sym dist-fls-def*)

context
begin

private lemma *instance-helper*:
fixes *a b c* :: ′*a fls*
assumes *neq*: *a*≠*b a*≠*c*

and *dist-ineq*: *dist a b* > *dist a c*
shows *fls-subdegree* (*a* − *b*) < *fls-subdegree* (*a* − *c*)
proof (
cases fls-subdegree (*a*−*b*) ≥ *0 fls-subdegree* (*a*−*c*) ≥ *0*
rule: *case-split*[*case-product case-split*]
)
case *True-True* **with** *neq dist-ineq* **show** *?thesis* **by** (*simp add*: *dist-fls-def*)
next
case *False-True* **with** *dist-ineq* **show** *?thesis* **by** (*simp add*: *dist-fls-def*)
next
case *False-False* **with** *neq dist-ineq* **show** *?thesis* **by** (*simp add*: *dist-fls-def*)
next
case *True-False*
with *neq*
have (*1*::*real*) > *2* ^ (*nat* (*fls-subdegree* (*a*−*b*)) + *nat* (−*fls-subdegree* (*a*−*c*)))
and *nat* (*fls-subdegree* (*a*−*b*)) + *nat* (−*fls-subdegree* (*a*−*c*)) =
nat (*fls-subdegree* (*a*−*b*) − *fls-subdegree* (*a*−*c*))
using *dist-ineq*
by (*simp-all add*: *dist-fls-def field-simps power-add*)
hence ¬ (*1*::*real*) < *2* ^ (*nat* (*fls-subdegree* (*a*−*b*) − *fls-subdegree* (*a*−*c*))) **by** *simp*
hence ¬ (*0* < *nat* (*fls-subdegree* (*a* − *b*) − *fls-subdegree* (*a* − *c*))) **by** *auto*
hence *fls-subdegree* (*a* − *b*) ≤ *fls-subdegree* (*a* − *c*) **by** *simp*
with *True-False* **show** *?thesis* **by** *simp*
qed

instance
proof
show *th*: *dist a b* = *0* ⟷ *a* = *b* **for** *a b* :: *'a fls*
by (*simp add*: *dist-fls-def split*: *if-split-asm*)
then have *th'*[*simp*]: *dist a a* = *0* **for** *a* :: *'a fls* **by** *simp*

fix *a b c* :: *'a fls*
consider *a* = *b* | *c* = *a* ∨ *c* = *b* | *a* ≠ *b a* ≠ *c b* ≠ *c* **by** *blast*
then show *dist a b* ≤ *dist a c* + *dist b c*
proof *cases*
case *1*
then show *?thesis* **by** (*simp add*: *dist-fls-def*)
next
case *2*
then show *?thesis*
by (*cases c* = *a*) (*simp-all add*: *th dist-fls-sym*)
next
case *neq*: *3*
have *False* **if** *dist a b* > *dist a c* + *dist b c*
proof −
from *neq* **have** *dist a b* > *0 dist b c* > *0 dist a c* > *0* **by** (*simp-all add*: *dist-fls-def*)
with *that* **have** *dist-ineq*: *dist a b* > *dist a c dist a b* > *dist b c* **by** *simp-all*

have *fls-subdegree* (*a* − *b*) < *fls-subdegree* (*a* − *c*)
and *fls-subdegree* (*a* − *b*) < *fls-subdegree* (*b* − *c*)
using *instance-helper*[*of a b c*] *instance-helper*[*of b a c*] *neq dist-ineq*
by (*simp-all add*: *dist-fls-sym fls-subdegree-minus-sym*)
hence (*a* − *c*) $$ *fls-subdegree* (*a* − *b*) = *0* **and** (*b* − *c*) $$ *fls-subdegree* (*a* − *b*) = *0*
by (*simp-all only*: *fls-eq0-below-subdegree*)
hence (*a* − *b*) $$ *fls-subdegree* (*a* − *b*) = *0* **by** *simp*
moreover from *neq* **have** (*a* − *b*) $$ *fls-subdegree* (*a* − *b*) ≠ *0*
by (*intro nth-fls-subdegree-nonzero*) *simp*
ultimately show *False* **by** *contradiction*
qed
thus *?thesis* **by** (*auto simp*: *not-le*[*symmetric*])
qed
qed (*rule open-fls-def′ uniformity-fls-def*)+

end
end

declare *uniformity-Abort*[**where** *′a*=*′a* :: *group-add fls*, *code*]

lemma *open-fls-def*:
open (*S* :: *′a*::*group-add fls set*) = (∀ *a* ∈ *S*. ∃ *r*. *r* >*0* ∧ {*y*. *dist y a* < *r*} ⊆ *S*)
unfolding *open-dist subset-eq* **by** *simp*

7.9 Notation

bundle *fps-syntax*
begin
notation *fls-nth* (**infixl** ‹$$› *75*)
end

unbundle *no fps-syntax*

end

8 The fraction field of any integral domain

theory *Fraction-Field*
imports *Main*
begin

8.1 General fractions construction

8.1.1 Construction of the type of fractions

context *idom* **begin**

definition *fractrel* :: *′a* × *′a* ⇒ *′a* ∗ *′a* ⇒ *bool* **where**

fractrel = (*λx y*. *snd x* ≠ *0* ∧ *snd y* ≠ *0* ∧ *fst x* ∗ *snd y* = *fst y* ∗ *snd x*)

lemma *fractrel-iff* [*simp*]:
fractrel x y ⟷ *snd x* ≠ *0* ∧ *snd y* ≠ *0* ∧ *fst x* ∗ *snd y* = *fst y* ∗ *snd x*
by (*simp add*: *fractrel-def*)

lemma *symp-fractrel*: *symp fractrel*
by (*simp add*: *symp-def*)

lemma *transp-fractrel*: *transp fractrel*
proof (*rule transpI*, *unfold split-paired-all*)
fix *a b a′ b′ a′′ b′′* :: ′*a*
assume *A*: *fractrel* (*a*, *b*) (*a′*, *b′*)
assume *B*: *fractrel* (*a′*, *b′*) (*a′′*, *b′′*)
have *b′* ∗ (*a* ∗ *b′′*) = *b′′* ∗ (*a* ∗ *b′*) **by** (*simp add*: *ac-simps*)
also from *A* **have** *a* ∗ *b′* = *a′* ∗ *b* **by** *auto*
also have *b′′* ∗ (*a′* ∗ *b*) = *b* ∗ (*a′* ∗ *b′′*) **by** (*simp add*: *ac-simps*)
also from *B* **have** *a′* ∗ *b′′* = *a′′* ∗ *b′* **by** *auto*
also have *b* ∗ (*a′′* ∗ *b′*) = *b′* ∗ (*a′′* ∗ *b*) **by** (*simp add*: *ac-simps*)
finally have *b′* ∗ (*a* ∗ *b′′*) = *b′* ∗ (*a′′* ∗ *b*) **.**
moreover from *B* **have** *b′* ≠ *0* **by** *auto*
ultimately have *a* ∗ *b′′* = *a′′* ∗ *b* **by** *simp*
with *A B* **show** *fractrel* (*a*, *b*) (*a′′*, *b′′*) **by** *auto*
qed

lemma *part-equivp-fractrel*: *part-equivp fractrel*
using - *symp-fractrel transp-fractrel*
by(*rule part-equivpI*)(*rule exI*[**where** *x*=(*0*, *1*)]; *simp*)

end

quotient-type (**overloaded**) ′*a fract* = ′*a* :: *idom* × ′*a* / *partial*: *fractrel*
by(*rule part-equivp-fractrel*)

8.1.2 Representation and basic operations

lift-definition *Fract* :: ′*a* :: *idom* ⇒ ′*a* ⇒ ′*a fract*
is *λa b*. *if b* = *0 then* (*0*, *1*) *else* (*a*, *b*)
by *simp*

lemma *Fract-cases* [*cases type*: *fract*]:
obtains (*Fract*) *a b* **where** *q* = *Fract a b b* ≠ *0*
by *transfer simp*

lemma *Fract-induct* [*case-names Fract*, *induct type*: *fract*]:
(⋀*a b*. *b* ≠ *0* ⟹ *P* (*Fract a b*)) ⟹ *P q*
by (*cases q*) *simp*

lemma *eq-fract*:

shows $\bigwedge a\ b\ c\ d.\ b \neq 0 \Longrightarrow d \neq 0 \Longrightarrow Fract\ a\ b = Fract\ c\ d \longleftrightarrow a * d = c * b$
and $\bigwedge a.\ Fract\ a\ 0 = Fract\ 0\ 1$
and $\bigwedge a\ c.\ Fract\ 0\ a = Fract\ 0\ c$
by(*transfer*; *simp*)+

instantiation *fract* :: (*idom*) *comm-ring-1*
begin

lift-definition *zero-fract* :: *'a fract* **is** (*0*, *1*) **by** *simp*

lemma *Zero-fract-def*: *0* = *Fract 0 1*
by *transfer simp*

lift-definition *one-fract* :: *'a fract* **is** (*1*, *1*) **by** *simp*

lemma *One-fract-def*: *1* = *Fract 1 1*
by *transfer simp*

lift-definition *plus-fract* :: *'a fract* ⇒ *'a fract* ⇒ *'a fract*
is $\lambda q\ r.\ (fst\ q * snd\ r + fst\ r * snd\ q,\ snd\ q * snd\ r)$
by(*auto simp add*: *algebra-simps*)

lemma *add-fract* [*simp*]:
$[\![\ b \neq 0;\ d \neq 0\ ]\!] \Longrightarrow Fract\ a\ b + Fract\ c\ d = Fract\ (a * d + c * b)\ (b * d)$
by *transfer simp*

lift-definition *uminus-fract* :: *'a fract* ⇒ *'a fract*
is $\lambda x.\ (-\ fst\ x,\ snd\ x)$
by *simp*

lemma *minus-fract* [*simp*]:
fixes *a b* :: *'a*::*idom*
shows $-\ Fract\ a\ b = Fract\ (-\ a)\ b$
by *transfer simp*

lemma *minus-fract-cancel* [*simp*]: $Fract\ (-\ a)\ (-\ b) = Fract\ a\ b$
by (*cases* $b = 0$) (*simp-all add*: *eq-fract*)

definition *diff-fract-def*: $q - r = q + -\ (r::'a\ fract)$

lemma *diff-fract* [*simp*]:
$[\![\ b \neq 0;\ d \neq 0\ ]\!] \Longrightarrow Fract\ a\ b - Fract\ c\ d = Fract\ (a * d - c * b)\ (b * d)$
by (*simp add*: *diff-fract-def*)

lift-definition *times-fract* :: *'a fract* ⇒ *'a fract* ⇒ *'a fract*
is $\lambda q\ r.\ (fst\ q * fst\ r,\ snd\ q * snd\ r)$
by(*simp add*: *algebra-simps*)

lemma *mult-fract* [*simp*]: $Fract\ (a::'a::idom)\ b * Fract\ c\ d = Fract\ (a * c)\ (b * d)$

by *transfer simp*

lemma *mult-fract-cancel*:
  $c \neq 0 \Longrightarrow$ *Fract* $(c * a)$ $(c * b)$ = *Fract a b*
by *transfer simp*

instance
proof
  fix *q r s* :: *'a fract*
  show $(q * r) * s = q * (r * s)$
    by (*cases q*, *cases r*, *cases s*) (*simp add*: *eq-fract algebra-simps*)
  show $q * r = r * q$
    by (*cases q*, *cases r*) (*simp add*: *eq-fract algebra-simps*)
  show $1 * q = q$
    by (*cases q*) (*simp add*: *One-fract-def eq-fract*)
  show $(q + r) + s = q + (r + s)$
    by (*cases q*, *cases r*, *cases s*) (*simp add*: *eq-fract algebra-simps*)
  show $q + r = r + q$
    by (*cases q*, *cases r*) (*simp add*: *eq-fract algebra-simps*)
  show $0 + q = q$
    by (*cases q*) (*simp add*: *Zero-fract-def eq-fract*)
  show $- q + q = 0$
    by (*cases q*) (*simp add*: *Zero-fract-def eq-fract*)
  show $q - r = q + - r$
    by (*cases q*, *cases r*) (*simp add*: *eq-fract*)
  show $(q + r) * s = q * s + r * s$
    by (*cases q*, *cases r*, *cases s*) (*simp add*: *eq-fract algebra-simps*)
  show (*0*::*'a fract*) $\neq$ *1*
    by (*simp add*: *Zero-fract-def One-fract-def eq-fract*)
qed

end

lemma *of-nat-fract*: *of-nat k* = *Fract* (*of-nat k*) *1*
  by (*induct k*) (*simp-all add*: *Zero-fract-def One-fract-def*)

lemma *Fract-of-nat-eq*: *Fract* (*of-nat k*) *1* = *of-nat k*
  by (*rule of-nat-fract* [*symmetric*])

lemma *fract-collapse*:
  Fract 0 k = *0*
  Fract 1 1 = *1*
  Fract k 0 = *0*
by(*transfer*; *simp*)+

lemma *fract-expand*:
  0 = *Fract 0 1*
  1 = *Fract 1 1*
  by (*simp-all add*: *fract-collapse*)

lemma *Fract-cases-nonzero*:
  obtains (*Fract*) *a b* **where** *q* = *Fract a b* **and** *b* ≠ *0* **and** *a* ≠ *0*
    | (*0*) *q* = *0*
proof (*cases q* = *0*)
  case *True*
  then show *thesis* **using** *0* **by** *auto*
next
  case *False*
  then obtain *a b* **where** *q* = *Fract a b* **and** *b* ≠ *0* **by** (*cases q*) *auto*
  with *False* **have** *0* ≠ *Fract a b* **by** *simp*
  with ‹*b* ≠ *0*› **have** *a* ≠ *0* **by** (*simp add*: *Zero-fract-def eq-fract*)
  with *Fract* ‹*q* = *Fract a b*› ‹*b* ≠ *0*› **show** *thesis* **by** *auto*
qed

8.1.3 The field of rational numbers

context *idom*
begin

subclass *ring-no-zero-divisors* **..**

end

instantiation *fract* :: (*idom*) *field*
begin

lift-definition *inverse-fract* :: *'a fract* ⇒ *'a fract*
  is λ*x*. *if fst x* = *0 then* (*0*, *1*) *else* (*snd x*, *fst x*)
by(*auto simp add*: *algebra-simps*)

lemma *inverse-fract* [*simp*]: *inverse* (*Fract a b*) = *Fract* (*b*::*'a*::*idom*) *a*
by *transfer simp*

definition *divide-fract-def*: *q div r* = *q* ∗ *inverse* (*r*:: *'a fract*)

lemma *divide-fract* [*simp*]: *Fract a b div Fract c d* = *Fract* (*a* ∗ *d*) (*b* ∗ *c*)
  by (*simp add*: *divide-fract-def*)

instance
proof
  fix *q* :: *'a fract*
  assume *q* ≠ *0*
  then show *inverse q* ∗ *q* = *1*
    by (*cases q rule*: *Fract-cases-nonzero*)
      (*simp-all add*: *fract-expand eq-fract mult.commute*)
next
  fix *q r* :: *'a fract*
  show *q div r* = *q* ∗ *inverse r* **by** (*simp add*: *divide-fract-def*)

next
  show *inverse 0* = (*0*:: *'a fract*)
    by (*simp add*: *fract-expand*) (*simp add*: *fract-collapse*)
qed

end

8.1.4 The ordered field of fractions over an ordered idom

instantiation *fract* :: (*linordered-idom*) *linorder*
begin

lemma *less-eq-fract-respect*:
  fixes $a\ b\ a'\ b'\ c\ d\ c'\ d'$ :: *'a*
  assumes *neq*: $b \neq 0\ \ b' \neq 0\ \ d \neq 0\ \ d' \neq 0$
  assumes *eq1*: $a * b' = a' * b$
  assumes *eq2*: $c * d' = c' * d$
  shows $((a * d) * (b * d) \leq (c * b) * (b * d)) \longleftrightarrow ((a' * d') * (b' * d') \leq (c' * b') * (b' * d'))$
proof −
  let *?le* = $\lambda a\ b\ c\ d.\ ((a * d) * (b * d) \leq (c * b) * (b * d))$
  {
    fix $a\ b\ c\ d\ x$ :: *'a*
    assume *x*: $x \neq 0$
    have *?le* $a\ b\ c\ d$ = *?le* $(a * x)\ (b * x)\ c\ d$
    proof −
      from *x* **have** $0 < x * x$
        by (*auto simp add*: *zero-less-mult-iff*)
      then have *?le* $a\ b\ c\ d$ =
          $((a * d) * (b * d) * (x * x) \leq (c * b) * (b * d) * (x * x))$
        by (*simp add*: *mult-le-cancel-right*)
      also have ... = *?le* $(a * x)\ (b * x)\ c\ d$
        by (*simp add*: *ac-simps*)
      finally show *?thesis* **.**
    qed
  } **note** *le-factor* = *this*

  let *?D* = $b * d$ **and** *?D'* = $b' * d'$
  from *neq* **have** *D*: *?D* $\neq 0$ **by** *simp*
  from *neq* **have** *?D'* $\neq 0$ **by** *simp*
  then have *?le* $a\ b\ c\ d$ = *?le* $(a * ?D')\ (b * ?D')\ c\ d$
    by (*rule le-factor*)
  also have ... = $((a * b') * ?D * ?D' * d * d' \leq (c * d') * ?D * ?D' * b * b')$
    by (*simp add*: *ac-simps*)
  also have ... = $((a' * b) * ?D * ?D' * d * d' \leq (c' * d) * ?D * ?D' * b * b')$
    by (*simp only*: *eq1 eq2*)
  also have ... = *?le* $(a' * ?D)\ (b' * ?D)\ c'\ d'$
    by (*simp add*: *ac-simps*)
  also from *D* **have** ... = *?le* $a'\ b'\ c'\ d'$

by (*rule le-factor* [*symmetric*])
finally show *?le a b c d* = *?le a′ b′ c′ d′* **.**
qed

lift-definition *less-eq-fract* :: ′*a fract* ⇒ ′*a fract* ⇒ *bool*
is $\lambda q\ r.\ (fst\ q * snd\ r) * (snd\ q * snd\ r) \leq (fst\ r * snd\ q) * (snd\ q * snd\ r)$
by (*clarsimp simp add*: *less-eq-fract-respect*)

definition *less-fract-def*: $z < (w{::}'a\ fract) \longleftrightarrow z \leq w \wedge \neg\ w \leq z$

lemma *le-fract* [*simp*]:
$[\![b \neq 0;\ d \neq 0]\!] \Longrightarrow Fract\ a\ b \leq Fract\ c\ d \longleftrightarrow (a * d) * (b * d) \leq (c * b) * (b * d)$
by *transfer simp*

lemma *less-fract* [*simp*]:
$[\![b \neq 0;\ d \neq 0]\!] \Longrightarrow Fract\ a\ b < Fract\ c\ d \longleftrightarrow (a * d) * (b * d) < (c * b) * (b * d)$
by (*simp add*: *less-fract-def less-le-not-le ac-simps*)

instance
proof
fix *q r s* :: ′*a fract*
assume $q \leq r$ **and** $r \leq s$
then show $q \leq s$
proof (*induct q*, *induct r*, *induct s*)
fix *a b c d e f* :: ′*a*
assume *neq*: $b \neq 0\ d \neq 0\ f \neq 0$
assume *1*: $Fract\ a\ b \leq Fract\ c\ d$
assume *2*: $Fract\ c\ d \leq Fract\ e\ f$
show $Fract\ a\ b \leq Fract\ e\ f$
proof −
from *neq* **obtain** *bb*: $0 < b * b$ **and** *dd*: $0 < d * d$ **and** *ff*: $0 < f * f$
by (*auto simp add*: *zero-less-mult-iff linorder-neq-iff*)
have $(a * d) * (b * d) * (f * f) \leq (c * b) * (b * d) * (f * f)$
proof −
from *neq 1* **have** $(a * d) * (b * d) \leq (c * b) * (b * d)$
by *simp*
with *ff* **show** *?thesis* **by** (*simp add*: *mult-le-cancel-right*)
qed
also have ... $= (c * f) * (d * f) * (b * b)$
by (*simp only*: *ac-simps*)
also have ... $\leq (e * d) * (d * f) * (b * b)$
proof −
from *neq 2* **have** $(c * f) * (d * f) \leq (e * d) * (d * f)$
by *simp*
with *bb* **show** *?thesis* **by** (*simp add*: *mult-le-cancel-right*)
qed
finally have $(a * f) * (b * f) * (d * d) \leq e * b * (b * f) * (d * d)$

        by (*simp only*: *ac-simps*)
      with *dd* **have** $(a * f) * (b * f) \leq (e * b) * (b * f)$
        by (*simp add*: *mult-le-cancel-right*)
      with *neq* **show** *?thesis* **by** *simp*
    qed
  qed
next
  fix *q r* :: *'a fract*
  assume $q \leq r$ **and** $r \leq q$
  then show $q = r$
  proof (*induct q*, *induct r*)
    fix *a b c d* :: *'a*
    assume *neq*: $b \neq 0$ $d \neq 0$
    assume *1*: *Fract a b* $\leq$ *Fract c d*
    assume *2*: *Fract c d* $\leq$ *Fract a b*
    show *Fract a b* = *Fract c d*
    proof −
      from *neq 1* **have** $(a * d) * (b * d) \leq (c * b) * (b * d)$
        by *simp*
      also have ... $\leq (a * d) * (b * d)$
      proof −
        from *neq 2* **have** $(c * b) * (d * b) \leq (a * d) * (d * b)$
          by *simp*
        then show *?thesis* **by** (*simp only*: *ac-simps*)
      qed
      finally have $(a * d) * (b * d) = (c * b) * (b * d)$ **.**
      moreover from *neq* **have** $b * d \neq 0$ **by** *simp*
      ultimately have $a * d = c * b$ **by** *simp*
      with *neq* **show** *?thesis* **by** (*simp add*: *eq-fract*)
    qed
  qed
next
  fix *q r* :: *'a fract*
  show $q \leq q$
    by (*induct q*) *simp*
  show $(q < r) = (q \leq r \land \neg\, r \leq q)$
    by (*simp only*: *less-fract-def*)
  show $q \leq r \lor r \leq q$
    by (*induct q*, *induct r*)
       (*simp add*: *mult.commute*, *rule linorder-linear*)
qed

end

instantiation *fract* :: (*linordered-idom*) *linordered-field*
begin

definition *abs-fract-def2*:
  $|q| = (\textit{if } q < 0 \textit{ then } -q \textit{ else } (q{::}'a\ fract))$

```
definition sgn-fract-def:
  sgn (q::'a fract) = (if q = 0 then 0 else if 0 < q then 1 else − 1)

theorem abs-fract [simp]: |Fract a b| = Fract |a| |b|
  unfolding abs-fract-def2 not-le [symmetric]
  by transfer (auto simp add: zero-less-mult-iff le-less)

instance proof
  fix q r s :: 'a fract
  assume q ≤ r
  then show s + q ≤ s + r
  proof (induct q, induct r, induct s)
    fix a b c d e f :: 'a
    assume neq: b ≠ 0 d ≠ 0 f ≠ 0
    assume le: Fract a b ≤ Fract c d
    show Fract e f + Fract a b ≤ Fract e f + Fract c d
    proof −
      let ?F = f ∗ f from neq have F: 0 < ?F
        by (auto simp add: zero-less-mult-iff)
      from neq le have (a ∗ d) ∗ (b ∗ d) ≤ (c ∗ b) ∗ (b ∗ d)
        by simp
      with F have (a ∗ d) ∗ (b ∗ d) ∗ ?F ∗ ?F ≤ (c ∗ b) ∗ (b ∗ d) ∗ ?F ∗ ?F
        by (simp add: mult-le-cancel-right)
      with neq show ?thesis by (simp add: field-simps)
    qed
  qed
next
  fix q r s :: 'a fract
  assume q < r and 0 < s
  then show s ∗ q < s ∗ r
  proof (induct q, induct r, induct s)
    fix a b c d e f :: 'a
    assume neq: b ≠ 0 d ≠ 0 f ≠ 0
    assume le: Fract a b < Fract c d
    assume gt: 0 < Fract e f
    show Fract e f ∗ Fract a b < Fract e f ∗ Fract c d
    proof −
      let ?E = e ∗ f and ?F = f ∗ f
      from neq gt have 0 < ?E
        by (auto simp add: Zero-fract-def order-less-le eq-fract)
      moreover from neq have 0 < ?F
        by (auto simp add: zero-less-mult-iff)
      moreover from neq le have (a ∗ d) ∗ (b ∗ d) < (c ∗ b) ∗ (b ∗ d)
        by simp
      ultimately have (a ∗ d) ∗ (b ∗ d) ∗ ?E ∗ ?F < (c ∗ b) ∗ (b ∗ d) ∗ ?E ∗ ?F
        by (simp add: mult-less-cancel-right)
      with neq show ?thesis
        by (simp add: ac-simps)
```

    qed
  qed
qed (*fact sgn-fract-def abs-fract-def2*)+

end

instantiation *fract* :: (*linordered-idom*) *distrib-lattice*
begin

definition *inf-fract-def*:
  (*inf* :: *'a fract* ⇒ *'a fract* ⇒ *'a fract*) = *min*

definition *sup-fract-def*:
  (*sup* :: *'a fract* ⇒ *'a fract* ⇒ *'a fract*) = *max*

instance
  by *standard* (*simp-all add*: *inf-fract-def sup-fract-def max-min-distrib2*)

end

lemma *fract-induct-pos* [*case-names Fract*]:
  fixes *P* :: *'a::linordered-idom fract* ⇒ *bool*
  assumes *step*: ⋀*a b*. *0* < *b* ⟹ *P* (*Fract a b*)
  shows *P q*
proof (*cases q*)
  case (*Fract a b*)
  {
    fix *a b* :: *'a*
    assume *b*: *b* < *0*
    have *P* (*Fract a b*)
    proof −
      from *b* **have** *0* < − *b* **by** *simp*
      then have *P* (*Fract* (− *a*) (− *b*))
        by (*rule step*)
      then show *P* (*Fract a b*)
        by (*simp add*: *order-less-imp-not-eq* [*OF b*])
    qed
  }
  with *Fract* **show** *P q*
    by (*auto simp add*: *linorder-neq-iff step*)
qed

lemma *zero-less-Fract-iff*: *0* < *b* ⟹ *0* < *Fract a b* ⟷ *0* < *a*
  by (*auto simp add*: *Zero-fract-def zero-less-mult-iff*)

lemma *Fract-less-zero-iff*: *0* < *b* ⟹ *Fract a b* < *0* ⟷ *a* < *0*
  by (*auto simp add*: *Zero-fract-def mult-less-0-iff*)

lemma *zero-le-Fract-iff*: *0* < *b* ⟹ *0* ≤ *Fract a b* ⟷ *0* ≤ *a*

by (*auto simp add*: *Zero-fract-def zero-le-mult-iff*)

lemma *Fract-le-zero-iff*: *0* < *b* ⟹ *Fract a b* ≤ *0* ⟷ *a* ≤ *0*
  by (*auto simp add*: *Zero-fract-def mult-le-0-iff*)

lemma *one-less-Fract-iff*: *0* < *b* ⟹ *1* < *Fract a b* ⟷ *b* < *a*
  by (*auto simp add*: *One-fract-def mult-less-cancel-right-disj*)

lemma *Fract-less-one-iff*: *0* < *b* ⟹ *Fract a b* < *1* ⟷ *a* < *b*
  by (*auto simp add*: *One-fract-def mult-less-cancel-right-disj*)

lemma *one-le-Fract-iff*: *0* < *b* ⟹ *1* ≤ *Fract a b* ⟷ *b* ≤ *a*
  by (*auto simp add*: *One-fract-def mult-le-cancel-right*)

lemma *Fract-le-one-iff*: *0* < *b* ⟹ *Fract a b* ≤ *1* ⟷ *a* ≤ *b*
  by (*auto simp add*: *One-fract-def mult-le-cancel-right*)

end

9 Fundamental Theorem of Algebra

theory *Fundamental-Theorem-Algebra*
imports *Polynomial Complex-Main*
begin

9.1 More lemmas about module of complex numbers

The triangle inequality for cmod

lemma *complex-mod-triangle-sub*: *cmod w* ≤ *cmod* (*w* + *z*) + *norm z*
  by (*metis add-diff-cancel norm-triangle-ineq4*)

