Isabelle/HOLCF — Higher-Order Logic of Computable Functions

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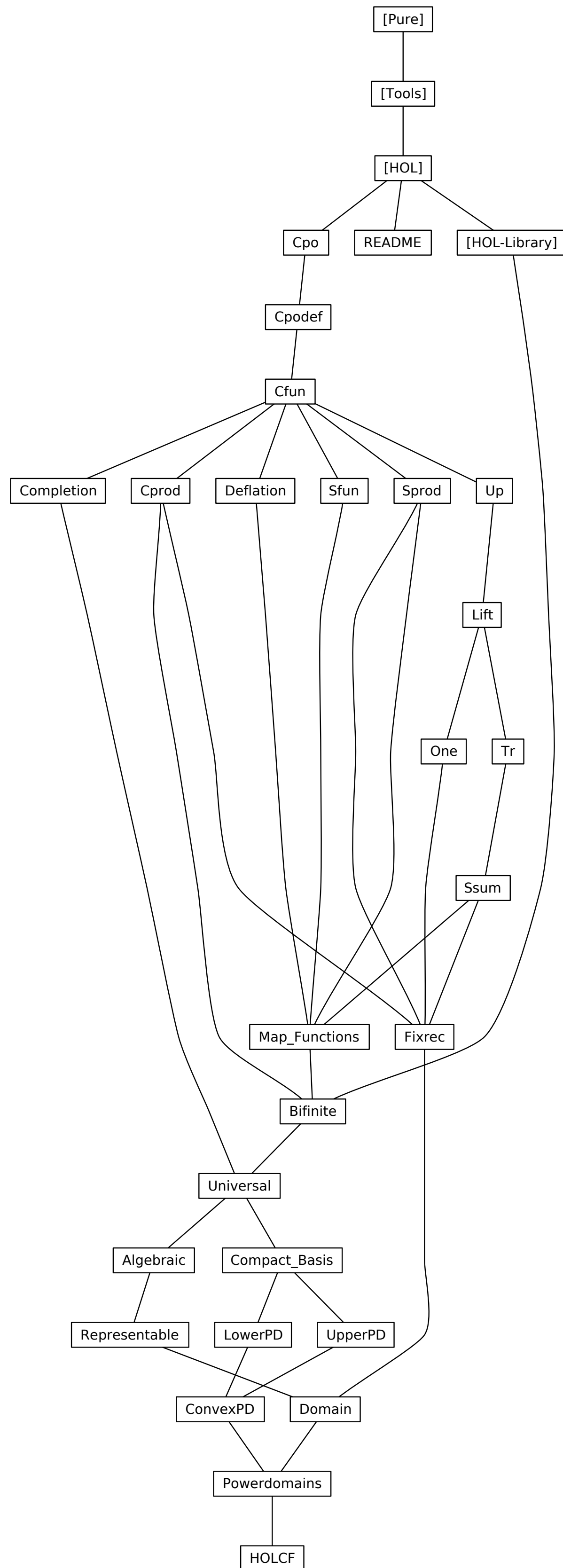
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Bifinite
Universal
Algebraic
Compact_Basis
Representable
LowerPD
UpperPD
ConvexPD
Domain
Powerdomains
HOLCF

```
theory Cpo
  imports Main
begin
```

1 Partial orders

```
declare [[typedef-overloaded]]
```

1.1 Type class for partial orders

```
class below =
  fixes below :: 'a ⇒ 'a ⇒ bool
begin

notation (ASCII)
  below (infix ‹<<› 50)

notation
  below (infix ‹⊑› 50)

abbreviation not-below :: 'a ⇒ 'a ⇒ bool  (infix ‹⋢› 50)
  where not-below x y ≡ ¬ below x y

notation (ASCII)
  not-below  (infix ‹~<<› 50)

lemma below-eq-trans: a ⊑ b ⟹ b = c ⟹ a ⊑ c
  by (rule subst)

lemma eq-below-trans: a = b ⟹ b ⊑ c ⟹ a ⊑ c
  by (rule ssubst)

end

class po = below +
  assumes below-refl [iff]: x ⊑ x
  assumes below-trans: x ⊑ y ⟹ y ⊑ z ⟹ x ⊑ z
  assumes below-antisym: x ⊑ y ⟹ y ⊑ x ⟹ x = y
begin

lemma eq-imp-below: x = y ⟹ x ⊑ y
  by simp

lemma box-below: a ⊑ b ⟹ c ⊑ a ⟹ b ⊑ d ⟹ c ⊑ d
  by (rule below-trans [OF below-trans])

lemma po-eq-conv: x = y ⟷ x ⊑ y ∧ y ⊑ x
```

by (*fast intro*!: *below-antisym*)

lemma *rev-below-trans*: $y \sqsubseteq z \Longrightarrow x \sqsubseteq y \Longrightarrow x \sqsubseteq z$
by (*rule below-trans*)

lemma *not-below2not-eq*: $x \not\sqsubseteq y \Longrightarrow x \neq y$
by *auto*

end

lemmas *HOLCF-trans-rules* [*trans*] =
below-trans
below-antisym
below-eq-trans
eq-below-trans

context *po*
begin

1.2 Upper bounds

definition *is-ub* :: *'a set* ⇒ *'a* ⇒ *bool* (**infix** ‹<|› *55*)
where $S <| x \longleftrightarrow (\forall y \in S.\ y \sqsubseteq x)$

lemma *is-ubI*: $(\bigwedge x.\ x \in S \Longrightarrow x \sqsubseteq u) \Longrightarrow S <| u$
by (*simp add*: *is-ub-def*)

lemma *is-ubD*: $[\![S <| u;\ x \in S]\!] \Longrightarrow x \sqsubseteq u$
by (*simp add*: *is-ub-def*)

lemma *ub-imageI*: $(\bigwedge x.\ x \in S \Longrightarrow f\ x \sqsubseteq u) \Longrightarrow (\lambda x.\ f\ x)\ `\ S <| u$
unfolding *is-ub-def* **by** *fast*

lemma *ub-imageD*: $[\![f\ `\ S <| u;\ x \in S]\!] \Longrightarrow f\ x \sqsubseteq u$
unfolding *is-ub-def* **by** *fast*

lemma *ub-rangeI*: $(\bigwedge i.\ S\ i \sqsubseteq x) \Longrightarrow range\ S <| x$
unfolding *is-ub-def* **by** *fast*

lemma *ub-rangeD*: $range\ S <| x \Longrightarrow S\ i \sqsubseteq x$
unfolding *is-ub-def* **by** *fast*

lemma *is-ub-empty* [*simp*]: $\{\} <| u$
unfolding *is-ub-def* **by** *fast*

lemma *is-ub-insert* [*simp*]: $(insert\ x\ A) <| y = (x \sqsubseteq y \wedge A <| y)$
unfolding *is-ub-def* **by** *fast*

lemma *is-ub-upward*: $[\![S <| x;\ x \sqsubseteq y]\!] \Longrightarrow S <| y$

unfolding *is-ub-def* **by** (*fast intro*: *below-trans*)

1.3 Least upper bounds

definition *is-lub* :: *'a set* ⇒ *'a* ⇒ *bool* (**infix** ‹<<|› *55*)
where $S <<| x \longleftrightarrow S <| x \wedge (\forall u.\ S <| u \longrightarrow x \sqsubseteq u)$

definition *lub* :: *'a set* ⇒ *'a*
where *lub S* = (*THE x. S* <<| *x*)

end

syntax (*ASCII*)
-BLub :: [*pttrn*, *'a set*, *'b*] ⇒ *'b* (‹(‹*indent=3 notation=*‹*binder LUB*››*LUB* -:-./ -)› [*0*,*0*, *10*] *10*)

syntax
-BLub :: [*pttrn*, *'a set*, *'b*] ⇒ *'b* (‹(‹*indent=3 notation=*‹*binder* ⨆››⨆-∈-./ -)› [*0*,*0*, *10*] *10*)

syntax-consts
-BLub ⇌ *lub*

translations
LUB x:*A*. *t* ⇌ *CONST lub* ((λ*x*. *t*) ‘ *A*)

context *po*
begin

abbreviation *Lub* (**binder** ‹⨆› *10*)
where $\bigsqcup n.\ t\ n \equiv lub\ (range\ t)$

notation (*ASCII*)
Lub (**binder** ‹*LUB* › *10*)

access to some definition as inference rule

lemma *is-lubD1*: $S <<| x \Longrightarrow S <| x$
unfolding *is-lub-def* **by** *fast*

lemma *is-lubD2*: $[\![S <<| x;\ S <| u]\!] \Longrightarrow x \sqsubseteq u$
unfolding *is-lub-def* **by** *fast*

lemma *is-lubI*: $[\![S <| x;\ \bigwedge u.\ S <| u \Longrightarrow x \sqsubseteq u]\!] \Longrightarrow S <<| x$
unfolding *is-lub-def* **by** *fast*

lemma *is-lub-below-iff*: $S <<| x \Longrightarrow x \sqsubseteq u \longleftrightarrow S <| u$
unfolding *is-lub-def is-ub-def* **by** (*metis below-trans*)

lubs are unique

lemma *is-lub-unique*: $S <<| x \Longrightarrow S <<| y \Longrightarrow x = y$
  unfolding *is-lub-def is-ub-def* **by** (*blast intro*: *below-antisym*)

technical lemmas about *lub* and $(<<|)$

lemma *is-lub-lub*: $M <<| x \Longrightarrow M <<| lub\ M$
  unfolding *lub-def* **by** (*rule theI* [*OF* - *is-lub-unique*])

lemma *lub-eqI*: $M <<| l \Longrightarrow lub\ M = l$
  by (*rule is-lub-unique* [*OF is-lub-lub*])

lemma *is-lub-singleton* [*simp*]: $\{x\} <<| x$
  by (*simp add*: *is-lub-def*)

lemma *lub-singleton* [*simp*]: $lub\ \{x\} = x$
  by (*rule is-lub-singleton* [*THEN lub-eqI*])

lemma *is-lub-bin*: $x \sqsubseteq y \Longrightarrow \{x, y\} <<| y$
  by (*simp add*: *is-lub-def*)

lemma *lub-bin*: $x \sqsubseteq y \Longrightarrow lub\ \{x, y\} = y$
  by (*rule is-lub-bin* [*THEN lub-eqI*])

lemma *is-lub-maximal*: $S <| x \Longrightarrow x \in S \Longrightarrow S <<| x$
  by (*erule is-lubI*, *erule* (*1*) *is-ubD*)

lemma *lub-maximal*: $S <| x \Longrightarrow x \in S \Longrightarrow lub\ S = x$
  by (*rule is-lub-maximal* [*THEN lub-eqI*])