9.2 Basic lemmas about polynomials

lemma *poly-bound-exists*:
  fixes *p* :: *'a*::{*comm-semiring-0*,*real-normed-div-algebra*} *poly*
  shows ∃ *m*. *m* > *0* ∧ (∀ *z*. *norm z* ≤ *r* ⟶ *norm* (*poly p z*) ≤ *m*)
proof (*induct p*)
  case *0*
  then show *?case* **by** (*rule exI*[**where** *x*=*1*]) *simp*
next
  case (*pCons c cs*)
  from *pCons.hyps* **obtain** *m* **where** *m*: ∀ *z*. *norm z* ≤ *r* ⟶ *norm* (*poly cs z*) ≤
m
    by *blast*
  let *?k* = *1* + *norm c* + |*r* ∗ *m*|
  have *kp*: *?k* > *0*
    using *abs-ge-zero*[*of r*∗*m*] *norm-ge-zero*[*of c*] **by** *arith*
  have *norm* (*poly* (*pCons c cs*) *z*) ≤ *?k* **if** *H*: *norm z* ≤ *r* **for** *z*

proof −
from *m H* **have** *th*: *norm* (*poly cs z*) ≤ *m*
by *blast*
from *H* **have** *rp*: *r* ≥ *0*
using *norm-ge-zero*[*of z*] **by** *arith*
have *norm* (*poly* (*pCons c cs*) *z*) ≤ *norm c* + *norm* (*z* ∗ *poly cs z*)
using *norm-triangle-ineq*[*of c z*∗ *poly cs z*] **by** *simp*
also have . . . ≤ *?k*
using *mult-mono*[*OF H th rp norm-ge-zero*[*of poly cs z*]]
by (*simp add*: *norm-mult*)
finally show *?thesis* **.**
qed
with *kp* **show** *?case* **by** *blast*
qed

Offsetting the variable in a polynomial gives another of same degree

definition *offset-poly* :: ′*a*::*comm-semiring-0 poly* ⇒ ′*a* ⇒ ′*a poly*
where *offset-poly p h* = *fold-coeffs* (λ*a q*. *smult h q* + *pCons a q*) *p 0*

lemma *offset-poly-0*: *offset-poly 0 h* = *0*
by (*simp add*: *offset-poly-def*)

lemma *offset-poly-pCons*:
offset-poly (*pCons a p*) *h* =
smult h (*offset-poly p h*) + *pCons a* (*offset-poly p h*)
by (*cases p* = *0* ∧ *a* = *0*) (*auto simp add*: *offset-poly-def*)

lemma *offset-poly-single* [*simp*]: *offset-poly* [:*a*:] *h* = [:*a*:]
by (*simp add*: *offset-poly-pCons offset-poly-0*)

lemma *poly-offset-poly*: *poly* (*offset-poly p h*) *x* = *poly p* (*h* + *x*)
by (*induct p*) (*auto simp add*: *offset-poly-0 offset-poly-pCons algebra-simps*)

lemma *offset-poly-eq-0-lemma*: *smult c p* + *pCons a p* = *0* =⇒ *p* = *0*
by (*induct p arbitrary*: *a*) (*simp*, *force*)

lemma *offset-poly-eq-0-iff* [*simp*]: *offset-poly p h* = *0* ←→ *p* = *0*
proof
show *offset-poly p h* = *0* =⇒ *p* = *0*
proof(*induction p*)
case *0*
then show *?case* **by** *blast*
next
case (*pCons a p*)
then show *?case*
by (*metis offset-poly-eq-0-lemma offset-poly-pCons offset-poly-single*)
qed
qed (*simp add*: *offset-poly-0*)

lemma *degree-offset-poly* [*simp*]: *degree* (*offset-poly p h*) = *degree p*
proof(*induction p*)
  case *0*
  then show *?case*
    by (*simp add*: *offset-poly-0*)
next
  case (*pCons a p*)
  have $p \neq 0 \Longrightarrow$ *degree* (*offset-poly* (*pCons a p*) *h*) = *Suc* (*degree p*)
    by (*metis degree-add-eq-right degree-pCons-eq degree-smult-le le-imp-less-Suc offset-poly-eq-0-iff offset-poly-pCons pCons.IH*)
  then show *?case*
    by *simp*
qed

definition *psize p* = (*if p* = *0 then 0 else Suc* (*degree p*))

lemma *psize-eq-0-iff* [*simp*]: *psize p* = *0* $\longleftrightarrow$ *p* = *0*
  unfolding *psize-def* **by** *simp*

lemma *poly-offset*:
  fixes *p* :: *'a*::*comm-ring-1 poly*
  shows $\exists$ *q*. *psize q* = *psize p* $\wedge$ ($\forall$ *x*. *poly q x* = *poly p* (*a* + *x*))
  by (*metis degree-offset-poly offset-poly-eq-0-iff poly-offset-poly psize-def*)

An alternative useful formulation of completeness of the reals

lemma *real-sup-exists*:
  assumes *ex*: $\exists$ *x*. *P x*
    and *bz*: $\exists$ *z*. $\forall$ *x*. *P x* $\longrightarrow$ *x* < *z*
  shows $\exists$ *s*::*real*. $\forall$ *y*. ($\exists$ *x*. *P x* $\wedge$ *y* < *x*) $\longleftrightarrow$ *y* < *s*
proof
  from *bz* **have** *bdd-above* (*Collect P*)
    by (*force intro*: *less-imp-le*)
  then show $\forall$ *y*. ($\exists$ *x*. *P x* $\wedge$ *y* < *x*) $\longleftrightarrow$ *y* < *Sup* (*Collect P*)
    using *ex bz* **by** (*subst less-cSup-iff*) *auto*
qed

9.3 Fundamental theorem of algebra

lemma *unimodular-reduce-norm*:
  assumes *md*: *cmod z* = *1*
  shows *cmod* (*z* + *1*) < *1* $\vee$ *cmod* (*z* − *1*) < *1* $\vee$ *cmod* (*z* + i) < *1* $\vee$ *cmod* (*z* − i) < *1*
proof −
  obtain *x y* **where** *z*: *z* = *Complex x y*
    by (*cases z*) *auto*
  from *md z* **have** *xy*: $x^2 + y^2 = 1$
    by (*simp add*: *cmod-def*)
  have *False* **if** *cmod* (*z* + *1*) $\geq$ *1 cmod* (*z* − *1*) $\geq$ *1 cmod* (*z* + i) $\geq$ *1 cmod* (*z* − i) $\geq$ *1*

proof −
from *that z xy* **have** ∗: $2 * x \leq 1$ $2 * x \geq -1$ $2 * y \leq 1$ $2 * y \geq -1$
by (*simp-all add*: *cmod-def power2-eq-square algebra-simps*)
then have $|2 * x| \leq 1$ $|2 * y| \leq 1$
by *simp-all*
then have $|2 * x|^2 \leq 1^2$ $|2 * y|^2 \leq 1^2$
by (*metis abs-square-le-1 one-power2 power2-abs*)+
with *xy* ∗ **show** *?thesis*
by (*smt* (*verit*, *best*) *four-x-squared square-le-1*)
qed
then show *?thesis*
by *force*
qed

Hence we can always reduce modulus of $1 + b\ z\hat{}n$ if nonzero

lemma *reduce-poly-simple*:
assumes *b*: $b \neq 0$
and *n*: $n \neq 0$
shows $\exists z.\ cmod\ (1 + b * z\hat{}n) < 1$
using *n*
proof (*induct n rule*: *nat-less-induct*)
fix *n*
assume *IH*: $\forall m<n.\ m \neq 0 \longrightarrow (\exists z.\ cmod\ (1 + b * z\ \hat{}\ m) < 1)$
assume *n*: $n \neq 0$
let $?P = \lambda z\ n.\ cmod\ (1 + b * z\ \hat{}\ n) < 1$
show $\exists z.\ ?P\ z\ n$
proof *cases*
assume *even n*
then obtain *m* **where** *m*: $n = 2 * m$ **and** $m \neq 0$ $m < n$
using *n* **by** *auto*
with *IH* **obtain** *z* **where** *z*: *?P z m*
by *blast*
from *z* **have** *?P* (*csqrt z*) *n*
by (*simp add*: *m power-mult*)
then show *?thesis* **..**
next
assume *odd n*
then have $\exists m.\ n = Suc\ (2 * m)$
by *presburger*+
then obtain *m* **where** *m*: $n = Suc\ (2 * m)$
by *blast*
have *0*: *cmod* (*complex-of-real* (*cmod b*) / *b*) = *1*
using *b* **by** (*simp add*: *norm-divide*)
have $\exists v.\ cmod\ (complex\text{-}of\text{-}real\ (cmod\ b) / b + v\hat{}n) < 1$
proof (*cases cmod* (*complex-of-real* (*cmod b*) / *b* + *1*) < *1*)
case *True*
then show *?thesis*
by (*metis power-one*)
next

case *F1*: *False*
show *?thesis*
proof (*cases cmod* (*complex-of-real* (*cmod b*) / *b* − *1*) < *1*)
case *True*
with ‹*odd n*› **show** *?thesis*
by (*metis add-uminus-conv-diff neg-one-odd-power*)
next
case *F2*: *False*
show *?thesis*
proof (*cases cmod* (*complex-of-real* (*cmod b*) / *b* + i) < *1*)
case *T1*: *True*
show *?thesis*
proof (*cases even m*)
case *True*
with *T1* **show** *?thesis*
by (*rule-tac x*=i **in** *exI*) (*simp add*: *m power-mult*)
next
case *False*
with *T1* **show** *?thesis*
by (*rule-tac x*=− i **in** *exI*) (*simp add*: *m power-mult*)
qed
next
case *False*
then have *lt1*: *cmod* (*of-real* (*cmod b*) / *b* − i) < *1*
using *0 F1 F2 unimodular-reduce-norm* **by** *blast*
show *?thesis*
proof (*cases even m*)
case *True*
with *m lt1* **show** *?thesis*
by (*rule-tac x*=− i **in** *exI*) (*simp add*: *power-mult*)
next
case *False*
with *m lt1* **show** *?thesis*
by (*rule-tac x*=i **in** *exI*) (*simp add*: *power-mult*)
qed
qed
qed
qed
then obtain *v* **where** *v*: *cmod* (*complex-of-real* (*cmod b*) / *b* + *v*^*n*) < *1*
by *blast*
let *?w* = *v* / *complex-of-real* (*root n* (*cmod b*))
from *odd-real-root-pow*[*OF* ‹*odd n*›, *of cmod b*]
have *1*: *?w* ^ *n* = *v*^*n* / *complex-of-real* (*cmod b*)
by (*simp add*: *power-divide of-real-power*[*symmetric*])
have *2*:*cmod* (*complex-of-real* (*cmod b*) / *b*) = *1*
using *b* **by** (*simp add*: *norm-divide*)
then have *3*: *cmod* (*complex-of-real* (*cmod b*) / *b*) ≥ *0*
by *simp*
have *4*: *cmod* (*complex-of-real* (*cmod b*) / *b*) *

```
      cmod (1 + b * (v ^ n / complex-of-real (cmod b))) <
      cmod (complex-of-real (cmod b) / b) * 1
    apply (simp only: norm-mult[symmetric] distrib-left)
    using b v
    apply (simp add: 2)
    done
  show ?thesis
    by (metis 1 mult-left-less-imp-less[OF 4 3])
 qed
qed
```

Bolzano-Weierstrass type property for closed disc in complex plane.

```
lemma metric-bound-lemma: cmod (x − y) ≤ |Re x − Re y| + |Im x − Im y|
 using real-sqrt-sum-squares-triangle-ineq[of Re x − Re y 0 0 Im x − Im y]
 unfolding cmod-def by simp

lemma Bolzano-Weierstrass-complex-disc:
 assumes r: ∀ n. cmod (s n) ≤ r
 shows ∃ f z. strict-mono (f :: nat ⇒ nat) ∧ (∀ e >0. ∃ N. ∀ n ≥ N. cmod (s (f
n) − z) < e)
proof −
 from seq-monosub[of Re ∘ s]
 obtain f where f: strict-mono f monoseq (λn. Re (s (f n)))
  unfolding o-def by blast
 from seq-monosub[of Im ∘ s ∘ f]
 obtain g where g: strict-mono g monoseq (λn. Im (s (f (g n))))
  unfolding o-def by blast
 let ?h = f ∘ g
 have r ≥ 0
  by (meson norm-ge-zero order-trans r)
 have ∀ n. r + 1 ≥ |Re (s n)|
  by (smt (verit, ccfv-threshold) abs-Re-le-cmod r)
 then have conv1: convergent (λn. Re (s (f n)))
  by (metis Bseq-monoseq-convergent f(2) BseqI′ real-norm-def)
 have ∀ n. r + 1 ≥ |Im (s n)|
  by (smt (verit) abs-Im-le-cmod r)
 then have conv2: convergent (λn. Im (s (f (g n))))
  by (metis Bseq-monoseq-convergent g(2) BseqI′ real-norm-def)

 obtain x where x: ∀ r>0. ∃ n0. ∀ n≥n0. |Re (s (f n)) − x| < r
  using conv1[unfolded convergent-def] LIMSEQ-iff real-norm-def by metis
 obtain y where y: ∀ r>0. ∃ n0. ∀ n≥n0. |Im (s (f (g n))) − y| < r
  using conv2[unfolded convergent-def] LIMSEQ-iff real-norm-def by metis
 let ?w = Complex x y
 from f(1) g(1) have hs: strict-mono ?h
  unfolding strict-mono-def by auto
 have ∃ N. ∀ n≥N. cmod (s (?h n) − ?w) < e if e > 0 for e
 proof −
  from that have e2: e/2 > 0
```

by *simp*
from *x y e2*
obtain *N1 N2* **where** *N1*: ∀ *n*≥*N1*. |*Re* (*s* (*f n*)) − *x*| < *e* / *2*
and *N2*: ∀ *n*≥*N2*. |*Im* (*s* (*f* (*g n*))) − *y*| < *e* / *2*
by *blast*
have *cmod* (*s* (*?h n*) − *?w*) < *e* **if** *n* ≥ *N1* + *N2* **for** *n*
proof −
from *that* **have** *nN1*: *g n* ≥ *N1* **and** *nN2*: *n* ≥ *N2*
using *seq-suble*[*OF g*(*1*), *of n*] **by** *arith*+
show *?thesis*
using *metric-bound-lemma*[*of s* (*f* (*g n*)) *?w*] *N1 N2 nN1 nN2* **by** *fastforce*
qed
then show *?thesis* **by** *blast*
qed
with *hs* **show** *?thesis* **by** *blast*
qed

Polynomial is continuous.

lemma *poly-cont*:
fixes *p* :: ′*a*::{*comm-semiring-0*,*real-normed-div-algebra*} *poly*
assumes *ep*: *e* > *0*
shows ∃ *d* >*0*. ∀ *w*. *0* < *norm* (*w* − *z*) ∧ *norm* (*w* − *z*) < *d* ⟶ *norm* (*poly p w* − *poly p z*) < *e*
proof −
obtain *q* **where** *degree q* = *degree p* **and** *q*: ⋀*w*. *poly p w* = *poly q* (*w* − *z*)
by (*metis add.commute degree-offset-poly diff-add-cancel poly-offset-poly*)
show *?thesis* **unfolding** *q*
proof (*induct q*)
case *0*
then show *?case*
using *ep* **by** *auto*
next
case (*pCons c cs*)
obtain *m* **where** *m*: *m* > *0 norm z* ≤ *1* ⟹ *norm* (*poly cs z*) ≤ *m* **for** *z*
using *poly-bound-exists*[*of 1 cs*] **by** *blast*
with *ep* **have** *em0*: *e*/*m* > *0*
by (*simp add*: *field-simps*)
obtain *d* **where** *d*: *d* > *0 d* < *1 d* < *e* / *m*
by (*meson em0 field-lbound-gt-zero zero-less-one*)
then have ⋀*w*. *norm* (*w* − *z*) < *d* ⟹ *norm* (*w* − *z*) ∗ *norm* (*poly cs* (*w* − *z*)) < *e*
by (*smt* (*verit*, *del-insts*) *m mult-left-mono norm-ge-zero pos-less-divide-eq*)
with *d* **show** *?case*
by (*force simp add*: *norm-mult*)
qed
qed

Hence a polynomial attains minimum on a closed disc in the complex plane.

lemma *poly-minimum-modulus-disc*: ∃ *z*. ∀ *w*. *cmod w* ≤ *r* ⟶ *cmod* (*poly p z*) ≤

cmod (*poly p w*)
proof −
  show *?thesis*
  proof (*cases r* ≥ *0*)
    case *False*
    then show *?thesis*
      by (*metis norm-ge-zero order.trans*)
  next
    case *True*
    then have *mth1*: ∃ *x z. cmod z* ≤ *r* ∧ *cmod* (*poly p z*) = − *x*
      by (*metis add.inverse-inverse norm-zero*)
    obtain *s* **where** *s*: ∀ *y.* (∃ *x.* (∃ *z. cmod z* ≤ *r* ∧ *cmod* (*poly p z*) = − *x*) ∧ *y* < *x*) ⟷ *y* < *s*
      by (*smt* (*verit, del-insts*) *real-sup-exists*[*OF mth1*] *norm-zero zero-less-norm-iff*)

    let *?m* = − *s*
    have *s1*: (∃ *z. cmod z* ≤ *r* ∧ − (− *cmod* (*poly p z*)) < *y*) ⟷ *?m* < *y* **for** *y*
      by (*metis add.inverse-inverse minus-less-iff s*)
    then have *s1m*: ⋀*z. cmod z* ≤ *r* ⟹ *cmod* (*poly p z*) ≥ *?m*
      by *force*
    have ∃ *z. cmod z* ≤ *r* ∧ *cmod* (*poly p z*) < − *s* + *1* / *real* (*Suc n*) **for** *n*
      using *s1*[*of ?m* + *1*/*real* (*Suc n*)] **by** *simp*
    then obtain *g* **where** *g*: ∀ *n. cmod* (*g n*) ≤ *r* ∀ *n. cmod* (*poly p* (*g n*)) <*?m* + *1* /*real*(*Suc n*)
      by *metis*
    from *Bolzano-Weierstrass-complex-disc*[*OF g*(*1*)]
    obtain *f*::*nat* ⇒ *nat* **and** *z* **where** *fz*: *strict-mono f* ∀ *e*>*0*. ∃ *N*. ∀ *n*≥*N*. *cmod* (*g* (*f n*) − *z*) < *e*
      by *blast*
    {
      fix *w*
      assume *wr*: *cmod w* ≤ *r*
      let *?e* = |*cmod* (*poly p z*) − *?m*|
      {
        assume *e*: *?e* > *0*
        then have *e2*: *?e*/*2* > *0*
          by *simp*
        with *poly-cont* **obtain** *d*
            where *d* > *0* **and** *d*: ⋀*w. 0*<*cmod* (*w* − *z*)∧ *cmod*(*w* − *z*) < *d* ⟶ *cmod*(*poly p w* − *poly p z*) < *?e*/*2*
          by *blast*
        have *1*: *cmod*(*poly p w* − *poly p z*) < *?e* / *2* **if** *w*: *cmod* (*w* − *z*) < *d* **for** *w*
          using *d*[*of w*] *w e* **by** (*cases w* = *z*) *simp-all*
        from *fz*(*2*) ‹*d* > *0*› **obtain** *N1* **where** *N1*: ∀ *n*≥*N1*. *cmod* (*g* (*f n*) − *z*) < *d*
          by *blast*
        from *reals-Archimedean2* **obtain** *N2* :: *nat* **where** *N2*: *2*/*?e* < *real N2*
          by *blast*
        have *2*: *cmod* (*poly p* (*g* (*f* (*N1* + *N2*))) − *poly p z*) < *?e*/*2*

using $N1\ 1$ **by** *auto*
have 0: $a < e2 \Longrightarrow |b - m| < e2 \Longrightarrow 2 * e2 \leq |b - m| + a \Longrightarrow False$
for *a b e2 m* :: *real*
by *arith*
from *seq-suble*[*OF fz*(*1*), *of N1* + *N2*]
have *00*: $?m + 1 / real (Suc (f (N1 + N2))) \leq ?m + 1 / real (Suc (N1 + N2))$
by (*simp add*: *frac-le*)
from *N2 e2 less-imp-inverse-less*[*of 2*/*?e real* (*Suc* (*N1* + *N2*))]
have $?e/2 > 1/ real (Suc (N1 + N2))$
by (*simp add*: *inverse-eq-divide*)
with *order-less-le-trans*[*OF - 00*]
have *1*: $|cmod (poly\ p\ (g\ (f\ (N1 + N2)))) - ?m| < ?e/2$
using *g s1* **by** (*smt* (*verit*))
with *0*[*OF 2*] **have** *False*
by (*smt* (*verit*) *field-sum-of-halves norm-triangle-ineq3*)
}
then have $?e = 0$
by *auto*
with *s1m*[*OF wr*] **have** $cmod (poly\ p\ z) \leq cmod (poly\ p\ w)$
by *simp*
}
then show *?thesis* **by** *blast*
qed
qed

Nonzero polynomial in z goes to infinity as z does.

lemma *poly-infinity*:
fixes *p*:: *'a*::{*comm-semiring-0*,*real-normed-div-algebra*} *poly*
assumes *ex*: $p \neq 0$
shows $\exists r. \forall z. r \leq norm\ z \longrightarrow d \leq norm (poly (pCons\ a\ p)\ z)$
using *ex*
proof (*induct p arbitrary*: *a d*)
case *0*
then show *?case* **by** *simp*
next
case (*pCons c cs a d*)
show *?case*
proof (*cases* $cs = 0$)
case *False*
with *pCons.hyps* **obtain** *r* **where** *r*: $\forall z. r \leq norm\ z \longrightarrow d + norm\ a \leq norm (poly (pCons\ c\ cs)\ z)$
by *blast*
let $?r = 1 + |r|$
have $d \leq norm (poly (pCons\ a\ (pCons\ c\ cs))\ z)$ **if** $1 + |r| \leq norm\ z$ **for** *z*
proof −
have $d \leq norm(z * poly (pCons\ c\ cs)\ z) - norm\ a$
by (*smt* (*verit*, *best*) *norm-ge-zero mult-less-cancel-right2 norm-mult r that*)
with *norm-diff-ineq add.commute*

```
      show ?thesis
        by (metis order.trans poly-pCons)
    qed
    then show ?thesis by blast
  next
    case True
    have d ≤ norm (poly (pCons a (pCons c cs)) z)
      if (|d| + norm a) / norm c ≤ norm z for z :: 'a
    proof -
      have |d| + norm a ≤ norm (z * c)
      by (metis that True norm-mult pCons.hyps(1) pos-divide-le-eq zero-less-norm-iff)
      also have ... ≤ norm (a + z * c) + norm a
        by (simp add: add.commute norm-add-leD)
      finally show ?thesis
        using True by auto
    qed
    then show ?thesis by blast
  qed
qed
```