1.4 Countable chains

definition *chain* :: $(nat \Rightarrow 'a) \Rightarrow bool$
  where — Here we use countable chains and I prefer to code them as functions!
  $chain\ Y = (\forall i.\ Y\ i \sqsubseteq Y\ (Suc\ i))$

lemma *chainI*: $(\bigwedge i.\ Y\ i \sqsubseteq Y\ (Suc\ i)) \Longrightarrow chain\ Y$
  unfolding *chain-def* **by** *fast*

lemma *chainE*: $chain\ Y \Longrightarrow Y\ i \sqsubseteq Y\ (Suc\ i)$
  unfolding *chain-def* **by** *fast*

chains are monotone functions

lemma *chain-mono-less*: $chain\ Y \Longrightarrow i < j \Longrightarrow Y\ i \sqsubseteq Y\ j$
  by (*erule less-Suc-induct*, *erule chainE*, *erule below-trans*)

lemma *chain-mono*: $chain\ Y \Longrightarrow i \le j \Longrightarrow Y\ i \sqsubseteq Y\ j$
  by (*cases* $i = j$) (*simp-all add*: *chain-mono-less*)

lemma *chain-shift*: $chain\ Y \Longrightarrow chain\ (\lambda i.\ Y\ (i + j))$
  by (*rule chainI*, *simp*, *erule chainE*)

technical lemmas about (least) upper bounds of chains

lemma *is-lub-rangeD1*: $\textit{range}\ S <<|\ x \Longrightarrow S\ i \sqsubseteq x$
by (*rule is-lubD1* [*THEN ub-rangeD*])

lemma *is-ub-range-shift*: $\textit{chain}\ S \Longrightarrow \textit{range}\ (\lambda i.\ S\ (i + j)) <|\ x = \textit{range}\ S <|\ x$
apply (*rule iffI*)
apply (*rule ub-rangeI*)
apply (*rule-tac* $y{=}S\ (i + j)$ **in** *below-trans*)
apply (*erule chain-mono*)
apply (*rule le-add1*)
apply (*erule ub-rangeD*)
apply (*rule ub-rangeI*)
apply (*erule ub-rangeD*)
done

lemma *is-lub-range-shift*: $\textit{chain}\ S \Longrightarrow \textit{range}\ (\lambda i.\ S\ (i + j)) <<|\ x = \textit{range}\ S <<|\ x$
by (*simp add*: *is-lub-def is-ub-range-shift*)

the lub of a constant chain is the constant

lemma *chain-const* [*simp*]: $\textit{chain}\ (\lambda i.\ c)$
by (*simp add*: *chainI*)

lemma *is-lub-const*: $\textit{range}\ (\lambda x.\ c) <<|\ c$
by (*blast dest*: *ub-rangeD intro*: *is-lubI ub-rangeI*)

lemma *lub-const* [*simp*]: $(\bigsqcup i.\ c) = c$
by (*rule is-lub-const* [*THEN lub-eqI*])

1.5 Finite chains

definition *max-in-chain* :: $\textit{nat} \Rightarrow (\textit{nat} \Rightarrow 'a) \Rightarrow \textit{bool}$
where — finite chains, needed for monotony of continuous functions
$\textit{max-in-chain}\ i\ C \longleftrightarrow (\forall j.\ i \leq j \longrightarrow C\ i = C\ j)$

definition *finite-chain* :: $(\textit{nat} \Rightarrow 'a) \Rightarrow \textit{bool}$
where $\textit{finite-chain}\ C = (\textit{chain}\ C \wedge (\exists i.\ \textit{max-in-chain}\ i\ C))$

results about finite chains

lemma *max-in-chainI*: $(\bigwedge j.\ i \leq j \Longrightarrow Y\ i = Y\ j) \Longrightarrow \textit{max-in-chain}\ i\ Y$
unfolding *max-in-chain-def* **by** *fast*

lemma *max-in-chainD*: $\textit{max-in-chain}\ i\ Y \Longrightarrow i \leq j \Longrightarrow Y\ i = Y\ j$
unfolding *max-in-chain-def* **by** *fast*

lemma *finite-chainI*: $\textit{chain}\ C \Longrightarrow \textit{max-in-chain}\ i\ C \Longrightarrow \textit{finite-chain}\ C$
unfolding *finite-chain-def* **by** *fast*

lemma *finite-chainE*: $[\![$*finite-chain C*; $\bigwedge i$. $[\![$*chain C*; *max-in-chain i C*$]\!] \Longrightarrow R]\!]$ $\Longrightarrow R$
  unfolding *finite-chain-def* **by** *fast*

lemma *lub-finch1*: *chain C* $\Longrightarrow$ *max-in-chain i C* $\Longrightarrow$ *range C* <<| *C i*
  apply (*rule is-lubI*)
   apply (*rule ub-rangeI*, *rename-tac j*)
   apply (*rule-tac x=i* **and** *y=j* **in** *linorder-le-cases*)
    apply (*drule* (*1*) *max-in-chainD*, *simp*)
   apply (*erule* (*1*) *chain-mono*)
  apply (*erule ub-rangeD*)
  done

lemma *lub-finch2*: *finite-chain C* $\Longrightarrow$ *range C* <<| *C* (*LEAST i*. *max-in-chain i C*)
  apply (*erule finite-chainE*)
  apply (*erule LeastI2* [**where** *Q*=λ*i*. *range C* <<| *C i*])
  apply (*erule* (*1*) *lub-finch1*)
  done