Hence polynomial's modulus attains its minimum somewhere.

```
lemma poly-minimum-modulus: ∃ z.∀ w. cmod (poly p z) ≤ cmod (poly p w)
proof (induct p)
  case 0
  then show ?case by simp
next
  case (pCons c cs)
  show ?case
  proof (cases cs = 0)
    case False
    from poly-infinity[OF False, of cmod (poly (pCons c cs) 0) c]
    obtain r where r: cmod (poly (pCons c cs) 0) ≤ cmod (poly (pCons c cs) z)
      if r ≤ cmod z for z
      by blast
    from poly-minimum-modulus-disc[of |r| pCons c cs] show ?thesis
      by (smt (verit, del-insts) order.trans linorder-linear r)
  qed (use pCons.hyps in auto)
qed
```

Constant function (non-syntactic characterization).

```
definition constant f ⟷ (∀ x y. f x = f y)

lemma nonconstant-length: ¬ constant (poly p) ⟹ psize p ≥ 2
  by (induct p) (auto simp: constant-def psize-def)

lemma poly-replicate-append: poly (monom 1 n * p) (x::'a::comm-ring-1) = x^n
* poly p x
  by (simp add: poly-monom)
```

Decomposition of polynomial, skipping zero coefficients after the first.

```
lemma poly-decompose-lemma:
  assumes nz: ¬ (∀ z. z ≠ 0 ⟶ poly p z = (0::'a::idom))
  shows ∃ k a q. a ≠ 0 ∧ Suc (psize q + k) = psize p ∧ (∀ z. poly p z = z^k * poly
(pCons a q) z)
  unfolding psize-def
  using nz
proof (induct p)
  case 0
  then show ?case by simp
next
  case (pCons c cs)
  show ?case
  proof (cases c = 0)
    case True
    from pCons.hyps pCons.prems True show ?thesis
      apply auto
      apply (rule-tac x=k+1 in exI)
      apply (rule-tac x=a in exI)
      apply clarsimp
      apply (rule-tac x=q in exI)
      apply auto
      done
  qed force
qed

lemma poly-decompose:
  fixes p :: 'a::idom poly
  assumes nc: ¬ constant (poly p)
  shows ∃ k a q. a ≠ 0 ∧ k ≠ 0 ∧
             psize q + k + 1 = psize p ∧
            (∀ z. poly p z = poly p 0 + z^k * poly (pCons a q) z)
  using nc
proof (induct p)
  case 0
  then show ?case
    by (simp add: constant-def)
next
  case (pCons c cs)
  have ¬ (∀ z. z ≠ 0 ⟶ poly cs z = 0)
    by (smt (verit) constant-def mult-eq-0-iff pCons.prems poly-pCons)
  from poly-decompose-lemma[OF this]
  obtain k a q where *: a ≠ 0 ∧
     Suc (psize q + k) = psize cs ∧ (∀ z. poly cs z = z ^ k * poly (pCons a q) z)
    by blast
  then have psize q + k + 2 = psize (pCons c cs)
    by (auto simp add: psize-def split: if-splits)
  then show ?case
    using * by force
qed
```

Fundamental theorem of algebra

theorem *fundamental-theorem-of-algebra*:
  assumes *nc*: ¬ *constant* (*poly p*)
  shows ∃ *z*::*complex*. *poly p z* = *0*
  using *nc*
proof (*induct psize p arbitrary*: *p rule*: *less-induct*)
  case *less*
  let *?p* = *poly p*
  let *?ths* = ∃ *z*. *?p z* = *0*

  from *nonconstant-length*[*OF less*(*2*)] **have** *n2*: *psize p* ≥ *2* **.**
  from *poly-minimum-modulus* **obtain** *c* **where** *c*: ∀ *w*. *cmod* (*?p c*) ≤ *cmod* (*?p w*)
    by *blast*

  show *?ths*
  proof (*cases ?p c* = *0*)
    case *True*
    then show *?thesis* **by** *blast*
  next
    case *False*
    obtain *q* **where** *q*: *psize q* = *psize p* ∀ *x*. *poly q x* = *?p* (*c* + *x*)
      using *poly-offset*[*of p c*] **by** *blast*
    then have *qnc*: ¬ *constant* (*poly q*)
      by (*metis* (*no-types*, *opaque-lifting*) *add.commute constant-def diff-add-cancel less.prems*)
    from *q*(*2*) **have** *pqc0*: *?p c* = *poly q 0*
      by *simp*
    from *c pqc0* **have** *cq0*: ∀ *w*. *cmod* (*poly q 0*) ≤ *cmod* (*?p w*)
      by *simp*
    let *?a0* = *poly q 0*
    from *False pqc0* **have** *a00*: *?a0* ≠ *0*
      by *simp*
    from *a00* **have** *qr*: ∀ *z*. *poly q z* = *poly* (*smult* (*inverse ?a0*) *q*) *z* ∗ *?a0*
      by *simp*
    let *?r* = *smult* (*inverse ?a0*) *q*
    have *lgqr*: *psize q* = *psize ?r*
      by (*simp add*: *a00 psize-def*)
    have *rnc*: ¬ *constant* (*poly ?r*)
      using *constant-def qnc qr* **by** *fastforce*
    have *r01*: *poly ?r 0* = *1*
      by (*simp add*: *a00*)
    have *mrmq-eq*: *cmod* (*poly ?r w*) < *1* ⟷ *cmod* (*poly q w*) < *cmod ?a0* **for** *w*
     by (*smt* (*verit*, *del-insts*) *a00 mult-less-cancel-right2 norm-mult qr zero-less-norm-iff*)
    from *poly-decompose*[*OF rnc*] **obtain** *k a s* **where**
      kas: *a* ≠ *0 k* ≠ *0 psize s* + *k* + *1* = *psize ?r*
        ∀ *z*. *poly ?r z* = *poly ?r 0* + *z*^*k*∗ *poly* (*pCons a s*) *z* **by** *blast*
    have ∃ *w*. *cmod* (*poly ?r w*) < *1*
    proof (*cases psize p* = *k* + *1*)

case *True*
with *kas q* **have** *s0*: *s = 0*
by (*simp add*: *lgqr*)
with *reduce-poly-simple kas* **show** *?thesis*
by (*metis mult.commute mult.right-neutral poly-1 poly-smult r01 smult-one*)
next
case *False* **note** *kn = this*
from *kn kas*(*3*) *q*(*1*) *lgqr* **have** *k1n*: *k + 1 < psize p*
by *simp*
have *01*: ¬ *constant* (*poly* (*pCons 1* (*monom a* (*k − 1*))))
unfolding *constant-def poly-pCons poly-monom*
by (*metis add-cancel-left-right kas*(*1*) *mult.commute mult-cancel-right2 power-one*)
have *02*: *k + 1 = psize* (*pCons 1* (*monom a* (*k − 1*)))
using *kas* **by** (*simp add*: *psize-def degree-monom-eq*)
from *less*(*1*) [*OF - 01*] *k1n 02*
obtain *w* **where** *w*: *1 + w^k ∗ a = 0*
by (*metis kas*(*2*) *mult.commute mult.left-commute poly-monom poly-pCons power-eq-if*)
from *poly-bound-exists*[*of cmod w s*] **obtain** *m* **where**
m: *m > 0* ∀ *z. cmod z ≤ cmod w* ⟶ *cmod* (*poly s z*) ≤ *m* **by** *blast*
have *w ≠ 0*
using *kas*(*2*) *w* **by** (*auto simp add*: *power-0-left*)
from *w* **have** *wm1*: *w^k ∗ a = − 1*
by (*simp add*: *add-eq-0-iff*)
have *inv0*: *0 < inverse* (*cmod w ^* (*k + 1*) ∗ *m*)
by (*simp add*: ‹*w ≠ 0*› *m*(*1*))
with *field-lbound-gt-zero*[*OF zero-less-one*] **obtain** *t* **where**
t: *t > 0 t < 1 t < inverse* (*cmod w ^* (*k + 1*) ∗ *m*) **by** *blast*
let *?ct = complex-of-real t*
let *?w = ?ct ∗ w*
have *1 + ?w^k ∗* (*a + ?w ∗ poly s ?w*) = *1 + ?ct^k ∗* (*w^k ∗ a*) + *?w^k ∗ ?w ∗ poly s ?w*
using *kas*(*1*) **by** (*simp add*: *algebra-simps power-mult-distrib*)
also have ... = *complex-of-real* (*1 − t^k*) + *?w^k ∗ ?w ∗ poly s ?w*
unfolding *wm1* **by** *simp*
finally have *cmod* (*1 + ?w^k ∗* (*a + ?w ∗ poly s ?w*)) =
cmod (*complex-of-real* (*1 − t^k*) + *?w^k ∗ ?w ∗ poly s ?w*)
by *metis*
with *norm-triangle-ineq*[*of complex-of-real* (*1 − t^k*) *?w^k ∗ ?w ∗ poly s ?w*]
have *11*: *cmod* (*1 + ?w^k ∗* (*a + ?w ∗ poly s ?w*)) ≤ |*1 − t^k*| + *cmod* (*?w^k ∗ ?w ∗ poly s ?w*)
unfolding *norm-of-real* **by** *simp*
have *ath*: ⋀*x t::real. 0 ≤ x* ⟹ *x < t* ⟹ *t ≤ 1* ⟹ |*1 − t*| + *x < 1*
by *arith*
have *tw*: *cmod ?w ≤ cmod w*
by (*smt* (*verit*) *mult-le-cancel-right2 norm-ge-zero norm-mult norm-of-real t*)
have *t ∗* (*cmod w ^* (*k + 1*) ∗ *m*) < *1*

by (*smt* (*verit*, *best*) *inv0 inverse-positive-iff-positive left-inverse mult-strict-right-mono t*(*3*))
with *zero-less-power*[*OF t*(*1*), *of k*] **have** *30*: $t\hat{}k * (t* (cmod\ w\ \hat{}\ (k + 1) * m)) < t\hat{}k$
by *simp*
have *cmod* ($?w\hat{}k * ?w * poly\ s\ ?w$) $= t\hat{}k * (t* (cmod\ w\ \hat{}\ (k + 1) * cmod\ (poly\ s\ ?w)))$
using ‹$w \neq 0$› *t*(*1*) **by** (*simp add*: *algebra-simps norm-power norm-mult*)
with *30* **have** *120*: *cmod* ($?w\hat{}k * ?w * poly\ s\ ?w$) $< t\hat{}k$
by (*smt* (*verit*, *ccfv-SIG*) *m*(*2*) *mult-left-mono norm-ge-zero t*(*1*) *tw zero-le-power*)
from *power-strict-mono*[*OF t*(*2*), *of k*] *t*(*1*) *kas*(*2*) **have** *121*: $t\hat{}k \leq 1$
by *auto*
from *ath*[*OF norm-ge-zero*[*of* $?w\hat{}k * ?w * poly\ s\ ?w$] *120 121*]
show *?thesis*
by (*smt* (*verit*) *11 kas*(*4*) *poly-pCons r01*)
qed
with *cq0 q*(*2*) **show** *?thesis*
by (*smt* (*verit*) *mrmq-eq*)
qed
qed

Alternative version with a syntactic notion of constant polynomial.

lemma *fundamental-theorem-of-algebra-alt*:
assumes *nc*: $\neg\ (\exists\, a\ l.\ a \neq 0 \wedge l = 0 \wedge p = pCons\ a\ l)$
shows $\exists\, z.\ poly\ p\ z = (0::complex)$
proof (*rule ccontr*)
assume *N*: $\nexists\, z.\ poly\ p\ z = 0$
then have ¬ *constant* (*poly p*)
unfolding *constant-def*
by (*metis* (*no-types*, *opaque-lifting*) *nc poly-pcompose pcompose-0′ pcompose-const poly-0-coeff-0*
poly-all-0-iff-0 poly-diff right-minus-eq)
then show *False*
using *N fundamental-theorem-of-algebra* **by** *blast*
qed

9.4 Nullstellensatz, degrees and divisibility of polynomials

lemma *nullstellensatz-lemma*:
fixes *p* :: *complex poly*
assumes $\forall\, x.\ poly\ p\ x = 0 \longrightarrow poly\ q\ x = 0$
and *degree* $p = n$
and $n \neq 0$
shows *p dvd* ($q\ \hat{}\ n$)
using *assms*
proof (*induct n arbitrary*: *p q rule*: *nat-less-induct*)
fix *n* :: *nat*
fix *p q* :: *complex poly*

```
assume IH: ∀ m<n. ∀ p q.
               (∀ x. poly p x = (0::complex) ⟶ poly q x = 0) ⟶
               degree p = m ⟶ m ≠ 0 ⟶ p dvd (q ^ m)
  and pq0: ∀ x. poly p x = 0 ⟶ poly q x = 0
  and dpn: degree p = n
  and n0: n ≠ 0
from dpn n0 have pne: p ≠ 0 by auto
show p dvd (q ^ n)
proof (cases ∃ a. poly p a = 0)
  case True
  then obtain a where a: poly p a = 0 ..
  have ?thesis if oa: order a p ≠ 0
  proof –
    let ?op = order a p
    from pne have ap: ([:– a, 1:] ^ ?op) dvd p ¬ [:– a, 1:] ^(Suc ?op) dvd p
      using order by blast+
    note oop = order-degree[OF pne, unfolded dpn]
    show ?thesis
    proof (cases q = 0)
      case True
      with n0 show ?thesis by (simp add: power-0-left)
    next
      case False
      from pq0[rule-format, OF a, unfolded poly-eq-0-iff-dvd]
      obtain r where r: q = [:– a, 1:] ∗ r by (rule dvdE)
      from ap(1) obtain s where s: p = [:– a, 1:] ^ ?op ∗ s
        by (rule dvdE)
      have sne: s ≠ 0
        using s pne by auto
      show ?thesis
      proof (cases degree s = 0)
        case True
        then obtain k where kpn: s = [:k:]
          by (cases s) (auto split: if-splits)
        from sne kpn have k: k ≠ 0 by simp
        let ?w = ([:1/k:] ∗ ([:–a,1:] ^(n – ?op))) ∗ (r ^ n)
        have q^n = [:– a, 1:] ^ n ∗ r ^ n
          using power-mult-distrib r by blast
        also have ... = [:– a, 1:] ^ order a p ∗ [:k:] ∗ ([:1 / k:] ∗ [:– a, 1:] ^(n
– order a p) ∗ r ^ n)
          using k oop [of a] by (simp flip: power-add)
        also have ... = p ∗ ?w
          by (metis s kpn)
        finally show ?thesis
          unfolding dvd-def by blast
      next
        case False
        with sne dpn s oa have dsn: degree s < n
        by (metis add-diff-cancel-right' degree-0 degree-linear-power degree-mult-eq
```

gr0I zero-less-diff)
have *poly r x = 0* **if** *h*: *poly s x = 0* **for** *x*
proof −
have *x ≠ a*
by (*metis ap*(*2*) *dvd-refl mult-dvd-mono poly-eq-0-iff-dvd power-Suc power-commutes s that*)
moreover have *poly p x = 0*
by (*metis* (*no-types*) *mult-eq-0-iff poly-mult s that*)
ultimately show *?thesis*
using *pq0 r* **by** *auto*
qed
with *False IH dsn* **obtain** *u* **where** *u*: *r ^* (*degree s*) = *s ∗ u*
by *blast*
then have *u′*: ⋀*x. poly s x ∗ poly u x = poly r x ^ degree s*
by (*simp only*: *poly-mult*[*symmetric*] *poly-power*[*symmetric*])
have *q^n* = [:− *a*, *1*:] *^ n ∗ r ^ n*
using *power-mult-distrib r* **by** *blast*
also have ... = [:− *a*, *1*:] *^ order a p ∗* (*s ∗ u ∗* ([:− *a*, *1*:] *^* (*n − order a p*) *∗ r ^* (*n − degree s*)))
by (*smt* (*verit*, *del-insts*) *s u mult-ac power-add add-diff-cancel-right′ degree-linear-power degree-mult-eq dpn mult-zero-left*)
also have ... = *p ∗* (*u ∗* ([:−*a*,*1*:] *^* (*n − ?op*))) *∗* (*r ^* (*n − degree s*))
using *s* **by** *force*
finally show *?thesis*
unfolding *dvd-def* **by** *auto*
qed
qed
qed
then show *?thesis*
using *a order-root pne* **by** *blast*
next
case *False*
then show *?thesis*
using *dpn n0 fundamental-theorem-of-algebra-alt*[*of p*]
by *fastforce*
qed
qed

lemma *nullstellensatz-univariate*:
(∀ *x. poly p x =* (*0::complex*) ⟶ *poly q x = 0*) ⟷
p dvd (*q ^* (*degree p*)) ∨ (*p = 0* ∧ *q = 0*)
proof −
consider *p = 0* | *p ≠ 0 degree p = 0* | *n* **where** *p ≠ 0 degree p = Suc n*
by (*cases degree p*) *auto*
then show *?thesis*
proof *cases*
case *p*: *1*
then have (∀ *x. poly p x =* (*0::complex*) ⟶ *poly q x = 0*) ⟷ *q = 0*
by (*auto simp add*: *poly-all-0-iff-0*)

with *p* **show** *?thesis*
  by *force*
next
  case *dp*: *2*
  then show *?thesis*
    by (*meson dvd-trans is-unit-iff-degree poly-eq-0-iff-dvd unit-imp-dvd*)
next
  case *dp*: *3*
  have *False* **if** *p dvd* (q ^ (*Suc n*)) *poly p x* = *0 poly q x* ≠ *0* **for** *x*
    by (*metis dvd-trans poly-eq-0-iff-dvd poly-power power-eq-0-iff that*)
  with *dp nullstellensatz-lemma*[*of p q degree p*] **show** *?thesis*
    by *auto*
qed
qed

Useful lemma

lemma *constant-degree*:
  fixes *p* :: *'a*::{*idom*,*ring-char-0*} *poly*
  shows *constant* (*poly p*) ⟷ *degree p* = *0* (**is** *?lhs* = *?rhs*)
proof
  show *?rhs* **if** *?lhs*
  proof −
    from *that*[*unfolded constant-def*, *rule-format*, *of - 0*]
    have *poly p* = *poly* [:*poly p 0*:]
      by *auto*
    then show *?thesis*
      by (*metis degree-pCons-0 poly-eq-poly-eq-iff*)
  qed
  show *?lhs* **if** *?rhs*
    unfolding *constant-def*
    by (*metis degree-eq-zeroE pcompose-const poly-0 poly-pcompose that*)
qed

lemma *complex-poly-decompose*:
  smult (*lead-coeff p*) (∏ *z*|*poly p z* = *0*. [:−*z*, *1*:] ^ *order z p*) = (*p* :: *complex poly*)
proof (*induction p rule*: *poly-root-order-induct*)
  case (*no-roots p*)
  show *?case*
  proof (*cases degree p* = *0*)
    case *False*
    hence ¬*constant* (*poly p*) **by** (*subst constant-degree*)
    with *fundamental-theorem-of-algebra* **and** *no-roots* **show** *?thesis* **by** *blast*
  qed (*auto elim*!: *degree-eq-zeroE*)
next
  case (*root p x n*)
  from *root* **have** ∗: {*z*. *poly* ([:− *x*, *1*:] ^ *n* ∗ *p*) *z* = *0*} = *insert x* {*z*. *poly p z* = *0*}
    by *auto*
  have *smult* (*lead-coeff* ([:−*x*, *1*:] ^ *n* ∗ *p*))

($\prod$ z|poly ([:−x,1:] ^ n ∗ p) z = 0. [:−z, 1:] ^ order z ([:− x, 1:] ^ n ∗ p)) =
[:− x, 1:] ^ order x ([:− x, 1:] ^ n ∗ p) ∗
smult (lead-coeff p) ($\prod$ z∈{z. poly p z = 0}. [:− z, 1:] ^ order z ([:− x, 1:] ^ n ∗ p))
by (subst ∗, subst prod.insert)
(insert root, auto intro: poly-roots-finite simp: mult-ac lead-coeff-mult lead-coeff-power)
also have order x ([:− x, 1:] ^ n ∗ p) = n
using root **by** (subst order-mult) (auto simp: order-power-n-n order-0I)
also have ($\prod$ z∈{z. poly p z = 0}. [:− z, 1:] ^ order z ([:− x, 1:] ^ n ∗ p)) =
($\prod$ z∈{z. poly p z = 0}. [:− z, 1:] ^ order z p)
proof (intro prod.cong refl, goal-cases)
case (1 y)
with root **have** order y ([:−x,1:] ^ n) = 0 **by** (intro order-0I) auto
thus ?case **using** root **by** (subst order-mult) auto
qed
also note root.IH
finally show ?case **.**
qed simp-all

instance complex :: alg-closed-field
by standard (use fundamental-theorem-of-algebra constant-degree neq0-conv **in** blast)

lemma size-proots-complex: size (proots (p :: complex poly)) = degree p
proof (cases p = 0)
case [simp]: False
show size (proots p) = degree p
by (subst (1 2) complex-poly-decompose [symmetric])
(simp add: proots-prod proots-power degree-prod-sum-eq degree-power-eq)
qed auto

lemma complex-poly-decompose-multiset:
smult (lead-coeff p) ($\prod$ x∈#proots p. [:−x, 1:]) = (p :: complex poly)
proof (cases p = 0)
case False
hence ($\prod$ x∈#proots p. [:−x, 1:]) = ($\prod$ x | poly p x = 0. [:−x, 1:] ^ order x p)
by (subst image-prod-mset-multiplicity) simp-all
also have smult (lead-coeff p) . . . = p
by (rule complex-poly-decompose)
finally show ?thesis **.**
qed auto

lemma complex-poly-decompose′:
obtains root **where** smult (lead-coeff p) ($\prod$ i<degree p. [:−root i, 1:]) = (p :: complex poly)
proof −
obtain roots **where** roots: mset roots = proots p
using ex-mset **by** blast

have *p* = *smult* (*lead-coeff p*) (∏ *x*∈#*proots p*. [:−*x*, *1*:])
  by (*rule complex-poly-decompose-multiset* [*symmetric*])
also have (∏ *x*∈#*proots p*. [:−*x*, *1*:]) = (∏ *x*←*roots*. [:−*x*, *1*:])
  by (*subst prod-mset-prod-list* [*symmetric*]) (*simp add*: *roots*)
also have ... = (∏ *i*<*length roots*. [:−*roots* ! *i*, *1*:])
  by (*subst prod.list-conv-set-nth*) (*auto simp*: *atLeast0LessThan*)
finally have *eq*: *p* = *smult* (*lead-coeff p*) (∏ *i*<*length roots*. [:−*roots* ! *i*, *1*:]) **.**
also have [*simp*]: *degree p* = *length roots*
  using *roots* **by** (*subst eq*) (*auto simp*: *degree-prod-sum-eq*)
finally show *?thesis* **by** (*intro that*[*of* λ*i*. *roots* ! *i*]) *auto*
qed

lemma *complex-poly-decompose-rsquarefree*:
  assumes *rsquarefree p*
  shows *smult* (*lead-coeff p*) (∏ *z*|*poly p z* = *0*. [:−*z*, *1*:]) = (*p* :: *complex poly*)
proof (*cases p* = *0*)
  case *False*
  have (∏ *z*|*poly p z* = *0*. [:−*z*, *1*:]) = (∏ *z*|*poly p z* = *0*. [:−*z*, *1*:] ^ *order z p*)
    using *assms False* **by** (*intro prod.cong*) (*auto simp*: *rsquarefree-root-order*)
  also have *smult* (*lead-coeff p*) ... = *p*
    by (*rule complex-poly-decompose*)
  finally show *?thesis* **.**
qed *auto*

Arithmetic operations on multivariate polynomials.