lemma *finch-imp-finite-range*: *finite-chain Y* $\Longrightarrow$ *finite* (*range Y*)
  apply (*erule finite-chainE*)
  apply (*rule-tac B*=*Y* ‘ {..*i*} **in** *finite-subset*)
   apply (*rule subsetI*)
   apply (*erule rangeE*, *rename-tac j*)
   apply (*rule-tac x=i* **and** *y=j* **in** *linorder-le-cases*)
    apply (*subgoal-tac Y j* = *Y i*, *simp*)
    apply (*simp add*: *max-in-chain-def*)
   apply *simp*
  apply *simp*
  done

lemma *finite-range-has-max*:
  fixes *f* :: *nat* $\Rightarrow$ *'a*
    and *r* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*
  assumes *mono*: $\bigwedge i\ j.\ i \le j \Longrightarrow r\ (f\ i)\ (f\ j)$
  assumes *finite-range*: *finite* (*range f*)
  shows $\exists k.\ \forall i.\ r\ (f\ i)\ (f\ k)$
proof (*intro exI allI*)
  fix *i* :: *nat*
  let *?j* = *LEAST k*. *f k* = *f i*
  let *?k* = *Max* ((λ*x*. *LEAST k*. *f k* = *x*) ‘ *range f*)
  have *?j* $\le$ *?k*
  proof (*rule Max-ge*)
    show *finite* ((λ*x*. *LEAST k*. *f k* = *x*) ‘ *range f*)
      using *finite-range* **by** (*rule finite-imageI*)
    show *?j* $\in$ (λ*x*. *LEAST k*. *f k* = *x*) ‘ *range f*
      by (*intro imageI rangeI*)
  qed

```
  hence r (f ?j) (f ?k)
    by (rule mono)
  also have f ?j = f i
    by (rule LeastI, rule refl)
  finally show r (f i) (f ?k) .
qed

lemma finite-range-imp-finch: chain Y ⟹ finite (range Y) ⟹ finite-chain Y
  apply (subgoal-tac ∃ k. ∀ i. Y i ⊑ Y k)
   apply (erule exE)
   apply (rule finite-chainI, assumption)
   apply (rule max-in-chainI)
   apply (rule below-antisym)
    apply (erule (1) chain-mono)
   apply (erule spec)
  apply (rule finite-range-has-max)
   apply (erule (1) chain-mono)
  apply assumption
  done

lemma bin-chain: x ⊑ y ⟹ chain (λi. if i=0 then x else y)
  by (rule chainI) simp

lemma bin-chainmax: x ⊑ y ⟹ max-in-chain (Suc 0) (λi. if i=0 then x else y)
  by (simp add: max-in-chain-def)

lemma is-lub-bin-chain: x ⊑ y ⟹ range (λi::nat. if i=0 then x else y) <<| y
  apply (frule bin-chain)
  apply (drule bin-chainmax)
  apply (drule (1) lub-finch1)
  apply simp
  done
```

the maximal element in a chain is its lub

```
lemma lub-chain-maxelem: Y i = c ⟹ ∀ i. Y i ⊑ c ⟹ lub (range Y) = c
  by (blast dest: ub-rangeD intro: lub-eqI is-lubI ub-rangeI)

end
```

2 Classes cpo and pcpo

2.1 Complete partial orders

The class cpo of chain complete partial orders

```
class cpo = po +
  assumes cpo: chain S ⟹ ∃ x. range S <<| x

default-sort cpo
```

context *cpo*
begin

in cpo’s everthing equal to THE lub has lub properties for every chain

lemma *cpo-lubI*: $chain\ S \Longrightarrow range\ S <<| (\bigsqcup i.\ S\ i)$
  by (*fast dest*: *cpo elim*: *is-lub-lub*)

lemma *thelubE*: $[\![chain\ S;\ (\bigsqcup i.\ S\ i) = l]\!] \Longrightarrow range\ S <<|\ l$
  by (*blast dest*: *cpo intro*: *is-lub-lub*)

Properties of the lub

lemma *is-ub-thelub*: $chain\ S \Longrightarrow S\ x \sqsubseteq (\bigsqcup i.\ S\ i)$
  by (*blast dest*: *cpo intro*: *is-lub-lub* [*THEN is-lub-rangeD1*])

lemma *is-lub-thelub*: $[\![chain\ S;\ range\ S <|\ x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
  by (*blast dest*: *cpo intro*: *is-lub-lub* [*THEN is-lubD2*])

lemma *lub-below-iff*: $chain\ S \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x \longleftrightarrow (\forall i.\ S\ i \sqsubseteq x)$
  by (*simp add*: *is-lub-below-iff* [*OF cpo-lubI*] *is-ub-def*)

lemma *lub-below*: $[\![chain\ S;\ \bigwedge i.\ S\ i \sqsubseteq x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
  by (*simp add*: *lub-below-iff*)

lemma *below-lub*: $[\![chain\ S;\ x \sqsubseteq S\ i]\!] \Longrightarrow x \sqsubseteq (\bigsqcup i.\ S\ i)$
  by (*erule below-trans*, *erule is-ub-thelub*)

lemma *lub-range-mono*: $[\![range\ X \subseteq range\ Y;\ chain\ Y;\ chain\ X]\!] \Longrightarrow (\bigsqcup i.\ X\ i) \sqsubseteq (\bigsqcup i.\ Y\ i)$
  apply (*erule lub-below*)
  apply (*subgoal-tac* $\exists j.\ X\ i = Y\ j$)
   apply *clarsimp*
   apply (*erule is-ub-thelub*)
  apply *auto*
  done

lemma *lub-range-shift*: $chain\ Y \Longrightarrow (\bigsqcup i.\ Y\ (i + j)) = (\bigsqcup i.\ Y\ i)$
  apply (*rule below-antisym*)
   apply (*rule lub-range-mono*)
     apply *fast*
    apply *assumption*
   apply (*erule chain-shift*)
  apply (*rule lub-below*)
   apply *assumption*
  apply (*rule-tac* $i=i$ **in** *below-lub*)
   apply (*erule chain-shift*)
  apply (*erule chain-mono*)
  apply (*rule le-add1*)
  done

```
lemma maxinch-is-thelub: chain Y ⟹ max-in-chain i Y = ((⨆i. Y i) = Y i)
  apply (rule iffI)
   apply (fast intro!: lub-eqI lub-finch1)
  apply (unfold max-in-chain-def)
  apply (safe intro!: below-antisym)
   apply (fast elim!: chain-mono)
  apply (drule sym)
  apply (force elim!: is-ub-thelub)
  done
```

the ⊑ relation between two chains is preserved by their lubs

```
lemma lub-mono: ⟦chain X; chain Y; ⋀i. X i ⊑ Y i⟧ ⟹ (⨆i. X i) ⊑ (⨆i. Y i)
  by (fast elim: lub-below below-lub)
```

the = relation between two chains is preserved by their lubs

```
lemma lub-eq: (⋀i. X i = Y i) ⟹ (⨆i. X i) = (⨆i. Y i)
  by simp

lemma ch2ch-lub:
  assumes 1: ⋀j. chain (λi. Y i j)
  assumes 2: ⋀i. chain (λj. Y i j)
  shows chain (λi. ⨆j. Y i j)
  apply (rule chainI)
  apply (rule lub-mono [OF 2 2])
  apply (rule chainE [OF 1])
  done

lemma diag-lub:
  assumes 1: ⋀j. chain (λi. Y i j)
  assumes 2: ⋀i. chain (λj. Y i j)
  shows (⨆i. ⨆j. Y i j) = (⨆i. Y i i)
proof (rule below-antisym)
  have 3: chain (λi. Y i i)
    apply (rule chainI)
    apply (rule below-trans)
     apply (rule chainE [OF 1])
    apply (rule chainE [OF 2])
    done
  have 4: chain (λi. ⨆j. Y i j)
    by (rule ch2ch-lub [OF 1 2])
  show (⨆i. ⨆j. Y i j) ⊑ (⨆i. Y i i)
    apply (rule lub-below [OF 4])
    apply (rule lub-below [OF 2])
    apply (rule below-lub [OF 3])
    apply (rule below-trans)
     apply (rule chain-mono [OF 1 max.cobounded1])
    apply (rule chain-mono [OF 2 max.cobounded2])
    done
```

```
  show (⨆i. Y i i) ⊑ (⨆i. ⨆j. Y i j)
    apply (rule lub-mono [OF 3 4])
    apply (rule is-ub-thelub [OF 2])
    done
qed

lemma ex-lub:
  assumes 1: ⋀j. chain (λi. Y i j)
  assumes 2: ⋀i. chain (λj. Y i j)
  shows (⨆i. ⨆j. Y i j) = (⨆j. ⨆i. Y i j)
  by (simp add: diag-lub 1 2)

end
```

2.2 Pointed cpos

The class pcpo of pointed cpos

```
class pcpo = cpo +
  assumes least: ∃x. ∀y. x ⊑ y
begin

definition bottom :: 'a  (‹⊥›)
  where bottom = (THE x. ∀y. x ⊑ y)

lemma minimal [iff]: ⊥ ⊑ x
  unfolding bottom-def
  apply (rule the1I2)
   apply (rule ex-ex1I)
    apply (rule least)
   apply (blast intro: below-antisym)
  apply simp
  done

end
```

Old "UU" syntax:

```
abbreviation (input) UU ≡ bottom
```

Simproc to rewrite $\bot = x$ to $x = \bot$.

```
setup ‹Reorient-Proc.add (fn Const- ‹bottom -› => true | - => false)›
simproc-setup reorient-bottom (⊥ = x) = ‹K Reorient-Proc.proc›
```

useful lemmas about $\bot$

```
lemma below-bottom-iff [simp]: x ⊑ ⊥ ⟷ x = ⊥
  by (simp add: po-eq-conv)

lemma eq-bottom-iff: x = ⊥ ⟷ x ⊑ ⊥
  by simp
```

lemma *bottomI*: $x \sqsubseteq \bot \Longrightarrow x = \bot$
  by (*subst eq-bottom-iff*)

lemma *lub-eq-bottom-iff*: *chain* $Y \Longrightarrow (\bigsqcup i.\ Y\ i) = \bot \longleftrightarrow (\forall i.\ Y\ i = \bot)$
  by (*simp only*: *eq-bottom-iff lub-below-iff*)