lemma *mpoly-base-conv*:
  fixes *x* :: ′*a*::*comm-ring-1*
  shows *0* = *poly 0 x c* = *poly* [:*c*:] *x x* = *poly* [:*0*,*1*:] *x*
  by *simp-all*

lemma *mpoly-norm-conv*:
  fixes *x* :: ′*a*::*comm-ring-1*
  shows *poly* [:*0*:] *x* = *poly 0 x poly* [:*poly 0 y*:] *x* = *poly 0 x*
  by *simp-all*

lemma *mpoly-sub-conv*:
  fixes *x* :: ′*a*::*comm-ring-1*
  shows *poly p x* − *poly q x* = *poly p x* + −*1* ∗ *poly q x*
  by *simp*

lemma *poly-pad-rule*: *poly p x* = *0* ⟹ *poly* (*pCons 0 p*) *x* = *0*
  by *simp*

lemma *poly-cancel-eq-conv*:
  fixes *x* :: ′*a*::*field*
  shows *x* = *0* ⟹ *a* ≠ *0* ⟹ *y* = *0* ⟷ *a* ∗ *y* − *b* ∗ *x* = *0*
  by *auto*

lemma *poly-divides-pad-rule*:
fixes *p*:: (*'a*::*comm-ring-1*) *poly*
assumes *pq*: *p dvd q*
shows *p dvd* (*pCons 0 q*)
by (*metis add-0 dvd-def mult-pCons-right pq smult-0-left*)

lemma *poly-divides-conv0*:
fixes *p*:: *'a*::*field poly*
assumes *lgpq*: *degree q* $<$ *degree p* **and** *lq*: $p \neq 0$
shows *p dvd q* $\longleftrightarrow$ $q = 0$
using *lgpq mod-poly-less* **by** *fastforce*

lemma *poly-divides-conv1*:
fixes *p* :: *'a*::*field poly*
assumes *a0*: $a \neq 0$
and *pp'*: *p dvd p'*
and *qrp'*: *smult a q* $-$ $p' = r$
shows *p dvd q* $\longleftrightarrow$ *p dvd r*
by (*metis a0 diff-add-cancel dvd-add-left-iff dvd-smult-iff pp' qrp'*)

lemma *basic-cqe-conv1*:
($\exists$ *x*. *poly p x* $= 0 \wedge$ *poly 0 x* $\neq 0$) $\longleftrightarrow$ *False*
($\exists$ *x*. *poly 0 x* $\neq 0$) $\longleftrightarrow$ *False*
($\exists$ *x*. *poly* [:*c*:] *x* $\neq 0$) $\longleftrightarrow$ $c \neq 0$
($\exists$ *x*. *poly 0 x* $= 0$) $\longleftrightarrow$ *True*
($\exists$ *x*. *poly* [:*c*:] *x* $= 0$) $\longleftrightarrow$ $c = 0$
by *simp-all*

lemma *basic-cqe-conv2*:
assumes *l*: $p \neq 0$
shows $\exists$ *x*. *poly* (*pCons a* (*pCons b p*)) *x* = (*0*::*complex*)
by (*meson fundamental-theorem-of-algebra-alt l pCons-eq-0-iff pCons-eq-iff*)

lemma *basic-cqe-conv-2b*: ($\exists$ *x*. *poly p x* $\neq$ (*0*::*complex*)) $\longleftrightarrow$ $p \neq 0$
by (*metis poly-all-0-iff-0*)

lemma *basic-cqe-conv3*:
fixes *p q* :: *complex poly*
assumes *l*: $p \neq 0$
shows ($\exists$ *x*. *poly* (*pCons a p*) *x* $= 0 \wedge$ *poly q x* $\neq 0$) $\longleftrightarrow \neg$ (*pCons a p*) *dvd* (*q* ^ *psize p*)
by (*metis degree-pCons-eq-if l nullstellensatz-univariate pCons-eq-0-iff psize-def*)

lemma *basic-cqe-conv4*:
fixes *p q* :: *complex poly*
assumes *h*: $\bigwedge$*x*. *poly* (*q* ^ *n*) *x* = *poly r x*
shows *p dvd* (*q* ^ *n*) $\longleftrightarrow$ *p dvd r*
by (*metis* (*no-types*) *basic-cqe-conv-2b h poly-diff right-minus-eq*)

lemma *poly-const-conv*:
fixes *x* :: *'a::comm-ring-1*
shows *poly* [:*c*:] *x* = *y* ⟷ *c* = *y*
by *simp*

end

theory *Group-Closure*
imports
Main
begin

context *ab-group-add*
begin

inductive-set *group-closure* :: *'a set* ⇒ *'a set* **for** *S*
where *base*: *s* ∈ *insert 0 S* ⟹ *s* ∈ *group-closure S*
| *diff*: *s* ∈ *group-closure S* ⟹ *t* ∈ *group-closure S* ⟹ *s* − *t* ∈ *group-closure S*

lemma *zero-in-group-closure* [*simp*]:
0 ∈ *group-closure S*
using *group-closure.base* [*of 0 S*] **by** *simp*

lemma *group-closure-minus-iff* [*simp*]:
− *s* ∈ *group-closure S* ⟷ *s* ∈ *group-closure S*
using *group-closure.diff* [*of 0 S s*] *group-closure.diff* [*of 0 S* − *s*] **by** *auto*

lemma *group-closure-add*:
s + *t* ∈ *group-closure S* **if** *s* ∈ *group-closure S* **and** *t* ∈ *group-closure S*
using *that group-closure.diff* [*of s S* − *t*] **by** *auto*

lemma *group-closure-empty* [*simp*]:
group-closure {} = {*0*}
by (*rule ccontr*) (*auto elim*: *group-closure.induct*)

lemma *group-closure-insert-zero* [*simp*]:
group-closure (*insert 0 S*) = *group-closure S*
by (*auto elim*: *group-closure.induct intro*: *group-closure.intros*)

end

context *comm-ring-1*
begin

lemma *group-closure-scalar-mult-left*:
of-nat n ∗ *s* ∈ *group-closure S* **if** *s* ∈ *group-closure S*
using *that* **by** (*induction n*) (*auto simp add*: *algebra-simps intro*: *group-closure-add*)

lemma *group-closure-scalar-mult-right*:
  s ∗ *of-nat n* ∈ *group-closure S* **if** *s* ∈ *group-closure S*
  using *that group-closure-scalar-mult-left* [*of s S n*] **by** (*simp add*: *ac-simps*)

end

lemma *group-closure-abs-iff* [*simp*]:
  |*s*| ∈ *group-closure S* ⟷ *s* ∈ *group-closure S* **for** *s* :: *int*
  by (*simp add*: *abs-if*)

lemma *group-closure-mult-left*:
  s ∗ *t* ∈ *group-closure S* **if** *s* ∈ *group-closure S* **for** *s t* :: *int*
proof −
  from *that group-closure-scalar-mult-right* [*of s S nat* |*t*|]
    have *s* ∗ *int* (*nat* |*t*|) ∈ *group-closure S*
    by (*simp only*:)
  then show *?thesis*
    by (*cases t* ≥ *0*) *simp-all*
qed

lemma *group-closure-mult-right*:
  s ∗ *t* ∈ *group-closure S* **if** *t* ∈ *group-closure S* **for** *s t* :: *int*
  using *that group-closure-mult-left* [*of t S s*] **by** (*simp add*: *ac-simps*)

context *idom*
begin

lemma *group-closure-mult-all-eq*:
  group-closure (*times k* ‘ *S*) = *times k* ‘ *group-closure S*
proof (*rule*; *rule*)
  fix *s*
  have ∗: *k* ∗ *a* + *k* ∗ *b* = *k* ∗ (*a* + *b*)
    k ∗ *a* − *k* ∗ *b* = *k* ∗ (*a* − *b*) **for** *a b*
    by (*simp-all add*: *algebra-simps*)
  assume *s* ∈ *group-closure* (*times k* ‘ *S*)
  then show *s* ∈ *times k* ‘ *group-closure S*
   by *induction* (*auto simp add*: ∗ *image-iff intro*: *group-closure.base group-closure.diff bexI* [*of - 0*])
next
  fix *s*
  assume *s* ∈ *times k* ‘ *group-closure S*
  then obtain *r* **where** *r*: *r* ∈ *group-closure S* **and** *s*: *s* = *k* ∗ *r*
    by *auto*
  from *r* **have** *k* ∗ *r* ∈ *group-closure* (*times k* ‘ *S*)
   by (*induction arbitrary*: *s*) (*auto simp add*: *algebra-simps intro*: *group-closure.intros*)
  with *s* **show** *s* ∈ *group-closure* (*times k* ‘ *S*)
    by *simp*
qed

end

lemma *Gcd-group-closure-eq-Gcd*:
  Gcd (*group-closure S*) = *Gcd S* **for** *S* :: *int set*
proof (*rule associated-eqI*)
  have *Gcd S dvd s* **if** $s \in$ *group-closure S* **for** *s*
    using *that* **by** *induction auto*
  then show *Gcd S dvd Gcd* (*group-closure S*)
    by *auto*
  have *Gcd* (*group-closure S*) *dvd s* **if** $s \in S$ **for** *s*
  proof −
    from *that* **have** $s \in$ *group-closure S*
      by (*simp add*: *group-closure.base*)
    then show *?thesis*
      by (*rule Gcd-dvd*)
  qed
  then show *Gcd* (*group-closure S*) *dvd Gcd S*
    by *auto*
qed *simp-all*

lemma *group-closure-sum*:
  fixes *S* :: *int set*
  assumes *X*: *finite X* $X \neq \{\}$ $X \subseteq S$
  shows ($\sum x \in X.\ a\ x * x$) $\in$ *group-closure S*
  using *X* **by** (*induction X rule*: *finite-ne-induct*)
    (*auto intro*: *group-closure-mult-right group-closure.base group-closure-add*)

lemma *Gcd-group-closure-in-group-closure*:
  Gcd (*group-closure S*) $\in$ *group-closure S* **for** *S* :: *int set*
proof (*cases* $S \subseteq \{0\}$)
  case *True*
  then have $S = \{\} \vee S = \{0\}$
    by *auto*
  then show *?thesis*
    by *auto*
next
  case *False*
  then obtain *s* **where** *s*: $s \neq 0$ $s \in S$
    by *auto*
  then have *s'*: $|s| \neq 0$ $|s| \in$ *group-closure S*
    by (*auto intro*: *group-closure.base*)
  define *m* **where** *m* = (*LEAST n*. $n > 0 \wedge$ *int n* $\in$ *group-closure S*)
  have $m > 0 \wedge$ *int m* $\in$ *group-closure S*
    unfolding *m-def*
    apply (*rule LeastI* [*of* - *nat* $|s|$])
    using *s'*
    by *simp*
  then have *m*: *int m* $\in$ *group-closure S* **and** $0 < m$
    by *auto*

have *Gcd* (*group-closure S*) = *int m*
proof (*rule associated-eqI*)
from *m* **show** *Gcd* (*group-closure S*) *dvd int m*
by (*rule Gcd-dvd*)
show *int m dvd Gcd* (*group-closure S*)
proof (*rule Gcd-greatest*)
fix *s*
assume *s*: *s* ∈ *group-closure S*
show *int m dvd s*
proof (*rule ccontr*)
assume ¬ *int m dvd s*
then have ∗: *0* < *s mod int m*
using ‹*0* < *m*› *le-less* **by** *fastforce*
have *m* ≤ *nat* (*s mod int m*)
proof (*subst m-def*, *rule Least-le*, *rule*)
from ∗ **show** *0* < *nat* (*s mod int m*)
by *simp*
from *minus-div-mult-eq-mod* [*symmetric*, *of s int m*]
have *s mod int m* = *s* − *s div int m* ∗ *int m*
by *auto*
also have *s* − *s div int m* ∗ *int m* ∈ *group-closure S*
by (*auto intro*: *group-closure.diff s group-closure-mult-right m*)
finally show *int* (*nat* (*s mod int m*)) ∈ *group-closure S*
by *simp*
qed
with ∗ **have** *int m* ≤ *s mod int m*
by *simp*
moreover have *s mod int m* < *int m*
using ‹*0* < *m*› **by** *simp*
ultimately show *False*
by *auto*
qed
qed
qed *simp-all*
with *m* **show** *?thesis*
by *simp*
qed

lemma *Gcd-in-group-closure*:
Gcd S ∈ *group-closure S* **for** *S* :: *int set*
using *Gcd-group-closure-in-group-closure* [*of S*]
by (*simp add*: *Gcd-group-closure-eq-Gcd*)

lemma *group-closure-eq*:
group-closure S = *range* (*times* (*Gcd S*)) **for** *S* :: *int set*
proof (*auto intro*: *Gcd-in-group-closure group-closure-mult-left*)
fix *s*
assume *s* ∈ *group-closure S*

then show *s* ∈ *range* (*times* (*Gcd S*))
proof *induction*
case (*base s*)
then have *Gcd S dvd s*
by (*auto intro*: *Gcd-dvd*)
then obtain *t* **where** *s* = *Gcd S* ∗ *t* **..**
then show *?case*
by *auto*
next
case (*diff s t*)
moreover have *Gcd S* ∗ *a* − *Gcd S* ∗ *b* = *Gcd S* ∗ (*a* − *b*) **for** *a b*
by (*simp add*: *algebra-simps*)
ultimately show *?case*
by *auto*
qed
qed

end

theory *Normalized-Fraction*
imports
Main
Euclidean-Algorithm
Fraction-Field
begin

lemma *unit-factor-1-imp-normalized*: *unit-factor x* = *1* ⟹ *normalize x* = *x*
using *unit-factor-mult-normalize* [*of x*] **by** *simp*

definition *quot-to-fract* :: *'a* × *'a* ⇒ *'a* :: *idom fract* **where**
quot-to-fract = (λ(*a*,*b*). *Fraction-Field.Fract a b*)

definition *normalize-quot* :: *'a* :: {*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} × *'a* ⇒ *'a* × *'a* **where**
normalize-quot =
(λ(*a*,*b*). *if b* = *0 then* (*0*,*1*) *else let d* = *gcd a b* ∗ *unit-factor b in* (*a div d*, *b div d*))

lemma *normalize-quot-zero* [*simp*]:
normalize-quot (*a*, *0*) = (*0*, *1*)
by (*simp add*: *normalize-quot-def*)

lemma *normalize-quot-proj*:
fst (*normalize-quot* (*a*, *b*)) = *a div* (*gcd a b* ∗ *unit-factor b*)
snd (*normalize-quot* (*a*, *b*)) = *normalize b div gcd a b* **if** *b* ≠ *0*
using *that* **by** (*simp-all add*: *normalize-quot-def Let-def mult.commute* [*of* - *unit-factor b*] *dvd-div-mult2-eq mult-unit-dvd-iff'*)

definition *normalized-fracts* :: (*'a* :: {*ring-gcd*,*idom-divide*} × *'a*) *set* **where**
normalized-fracts = {(*a*,*b*). *coprime a b* ∧ *unit-factor b* = *1*}

lemma *not-normalized-fracts-0-denom* [*simp*]: (*a*, *0*) ∉ *normalized-fracts*
by (*auto simp*: *normalized-fracts-def*)

lemma *unit-factor-snd-normalize-quot* [*simp*]:
unit-factor (*snd* (*normalize-quot x*)) = *1*
by (*simp add*: *normalize-quot-def case-prod-unfold Let-def dvd-unit-factor-div mult-unit-dvd-iff unit-factor-mult unit-factor-gcd*)

lemma *snd-normalize-quot-nonzero* [*simp*]: *snd* (*normalize-quot x*) ≠ *0*
using *unit-factor-snd-normalize-quot*[*of x*]
by (*auto simp del*: *unit-factor-snd-normalize-quot*)

lemma *normalize-quot-aux*:
fixes *a b*
assumes *b* ≠ *0*
defines *d* ≡ *gcd a b* ∗ *unit-factor b*
shows *a* = *fst* (*normalize-quot* (*a*,*b*)) ∗ *d b* = *snd* (*normalize-quot* (*a*,*b*)) ∗ *d*
d dvd a d dvd b d ≠ *0*
proof −
from *assms* **show** *d dvd a d dvd b*
by (*simp-all add*: *d-def mult-unit-dvd-iff*)
thus *a* = *fst* (*normalize-quot* (*a*,*b*)) ∗ *d b* = *snd* (*normalize-quot* (*a*,*b*)) ∗ *d d* ≠ *0*
by (*auto simp*: *normalize-quot-def Let-def d-def* ‹*b* ≠ *0*›)
qed

lemma *normalize-quotE*:
assumes *b* ≠ *0*
obtains *d* **where** *a* = *fst* (*normalize-quot* (*a*,*b*)) ∗ *d b* = *snd* (*normalize-quot* (*a*,*b*)) ∗ *d*
d dvd a d dvd b d ≠ *0*
using *that*[*OF normalize-quot-aux*[*OF assms*]] **.**

lemma *normalize-quotE'*:
assumes *snd x* ≠ *0*
obtains *d* **where** *fst x* = *fst* (*normalize-quot x*) ∗ *d snd x* = *snd* (*normalize-quot x*) ∗ *d*
d dvd fst x d dvd snd x d ≠ *0*
proof −
from *normalize-quotE*[*OF assms*, *of fst x*] **obtain** *d* **where**
fst x = *fst* (*normalize-quot* (*fst x*, *snd x*)) ∗ *d*
snd x = *snd* (*normalize-quot* (*fst x*, *snd x*)) ∗ *d*
d dvd fst x
d dvd snd x
d ≠ *0* **.**
then show *?thesis* **unfolding** *prod.collapse* **by** (*intro that*[*of d*])

qed

lemma *coprime-normalize-quot*:
coprime (*fst* (*normalize-quot x*)) (*snd* (*normalize-quot x*))
by (*simp add*: *normalize-quot-def case-prod-unfold div-mult-unit2*)
(*metis coprime-mult-self-right-iff div-gcd-coprime unit-div-mult-self unit-factor-is-unit*)

lemma *normalize-quot-in-normalized-fracts* [*simp*]: *normalize-quot x* ∈ *normalized-fracts*
by (*simp add*: *normalized-fracts-def coprime-normalize-quot case-prod-unfold*)

lemma *normalize-quot-eq-iff*:
assumes $b \neq 0$ $d \neq 0$
shows *normalize-quot* (*a*,*b*) = *normalize-quot* (*c*,*d*) ⟷ $a * d = b * c$
proof −
define *x y* **where** *x* = *normalize-quot* (*a*,*b*) **and** *y* = *normalize-quot* (*c*,*d*)
from *normalize-quotE*[*OF assms*(*1*), *of a*] *normalize-quotE*[*OF assms*(*2*), *of c*]
obtain *d1 d2*
where *a* = *fst x* ∗ *d1* *b* = *snd x* ∗ *d1* *c* = *fst y* ∗ *d2* *d* = *snd y* ∗ *d2* *d1* ≠ *0* *d2* ≠ *0*
unfolding *x-def y-def* **by** *metis*
hence $a * d = b * c$ ⟷ *fst x* ∗ *snd y* = *snd x* ∗ *fst y* **by** *simp*
also have ... ⟷ *fst x* = *fst y* ∧ *snd x* = *snd y*
by (*intro coprime-crossproduct'*) (*simp-all add*: *x-def y-def coprime-normalize-quot*)
also have ... ⟷ $x = y$ **using** *prod-eqI* **by** *blast*
finally show $x = y$ ⟷ $a * d = b * c$ **..**
qed

lemma *normalize-quot-eq-iff'*:
assumes *snd x* ≠ *0* *snd y* ≠ *0*
shows *normalize-quot x* = *normalize-quot y* ⟷ *fst x* ∗ *snd y* = *snd x* ∗ *fst y*
using *assms* **by** (*cases x*, *cases y*, *hypsubst*) (*subst normalize-quot-eq-iff*, *simp-all*)

lemma *normalize-quot-id*: *x* ∈ *normalized-fracts* ⟹ *normalize-quot x* = *x*
by (*auto simp*: *normalized-fracts-def normalize-quot-def case-prod-unfold*)

lemma *normalize-quot-idem* [*simp*]: *normalize-quot* (*normalize-quot x*) = *normalize-quot x*
by (*rule normalize-quot-id*) *simp-all*

lemma *fractrel-iff-normalize-quot-eq*:
fractrel x y ⟷ *normalize-quot x* = *normalize-quot y* ∧ *snd x* ≠ *0* ∧ *snd y* ≠ *0*
by (*cases x*, *cases y*) (*auto simp*: *fractrel-def normalize-quot-eq-iff*)

lemma *fractrel-normalize-quot-left*:
assumes *snd x* ≠ *0*
shows *fractrel* (*normalize-quot x*) *y* ⟷ *fractrel x y*
using *assms* **by** (*subst* (*1 2*) *fractrel-iff-normalize-quot-eq*) *auto*

lemma *fractrel-normalize-quot-right*:
  assumes *snd x* ≠ *0*
  shows *fractrel y* (*normalize-quot x*) ⟷ *fractrel y x*
  using *assms* **by** (*subst* (*1 2*) *fractrel-iff-normalize-quot-eq*) *auto*

lift-definition *quot-of-fract* ::
  'a :: {*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *fract* ⇒ *'a* × *'a*
    is *normalize-quot*
  by (*subst* (*asm*) *fractrel-iff-normalize-quot-eq*) *simp-all*

lemma *quot-to-fract-quot-of-fract* [*simp*]: *quot-to-fract* (*quot-of-fract x*) = *x*
  unfolding *quot-to-fract-def*
proof *transfer*
  fix *x* :: *'a* × *'a* **assume** *rel*: *fractrel x x*
  define *x'* **where** *x'* = *normalize-quot x*
  obtain *a b* **where** [*simp*]: *x* = (*a*, *b*) **by** (*cases x*)
  from *rel* **have** *b* ≠ *0* **by** *simp*
  from *normalize-quotE*[*OF this*, *of a*] **obtain** *d*
    where
      a = *fst* (*normalize-quot* (*a*, *b*)) ∗ *d*
      b = *snd* (*normalize-quot* (*a*, *b*)) ∗ *d*
      d dvd a
      d dvd b
      d ≠ *0* **.**
  hence *a* = *fst x'* ∗ *d b* = *snd x'* ∗ *d d* ≠ *0 snd x'* ≠ *0* **by** (*simp-all add*: *x'-def*)
  thus *fractrel* (*case x' of* (*a*, *b*) ⇒ *if b* = *0 then* (*0*, *1*) *else* (*a*, *b*)) *x*
    by (*auto simp add*: *case-prod-unfold*)
qed

lemma *quot-of-fract-quot-to-fract*: *quot-of-fract* (*quot-to-fract x*) = *normalize-quot*
x
proof (*cases snd x* = *0*)
  case *True*
  thus *?thesis* **unfolding** *quot-to-fract-def*
    by *transfer* (*simp add*: *case-prod-unfold normalize-quot-def*)
next
  case *False*
  thus *?thesis* **unfolding** *quot-to-fract-def* **by** *transfer* (*simp add*: *case-prod-unfold*)
qed

lemma *quot-of-fract-quot-to-fract'*:
  x ∈ *normalized-fracts* ⟹ *quot-of-fract* (*quot-to-fract x*) = *x*
  unfolding *quot-to-fract-def* **by** *transfer* (*auto simp*: *normalize-quot-id*)

lemma *quot-of-fract-in-normalized-fracts* [*simp*]: *quot-of-fract x* ∈ *normalized-fracts*
  by *transfer simp*

lemma *normalize-quotI*:

assumes *a* ∗ *d* = *b* ∗ *c b* ≠ *0* (*c*, *d*) ∈ *normalized-fracts*
shows *normalize-quot* (*a*, *b*) = (*c*, *d*)
proof −
from *assms* **have** *normalize-quot* (*a*, *b*) = *normalize-quot* (*c*, *d*)
by (*subst normalize-quot-eq-iff*) *auto*
also have . . . = (*c*, *d*) **by** (*intro normalize-quot-id*) *fact*
finally show *?thesis* **.**
qed

lemma *td-normalized-fract*:
type-definition quot-of-fract quot-to-fract normalized-fracts
by *standard* (*simp-all add*: *quot-of-fract-quot-to-fract′*)

lemma *quot-of-fract-add-aux*:
assumes *snd x* ≠ *0 snd y* ≠ *0*
shows (*fst x* ∗ *snd y* + *fst y* ∗ *snd x*) ∗ (*snd* (*normalize-quot x*) ∗ *snd* (*normalize-quot y*)) =
snd x ∗ *snd y* ∗ (*fst* (*normalize-quot x*) ∗ *snd* (*normalize-quot y*) +
snd (*normalize-quot x*) ∗ *fst* (*normalize-quot y*))
proof −
from *normalize-quotE′*[*OF assms*(*1*)] **obtain** *d*
where *d*:
fst x = *fst* (*normalize-quot x*) ∗ *d*
snd x = *snd* (*normalize-quot x*) ∗ *d*
d dvd fst x
d dvd snd x
d ≠ *0* **.**
from *normalize-quotE′*[*OF assms*(*2*)] **obtain** *e*
where *e*:
fst y = *fst* (*normalize-quot y*) ∗ *e*
snd y = *snd* (*normalize-quot y*) ∗ *e*
e dvd fst y
e dvd snd y
e ≠ *0* **.**
show *?thesis* **by** (*simp-all add*: *d e algebra-simps*)
qed

locale *fract-as-normalized-quot*
begin
setup-lifting *td-normalized-fract*
end

lemma *quot-of-fract-add*:
quot-of-fract (*x* + *y*) =
(*let* (*a*,*b*) = *quot-of-fract x*; (*c*,*d*) = *quot-of-fract y*
in *normalize-quot* (*a* ∗ *d* + *b* ∗ *c*, *b* ∗ *d*))
by *transfer* (*insert quot-of-fract-add-aux*,

simp-all add: *Let-def case-prod-unfold normalize-quot-eq-iff*)

lemma *quot-of-fract-uminus*:
quot-of-fract (−*x*) = (*let* (*a*,*b*) = *quot-of-fract x in* (−*a*, *b*))
by *transfer* (*auto simp*: *case-prod-unfold Let-def normalize-quot-def dvd-neg-div mult-unit-dvd-iff*)

lemma *quot-of-fract-diff*:
quot-of-fract (*x* − *y*) =
(*let* (*a*,*b*) = *quot-of-fract x*; (*c*,*d*) = *quot-of-fract y*
in normalize-quot (*a* ∗ *d* − *b* ∗ *c*, *b* ∗ *d*)) (**is** - = *?rhs*)
proof −
have *x* − *y* = *x* + −*y* **by** *simp*
also have *quot-of-fract* ... = *?rhs*
by (*simp only*: *quot-of-fract-add quot-of-fract-uminus Let-def case-prod-unfold*) *simp-all*
finally show *?thesis* **.**
qed

lemma *normalize-quot-mult-coprime*:
assumes *coprime a b coprime c d unit-factor b* = *1 unit-factor d* = *1*
defines *e* ≡ *fst* (*normalize-quot* (*a*, *d*)) **and** *f* ≡ *snd* (*normalize-quot* (*a*, *d*))
and *g* ≡ *fst* (*normalize-quot* (*c*, *b*)) **and** *h* ≡ *snd* (*normalize-quot* (*c*, *b*))
shows *normalize-quot* (*a* ∗ *c*, *b* ∗ *d*) = (*e* ∗ *g*, *f* ∗ *h*)
proof (*rule normalize-quotI*)
from *assms* **have** *gcd a b* = *1 gcd c d* = *1*
by *simp-all*
from *assms* **have** *b* ≠ *0 d* ≠ *0* **by** *auto*
with *assms* **have** *normalize b* = *b normalize d* = *d*
by (*auto intro*: *normalize-unit-factor-eqI*)
from *normalize-quotE* [*OF* ‹*b* ≠ *0*›, *of c*] **obtain** *k*
where
c = *fst* (*normalize-quot* (*c*, *b*)) ∗ *k*
b = *snd* (*normalize-quot* (*c*, *b*)) ∗ *k*
k dvd c k dvd b k ≠ *0* **.**
note *k* = *this* [*folded* ‹*gcd a b* = *1*› ‹*gcd c d* = *1*› *assms*(*3*) *assms*(*4*)]
from *normalize-quotE* [*OF* ‹*d* ≠ *0*›, *of a*] **obtain** *l*
where *a* = *fst* (*normalize-quot* (*a*, *d*)) ∗ *l*
d = *snd* (*normalize-quot* (*a*, *d*)) ∗ *l*
l dvd a l dvd d l ≠ *0* **.**
note *l* = *this* [*folded* ‹*gcd a b* = *1*› ‹*gcd c d* = *1*› *assms*(*3*) *assms*(*4*)]
from *k l* **show** *a* ∗ *c* ∗ (*f* ∗ *h*) = *b* ∗ *d* ∗ (*e* ∗ *g*)
by (*metis e-def f-def g-def h-def mult.commute mult.left-commute*)
from *assms* **have** [*simp*]: *unit-factor f* = *1 unit-factor h* = *1*
by *simp-all*
from *assms* **have** *coprime e f coprime g h* **by** (*simp-all add*: *coprime-normalize-quot*)
with *k l assms*(*1*,*2*) ‹*b* ≠ *0*› ‹*d* ≠ *0*› ‹*unit-factor b* = *1*› ‹*unit-factor d* = *1*›
‹*normalize b* = *b*› ‹*normalize d* = *d*›
show (*e* ∗ *g*, *f* ∗ *h*) ∈ *normalized-fracts*

by (*simp add*: *normalized-fracts-def unit-factor-mult e-def f-def g-def h-def coprime-normalize-quot dvd-unit-factor-div unit-factor-gcd*)
(*metis coprime-mult-left-iff coprime-mult-right-iff*)
qed (*insert assms*(*3*,*4*), *auto*)

lemma *normalize-quot-mult*:
assumes *snd x* ≠ *0 snd y* ≠ *0*
shows *normalize-quot* (*fst x* ∗ *fst y*, *snd x* ∗ *snd y*) = *normalize-quot*
(*fst* (*normalize-quot x*) ∗ *fst* (*normalize-quot y*),
snd (*normalize-quot x*) ∗ *snd* (*normalize-quot y*))
proof −
from *normalize-quotE′*[*OF assms*(*1*)] **obtain** *d* **where** *d*:
fst x = *fst* (*normalize-quot x*) ∗ *d*
snd x = *snd* (*normalize-quot x*) ∗ *d*
d dvd fst x
d dvd snd x
d ≠ *0* .
from *normalize-quotE′*[*OF assms*(*2*)] **obtain** *e* **where** *e*:
fst y = *fst* (*normalize-quot y*) ∗ *e*
snd y = *snd* (*normalize-quot y*) ∗ *e*
e dvd fst y
e dvd snd y
e ≠ *0* .
show *?thesis* **by** (*simp-all add*: *d e algebra-simps normalize-quot-eq-iff*)
qed

lemma *quot-of-fract-mult*:
quot-of-fract (*x* ∗ *y*) =
(*let* (*a*,*b*) = *quot-of-fract x*; (*c*,*d*) = *quot-of-fract y*;
(*e*,*f*) = *normalize-quot* (*a*,*d*); (*g*,*h*) = *normalize-quot* (*c*,*b*)
in (*e*∗*g*, *f*∗*h*))
by *transfer*
(*simp add*: *split-def Let-def coprime-normalize-quot normalize-quot-mult normalize-quot-mult-coprime*)

lemma *normalize-quot-0* [*simp*]:
normalize-quot (*0*, *x*) = (*0*, *1*) *normalize-quot* (*x*, *0*) = (*0*, *1*)
by (*simp-all add*: *normalize-quot-def*)

lemma *normalize-quot-eq-0-iff* [*simp*]: *fst* (*normalize-quot x*) = *0* ⟷ *fst x* = *0* ∨ *snd x* = *0*
by (*auto simp*: *normalize-quot-def case-prod-unfold Let-def div-mult-unit2 dvd-div-eq-0-iff*)

lemma *fst-quot-of-fract-0-imp*: *fst* (*quot-of-fract x*) = *0* ⟹ *snd* (*quot-of-fract x*) = *1*
by *transfer auto*

lemma *normalize-quot-swap*:
assumes *a* ≠ *0 b* ≠ *0*

defines *a′* ≡ *fst* (*normalize-quot* (*a*, *b*)) **and** *b′* ≡ *snd* (*normalize-quot* (*a*, *b*))
shows *normalize-quot* (*b*, *a*) = (*b′ div unit-factor a′*, *a′ div unit-factor a′*)
proof (*rule normalize-quotI*)
from *normalize-quotE*[*OF assms*(*2*), *of a*] **obtain** *d* **where**
a = *fst* (*normalize-quot* (*a*, *b*)) ∗ *d*
b = *snd* (*normalize-quot* (*a*, *b*)) ∗ *d*
d dvd a d dvd b d ≠ *0* **.**
note *d* = *this* [*folded assms*(*3*,*4*)]
show *b* ∗ (*a′ div unit-factor a′*) = *a* ∗ (*b′ div unit-factor a′*)
using *assms*(*1*,*2*) *d*
by (*simp add*: *div-unit-factor* [*symmetric*] *unit-div-mult-swap mult-ac del*: *div-unit-factor*)
have *coprime a′ b′* **by** (*simp add*: *a′-def b′-def coprime-normalize-quot*)
thus (*b′ div unit-factor a′*, *a′ div unit-factor a′*) ∈ *normalized-fracts*
using *assms*(*1*,*2*) *d*
by (*auto simp add*: *normalized-fracts-def ac-simps dvd-div-unit-iff elim*: *coprime-imp-coprime*)
qed *fact*+

lemma *quot-of-fract-inverse*:
quot-of-fract (*inverse x*) =
(*let* (*a*,*b*) = *quot-of-fract x*; *d* = *unit-factor a*
in if d = *0 then* (*0*, *1*) *else* (*b div d*, *a div d*))
proof (*transfer*, *goal-cases*)
case (*1 x*)
from *normalize-quot-swap*[*of fst x snd x*] **show** *?case*
by (*auto simp*: *Let-def case-prod-unfold*)
qed

lemma *normalize-quot-div-unit-left*:
fixes *x y u*
assumes *is-unit u*
defines *x′* ≡ *fst* (*normalize-quot* (*x*, *y*)) **and** *y′* ≡ *snd* (*normalize-quot* (*x*, *y*))
shows *normalize-quot* (*x div u*, *y*) = (*x′ div u*, *y′*)
proof (*cases y* = *0*)
case *False*
define *v* **where** *v* = *1 div u*
with ‹*is-unit u*› **have** *is-unit v* **and** *u*: ⋀*a*. *a div u* = *a* ∗ *v*
by *simp-all*
from ‹*is-unit v*› **have** *coprime v* = *top*
by (*simp add*: *fun-eq-iff is-unit-left-imp-coprime*)
from *normalize-quotE*[*OF False*, *of x*] **obtain** *d* **where**
x = *fst* (*normalize-quot* (*x*, *y*)) ∗ *d*
y = *snd* (*normalize-quot* (*x*, *y*)) ∗ *d*
d dvd x d dvd y d ≠ *0* **.**
note *d* = *this*[*folded assms*(*2*,*3*)]
from *assms* **have** *coprime x′ y′ unit-factor y′* = *1*
by (*simp-all add*: *coprime-normalize-quot*)
with *d* ‹*coprime v* = *top*› **have** *normalize-quot* (*x* ∗ *v*, *y*) = (*x′* ∗ *v*, *y′*)

by (*auto simp*: *normalized-fracts-def intro*: *normalize-quotI*)
then show *?thesis*
by (*simp add*: *u*)
qed (*simp-all add*: *assms*)

lemma *normalize-quot-div-unit-right*:
fixes *x y u*
assumes *is-unit u*
defines $x' \equiv$ *fst* (*normalize-quot* (*x*, *y*)) **and** $y' \equiv$ *snd* (*normalize-quot* (*x*, *y*))
shows *normalize-quot* (*x*, *y div u*) = ($x' * u$, y')
proof (*cases y* = *0*)
case *False*
from *normalize-quotE*[*OF this*, *of x*]
obtain *d* **where** *d*:
x = *fst* (*normalize-quot* (*x*, *y*)) ∗ *d*
y = *snd* (*normalize-quot* (*x*, *y*)) ∗ *d*
d dvd x d dvd y d ≠ *0* **.**
note *d* = *this*[*folded assms*(*2*,*3*)]
from *assms* **have** *coprime* x' y' *unit-factor* y' = *1* **by** (*simp-all add*: *coprime-normalize-quot*)
with *d* ‹*is-unit u*› **show** *?thesis*
by (*auto simp add*: *normalized-fracts-def is-unit-left-imp-coprime unit-div-eq-0-iff*
intro: *normalize-quotI*)
qed (*simp-all add*: *assms*)

lemma *normalize-quot-normalize-left*:
fixes *x y u*
defines $x' \equiv$ *fst* (*normalize-quot* (*x*, *y*)) **and** $y' \equiv$ *snd* (*normalize-quot* (*x*, *y*))
shows *normalize-quot* (*normalize x*, *y*) = (x' *div unit-factor x*, y')
using *normalize-quot-div-unit-left*[*of unit-factor x x y*]
by (*cases x* = *0*) (*simp-all add*: *assms*)

lemma *normalize-quot-normalize-right*:
fixes *x y u*
defines $x' \equiv$ *fst* (*normalize-quot* (*x*, *y*)) **and** $y' \equiv$ *snd* (*normalize-quot* (*x*, *y*))
shows *normalize-quot* (*x*, *normalize y*) = (x' ∗ *unit-factor y*, y')
using *normalize-quot-div-unit-right*[*of unit-factor y x y*]
by (*cases y* = *0*) (*simp-all add*: *assms*)

lemma *quot-of-fract-0* [*simp*]: *quot-of-fract 0* = (*0*, *1*)
by *transfer auto*

lemma *quot-of-fract-1* [*simp*]: *quot-of-fract 1* = (*1*, *1*)
by *transfer* (*rule normalize-quotI*, *simp-all add*: *normalized-fracts-def*)

lemma *quot-of-fract-divide*:
quot-of-fract (*x* / *y*) = (*if y* = *0 then* (*0*, *1*) *else*
(*let* (*a*,*b*) = *quot-of-fract x*; (*c*,*d*) = *quot-of-fract y*;
(*e*,*f*) = *normalize-quot* (*a*,*c*); (*g*,*h*) = *normalize-quot* (*d*,*b*)
in (*e* ∗ *g*, *f* ∗ *h*))) (**is** - = *?rhs*)

proof (*cases y = 0*)
  case *False*
  hence *A*: *fst* (*quot-of-fract y*) ≠ *0* **by** *transfer auto*
  have *x* / *y* = *x* ∗ *inverse y* **by** (*simp add*: *divide-inverse*)
  also from *False A* **have** *quot-of-fract* . . . = *?rhs*
    by (*simp only*: *quot-of-fract-mult quot-of-fract-inverse*)
      (*simp-all add*: *Let-def case-prod-unfold fst-quot-of-fract-0-imp*
        normalize-quot-div-unit-left normalize-quot-div-unit-right
        normalize-quot-normalize-right normalize-quot-normalize-left)
  finally show *?thesis* **.**
qed *simp-all*

lemma *snd-quot-of-fract-nonzero* [*simp*]: *snd* (*quot-of-fract x*) ≠ *0*
  by *transfer simp*

lemma *Fract-quot-of-fract* [*simp*]: *Fract* (*fst* (*quot-of-fract x*)) (*snd* (*quot-of-fract x*)) = *x*
  by *transfer* (*simp del*: *fractrel-iff*, *subst fractrel-normalize-quot-left*, *simp*)

lemma *snd-quot-of-fract-Fract-whole*:
  assumes *y dvd x*
  shows *snd* (*quot-of-fract* (*Fract x y*)) = *1*
  using *assms* **by** *transfer* (*auto simp*: *normalize-quot-def Let-def gcd-proj2-if-dvd*)

lemma *fst-quot-of-fract-eq-0-iff* [*simp*]: *fst* (*quot-of-fract x*) = *0* ⟷ *x* = *0*
  by *transfer simp*

lemma *coprime-quot-of-fract*:
  coprime (*fst* (*quot-of-fract x*)) (*snd* (*quot-of-fract x*))
  by *transfer* (*simp add*: *coprime-normalize-quot*)

lemma *unit-factor-snd-quot-of-fract*: *unit-factor* (*snd* (*quot-of-fract x*)) = *1*
  using *quot-of-fract-in-normalized-fracts*[*of x*]
  by (*simp add*: *normalized-fracts-def case-prod-unfold*)

lemma *normalize-snd-quot-of-fract*: *normalize* (*snd* (*quot-of-fract x*)) = *snd* (*quot-of-fract x*)
  by (*intro unit-factor-1-imp-normalized unit-factor-snd-quot-of-fract*)

end

10 n-th powers and roots of naturals

theory *Nth-Powers*
  imports *Primes*
begin

10.1 The set of n-th powers

definition *is-nth-power* :: *nat* ⇒ *'a* :: *monoid-mult* ⇒ *bool* **where**
is-nth-power n x ⟷ (∃ *y*. *x* = *y* ^ *n*)

lemma *is-nth-power-nth-power* [*simp*, *intro*]: *is-nth-power n* (*x* ^ *n*)
by (*auto simp add*: *is-nth-power-def*)

lemma *is-nth-powerI* [*intro?*]: *x* = *y* ^ *n* ⟹ *is-nth-power n x*
by (*auto simp*: *is-nth-power-def*)

lemma *is-nth-powerE*: *is-nth-power n x* ⟹ (⋀*y*. *x* = *y* ^ *n* ⟹ *P*) ⟹ *P*
by (*auto simp*: *is-nth-power-def*)

abbreviation *is-square* **where** *is-square* ≡ *is-nth-power 2*

lemma *is-zeroth-power* [*simp*]: *is-nth-power 0 x* ⟷ *x* = *1*
by (*simp add*: *is-nth-power-def*)

lemma *is-first-power* [*simp*]: *is-nth-power 1 x*
by (*simp add*: *is-nth-power-def*)

lemma *is-first-power'* [*simp*]: *is-nth-power* (*Suc 0*) *x*
by (*simp add*: *is-nth-power-def*)

lemma *is-nth-power-0* [*simp*]: *n* > *0* ⟹ *is-nth-power n* (*0* :: *'a* :: *semiring-1*)
by (*auto simp*: *is-nth-power-def power-0-left intro*!: *exI*[*of - 0*])

lemma *is-nth-power-0-iff* [*simp*]: *is-nth-power n* (*0* :: *'a* :: *semiring-1*) ⟷ *n* > *0*
by (*cases n*) *auto*

lemma *is-nth-power-1* [*simp*]: *is-nth-power n 1*
by (*auto simp*: *is-nth-power-def intro*!: *exI*[*of - 1*])

lemma *is-nth-power-Suc-0* [*simp*]: *is-nth-power n* (*Suc 0*)
by (*simp add*: *One-nat-def* [*symmetric*] *del*: *One-nat-def*)

lemma *is-nth-power-conv-multiplicity*:
fixes *x* :: *'a* :: {*factorial-semiring*, *normalization-semidom-multiplicative*}
assumes *n* > *0*
shows *is-nth-power n* (*normalize x*) ⟷ (∀ *p*. *prime p* ⟶ *n dvd multiplicity p x*)
proof (*cases x* = *0*)
case *False*
show *?thesis*
proof (*safe intro*!: *is-nth-powerI elim*!: *is-nth-powerE*)
fix *y p* :: *'a* **assume** ∗: *normalize x* = *y* ^ *n prime p*
with *assms* **and** *False* **have** [*simp*]: *y* ≠ *0* **by** (*auto simp*: *power-0-left*)
have *multiplicity p x* = *multiplicity p* (*y* ^ *n*)

by (*subst* ∗(*1*) [*symmetric*]) *simp*
with *False* **and** ∗ **and** *assms* **show** *n dvd multiplicity p x*
by (*auto simp*: *prime-elem-multiplicity-power-distrib*)
next
assume ∗: ∀ *p*. *prime p* ⟶ *n dvd multiplicity p x*
have *multiplicity p* ((∏ *p*∈*prime-factors x*. *p* ^ (*multiplicity p x div n*)) ^ *n*) = *multiplicity p x* **if** *prime p* **for** *p*
proof −
from *that* **and** ∗ **have** *n dvd multiplicity p x* **by** *blast*
have *multiplicity p x* = *0* **if** *p* ∉ *prime-factors x*
using *that* **and** ‹*prime p*› **by** (*simp add*: *prime-factors-multiplicity*)
with *that* **and** ∗ **and** *assms* **show** *?thesis* **unfolding** *prod-power-distrib power-mult* [*symmetric*]
by (*subst multiplicity-prod-prime-powers*) (*auto simp*: *in-prime-factors-imp-prime elim*: *dvdE*)
qed
with *assms False*
have *normalize x* = *normalize* ((∏ *p*∈*prime-factors x*. *p* ^ (*multiplicity p x div n*)) ^ *n*)
by (*intro multiplicity-eq-imp-eq*) (*auto simp*: *multiplicity-prod-prime-powers*)
thus *normalize x* = *normalize* (∏ *p*∈*prime-factors x*. *p* ^ (*multiplicity p x div n*)) ^ *n*
by (*simp add*: *normalize-power*)
qed
qed (*insert assms*, *auto*)

lemma *is-nth-power-conv-multiplicity-nat*:
assumes *n* > *0*
shows *is-nth-power n* (*x* :: *nat*) ⟷ (∀ *p*. *prime p* ⟶ *n dvd multiplicity p x*)
using *is-nth-power-conv-multiplicity*[*OF assms*, *of x*] **by** *simp*

lemma *is-nth-power-mult*:
assumes *is-nth-power n a is-nth-power n b*
shows *is-nth-power n* (*a* ∗ *b* :: ′*a* :: *comm-monoid-mult*)
proof −
from *assms* **obtain** *a*′ *b*′ **where** *a* = *a*′ ^ *n b* = *b*′ ^ *n* **by** (*auto elim*!: *is-nth-powerE*)
hence *a* ∗ *b* = (*a*′ ∗ *b*′) ^ *n* **by** (*simp add*: *power-mult-distrib*)
thus *?thesis* **by** (*rule is-nth-powerI*)
qed

lemma *is-nth-power-mult-coprime-natD*:
fixes *a b* :: *nat*
assumes *coprime a b is-nth-power n* (*a* ∗ *b*) *a* > *0 b* > *0*
shows *is-nth-power n a is-nth-power n b*
proof −
have *A*: *is-nth-power n a* **if** *coprime a b is-nth-power n* (*a* ∗ *b*) *a* ≠ *0 b* ≠ *0 n* > *0*
for *a b* :: *nat* **unfolding** *is-nth-power-conv-multiplicity-nat*[*OF* ‹*n* > *0*›]

proof *safe*
  fix *p* :: *nat* **assume** *p*: *prime p*
  from ‹*coprime a b*› **have** ¬(*p dvd a* ∧ *p dvd b*)
    using *coprime-common-divisor-nat*[*of a b p*] *p* **by** *auto*
  moreover from *that* **and** *p*
    have *n dvd multiplicity p a* + *multiplicity p b*
   by (*auto simp*: *is-nth-power-conv-multiplicity-nat prime-elem-multiplicity-mult-distrib*)
  ultimately show *n dvd multiplicity p a*
    by (*auto simp*: *not-dvd-imp-multiplicity-0*)
qed
from *A* [*of a b*] *assms* **show** *is-nth-power n a*
  by (*cases n* = *0*) *simp-all*
from *A* [*of b a*] *assms* **show** *is-nth-power n b*
  by (*cases n* = *0*) (*simp-all add*: *ac-simps*)
qed

lemma *is-nth-power-mult-coprime-nat-iff*:
  fixes *a b* :: *nat*
  assumes *coprime a b*
  shows *is-nth-power n* (*a* ∗ *b*) ⟷ *is-nth-power n a* ∧*is-nth-power n b*
  using *assms*
  by (*cases a* = *0*; *cases b* = *0*)
    (*auto intro*: *is-nth-power-mult dest*: *is-nth-power-mult-coprime-natD*[*of a b n*]
        simp del: *One-nat-def*)

lemma *is-nth-power-prime-power-nat-iff*:
  fixes *p* :: *nat* **assumes** *prime p*
  shows *is-nth-power n* (*p* ^ *k*) ⟷ *n dvd k*
  using *assms*
  by (*cases n* > *0*)
   (*auto simp*: *is-nth-power-conv-multiplicity-nat prime-elem-multiplicity-power-distrib*)

lemma *is-nth-power-nth-power'*:
  assumes *n dvd n'*
  shows *is-nth-power n* (*m* ^ *n'*)
proof −
  from *assms* **have** *n'* = *n' div n* ∗ *n* **by** *simp*
  also have *m* ^ ... = (*m* ^ (*n' div n*)) ^ *n* **by** (*simp add*: *power-mult*)
  also have *is-nth-power n* ... **by** *simp*
  finally show *?thesis* **.**
qed

definition *is-nth-power-nat* :: *nat* ⇒ *nat* ⇒ *bool*
  where [*code-abbrev*]: *is-nth-power-nat* = *is-nth-power*

lemma *is-nth-power-nat-code* [*code*]:
  is-nth-power-nat n m =
    (*if n* = *0 then m* = *1*
     else if m = *0 then n* > *0*

else if n = 1 then True
else $(\exists k\in\{1..m\}.\ k\ \hat{}\ n = m))$
by (*auto simp*: *is-nth-power-nat-def is-nth-power-def power-eq-iff-eq-base self-le-power*)