2.3 Chain-finite and flat cpos

further useful classes for HOLCF domains

class *chfin* = *po* +
  assumes *chfin*: *chain* $Y \Longrightarrow \exists n.$ *max-in-chain* $n\ Y$
begin

subclass *cpo*
  apply *standard*
  apply (*frule chfin*)
  apply (*blast intro*: *lub-finch1*)
  done

lemma *chfin2finch*: *chain* $Y \Longrightarrow$ *finite-chain* Y
  by (*simp add*: *chfin finite-chain-def*)

end

class *flat* = *pcpo* +
  assumes *ax-flat*: $x \sqsubseteq y \Longrightarrow x = \bot \lor x = y$
begin

subclass *chfin*
proof
  fix Y
  assume $*$: *chain* Y
  show $\exists n.$ *max-in-chain* $n\ Y$
    apply (*unfold max-in-chain-def*)
    apply (*cases* $\forall i.\ Y\ i = \bot$)
     apply *simp*
    apply *simp*
    apply (*erule exE*)
    apply (*rule-tac x=i* **in** *exI*)
    apply *clarify*
    using $*$ **apply** (*blast dest*: *chain-mono ax-flat*)
    done
qed

lemma *flat-below-iff*: $x \sqsubseteq y \longleftrightarrow x = \bot \lor x = y$
  by (*safe dest!*: *ax-flat*)

lemma *flat-eq*: $a \neq \bot \Longrightarrow a \sqsubseteq b = (a = b)$

by (*safe dest*!: *ax-flat*)

end

2.4 Discrete cpos

class *discrete-cpo* = *below* +
assumes *discrete-cpo* [*simp*]: $x \sqsubseteq y \longleftrightarrow x = y$
begin

subclass *po*
by *standard simp-all*

In a discrete cpo, every chain is constant

lemma *discrete-chain-const*:
assumes *S*: *chain S*
shows $\exists x.\ S = (\lambda i.\ x)$
proof (*intro exI ext*)
fix *i* :: *nat*
from *S le0* **have** $S\ 0 \sqsubseteq S\ i$ **by** (*rule chain-mono*)
then have $S\ 0 = S\ i$ **by** *simp*
then show $S\ i = S\ 0$ **by** (*rule sym*)
qed

subclass *chfin*
proof
fix *S* :: *nat* ⇒ *'a*
assume *S*: *chain S*
then have $\exists x.\ S = (\lambda i.\ x)$
by (*rule discrete-chain-const*)
then have *max-in-chain 0 S*
by (*auto simp*: *max-in-chain-def*)
then show $\exists i.$ *max-in-chain i S* **..**
qed

end

3 Continuity and monotonicity

3.1 Definitions

definition *monofun* :: (*'a*::*po* ⇒ *'b*::*po*) ⇒ *bool* — monotonicity
where *monofun f* ⟷ ($\forall x\ y.\ x \sqsubseteq y \longrightarrow f\ x \sqsubseteq f\ y$)

definition *cont* :: (*'a* ⇒ *'b*) ⇒ *bool*
where *cont f* = ($\forall Y.$ *chain Y* ⟶ *range* ($\lambda i.\ f\ (Y\ i)$) <<| *f* ($\bigsqcup i.\ Y\ i$))

lemma *contI*: ($\bigwedge Y.$ *chain Y* ⟹ *range* ($\lambda i.\ f\ (Y\ i)$) <<| *f* ($\bigsqcup i.\ Y\ i$)) ⟹ *cont f*
by (*simp add*: *cont-def*)

lemma *contE*: $cont\ f \Longrightarrow chain\ Y \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i)$
  by (*simp add*: *cont-def*)

lemma *monofunI*: $(\bigwedge x\ y.\ x \sqsubseteq y \Longrightarrow f\ x \sqsubseteq f\ y) \Longrightarrow monofun\ f$
  by (*simp add*: *monofun-def*)

lemma *monofunE*: $monofun\ f \Longrightarrow x \sqsubseteq y \Longrightarrow f\ x \sqsubseteq f\ y$
  by (*simp add*: *monofun-def*)

3.2 Equivalence of alternate definition

monotone functions map chains to chains

lemma *ch2ch-monofun*: $monofun\ f \Longrightarrow chain\ Y \Longrightarrow chain\ (\lambda i.\ f\ (Y\ i))$
  apply (*rule chainI*)
  apply (*erule monofunE*)
  apply (*erule chainE*)
  done

monotone functions map upper bound to upper bounds

lemma *ub2ub-monofun*: $monofun\ f \Longrightarrow range\ Y <|\ u \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <|\ f\ u$
  apply (*rule ub-rangeI*)
  apply (*erule monofunE*)
  apply (*erule ub-rangeD*)
  done

a lemma about binary chains

lemma *binchain-cont*: $cont\ f \Longrightarrow x \sqsubseteq y \Longrightarrow range\ (\lambda i{::}nat.\ f\ (if\ i = 0\ then\ x\ else\ y)) <<|\ f\ y$
  apply (*subgoal-tac* $f\ (\bigsqcup i{::}nat.\ if\ i = 0\ then\ x\ else\ y) = f\ y$)
   apply (*erule subst*)
   apply (*erule contE*)
   apply (*erule bin-chain*)
  apply (*rule-tac f=f* **in** *arg-cong*)
  apply (*erule is-lub-bin-chain* [*THEN lub-eqI*])
  done

continuity implies monotonicity

lemma *cont2mono*: $cont\ f \Longrightarrow monofun\ f$
  apply (*rule monofunI*)
  apply (*drule* (*1*) *binchain-cont*)
  apply (*drule-tac i=0* **in** *is-lub-rangeD1*)
  apply *simp*
  done

lemmas *cont2monofunE* = *cont2mono* [*THEN monofunE*]

lemmas *ch2ch-cont* = *cont2mono* [*THEN ch2ch-monofun*]

continuity implies preservation of lubs

lemma *cont2contlubE*: *cont f* $\Longrightarrow$ *chain Y* $\Longrightarrow$ f ($\bigsqcup i.\ Y\ i$) = ($\bigsqcup i.\ f\ (Y\ i)$)
  apply (*rule lub-eqI* [*symmetric*])
  apply (*erule* (*1*) *contE*)
  done

lemma *contI2*:
  fixes f :: $'a \Rightarrow 'b$
  assumes *mono*: *monofun f*
  assumes *below*: $\bigwedge Y.\ [\![chain\ Y;\ chain\ (\lambda i.\ f\ (Y\ i))]\!] \Longrightarrow f\ (\bigsqcup i.\ Y\ i) \sqsubseteq (\bigsqcup i.\ f$
$(Y\ i))$
  shows *cont f*
proof (*rule contI*)
  fix Y :: $nat \Rightarrow 'a$
  assume Y: *chain Y*
  with *mono* **have** *fY*: *chain* ($\lambda i.\ f\ (Y\ i)$)
    by (*rule ch2ch-monofun*)
  have ($\bigsqcup i.\ f\ (Y\ i)$) = f ($\bigsqcup i.\ Y\ i$)
    apply (*rule below-antisym*)
     apply (*rule lub-below* [*OF fY*])
     apply (*rule monofunE* [*OF mono*])
     apply (*rule is-ub-thelub* [*OF Y*])
    apply (*rule below* [*OF Y fY*])
    done
  with *fY* **show** *range* ($\lambda i.\ f\ (Y\ i)$) <<| f ($\bigsqcup i.\ Y\ i$)
    by (*rule thelubE*)
qed

3.3 Collection of continuity rules

named-theorems *cont2cont continuity intro rule*

3.4 Continuity of basic functions

The identity function is continuous

lemma *cont-id* [*simp*, *cont2cont*]: *cont* ($\lambda x.\ x$)
  apply (*rule contI*)
  apply (*erule cpo-lubI*)
  done

constant functions are continuous

lemma *cont-const* [*simp*, *cont2cont*]: *cont* ($\lambda x.\ c$)
  using *is-lub-const* **by** (*rule contI*)

application of functions is continuous

lemma *cont-apply*:

fixes f :: $'a \Rightarrow 'b \Rightarrow 'c$ **and** t :: $'a \Rightarrow 'b$
assumes *1*: *cont* $(\lambda x.\ t\ x)$
assumes *2*: $\bigwedge x.$ *cont* $(\lambda y.\ f\ x\ y)$
assumes *3*: $\bigwedge y.$ *cont* $(\lambda x.\ f\ x\ y)$
shows *cont* $(\lambda x.\ (f\ x)\ (t\ x))$
proof (*rule contI2* [*OF monofunI*])
fix $x\ y$ :: $'a$
assume $x \sqsubseteq y$
then show $f\ x\ (t\ x) \sqsubseteq f\ y\ (t\ y)$
by (*auto intro*: *cont2monofunE* [*OF 1*]
cont2monofunE [*OF 2*]
cont2monofunE [*OF 3*]
below-trans)
next
fix Y :: $nat \Rightarrow 'a$
assume *chain Y*
then show $f\ (\bigsqcup i.\ Y\ i)\ (t\ (\bigsqcup i.\ Y\ i)) \sqsubseteq (\bigsqcup i.\ f\ (Y\ i)\ (t\ (Y\ i)))$
by (*simp only*: *cont2contlubE* [*OF 1*] *ch2ch-cont* [*OF 1*]
cont2contlubE [*OF 2*] *ch2ch-cont* [*OF 2*]
cont2contlubE [*OF 3*] *ch2ch-cont* [*OF 3*]
diag-lub below-refl)
qed

lemma *cont-compose*: *cont c* $\Longrightarrow$ *cont* $(\lambda x.\ f\ x)$ $\Longrightarrow$ *cont* $(\lambda x.\ c\ (f\ x))$
by (*rule cont-apply* [*OF - - cont-const*])

Least upper bounds preserve continuity

lemma *cont2cont-lub* [*simp*]:
assumes *chain*: $\bigwedge x.$ *chain* $(\lambda i.\ F\ i\ x)$
and *cont*: $\bigwedge i.$ *cont* $(\lambda x.\ F\ i\ x)$
shows *cont* $(\lambda x.\ \bigsqcup i.\ F\ i\ x)$
apply (*rule contI2*)
apply (*simp add*: *monofunI cont2monofunE* [*OF cont*] *lub-mono chain*)
apply (*simp add*: *cont2contlubE* [*OF cont*])
apply (*simp add*: *diag-lub ch2ch-cont* [*OF cont*] *chain*)
done

if-then-else is continuous

lemma *cont-if* [*simp*, *cont2cont*]: *cont f* $\Longrightarrow$ *cont g* $\Longrightarrow$ *cont* $(\lambda x.$ *if b then f x else g x*$)$
by (*induct b*) *simp-all*

3.5 Finite chains and flat pcpos

Monotone functions map finite chains to finite chains.

lemma *monofun-finch2finch*: *monofun f* $\Longrightarrow$ *finite-chain Y* $\Longrightarrow$ *finite-chain* $(\lambda n.\ f\ (Y\ n))$
by (*force simp add*: *finite-chain-def ch2ch-monofun max-in-chain-def*)

The same holds for continuous functions.