10.2 The n-root of a natural number

definition *nth-root-nat* :: *nat* ⇒ *nat* ⇒ *nat* **where**
nth-root-nat k n = (*if k = 0 then 0 else Max* {*m. m* ^ *k* ≤ *n*})

lemma *zeroth-root-nat* [*simp*]: *nth-root-nat 0 n = 0*
by (*simp add*: *nth-root-nat-def*)

lemma *nth-root-nat-aux1*:
assumes *k* > *0*
shows {*m*::*nat*. *m* ^ *k* ≤ *n*} ⊆ {..*n*}
proof *safe*
fix *m* **assume** *m* ^ *k* ≤ *n*
show *m* ≤ *n*
proof (*cases m = 0*)
case *False*
with *assms* **have** *m* ^ *1* ≤ *m* ^ *k* **by** (*intro power-increasing*) *simp-all*
also note ‹*m* ^ *k* ≤ *n*›
finally show *?thesis* **by** *simp*
qed *simp-all*
qed

lemma *nth-root-nat-aux2*:
assumes *k* > *0*
shows *finite* {*m*::*nat*. *m* ^ *k* ≤ *n*} {*m*::*nat*. *m* ^ *k* ≤ *n*} ≠ {}
proof −
from *assms* **have** {*m*. *m* ^ *k* ≤ *n*} ⊆ {..*n*} **by** (*rule nth-root-nat-aux1*)
moreover have *finite* {..*n*} **by** *simp*
ultimately show *finite* {*m*::*nat*. *m* ^ *k* ≤ *n*} **by** (*rule finite-subset*)
next
from *assms* **show** {*m*::*nat*. *m* ^ *k* ≤ *n*} ≠ {} **by** (*auto intro*!: *exI*[*of* - *0*] *simp*: *power-0-left*)
qed

lemma
assumes *k* > *0*
shows *nth-root-nat-power-le*: *nth-root-nat k n* ^ *k* ≤ *n*
and *nth-root-nat-ge*: *x* ^ *k* ≤ *n* ⟹ *x* ≤ *nth-root-nat k n*
using *Max-in*[*OF nth-root-nat-aux2*[*OF assms*], *of n*]
Max-ge[*OF nth-root-nat-aux2*(*1*)[*OF assms*], *of x n*] *assms*
by (*auto simp*: *nth-root-nat-def*)

lemma *nth-root-nat-less*:
assumes *k* > *0* *x* ^ *k* > *n*
shows *nth-root-nat k n* < *x*

proof −
  from ‹$k > 0$› **have** *nth-root-nat k n* $\hat{}$ $k \leq n$ **by** (*rule nth-root-nat-power-le*)
  also have $n < x$ $\hat{}$ k **by** *fact*
  finally show *?thesis* **by** (*rule power-less-imp-less-base*) *simp-all*
qed

lemma *nth-root-nat-unique*:
  assumes m $\hat{}$ $k \leq n$ $(m + 1)$ $\hat{}$ $k > n$
  shows *nth-root-nat k n* = m
proof (*cases* $k > 0$)
  case *True*
  from *nth-root-nat-less*[*OF* ‹$k > 0$› *assms*(*2*)]
    have *nth-root-nat k n* $\leq m$ **by** *simp*
  moreover from ‹$k > 0$› **and** *assms*(*1*) **have** *nth-root-nat k n* $\geq m$
    by (*intro nth-root-nat-ge*)
  ultimately show *?thesis* **by** (*rule antisym*)
qed (*insert assms*, *auto*)

lemma *nth-root-nat-0* [*simp*]: *nth-root-nat k 0* = *0* **by** (*simp add*: *nth-root-nat-def*)
lemma *nth-root-nat-1* [*simp*]: $k > 0 \Longrightarrow$ *nth-root-nat k 1* = *1*
  by (*rule nth-root-nat-unique*) (*auto simp del*: *One-nat-def*)
lemma *nth-root-nat-Suc-0* [*simp*]: $k > 0 \Longrightarrow$ *nth-root-nat k* (*Suc 0*) = *Suc 0*
  using *nth-root-nat-1* **by** (*simp del*: *nth-root-nat-1*)

lemma *first-root-nat* [*simp*]: *nth-root-nat 1 n* = n
  by (*intro nth-root-nat-unique*) *auto*

lemma *first-root-nat′* [*simp*]: *nth-root-nat* (*Suc 0*) $n = n$
  by (*intro nth-root-nat-unique*) *auto*

lemma *nth-root-nat-code-naive′*:
  nth-root-nat k n = (*if* $k = 0$ *then 0 else Max* (*Set.filter* ($\lambda m.\ m$ $\hat{}$ $k \leq n$) $\{..n\}$))
proof (*cases* $k > 0$)
  case *True*
  hence $\{m.\ m$ $\hat{}$ $k \leq n\} \subseteq \{..n\}$ **by** (*rule nth-root-nat-aux1*)
  hence *Set.filter* ($\lambda m.\ m$ $\hat{}$ $k \leq n$) $\{..n\} = \{m.\ m$ $\hat{}$ $k \leq n\}$
    by (*auto simp*: *Set.filter-def*)
  with *True* **show** *?thesis* **by** (*simp add*: *nth-root-nat-def Set.filter-def*)
qed *simp*

function *nth-root-nat-aux* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* **where**
  nth-root-nat-aux m k acc n =
    (*let acc′* = $(k + 1)$ $\hat{}$ m
     in if $k \geq n \vee acc' > n$ *then k else nth-root-nat-aux m* (*k+1*) *acc′ n*)
  by *auto*
termination by (*relation measure* (λ(-,*k*,-,*n*). $n - k$), *goal-cases*) *auto*

lemma *nth-root-nat-aux-le*:

assumes *k* ^ *m* ≤ *n m* > *0*
shows *nth-root-nat-aux m k* (*k* ^ *m*) *n* ^ *m* ≤ *n*
using *assms*
by (*induction m k k* ^ *m n rule*: *nth-root-nat-aux.induct*) (*auto simp*: *Let-def*)

lemma *nth-root-nat-aux-gt*:
assumes *m* > *0*
shows (*nth-root-nat-aux m k* (*k* ^ *m*) *n* + *1*) ^ *m* > *n*
using *assms*
proof (*induction m k k* ^ *m n rule*: *nth-root-nat-aux.induct*)
case (*1 m k n*)
have *n* < *Suc k* ^ *m* **if** *n* ≤ *k*
proof −
note *that*
also have *k* < *Suc k* ^ *1* **by** *simp*
also from ‹*m* > *0*› **have** . . . ≤ *Suc k* ^ *m* **by** (*intro power-increasing*) *simp-all*
finally show *?thesis* **.**
qed
with *1* **show** *?case* **by** (*auto simp*: *Let-def*)
qed

lemma *nth-root-nat-aux-correct*:
assumes *k* ^ *m* ≤ *n m* > *0*
shows *nth-root-nat-aux m k* (*k* ^ *m*) *n* = *nth-root-nat m n*
by (*rule sym*, *intro nth-root-nat-unique nth-root-nat-aux-le nth-root-nat-aux-gt assms*)

lemma *nth-root-nat-naive-code* [*code*]:
nth-root-nat m n = (*if m* = *0* ∨ *n* = *0 then 0 else if m* = *1* ∨ *n* = *1 then n else nth-root-nat-aux m 1 1 n*)
using *nth-root-nat-aux-correct*[*of 1 m n*] **by** *auto*

lemma *nth-root-nat-nth-power* [*simp*]: *k* > *0* ⟹ *nth-root-nat k* (*n* ^ *k*) = *n*
by (*intro nth-root-nat-unique order.refl power-strict-mono*) *simp-all*

lemma *nth-root-nat-nth-power′*:
assumes *k* > *0 k dvd m*
shows *nth-root-nat k* (*n* ^ *m*) = *n* ^ (*m div k*)
proof −
from *assms* **have** *m* = (*m div k*) ∗ *k* **by** *simp*
also have *n* ^ . . . = (*n* ^ (*m div k*)) ^ *k* **by** (*simp add*: *power-mult*)
also from *assms* **have** *nth-root-nat k* . . . = *n* ^ (*m div k*) **by** *simp*
finally show *?thesis* **.**
qed

lemma *nth-root-nat-mono*:
assumes *m* ≤ *n*
shows *nth-root-nat k m* ≤ *nth-root-nat k n*

proof (*cases k = 0*)
  case *False*
  with *assms* **show** *?thesis* **unfolding** *nth-root-nat-def*
    using *nth-root-nat-aux2*[*of k m*] *nth-root-nat-aux2*[*of k n*]
    by (*auto intro*!: *Max-mono*)
qed *auto*

end

11 Polynomials, fractions and rings

theory *Polynomial-Factorial*
imports
  Complex-Main
  Polynomial
  Normalized-Fraction
begin

11.1 Lifting elements into the field of fractions

definition *to-fract* :: *'a* :: *idom* ⇒ *'a fract*
  where *to-fract x* = *Fract x 1*
  — FIXME: more idiomatic name, abbreviation

lemma *to-fract-0* [*simp*]: *to-fract 0* = *0*
  by (*simp add*: *to-fract-def eq-fract Zero-fract-def*)

lemma *to-fract-1* [*simp*]: *to-fract 1* = *1*
  by (*simp add*: *to-fract-def eq-fract One-fract-def*)

lemma *to-fract-add* [*simp*]: *to-fract* (*x* + *y*) = *to-fract x* + *to-fract y*
  by (*simp add*: *to-fract-def*)

lemma *to-fract-diff* [*simp*]: *to-fract* (*x* − *y*) = *to-fract x* − *to-fract y*
  by (*simp add*: *to-fract-def*)

lemma *to-fract-uminus* [*simp*]: *to-fract* (−*x*) = −*to-fract x*
  by (*simp add*: *to-fract-def*)

lemma *to-fract-mult* [*simp*]: *to-fract* (*x* ∗ *y*) = *to-fract x* ∗ *to-fract y*
  by (*simp add*: *to-fract-def*)

lemma *to-fract-eq-iff* [*simp*]: *to-fract x* = *to-fract y* ⟷ *x* = *y*
  by (*simp add*: *to-fract-def eq-fract*)

lemma *to-fract-eq-0-iff* [*simp*]: *to-fract x* = *0* ⟷ *x* = *0*
  by (*simp add*: *to-fract-def Zero-fract-def eq-fract*)

lemma *to-fract-quot-of-fract*:

assumes *snd* (*quot-of-fract x*) = *1*
shows *to-fract* (*fst* (*quot-of-fract x*)) = *x*
proof −
have *x* = *Fract* (*fst* (*quot-of-fract x*)) (*snd* (*quot-of-fract x*)) **by** *simp*
also note *assms*
finally show *?thesis* **by** (*simp add*: *to-fract-def*)
qed

lemma *Fract-conv-to-fract*: *Fract a b* = *to-fract a* / *to-fract b*
by (*simp add*: *to-fract-def*)

lemma *quot-of-fract-to-fract* [*simp*]: *quot-of-fract* (*to-fract x*) = (*x*, *1*)
unfolding *to-fract-def* **by** *transfer* (*simp add*: *normalize-quot-def*)

lemma *snd-quot-of-fract-to-fract* [*simp*]: *snd* (*quot-of-fract* (*to-fract x*)) = *1*
unfolding *to-fract-def* **by** (*rule snd-quot-of-fract-Fract-whole*) *simp-all*

11.2 Lifting polynomial coefficients to the field of fractions

abbreviation (*input*) *fract-poly* :: ‹′*a*::*idom poly* ⇒ ′*a fract poly*›
where *fract-poly* ≡ *map-poly to-fract*

abbreviation (*input*) *unfract-poly* :: ‹′*a*::{*ring-gcd*,*semiring-gcd-mult-normalize*,*idom-divide*} *fract poly* ⇒ ′*a poly*›
where *unfract-poly* ≡ *map-poly* (*fst* ∘ *quot-of-fract*)

lemma *fract-poly-smult* [*simp*]: *fract-poly* (*smult c p*) = *smult* (*to-fract c*) (*fract-poly p*)
by (*simp add*: *smult-conv-map-poly map-poly-map-poly o-def*)

lemma *fract-poly-0* [*simp*]: *fract-poly 0* = *0*
by (*simp add*: *poly-eqI coeff-map-poly*)

lemma *fract-poly-1* [*simp*]: *fract-poly 1* = *1*
by (*simp add*: *map-poly-pCons*)

lemma *fract-poly-add* [*simp*]:
fract-poly (*p* + *q*) = *fract-poly p* + *fract-poly q*
by (*intro poly-eqI*) (*simp-all add*: *coeff-map-poly*)

lemma *fract-poly-diff* [*simp*]:
fract-poly (*p* − *q*) = *fract-poly p* − *fract-poly q*
by (*intro poly-eqI*) (*simp-all add*: *coeff-map-poly*)

lemma *to-fract-sum* [*simp*]: *to-fract* (*sum f A*) = *sum* (λ*x*. *to-fract* (*f x*)) *A*
by (*cases finite A*, *induction A rule*: *finite-induct*) *simp-all*

lemma *fract-poly-mult* [*simp*]:
fract-poly (*p* ∗ *q*) = *fract-poly p* ∗ *fract-poly q*

by (*intro poly-eqI*) (*simp-all add*: *coeff-map-poly coeff-mult*)

lemma *fract-poly-eq-iff* [*simp*]: *fract-poly p* = *fract-poly q* ⟷ *p* = *q*
by (*auto simp*: *poly-eq-iff coeff-map-poly*)

lemma *fract-poly-eq-0-iff* [*simp*]: *fract-poly p* = *0* ⟷ *p* = *0*
using *fract-poly-eq-iff*[*of p 0*] **by** (*simp del*: *fract-poly-eq-iff*)

lemma *fract-poly-dvd*: *p dvd q* ⟹ *fract-poly p dvd fract-poly q*
by *auto*

lemma *prod-mset-fract-poly*:
(∏ *x*∈#*A*. *map-poly to-fract* (*f x*)) = *fract-poly* (*prod-mset* (*image-mset f A*))
by (*induct A*) (*simp-all add*: *ac-simps*)

lemma *is-unit-fract-poly-iff*:
p dvd 1 ⟷ *fract-poly p dvd 1* ∧ *content p* = *1*
proof *safe*
assume *A*: *p dvd 1*
with *fract-poly-dvd* [*of p 1*] **show** *is-unit* (*fract-poly p*)
by *simp*
from *A* **show** *content p* = *1*
by (*auto simp*: *is-unit-poly-iff normalize-1-iff*)
next
assume *A*: *fract-poly p dvd 1* **and** *B*: *content p* = *1*
from *A* **obtain** *c* **where** *c*: *fract-poly p* = [:*c*:] **by** (*auto simp*: *is-unit-poly-iff*)
{
fix *n* :: *nat* **assume** *n* > *0*
have *to-fract* (*coeff p n*) = *coeff* (*fract-poly p*) *n* **by** (*simp add*: *coeff-map-poly*)
also note *c*
also from ‹*n* > *0*› **have** *coeff* [:*c*:] *n* = *0* **by** (*simp add*: *coeff-pCons split*: *nat.splits*)
finally have *coeff p n* = *0* **by** *simp*
}
hence *degree p* ≤ *0* **by** (*intro degree-le*) *simp-all*
with *B* **show** *p dvd 1* **by** (*auto simp*: *is-unit-poly-iff normalize-1-iff elim*!: *degree-eq-zeroE*)
qed

lemma *fract-poly-is-unit*: *p dvd 1* ⟹ *fract-poly p dvd 1*
using *fract-poly-dvd*[*of p 1*] **by** *simp*

lemma *fract-poly-smult-eqE*:
fixes *c* :: *'a* :: {*idom-divide*,*ring-gcd*,*semiring-gcd-mult-normalize*} *fract*
assumes *fract-poly p* = *smult c* (*fract-poly q*)
obtains *a b*
where *c* = *to-fract b* / *to-fract a smult a p* = *smult b q coprime a b normalize a* = *a*
proof −

define *a b* **where** *a* = *fst* (*quot-of-fract c*) **and** *b* = *snd* (*quot-of-fract c*)
have *smult* (*to-fract a*) (*fract-poly q*) = *smult* (*to-fract b*) (*fract-poly p*)
by (*subst smult-eq-iff*) (*simp-all add*: *a-def b-def Fract-conv-to-fract* [*symmetric*] *assms*)
hence *fract-poly* (*smult a q*) = *fract-poly* (*smult b p*) **by** (*simp del*: *fract-poly-eq-iff*)
hence *smult b p* = *smult a q* **by** (*simp only*: *fract-poly-eq-iff*)
moreover have *c* = *to-fract a* / *to-fract b coprime b a normalize b* = *b*
by (*simp-all add*: *a-def b-def coprime-quot-of-fract* [*of c*] *ac-simps*
normalize-snd-quot-of-fract Fract-conv-to-fract [*symmetric*])
ultimately show *?thesis* **by** (*intro that*[*of a b*])
qed

11.3 Fractional content

abbreviation (*input*) *Lcm-coeff-denoms*
:: *'a* :: {*semiring-Gcd*,*idom-divide*,*ring-gcd*,*semiring-gcd-mult-normalize*} *fract poly* ⇒ *'a*
where *Lcm-coeff-denoms p* ≡ *Lcm* (*snd* ‘ *quot-of-fract* ‘ *set* (*coeffs p*))

definition *fract-content* ::
'a :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *fract poly* ⇒ *'a fract* **where**
fract-content p =
(*let d* = *Lcm-coeff-denoms p in Fract* (*content* (*unfract-poly* (*smult* (*to-fract d*) *p*))) *d*)

definition *primitive-part-fract* ::
'a :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *fract poly* ⇒ *'a poly* **where**
primitive-part-fract p =
primitive-part (*unfract-poly* (*smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*))

lemma *primitive-part-fract-0* [*simp*]: *primitive-part-fract 0* = *0*
by (*simp add*: *primitive-part-fract-def*)

lemma *fract-content-eq-0-iff* [*simp*]:
fract-content p = *0* ⟷ *p* = *0*
unfolding *fract-content-def Let-def Zero-fract-def*
by (*subst eq-fract*) (*auto simp*: *Lcm-0-iff map-poly-eq-0-iff*)

lemma *content-primitive-part-fract* [*simp*]:
fixes *p* :: *'a* :: {*semiring-gcd-mult-normalize*,
factorial-semiring, *ring-gcd*, *semiring-Gcd*,*idom-divide*} *fract poly*
shows *p* ≠ *0* ⟹ *content* (*primitive-part-fract p*) = *1*
unfolding *primitive-part-fract-def*
by (*rule content-primitive-part*)
(*auto simp*: *primitive-part-fract-def map-poly-eq-0-iff Lcm-0-iff*)

lemma *content-times-primitive-part-fract*:

smult (*fract-content p*) (*fract-poly* (*primitive-part-fract p*)) = *p*
proof −
define *p′* **where** *p′* = *unfract-poly* (*smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*)
have *fract-poly p′* =
map-poly (*to-fract* ∘ *fst* ∘ *quot-of-fract*) (*smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*)
unfolding *primitive-part-fract-def p′-def*
by (*subst map-poly-map-poly*) (*simp-all add*: *o-assoc*)
also have ... = *smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*
proof (*intro map-poly-idI*, *unfold o-apply*)
fix *c* **assume** *c* ∈ *set* (*coeffs* (*smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*))
then obtain *c′* **where** *c*: *c′* ∈ *set* (*coeffs p*) *c* = *to-fract* (*Lcm-coeff-denoms p*) ∗ *c′*
by (*auto simp add*: *Lcm-0-iff coeffs-smult split*: *if-splits*)
note *c*(*2*)
also have *c′* = *Fract* (*fst* (*quot-of-fract c′*)) (*snd* (*quot-of-fract c′*))
by *simp*
also have *to-fract* (*Lcm-coeff-denoms p*) ∗ ... =
Fract (*Lcm-coeff-denoms p* ∗ *fst* (*quot-of-fract c′*)) (*snd* (*quot-of-fract c′*))
unfolding *to-fract-def* **by** (*subst mult-fract*) *simp-all*
also have *snd* (*quot-of-fract* ...) = *1*
by (*intro snd-quot-of-fract-Fract-whole dvd-mult2 dvd-Lcm*) (*insert c*(*1*), *auto*)
finally show *to-fract* (*fst* (*quot-of-fract c*)) = *c*
by (*rule to-fract-quot-of-fract*)
qed
also have *p′* = *smult* (*content p′*) (*primitive-part p′*)
by (*rule content-times-primitive-part* [*symmetric*])
also have *primitive-part p′* = *primitive-part-fract p*
by (*simp add*: *primitive-part-fract-def p′-def*)
also have *fract-poly* (*smult* (*content p′*) (*primitive-part-fract p*)) =
smult (*to-fract* (*content p′*)) (*fract-poly* (*primitive-part-fract p*)) **by** *simp*
finally have *smult* (*to-fract* (*content p′*)) (*fract-poly* (*primitive-part-fract p*)) =
smult (*to-fract* (*Lcm-coeff-denoms p*)) *p* **.**
thus *?thesis*
by (*subst* (*asm*) *smult-eq-iff*)
(*auto simp add*: *Let-def p′-def Fract-conv-to-fract field-simps Lcm-0-iff fract-content-def*)
qed

lemma *fract-content-fract-poly* [*simp*]: *fract-content* (*fract-poly p*) = *to-fract* (*content p*)
proof −
have *Lcm-coeff-denoms* (*fract-poly p*) = *1*
by (*auto simp*: *set-coeffs-map-poly*)
hence *fract-content* (*fract-poly p*) =
to-fract (*content* (*map-poly* (*fst* ∘ *quot-of-fract* ∘ *to-fract*) *p*))
by (*simp add*: *fract-content-def to-fract-def fract-collapse map-poly-map-poly*

del: *Lcm-1-iff*)
also have *map-poly* (*fst* ∘ *quot-of-fract* ∘ *to-fract*) *p* = *p*
by (*intro map-poly-idI*) *simp-all*
finally show *?thesis* **.**
qed

lemma *content-decompose-fract*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,
semiring-gcd-mult-normalize} *fract poly*
obtains *c p'* **where** *p* = *smult c* (*map-poly to-fract p'*) *content p'* = *1*
proof (*cases p* = *0*)
case *True*
hence *p* = *smult 0* (*map-poly to-fract 1*) *content 1* = *1* **by** *simp-all*
thus *?thesis* **..**
next
case *False*
thus *?thesis*
by (*rule that*[*OF content-times-primitive-part-fract* [*symmetric*] *content-primitive-part-fract*])
qed

lemma *fract-poly-dvdD*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,
semiring-gcd-mult-normalize} *poly*
assumes *fract-poly p dvd fract-poly q content p* = *1*
shows *p dvd q*
proof −
from *assms*(*1*) **obtain** *r* **where** *r*: *fract-poly q* = *fract-poly p* ∗ *r* **by** (*erule dvdE*)
from *content-decompose-fract*[*of r*]
obtain *c r'* **where** *r'*: *r* = *smult c* (*map-poly to-fract r'*) *content r'* = *1* **.**
from *r r'* **have** *eq*: *fract-poly q* = *smult c* (*fract-poly* (*p* ∗ *r'*)) **by** *simp*
from *fract-poly-smult-eqE*[*OF this*] **obtain** *a b*
where *ab*:
c = *to-fract b* / *to-fract a*
smult a q = *smult b* (*p* ∗ *r'*)
coprime a b
normalize a = *a* **.**
have *content* (*smult a q*) = *content* (*smult b* (*p* ∗ *r'*)) **by** (*simp only*: *ab*(*2*))
hence *eq'*: *normalize b* = *a* ∗ *content q* **by** (*simp add*: *assms content-mult r' ab*(*4*))
have *1* = *gcd a* (*normalize b*) **by** (*simp add*: *ab*)
also note *eq'*
also have *gcd a* (*a* ∗ *content q*) = *a* **by** (*simp add*: *gcd-proj1-if-dvd ab*(*4*))
finally have [*simp*]: *a* = *1* **by** *simp*
from *eq ab* **have** *q* = *p* ∗ ([:*b*:] ∗ *r'*) **by** *simp*
thus *?thesis* **by** (*rule dvdI*)
qed

11.4 Polynomials over a field are a Euclidean ring

context
begin

interpretation *field-poly*:
  normalization-euclidean-semiring-multiplicative **where** *zero = 0 :: 'a :: field poly*
    and *one = 1* **and** *plus = plus* **and** *minus = minus*
    and *times = times*
    and *normalize = λp. smult (inverse (lead-coeff p)) p*
    and *unit-factor = λp. [:lead-coeff p:]*
    and *euclidean-size = λp. if p = 0 then 0 else 2 ^ degree p*
    and *divide = divide* **and** *modulo = modulo*
  rewrites *dvd.dvd (times :: 'a poly ⇒ -) = Rings.dvd*
    and *comm-monoid-mult.prod-mset times 1 = prod-mset*
    and *comm-semiring-1.irreducible times 1 0 = irreducible*
    and *comm-semiring-1.prime-elem times 1 0 = prime-elem*
proof −
  show *dvd.dvd (times :: 'a poly ⇒ -) = Rings.dvd*
    by (*simp add*: *dvd-dict*)
  show *comm-monoid-mult.prod-mset times 1 = prod-mset*
    by (*simp add*: *prod-mset-dict*)
  show *comm-semiring-1.irreducible times 1 0 = irreducible*
    by (*simp add*: *irreducible-dict*)
  show *comm-semiring-1.prime-elem times 1 0 = prime-elem*
    by (*simp add*: *prime-elem-dict*)
  show *class.normalization-euclidean-semiring-multiplicative divide plus minus (0 :: 'a poly) times 1*
    modulo (λp. if p = 0 then 0 else 2 ^ degree p)
    (λp. [:lead-coeff p:]) (λp. smult (inverse (lead-coeff p)) p)
  proof (*standard*, *fold dvd-dict*)
    fix *p :: 'a poly*
    show *[:lead-coeff p:] ∗ smult (inverse (lead-coeff p)) p = p*
      by (*cases p = 0*) *simp-all*
  next
    fix *p :: 'a poly* **assume** *is-unit p*
    then show *[:lead-coeff p:] = p*
      by (*elim is-unit-polyE*) (*auto simp*: *monom-0 one-poly-def field-simps*)
  next
    fix *p :: 'a poly* **assume** *p ≠ 0*
    then show *is-unit [:lead-coeff p:]*
      by (*simp add*: *is-unit-pCons-iff*)
  next
    fix *a b :: 'a poly* **assume** *is-unit a*
    thus *[:lead-coeff (a ∗ b):] = a ∗ [:lead-coeff b:]*
      by (*auto elim*!: *is-unit-polyE*)
  qed (*auto simp*: *lead-coeff-mult Rings.div-mult-mod-eq intro*!: *degree-mod-less' degree-mult-right-le*)
qed

lemma *field-poly-irreducible-imp-prime*:
prime-elem p **if** *irreducible p* **for** *p* :: *'a* :: *field poly*
using *that* **by** (*fact field-poly.irreducible-imp-prime-elem*)

lemma *field-poly-prod-mset-prime-factorization*:
prod-mset (*field-poly.prime-factorization p*) = *smult* (*inverse* (*lead-coeff p*)) *p*
if $p \neq 0$ **for** *p* :: *'a* :: *field poly*
using *that* **by** (*fact field-poly.prod-mset-prime-factorization*)

lemma *field-poly-in-prime-factorization-imp-prime*:
prime-elem p **if** *p* ∈# *field-poly.prime-factorization x*
for *p* :: *'a* :: *field poly*
by (*rule field-poly.prime-imp-prime-elem*, *rule field-poly.in-prime-factors-imp-prime*)
(*fact that*)

11.5 Primality and irreducibility in polynomial rings

lemma *nonconst-poly-irreducible-iff*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *poly*
assumes *degree* $p \neq 0$
shows *irreducible p* ⟷ *irreducible* (*fract-poly p*) ∧ *content p* = *1*
proof *safe*
assume *p*: *irreducible p*

from *content-decompose*[*of p*] **obtain** *p'* **where** *p'*: *p* = *smult* (*content p*) *p'* *content p'* = *1* **.**
hence *p* = [:*content p*:] ∗ *p'* **by** *simp*
from *p this* **have** [:*content p*:] *dvd 1* ∨ *p' dvd 1* **by** (*rule irreducibleD*)
moreover have ¬*p' dvd 1*
proof
assume *p' dvd 1*
hence *degree p* = *0* **by** (*subst p'*) (*auto simp*: *is-unit-poly-iff*)
with *assms* **show** *False* **by** *contradiction*
qed
ultimately show [*simp*]: *content p* = *1* **by** (*simp add*: *is-unit-const-poly-iff*)

show *irreducible* (*map-poly to-fract p*)
proof (*rule irreducibleI*)
have *fract-poly p* = *0* ⟷ *p* = *0* **by** (*intro map-poly-eq-0-iff*) *auto*
with *assms* **show** *map-poly to-fract p* ≠ *0* **by** *auto*
next
show ¬*is-unit* (*fract-poly p*)
proof
assume *is-unit* (*map-poly to-fract p*)
hence *degree* (*map-poly to-fract p*) = *0*
by (*auto simp*: *is-unit-poly-iff*)
hence *degree p* = *0* **by** (*simp add*: *degree-map-poly*)
with *assms* **show** *False* **by** *contradiction*

```
  qed
 next
  fix q r assume qr: fract-poly p = q * r
  from content-decompose-fract[of q]
  obtain cg q' where q: q = smult cg (map-poly to-fract q') content q' = 1 .
  from content-decompose-fract[of r]
  obtain cr r' where r: r = smult cr (map-poly to-fract r') content r' = 1 .
  from qr q r p have nz: cg ≠ 0 cr ≠ 0 by auto
  from qr have eq: fract-poly p = smult (cr * cg) (fract-poly (q' * r'))
    by (simp add: q r)
  from fract-poly-smult-eqE[OF this] obtain a b
    where ab: cr * cg = to-fract b / to-fract a
      smult a p = smult b (q' * r') coprime a b normalize a = a .
  hence content (smult a p) = content (smult b (q' * r')) by (simp only:)
  with ab(4) have a: a = normalize b by (simp add: content-mult q r)
  then have normalize b = gcd a b
    by simp
  with ‹coprime a b› have normalize b = 1
    by simp
  then have a = 1 is-unit b
    by (simp-all add: a normalize-1-iff)

  note eq
  also from ab(1) ‹a = 1› have cr * cg = to-fract b by simp
   also have smult ... (fract-poly (q' * r')) = fract-poly (smult b (q' * r')) by
simp
  finally have p = ([:b:] * q') * r' by (simp del: fract-poly-smult)
  from p and this have ([:b:] * q') dvd 1 ∨ r' dvd 1 by (rule irreducibleD)
 hence q' dvd 1 ∨ r' dvd 1 by (auto dest: dvd-mult-right simp del: mult-pCons-left)
  hence fract-poly q' dvd 1 ∨ fract-poly r' dvd 1 by (auto simp: fract-poly-is-unit)
  with q r show is-unit q ∨ is-unit r
    by (auto simp add: is-unit-smult-iff dvd-field-iff nz)
 qed

next

 assume irred: irreducible (fract-poly p) and primitive: content p = 1
 show irreducible p
 proof (rule irreducibleI)
  from irred show p ≠ 0 by auto
 next
  from irred show ¬p dvd 1
    by (auto simp: irreducible-def dest: fract-poly-is-unit)
 next
  fix q r assume qr: p = q * r
  hence fract-poly p = fract-poly q * fract-poly r by simp
  from irred and this have fract-poly q dvd 1 ∨ fract-poly r dvd 1
    by (rule irreducibleD)
  with primitive qr show q dvd 1 ∨ r dvd 1
```

by (*auto simp*: *content-prod-eq-1-iff is-unit-fract-poly-iff*)
qed
qed

lemma *irreducible-imp-prime-poly*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *poly*
assumes *irreducible p*
shows *prime-elem p*
proof (*cases degree p* = *0*)
case *True*
with *assms* **show** *?thesis*
by (*auto simp*: *prime-elem-const-poly-iff irreducible-const-poly-iff*
intro!: *irreducible-imp-prime-elem elim*!: *degree-eq-zeroE*)
next
case *False*
from *assms False* **have** *irred*: *irreducible* (*fract-poly p*) **and** *primitive*: *content p* = *1*
by (*simp-all add*: *nonconst-poly-irreducible-iff*)
from *irred* **have** *prime*: *prime-elem* (*fract-poly p*) **by** (*rule field-poly-irreducible-imp-prime*)
show *?thesis*
proof (*rule prime-elemI*)
fix *q r* **assume** *p dvd q* ∗ *r*
hence *fract-poly p dvd fract-poly* (*q* ∗ *r*) **by** (*rule fract-poly-dvd*)
hence *fract-poly p dvd fract-poly q* ∗ *fract-poly r* **by** *simp*
from *prime* **and** *this* **have** *fract-poly p dvd fract-poly q* ∨ *fract-poly p dvd fract-poly r*
by (*rule prime-elem-dvd-multD*)
with *primitive* **show** *p dvd q* ∨ *p dvd r* **by** (*auto dest*: *fract-poly-dvdD*)
qed (*insert assms*, *auto simp*: *irreducible-def*)
qed

lemma *degree-primitive-part-fract* [*simp*]:
degree (*primitive-part-fract p*) = *degree p*
proof −
have *p* = *smult* (*fract-content p*) (*fract-poly* (*primitive-part-fract p*))
by (*simp add*: *content-times-primitive-part-fract*)
also have *degree* . . . = *degree* (*primitive-part-fract p*)
by (*auto simp*: *degree-map-poly*)
finally show *?thesis* **..**
qed

lemma *irreducible-primitive-part-fract*:
fixes *p* :: *'a* :: {*idom-divide*, *ring-gcd*, *factorial-semiring*, *semiring-Gcd*,*semiring-gcd-mult-normalize*} *fract poly*
assumes *irreducible p*
shows *irreducible* (*primitive-part-fract p*)
proof −
from *assms* **have** *deg*: *degree* (*primitive-part-fract p*) ≠ *0*

by (*intro notI*)
(*auto elim*!: *degree-eq-zeroE simp*: *irreducible-def is-unit-poly-iff dvd-field-iff*)
hence [*simp*]: *p* ≠ *0* **by** *auto*

note ‹*irreducible p*›
also have *p* = [:*fract-content p*:] ∗ *fract-poly* (*primitive-part-fract p*)
by (*simp add*: *content-times-primitive-part-fract*)
also have *irreducible* ... ⟷ *irreducible* (*fract-poly* (*primitive-part-fract p*))
by (*intro irreducible-mult-unit-left*) (*simp-all add*: *is-unit-poly-iff dvd-field-iff*)
finally show *?thesis* **using** *deg*
by (*simp add*: *nonconst-poly-irreducible-iff*)
qed

lemma *prime-elem-primitive-part-fract*:
fixes *p* :: ′*a* :: {*idom-divide*, *ring-gcd*, *factorial-semiring*, *semiring-Gcd*,*semiring-gcd-mult-normalize*}
fract poly
shows *irreducible p* ⟹ *prime-elem* (*primitive-part-fract p*)
by (*intro irreducible-imp-prime-poly irreducible-primitive-part-fract*)

lemma *irreducible-linear-field-poly*:
fixes *a b* :: ′*a*::*field*
assumes *b* ≠ *0*
shows *irreducible* [:*a*,*b*:]
proof (*rule irreducibleI*)
fix *p q* **assume** *pq*: [:*a*,*b*:] = *p* ∗ *q*
also from *pq assms* **have** *degree* ... = *degree p* + *degree q*
by (*intro degree-mult-eq*) *auto*
finally have *degree p* = *0* ∨ *degree q* = *0* **using** *assms* **by** *auto*
with *assms pq* **show** *is-unit p* ∨ *is-unit q*
by (*auto simp*: *is-unit-const-poly-iff dvd-field-iff elim*!: *degree-eq-zeroE*)
qed (*insert assms*, *auto simp*: *is-unit-poly-iff*)

lemma *prime-elem-linear-field-poly*:
(*b* :: ′*a* :: *field*) ≠ *0* ⟹ *prime-elem* [:*a*,*b*:]
by (*rule field-poly-irreducible-imp-prime*, *rule irreducible-linear-field-poly*)

lemma *irreducible-linear-poly*:
fixes *a b* :: ′*a*::{*idom-divide*,*ring-gcd*,*factorial-semiring*,*semiring-Gcd*,*semiring-gcd-mult-normalize*}
shows *b* ≠ *0* ⟹ *coprime a b* ⟹ *irreducible* [:*a*,*b*:]
by (*auto intro*!: *irreducible-linear-field-poly*
simp: *nonconst-poly-irreducible-iff content-def map-poly-pCons*)

lemma *prime-elem-linear-poly*:
fixes *a b* :: ′*a*::{*idom-divide*,*ring-gcd*,*factorial-semiring*,*semiring-Gcd*,*semiring-gcd-mult-normalize*}
shows *b* ≠ *0* ⟹ *coprime a b* ⟹ *prime-elem* [:*a*,*b*:]
by (*rule irreducible-imp-prime-poly*, *rule irreducible-linear-poly*)

11.6 Prime factorisation of polynomials

lemma *poly-prime-factorization-exists-content-1*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *poly*
assumes $p \neq 0$ *content* $p = 1$
shows $\exists A. (\forall p.\ p \in\# A \longrightarrow$ *prime-elem* $p) \wedge$ *prod-mset* $A =$ *normalize* p
proof −
let *?P* = *field-poly.prime-factorization* (*fract-poly p*)
define *c* **where** *c* = *prod-mset* (*image-mset fract-content ?P*)
define *c'* **where** *c'* = *c* ∗ *to-fract* (*lead-coeff p*)
define *e* **where** *e* = *prod-mset* (*image-mset primitive-part-fract ?P*)
define *A* **where** *A* = *image-mset* (*normalize* ∘ *primitive-part-fract*) *?P*
have *content e* = (∏ *x*∈#*field-poly.prime-factorization* (*map-poly to-fract p*).
content (*primitive-part-fract x*))
by (*simp add*: *e-def content-prod-mset multiset.map-comp o-def*)
also have *image-mset* (λ*x*. *content* (*primitive-part-fract x*)) *?P* = *image-mset* (λ-. *1*) *?P*
by (*intro image-mset-cong content-primitive-part-fract*) *auto*
finally have *content-e*: *content e* = *1*
by *simp*

from ‹*p* ≠ *0*› **have** *fract-poly p* = [:*lead-coeff* (*fract-poly p*):] ∗
smult (*inverse* (*lead-coeff* (*fract-poly p*))) (*fract-poly p*)
by *simp*
also have [:*lead-coeff* (*fract-poly p*):] = [:*to-fract* (*lead-coeff p*):]
by (*simp add*: *monom-0 degree-map-poly coeff-map-poly*)
also from *assms* **have** *smult* (*inverse* (*lead-coeff* (*fract-poly p*))) (*fract-poly p*) = *prod-mset ?P*
by (*subst field-poly-prod-mset-prime-factorization*) *simp-all*
also have . . . = *prod-mset* (*image-mset id ?P*) **by** *simp*
also have *image-mset id ?P* =
image-mset (λ*x*. [:*fract-content x*:] ∗ *fract-poly* (*primitive-part-fract x*)) *?P*
by (*intro image-mset-cong*) (*auto simp*: *content-times-primitive-part-fract*)
also have *prod-mset* . . . = *smult c* (*fract-poly e*)
by (*subst prod-mset.distrib*) (*simp-all add*: *prod-mset-fract-poly prod-mset-const-poly c-def e-def*)
also have [:*to-fract* (*lead-coeff p*):] ∗ . . . = *smult c'* (*fract-poly e*)
by (*simp add*: *c'-def*)
finally have *eq*: *fract-poly p* = *smult c'* (*fract-poly e*) **.**
also obtain *b* **where** *b*: *c'* = *to-fract b is-unit b*
proof −
from *fract-poly-smult-eqE*[*OF eq*]
obtain *a b* **where** *ab*:
c' = *to-fract b* / *to-fract a*
smult a p = *smult b e*
coprime a b
normalize a = *a* **.**
from *ab*(*2*) **have** *content* (*smult a p*) = *content* (*smult b e*) **by** (*simp only*:)

with *assms content-e* **have** *a* = *normalize b* **by** (*simp add*: *ab*(*4*))
with *ab* **have** *ab'*: *a* = *1 is-unit b*
by (*simp-all add*: *normalize-1-iff*)
with *ab ab'* **have** *c'* = *to-fract b* **by** *auto*
from *this* **and** ‹*is-unit b*› **show** *?thesis* **by** (*rule that*)
qed
hence *smult c'* (*fract-poly e*) = *fract-poly* (*smult b e*) **by** *simp*
finally have *p* = *smult b e* **by** (*simp only*: *fract-poly-eq-iff*)
hence *p* = [:*b*:] ∗ *e* **by** *simp*
with *b* **have** *normalize p* = *normalize e*
by (*simp only*: *normalize-mult*) (*simp add*: *is-unit-normalize is-unit-poly-iff*)
also have *normalize e* = *prod-mset A*
by (*simp add*: *multiset.map-comp e-def A-def normalize-prod-mset*)
finally have *prod-mset A* = *normalize p* **..**

have *prime-elem p* **if** *p* ∈# *A* **for** *p*
using *that* **by** (*auto simp*: *A-def prime-elem-primitive-part-fract prime-elem-imp-irreducible*

dest!: *field-poly-in-prime-factorization-imp-prime*)
from *this* **and** ‹*prod-mset A* = *normalize p*› **show** *?thesis*
by (*intro exI*[*of - A*]) *blast*
qed

lemma *poly-prime-factorization-exists*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *poly*
assumes *p* ≠ *0*
shows ∃ *A*. (∀ *p*. *p* ∈# *A* ⟶ *prime-elem p*) ∧ *normalize* (*prod-mset A*) = *normalize p*
proof −
define *B* **where** *B* = *image-mset* (λ*x*. [:*x*:]) (*prime-factorization* (*content p*))
have ∃ *A*. (∀ *p*. *p* ∈# *A* ⟶ *prime-elem p*) ∧ *prod-mset A* = *normalize* (*primitive-part p*)
by (*rule poly-prime-factorization-exists-content-1*) (*insert assms*, *simp-all*)
then obtain *A* **where** *A*: ∀ *p*. *p* ∈# *A* ⟶ *prime-elem p* $\prod_{\#}$ *A* = *normalize* (*primitive-part p*)
by *blast*
have *normalize* (*prod-mset* (*A* + *B*)) = *normalize* (*prod-mset A* ∗ *normalize* (*prod-mset B*))
by *simp*
also from *assms* **have** *normalize* (*prod-mset B*) = *normalize* [:*content p*:]
by (*simp add*: *prod-mset-const-poly normalize-const-poly prod-mset-prime-factorization-weak B-def*)
also have *prod-mset A* = *normalize* (*primitive-part p*)
using *A* **by** *simp*
finally have *normalize* (*prod-mset* (*A* + *B*)) = *normalize* (*primitive-part p* ∗ [:*content p*:])
by *simp*
moreover have ∀ *p*. *p* ∈# *B* ⟶ *prime-elem p*

by (*auto simp*: *B-def intro*!: *lift-prime-elem-poly dest*: *in-prime-factors-imp-prime*)
ultimately show *?thesis* **using** *A* **by** (*intro exI*[*of* - *A* + *B*]) (*auto*)
qed

end

11.7 Typeclass instances

instance *poly* :: ({*factorial-ring-gcd*,*semiring-gcd-mult-normalize*}) *factorial-semiring*
by *standard* (*rule poly-prime-factorization-exists*)

instantiation *poly* :: ({*factorial-ring-gcd*, *semiring-gcd-mult-normalize*}) *factorial-ring-gcd*
begin

definition *gcd-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly* **where**
[*code del*]: *gcd-poly* = *gcd-factorial*

definition *lcm-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly* **where**
[*code del*]: *lcm-poly* = *lcm-factorial*

definition *Gcd-poly* :: *'a poly set* ⇒ *'a poly* **where**
[*code del*]: *Gcd-poly* = *Gcd-factorial*

definition *Lcm-poly* :: *'a poly set* ⇒ *'a poly* **where**
[*code del*]: *Lcm-poly* = *Lcm-factorial*

instance by *standard* (*simp-all add*: *gcd-poly-def lcm-poly-def Gcd-poly-def Lcm-poly-def*)

end

instance *poly* :: ({*factorial-ring-gcd*, *semiring-gcd-mult-normalize*}) *semiring-gcd-mult-normalize*
..

instance *poly* :: ({*field*,*factorial-ring-gcd*,*semiring-gcd-mult-normalize*})
normalization-euclidean-semiring **..**

instance *poly* :: ({*field*, *normalization-euclidean-semiring*, *factorial-ring-gcd*,
semiring-gcd-mult-normalize}) *euclidean-ring-gcd*
by (*rule euclidean-ring-gcd-class.intro*, *rule factorial-euclidean-semiring-gcdI*) *standard*

instance *poly* :: ({*field*, *normalization-euclidean-semiring*, *factorial-ring-gcd*,
semiring-gcd-mult-normalize}) *factorial-semiring-multiplicative* **..**

11.8 Polynomial GCD

lemma *gcd-poly-decompose*:
fixes *p q* :: *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*
shows *gcd p q* =

smult (*gcd* (*content p*) (*content q*)) (*gcd* (*primitive-part p*) (*primitive-part q*))
proof (*rule sym*, *rule gcdI*)
have [:*gcd* (*content p*) (*content q*):] ∗ *gcd* (*primitive-part p*) (*primitive-part q*) *dvd*
[:*content p*:] ∗ *primitive-part p* **by** (*intro mult-dvd-mono*) *simp-all*
thus *smult* (*gcd* (*content p*) (*content q*)) (*gcd* (*primitive-part p*) (*primitive-part q*)) *dvd p*
by *simp*
next
have [:*gcd* (*content p*) (*content q*):] ∗ *gcd* (*primitive-part p*) (*primitive-part q*) *dvd*
[:*content q*:] ∗ *primitive-part q* **by** (*intro mult-dvd-mono*) *simp-all*
thus *smult* (*gcd* (*content p*) (*content q*)) (*gcd* (*primitive-part p*) (*primitive-part q*)) *dvd q*
by *simp*
next
fix *d* **assume** *d dvd p d dvd q*
hence [:*content d*:] ∗ *primitive-part d dvd*
[:*gcd* (*content p*) (*content q*):] ∗ *gcd* (*primitive-part p*) (*primitive-part q*)
by (*intro mult-dvd-mono*) *auto*
thus *d dvd smult* (*gcd* (*content p*) (*content q*)) (*gcd* (*primitive-part p*) (*primitive-part q*))
by *simp*
qed (*auto simp*: *normalize-smult*)

lemma *gcd-poly-pseudo-mod*:
fixes *p q* :: *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*
assumes *nz*: $q \neq 0$ **and** *prim*: *content p* = *1 content q* = *1*
shows *gcd p q* = *gcd q* (*primitive-part* (*pseudo-mod p q*))
proof −
define *r s* **where** *r* = *fst* (*pseudo-divmod p q*) **and** *s* = *snd* (*pseudo-divmod p q*)
define *a* **where** *a* = [:*coeff q* (*degree q*) ^ (*Suc* (*degree p*) − *degree q*):]
have [*simp*]: *primitive-part a* = *unit-factor a*
by (*simp add*: *a-def unit-factor-poly-def unit-factor-power monom-0*)
from *nz* **have** [*simp*]: $a \neq 0$ **by** (*auto simp*: *a-def*)

have *rs*: *pseudo-divmod p q* = (*r*, *s*) **by** (*simp add*: *r-def s-def*)
have *gcd* (*q* ∗ *r* + *s*) *q* = *gcd q s*
using *gcd-add-mult*[*of q r s*] **by** (*simp add*: *gcd.commute add-ac mult-ac*)
with *pseudo-divmod*(*1*)[*OF nz rs*]
have *gcd* (*p* ∗ *a*) *q* = *gcd q s* **by** (*simp add*: *a-def*)
also from *prim* **have** *gcd* (*p* ∗ *a*) *q* = *gcd p q*
by (*subst gcd-poly-decompose*)
(*auto simp*: *primitive-part-mult gcd-mult-unit1 primitive-part-prim*
simp del: *mult-pCons-right*)
also from *prim* **have** *gcd q s* = *gcd q* (*primitive-part s*)

    by (*subst gcd-poly-decompose*) (*simp-all add*: *primitive-part-prim*)
  also have *s* = *pseudo-mod p q* **by** (*simp add*: *s-def pseudo-mod-def*)
  finally show *?thesis* **.**
qed

lemma *degree-pseudo-mod-less*:
  assumes *q* ≠ *0 pseudo-mod p q* ≠ *0*
  shows *degree* (*pseudo-mod p q*) < *degree q*
  using *pseudo-mod*(*2*)[*of q p*] *assms* **by** *auto*

function *gcd-poly-code-aux* :: *'a* :: *factorial-ring-gcd poly* ⇒ *'a poly* ⇒ *'a poly*
where
  gcd-poly-code-aux p q =
    (*if q* = *0 then normalize p else gcd-poly-code-aux q* (*primitive-part* (*pseudo-mod p q*)))
by *auto*
termination
  by (*relation measure* ((λ*p. if p* = *0 then 0 else Suc* (*degree p*)) ∘ *snd*))
    (*auto simp*: *degree-pseudo-mod-less*)

declare *gcd-poly-code-aux.simps* [*simp del*]

lemma *gcd-poly-code-aux-correct*:
  assumes *content p* = *1 q* = *0* ∨ *content q* = *1*
  shows *gcd-poly-code-aux p q* = *gcd p q*
  using *assms*
proof (*induction p q rule*: *gcd-poly-code-aux.induct*)
  case (*1 p q*)
  show *?case*
  proof (*cases q* = *0*)
    case *True*
    thus *?thesis* **by** (*subst gcd-poly-code-aux.simps*) *auto*
  next
    case *False*
    hence *gcd-poly-code-aux p q* = *gcd-poly-code-aux q* (*primitive-part* (*pseudo-mod p q*))
      by (*subst gcd-poly-code-aux.simps*) *simp-all*
    also from *1.prems False*
      have *primitive-part* (*pseudo-mod p q*) = *0* ∨
            content (*primitive-part* (*pseudo-mod p q*)) = *1*
      by (*cases pseudo-mod p q* = *0*) *auto*
    with *1.prems False*
      have *gcd-poly-code-aux q* (*primitive-part* (*pseudo-mod p q*)) =
            gcd q (*primitive-part* (*pseudo-mod p q*))
      by (*intro 1*) *simp-all*
    also from *1.prems False*
      have . . . = *gcd p q* **by** (*intro gcd-poly-pseudo-mod* [*symmetric*]) *auto*
    finally show *?thesis* **.**
  qed

qed

definition *gcd-poly-code*
:: *'a* :: *factorial-ring-gcd poly* ⇒ *'a poly* ⇒ *'a poly*
where *gcd-poly-code p q* =
(*if p* = *0 then normalize q else if q* = *0 then normalize p else*
smult (*gcd* (*content p*) (*content q*))
(*gcd-poly-code-aux* (*primitive-part p*) (*primitive-part q*)))

lemma *gcd-poly-code* [*code*]: *gcd p q* = *gcd-poly-code p q*
by (*simp add*: *gcd-poly-code-def gcd-poly-code-aux-correct gcd-poly-decompose* [*symmetric*])

lemma *lcm-poly-code* [*code*]:
fixes *p q* :: *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*
shows *lcm p q* = *normalize* (*p* ∗ *q div gcd p q*)
by (*fact lcm-gcd*)

lemmas *Gcd-poly-set-eq-fold* [*code*] =
Gcd-set-eq-fold [**where** *?'a* = *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*]
lemmas *Lcm-poly-set-eq-fold* [*code*] =
Lcm-set-eq-fold [**where** *?'a* = *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*]

Example: *Lcm* {[:*1*, *2*, *3*:], [:*2*, *3*, *4*:]} = [:[:*2*:], [:*7*:], [:*16*:], [:*17*:], [:*12*:]:]

end

12 Squarefreeness

theory *Squarefree*
imports *Primes*
begin

definition *squarefree* :: *'a* :: *comm-monoid-mult* ⇒ *bool* **where**
squarefree n ⟷ (∀ *x*. *x* ^ *2 dvd n* ⟶ *x dvd 1*)

lemma *squarefreeI*: (⋀*x*. *x* ^ *2 dvd n* ⟹ *x dvd 1*) ⟹ *squarefree n*
by (*auto simp*: *squarefree-def*)

lemma *squarefreeD*: *squarefree n* ⟹ *x* ^ *2 dvd n* ⟹ *x dvd 1*
by (*auto simp*: *squarefree-def*)

lemma *not-squarefreeI*: *x* ^ *2 dvd n* ⟹ ¬*x dvd 1* ⟹ ¬*squarefree n*
by (*auto simp*: *squarefree-def*)

lemma *not-squarefreeE* [*case-names square-dvd*]:
¬*squarefree n* ⟹ (⋀*x*. *x* ^ *2 dvd n* ⟹ ¬*x dvd 1* ⟹ *P*) ⟹ *P*

by (*auto simp*: *squarefree-def*)

lemma *not-squarefree-0* [*simp*]: ¬*squarefree* (*0* :: *'a* :: *comm-semiring-1*)
by (*rule not-squarefreeI*[*of 0*]) *auto*

lemma *squarefree-factorial-semiring*:
assumes $n \neq 0$
shows *squarefree* (*n* :: *'a* :: *factorial-semiring*) ⟷ (∀ *p*. *prime p* ⟶ ¬*p* ^ *2*
dvd n)
unfolding *squarefree-def*
proof *safe*
assume ∗: ∀ *p*. *prime p* ⟶ ¬*p* ^ *2 dvd n*
fix *x* :: *'a* **assume** *x*: *x* ^ *2 dvd n*
{
assume ¬*is-unit x*
moreover from *assms* **and** *x* **have** $x \neq 0$ **by** *auto*
ultimately obtain *p* **where** *p dvd x prime p*
using *prime-divisor-exists* **by** *blast*
with ∗ **have** ¬*p* ^ *2 dvd n* **by** *blast*
moreover from ‹*p dvd x*› **have** *p* ^ *2 dvd x* ^ *2* **by** (*rule dvd-power-same*)
ultimately have ¬*x* ^ *2 dvd n* **by** (*blast dest*: *dvd-trans*)
with *x* **have** *False* **by** *contradiction*
}
thus *is-unit x* **by** *blast*
qed *auto*

lemma *squarefree-factorial-semiring'*:
assumes $n \neq 0$
shows *squarefree* (*n* :: *'a* :: *factorial-semiring*) ⟷
(∀ *p*∈*prime-factors n*. *multiplicity p n* = *1*)
proof (*subst squarefree-factorial-semiring* [*OF assms*], *safe*)
fix *p* **assume** ∀ *p*∈#*prime-factorization n*. *multiplicity p n* = *1 prime p p*^*2 dvd*
n
with *assms* **show** *False*
by (*cases p dvd n*)
(*auto simp*: *prime-factors-dvd power-dvd-iff-le-multiplicity not-dvd-imp-multiplicity-0*)
qed (*auto intro*!: *multiplicity-eqI simp*: *power2-eq-square* [*symmetric*])

lemma *squarefree-factorial-semiring''*:
assumes $n \neq 0$
shows *squarefree* (*n* :: *'a* :: *factorial-semiring*) ⟷
(∀ *p*. *prime p* ⟶ *multiplicity p n* ≤ *1*)
by (*subst squarefree-factorial-semiring'*[*OF assms*]) (*auto simp*: *prime-factors-multiplicity*)

lemma *squarefree-unit* [*simp*]: *is-unit n* ⟹ *squarefree n*
proof (*rule squarefreeI*)
fix *x* **assume** *x*^*2 dvd n n dvd 1*
hence *is-unit* (*x*^*2*) **by** (*rule dvd-unit-imp-unit*)
thus *is-unit x* **by** (*simp add*: *is-unit-power-iff*)

qed

lemma *squarefree-1* [*simp*]: *squarefree* (*1* :: *'a* :: *algebraic-semidom*)
  by *simp*

lemma *squarefree-minus* [*simp*]: *squarefree* ($-n$ :: *'a* :: *comm-ring-1*) $\longleftrightarrow$ *square-free n*
  by (*simp add*: *squarefree-def*)

lemma *squarefree-mono*: *a dvd b* $\Longrightarrow$ *squarefree b* $\Longrightarrow$ *squarefree a*
  by (*auto simp*: *squarefree-def intro*: *dvd-trans*)

lemma *squarefree-multD*:
  assumes *squarefree* ($a * b$)
  shows *squarefree a squarefree b*
  by (*rule squarefree-mono*[*OF - assms*], *simp*)+

lemma *squarefree-prime-elem*:
  assumes *prime-elem* (*p* :: *'a* :: *factorial-semiring*)
  shows *squarefree p*
proof −
  from *assms* **have** $p \neq 0$ **by** *auto*
  show *?thesis*
  proof (*subst squarefree-factorial-semiring* [*OF* ‹$p \neq 0$›]; *safe*)
    fix *q* **assume** ∗: *prime q q^2 dvd p*
    with *assms* **have** *multiplicity q p* ≥ 2 **by** (*intro multiplicity-geI*) *auto*
    thus *False* **using** *assms* ‹*prime q*› *prime-multiplicity-other*[*of q normalize p*]
      by (*cases q* = *normalize p*) *simp-all*
  qed
qed

lemma *squarefree-prime*:
  assumes *prime* (*p* :: *'a* :: *factorial-semiring*)
  shows *squarefree p*
  using *assms* **by** (*intro squarefree-prime-elem*) *auto*

lemma *squarefree-mult-coprime*:
  fixes *a b* :: *'a* :: *factorial-semiring-gcd*
  assumes *coprime a b squarefree a squarefree b*
  shows *squarefree* ($a * b$)
proof −
  from *assms* **have** *nz*: $a * b \neq 0$ **by** *auto*
  show *?thesis* **unfolding** *squarefree-factorial-semiring'*[*OF nz*]
  proof
    fix *p* **assume** *p*: $p \in$ *prime-factors* ($a * b$)
    with *nz* **have** *prime p*
      by (*simp add*: *prime-factors-dvd*)
    have ¬ (*p dvd a* ∧ *p dvd b*)
    proof

assume *p dvd a* ∧ *p dvd b*
with ‹*coprime a b*› **have** *is-unit p*
by (*auto intro*: *coprime-common-divisor*)
with ‹*prime p*› **show** *False*
by *simp*
qed
moreover from *p* **have** *p dvd a* ∨ *p dvd b* **using** *nz*
by (*auto simp*: *prime-factors-dvd prime-dvd-mult-iff*)
ultimately show *multiplicity p* (*a* ∗ *b*) = *1* **using** *nz p assms*(*2*,*3*)
by (*auto simp*: *prime-elem-multiplicity-mult-distrib prime-factors-multiplicity not-dvd-imp-multiplicity-0 squarefree-factorial-semiring′*)
qed
qed

lemma *squarefree-prod-coprime*:
fixes *f* :: ′*a* ⇒ ′*b* :: *factorial-semiring-gcd*
assumes ⋀*a b*. *a* ∈ *A* ⟹ *b* ∈ *A* ⟹ *a* ≠ *b* ⟹ *coprime* (*f a*) (*f b*)
assumes ⋀*a*. *a* ∈ *A* ⟹ *squarefree* (*f a*)
shows *squarefree* (*prod f A*)
using *assms*
by (*induction A rule*: *infinite-finite-induct*)
(*auto intro*!: *squarefree-mult-coprime prod-coprime-right*)

lemma *squarefree-powerD*: *m* > *0* ⟹ *squarefree* (*n* ^ *m*) ⟹ *squarefree n*
by (*cases m*) (*auto dest*: *squarefree-multD*)

lemma *squarefree-power-iff*:
squarefree (*n* ^ *m*) ⟷ *m* = *0* ∨ *is-unit n* ∨ (*squarefree n* ∧ *m* = *1*)
proof *safe*
assume *squarefree* (*n* ^ *m*) *m* > *0* ¬*is-unit n*
show *m* = *1*
proof (*rule ccontr*)
assume *m* ≠ *1*
with ‹*m* > *0*› **have** *n* ^ *2 dvd n* ^ *m* **by** (*intro le-imp-power-dvd*) *auto*
from *this* **and** ‹¬*is-unit n*› **have** ¬*squarefree* (*n* ^ *m*) **by** (*rule not-squarefreeI*)
with ‹*squarefree* (*n* ^ *m*)› **show** *False* **by** *contradiction*
qed
qed (*auto simp*: *is-unit-power-iff dest*: *squarefree-powerD*)

definition *squarefree-nat* :: *nat* ⇒ *bool* **where**
[*code-abbrev*]: *squarefree-nat* = *squarefree*

lemma *squarefree-nat-code-naive* [*code*]:
squarefree-nat n ⟷ *n* ≠ *0* ∧ (∀ *k*∈{*2*..*n*}. ¬*k* ^ *2 dvd n*)
proof *safe*
assume ∗: ∀ *k*∈{*2*..*n*}. ¬ k^2 *dvd n* **and** *n*: *n* > *0*
show *squarefree-nat n* **unfolding** *squarefree-nat-def*
proof (*rule squarefreeI*)
fix *k* **assume** *k*: *k* ^ *2 dvd n*

have *k dvd n* **by** (*rule dvd-trans*[*OF - k*]) *auto*
with *n* **have** $k \le n$ **by** (*intro dvd-imp-le*)
with *bspec*[*OF* ∗, *of k*] *k* **have** $\neg k > 1$ **by** (*intro notI*) *auto*
moreover from *k* **and** *n* **have** $k \neq 0$ **by** (*intro notI*) *auto*
ultimately have $k = 1$ **by** *presburger*
thus *is-unit k* **by** *simp*
qed
qed (*auto simp*: *squarefree-nat-def squarefree-def intro*!: *Nat.gr0I*)

definition *square-part* :: *'a* :: *factorial-semiring* ⇒ *'a* **where**
square-part n = (*if n* = *0 then 0 else*
normalize (∏ *p*∈*prime-factors n*. *p* ^ (*multiplicity p n div 2*)))

lemma *square-part-nonzero*:
$n \neq 0 \Longrightarrow$ *square-part n* = *normalize* (∏ *p*∈*prime-factors n*. *p* ^ (*multiplicity p n div 2*))
by (*simp add*: *square-part-def*)

lemma *square-part-0* [*simp*]: *square-part 0* = *0*
by (*simp add*: *square-part-def*)

lemma *square-part-unit* [*simp*]: *is-unit x* ⟹ *square-part x* = *1*
by (*auto simp*: *square-part-def prime-factorization-unit*)

lemma *square-part-1* [*simp*]: *square-part 1* = *1*
by *simp*

lemma *square-part-0-iff* [*simp*]: *square-part n* = *0* ⟷ *n* = *0*
by (*simp add*: *square-part-def*)

lemma *normalize-uminus* [*simp*]:
normalize (−*x* :: *'a* :: {*normalization-semidom*, *comm-ring-1*}) = *normalize x*
by (*rule associatedI*) *auto*

lemma *multiplicity-uminus-right* [*simp*]:
multiplicity (*x* :: *'a* :: {*factorial-semiring*, *comm-ring-1*}) (−*y*) = *multiplicity x y*
proof −
have *multiplicity x* (−*y*) = *multiplicity x* (*normalize* (−*y*))
by (*rule multiplicity-normalize-right* [*symmetric*])
also have ... = *multiplicity x y* **by** *simp*
finally show *?thesis* **.**
qed

lemma *multiplicity-uminus-left* [*simp*]:
multiplicity (−*x* :: *'a* :: {*factorial-semiring*, *comm-ring-1*}) *y* = *multiplicity x y*
proof −
have *multiplicity* (−*x*) *y* = *multiplicity* (*normalize* (−*x*)) *y*

by (*rule multiplicity-normalize-left* [*symmetric*])
also have ... = *multiplicity x y* **by** *simp*
finally show *?thesis* **.**
qed

lemma *prime-factorization-uminus* [*simp*]:
prime-factorization (−*x* :: ′*a* :: {*factorial-semiring*, *comm-ring-1*}) = *prime-factorization x*
by (*rule prime-factorization-cong*) *simp-all*

lemma *square-part-uminus* [*simp*]:
square-part (−*x* :: ′*a* :: {*factorial-semiring*, *comm-ring-1*}) = *square-part x*
by (*simp add*: *square-part-def*)

lemma *prime-multiplicity-square-part*:
assumes *prime p*
shows *multiplicity p* (*square-part n*) = *multiplicity p n div 2*
proof (*cases n* = *0*)
case *False*
thus *?thesis* **unfolding** *square-part-nonzero*[*OF False*] *multiplicity-normalize-right*
using *finite-prime-divisors*[*of n*] *assms*
by (*subst multiplicity-prod-prime-powers*)
(*auto simp*: *not-dvd-imp-multiplicity-0 prime-factors-dvd multiplicity-prod-prime-powers*)
qed *auto*

lemma *square-part-square-dvd* [*simp*, *intro*]: *square-part n* ^ *2 dvd n*
proof (*cases n* = *0*)
case *False*
thus *?thesis*
by (*intro multiplicity-le-imp-dvd*)
(*auto simp*: *prime-multiplicity-square-part prime-elem-multiplicity-power-distrib*)
qed *auto*

lemma *prime-multiplicity-le-imp-dvd*:
assumes *x* ≠ *0 y* ≠ *0*
shows *x dvd y* ⟷ (∀ *p*. *prime p* ⟶ *multiplicity p x* ≤ *multiplicity p y*)
using *assms* **by** (*auto intro*: *multiplicity-le-imp-dvd dvd-imp-multiplicity-le*)

lemma *dvd-square-part-iff*: *x dvd square-part n* ⟷ *x* ^ *2 dvd n*
proof (*cases x* = *0*; *cases n* = *0*)
assume *nz*: *x* ≠ *0 n* ≠ *0*
thus *?thesis*
by (*subst* (*1 2*) *prime-multiplicity-le-imp-dvd*)
(*auto simp*: *prime-multiplicity-square-part prime-elem-multiplicity-power-distrib*)
qed *auto*

definition *squarefree-part* :: ′*a* :: *factorial-semiring* ⇒ ′*a* **where**
squarefree-part n = (*if n* = *0 then 1 else n div square-part n* ^ *2*)

lemma *squarefree-part-0* [*simp*]: *squarefree-part 0* = *1*
 by (*simp add*: *squarefree-part-def*)

lemma *squarefree-part-unit* [*simp*]: *is-unit n* ⟹ *squarefree-part n* = *n*
 by (*auto simp add*: *squarefree-part-def*)

lemma *squarefree-part-1* [*simp*]: *squarefree-part 1* = *1*
 by *simp*

lemma *squarefree-decompose*: *n* = *squarefree-part n* ∗ *square-part n* ^ *2*
 by (*simp add*: *squarefree-part-def*)

lemma *squarefree-part-uminus* [*simp*]:
 assumes *x* ≠ *0*
 shows *squarefree-part* (−*x* :: ′*a* :: {*factorial-semiring*, *comm-ring-1*}) = −*squarefree-part x*
proof −
 have −(*squarefree-part x* ∗ *square-part x* ^ *2*) = −*x*
  by (*subst squarefree-decompose* [*symmetric*]) *auto*
 also have ... = *squarefree-part* (−*x*) ∗ *square-part* (−*x*) ^ *2* **by** (*rule squarefree-decompose*)
 finally have (− *squarefree-part x*) ∗ *square-part x* ^ *2* =
   squarefree-part (−*x*) ∗ *square-part x* ^ *2* **by** *simp*
 thus *?thesis* **using** *assms* **by** (*subst* (*asm*) *mult-right-cancel*) *auto*
qed

lemma *squarefree-part-nonzero* [*simp*]: *squarefree-part n* ≠ *0*
 using *squarefree-decompose*[*of n*] **by** (*cases n* ≠ *0*) *auto*

lemma *prime-multiplicity-squarefree-part*:
 assumes *prime p*
 shows *multiplicity p* (*squarefree-part n*) = *multiplicity p n mod 2*
proof (*cases n* = *0*)
 case *False*
 hence *n*: *n* ≠ *0* **by** *auto*
 have *multiplicity p n mod 2* + *2* ∗ (*multiplicity p n div 2*) = *multiplicity p n* **by** *simp*
 also have ... = *multiplicity p* (*squarefree-part n* ∗ *square-part n* ^ *2*)
  by (*subst squarefree-decompose*[*of n*]) *simp*
 also from *assms n* **have** ... = *multiplicity p* (*squarefree-part n*) + *2* ∗ (*multiplicity p n div 2*)
  by (*subst prime-elem-multiplicity-mult-distrib*)
   (*auto simp*: *prime-elem-multiplicity-power-distrib prime-multiplicity-square-part*)
 finally show *?thesis* **by** (*subst* (*asm*) *add-right-cancel*) *simp*
qed *auto*

lemma *prime-multiplicity-squarefree-part-le-Suc-0* [*intro*]:
 assumes *prime p*

shows *multiplicity p* (*squarefree-part n*) $\leq$ *Suc 0*
by (*simp add*: *assms prime-multiplicity-squarefree-part*)

lemma *squarefree-squarefree-part* [*simp*, *intro*]: *squarefree* (*squarefree-part n*)
by (*subst squarefree-factorial-semiring''*)
(*auto simp*: *prime-multiplicity-squarefree-part-le-Suc-0*)

lemma *squarefree-decomposition-unique*:
assumes *square-part m* = *square-part n*
assumes *squarefree-part m* = *squarefree-part n*
shows *m* = *n*
by (*subst* (*1 2*) *squarefree-decompose*) (*simp-all add*: *assms*)

lemma *normalize-square-part* [*simp*]: *normalize* (*square-part x*) = *square-part x*
by (*simp add*: *square-part-def*)

lemma *square-part-even-power'*: *square-part* (*x* ^ (*2* ∗ *n*)) = *normalize* (*x* ^ *n*)
proof (*cases x* = *0*)
case *False*
have *normalize* (*square-part* (*x* ^ (*2* ∗ *n*))) = *normalize* (*x* ^ *n*) **using** *False*
by (*intro multiplicity-eq-imp-eq*)
(*auto simp*: *prime-multiplicity-square-part prime-elem-multiplicity-power-distrib*)
thus *?thesis* **by** *simp*
qed (*auto simp*: *power-0-left*)

lemma *square-part-even-power*: *even n* $\Longrightarrow$ *square-part* (*x* ^ *n*) = *normalize* (*x* ^ (*n div 2*))
by (*subst square-part-even-power'* [*symmetric*]) *auto*

lemma *square-part-odd-power'*: *square-part* (*x* ^ (*Suc* (*2* ∗ *n*))) = *normalize* (*x* ^ *n* ∗ *square-part x*)
proof (*cases x* = *0*)
case *False*
have *normalize* (*square-part* (*x* ^ (*Suc* (*2* ∗ *n*)))) = *normalize* (*square-part x* ∗ *x* ^ *n*)
proof (*rule multiplicity-eq-imp-eq*, *goal-cases*)
case (*3 p*)
hence *multiplicity p* (*square-part* (*x* ^ *Suc* (*2* ∗ *n*))) =
(*2* ∗ (*n* ∗ *multiplicity p x*) + *multiplicity p x*) *div 2*
by (*subst prime-multiplicity-square-part*)
(*auto simp*: *False prime-elem-multiplicity-power-distrib algebra-simps simp del*: *power-Suc*)
also from *3 False* **have** ... = *multiplicity p* (*square-part x* ∗ *x* ^ *n*)
by (*subst div-mult-self4*) (*auto simp*: *prime-multiplicity-square-part prime-elem-multiplicity-mult-distrib prime-elem-multiplicity-power-distrib*)
finally show *?case* **.**
qed (*insert False*, *auto*)
thus *?thesis* **by** (*simp add*: *mult-ac*)
qed *auto*

lemma *square-part-odd-power*:
  odd n ⟹ *square-part* (*x* ^ *n*) = *normalize* (*x* ^ (*n div 2*) ∗ *square-part x*)
  by (*subst square-part-odd-power′* [*symmetric*]) *auto*

end

13 Pieces of computational Algebra

theory *Computational-Algebra*
imports
  Euclidean-Algorithm
  Factorial-Ring
  Formal-Laurent-Series
  Fraction-Field
  Fundamental-Theorem-Algebra
  Group-Closure
  Normalized-Fraction
  Nth-Powers
  Polynomial-FPS
  Polynomial
  Polynomial-Factorial
  Primes
  Squarefree
begin

end

theory *Field-as-Ring*
imports
  Complex-Main
  Euclidean-Algorithm
begin

context *field*
begin

subclass *idom-divide* **..**

definition *normalize-field* :: ′*a* ⇒ ′*a*
  where [*simp*]: *normalize-field x* = (*if x* = *0 then 0 else 1*)
definition *unit-factor-field* :: ′*a* ⇒ ′*a*
  where [*simp*]: *unit-factor-field x* = *x*
definition *euclidean-size-field* :: ′*a* ⇒ *nat*
  where [*simp*]: *euclidean-size-field x* = (*if x* = *0 then 0 else 1*)
definition *mod-field* :: ′*a* ⇒ ′*a* ⇒ ′*a*
  where [*simp*]: *mod-field x y* = (*if y* = *0 then x else 0*)

end

instantiation *real* ::
{*unique-euclidean-ring*, *normalization-euclidean-semiring*, *normalization-semidom-multiplicative*}
begin

definition [*simp*]: *normalize-real* = (*normalize-field* :: *real* ⇒ -)
definition [*simp*]: *unit-factor-real* = (*unit-factor-field* :: *real* ⇒ -)
definition [*simp*]: *modulo-real* = (*mod-field* :: *real* ⇒ -)
definition [*simp*]: *euclidean-size-real* = (*euclidean-size-field* :: *real* ⇒ -)
definition [*simp*]: *division-segment* (*x* :: *real*) = *1*

instance
by *standard*
(*simp-all add*: *dvd-field-iff field-split-simps split*: *if-splits*)

end

instantiation *real* :: *euclidean-ring-gcd*
begin

definition *gcd-real* :: *real* ⇒ *real* ⇒ *real* **where**
gcd-real = *Euclidean-Algorithm.gcd*
definition *lcm-real* :: *real* ⇒ *real* ⇒ *real* **where**
lcm-real = *Euclidean-Algorithm.lcm*
definition *Gcd-real* :: *real set* ⇒ *real* **where**
Gcd-real = *Euclidean-Algorithm.Gcd*
definition *Lcm-real* :: *real set* ⇒ *real* **where**
Lcm-real = *Euclidean-Algorithm.Lcm*

instance by *standard* (*simp-all add*: *gcd-real-def lcm-real-def Gcd-real-def Lcm-real-def*)

end

instance *real* :: *field-gcd* **..**

instantiation *rat* ::
{*unique-euclidean-ring*, *normalization-euclidean-semiring*, *normalization-semidom-multiplicative*}
begin

definition [*simp*]: *normalize-rat* = (*normalize-field* :: *rat* ⇒ -)
definition [*simp*]: *unit-factor-rat* = (*unit-factor-field* :: *rat* ⇒ -)
definition [*simp*]: *modulo-rat* = (*mod-field* :: *rat* ⇒ -)
definition [*simp*]: *euclidean-size-rat* = (*euclidean-size-field* :: *rat* ⇒ -)
definition [*simp*]: *division-segment* (*x* :: *rat*) = *1*

instance
by *standard*

(*simp-all add*: *dvd-field-iff field-split-simps split*: *if-splits*)

end

instantiation *rat* :: *euclidean-ring-gcd*
begin

definition *gcd-rat* :: *rat* ⇒ *rat* ⇒ *rat* **where**
gcd-rat = *Euclidean-Algorithm.gcd*
definition *lcm-rat* :: *rat* ⇒ *rat* ⇒ *rat* **where**
lcm-rat = *Euclidean-Algorithm.lcm*
definition *Gcd-rat* :: *rat set* ⇒ *rat* **where**
Gcd-rat = *Euclidean-Algorithm.Gcd*
definition *Lcm-rat* :: *rat set* ⇒ *rat* **where**
Lcm-rat = *Euclidean-Algorithm.Lcm*

instance by *standard* (*simp-all add*: *gcd-rat-def lcm-rat-def Gcd-rat-def Lcm-rat-def*)

end

instance *rat* :: *field-gcd* **..**

instantiation *complex* ::
{*unique-euclidean-ring*, *normalization-euclidean-semiring*, *normalization-semidom-multiplicative*}
begin

definition [*simp*]: *normalize-complex* = (*normalize-field* :: *complex* ⇒ -)
definition [*simp*]: *unit-factor-complex* = (*unit-factor-field* :: *complex* ⇒ -)
definition [*simp*]: *modulo-complex* = (*mod-field* :: *complex* ⇒ -)
definition [*simp*]: *euclidean-size-complex* = (*euclidean-size-field* :: *complex* ⇒ -)
definition [*simp*]: *division-segment* (*x* :: *complex*) = *1*

instance
by *standard*
(*simp-all add*: *dvd-field-iff field-split-simps split*: *if-splits*)

end

instantiation *complex* :: *euclidean-ring-gcd*
begin

definition *gcd-complex* :: *complex* ⇒ *complex* ⇒ *complex* **where**
gcd-complex = *Euclidean-Algorithm.gcd*
definition *lcm-complex* :: *complex* ⇒ *complex* ⇒ *complex* **where**
lcm-complex = *Euclidean-Algorithm.lcm*
definition *Gcd-complex* :: *complex set* ⇒ *complex* **where**
Gcd-complex = *Euclidean-Algorithm.Gcd*
definition *Lcm-complex* :: *complex set* ⇒ *complex* **where**

Lcm-complex = *Euclidean-Algorithm.Lcm*

instance by *standard* (*simp-all add*: *gcd-complex-def lcm-complex-def Gcd-complex-def Lcm-complex-def*)

end

instance *complex* :: *field-gcd* **..**

end

References

[1] K. J. Nowak. Some elementary proofs of Puiseuxs theorems. *Univ. Iagel. Acta Math*, 38:279–282, 2000.