lemma *cont-finch2finch*: *cont f* $\Longrightarrow$ *finite-chain Y* $\Longrightarrow$ *finite-chain* ($\lambda n.\ f\ (Y\ n)$)
by (*rule cont2mono* [*THEN monofun-finch2finch*])

All monotone functions with chain-finite domain are continuous.

lemma *chfindom-monofun2cont*: *monofun f* $\Longrightarrow$ *cont f*
for f :: *'a::chfin* $\Rightarrow$ *'b*
apply (*erule contI2*)
apply (*frule chfin2finch*)
apply (*clarsimp simp add*: *finite-chain-def*)
apply (*subgoal-tac max-in-chain i* ($\lambda i.\ f\ (Y\ i)$))
apply (*simp add*: *maxinch-is-thelub ch2ch-monofun*)
apply (*force simp add*: *max-in-chain-def*)
done

All strict functions with flat domain are continuous.

lemma *flatdom-strict2mono*: $f\ \bot = \bot \Longrightarrow$ *monofun f*
for f :: *'a::flat* $\Rightarrow$ *'b::pcpo*
apply (*rule monofunI*)
apply (*drule ax-flat*)
apply *auto*
done

lemma *flatdom-strict2cont*: $f\ \bot = \bot \Longrightarrow$ *cont f*
for f :: *'a::flat* $\Rightarrow$ *'b::pcpo*
by (*rule flatdom-strict2mono* [*THEN chfindom-monofun2cont*])

All functions with discrete domain are continuous.

lemma *cont-discrete-cpo* [*simp*, *cont2cont*]: *cont f*
for f :: *'a::discrete-cpo* $\Rightarrow$ *'b*
apply (*rule contI*)
apply (*drule discrete-chain-const*, *clarify*)
apply *simp*
done

4 Admissibility and compactness

4.1 Definitions

context *cpo*
begin

definition *adm* :: (*'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool*
where *adm P* $\longleftrightarrow$ ($\forall Y.$ *chain Y* $\longrightarrow$ ($\forall i.\ P\ (Y\ i)$) $\longrightarrow P\ (\bigsqcup i.\ Y\ i)$)

lemma *admI*: ($\bigwedge Y.$ $[\![$*chain Y*; $\forall i.\ P\ (Y\ i)]\!] \Longrightarrow P\ (\bigsqcup i.\ Y\ i)$) $\Longrightarrow$ *adm P*
unfolding *adm-def* **by** *fast*

lemma *admD*: $adm\ P \Longrightarrow chain\ Y \Longrightarrow (\bigwedge i.\ P\ (Y\ i)) \Longrightarrow P\ (\bigsqcup i.\ Y\ i)$
unfolding *adm-def* **by** *fast*

lemma *admD2*: $adm\ (\lambda x.\ \neg\ P\ x) \Longrightarrow chain\ Y \Longrightarrow P\ (\bigsqcup i.\ Y\ i) \Longrightarrow \exists\, i.\ P\ (Y\ i)$
unfolding *adm-def* **by** *fast*

lemma *triv-admI*: $\forall\, x.\ P\ x \Longrightarrow adm\ P$
by (*rule admI*) (*erule spec*)

end

4.2 Admissibility on chain-finite types

For chain-finite (easy) types every formula is admissible.

lemma *adm-chfin* [*simp*]: *adm P* **for** $P :: 'a{::}chfin \Rightarrow bool$
by (*rule admI*, *frule chfin*, *auto simp add*: *maxinch-is-thelub*)

4.3 Admissibility of special formulae and propagation

context *cpo*
begin

lemma *adm-const* [*simp*]: $adm\ (\lambda x.\ t)$
by (*rule admI*, *simp*)

lemma *adm-conj* [*simp*]: $adm\ (\lambda x.\ P\ x) \Longrightarrow adm\ (\lambda x.\ Q\ x) \Longrightarrow adm\ (\lambda x.\ P\ x \wedge Q\ x)$
by (*fast intro*: *admI elim*: *admD*)

lemma *adm-all* [*simp*]: $(\bigwedge y.\ adm\ (\lambda x.\ P\ x\ y)) \Longrightarrow adm\ (\lambda x.\ \forall\, y.\ P\ x\ y)$
by (*fast intro*: *admI elim*: *admD*)

lemma *adm-ball* [*simp*]: $(\bigwedge y.\ y \in A \Longrightarrow adm\ (\lambda x.\ P\ x\ y)) \Longrightarrow adm\ (\lambda x.\ \forall\, y{\in}A.\ P\ x\ y)$
by (*fast intro*: *admI elim*: *admD*)

Admissibility for disjunction is hard to prove. It requires 2 lemmas.

lemma *adm-disj-lemma1*:
assumes *adm*: *adm P*
assumes *chain*: *chain Y*
assumes *P*: $\forall\, i.\ \exists\, j{\geq}i.\ P\ (Y\ j)$
shows $P\ (\bigsqcup i.\ Y\ i)$
proof −
define *f* **where** $f\ i = (LEAST\ j.\ i \leq j \wedge P\ (Y\ j))$ **for** *i*
have *chain'*: $chain\ (\lambda i.\ Y\ (f\ i))$
unfolding *f-def*
apply (*rule chainI*)
apply (*rule chain-mono* [*OF chain*])
apply (*rule Least-le*)

apply (*rule LeastI2-ex*)
apply (*simp-all add*: *P*)
done
have *f1*: $\bigwedge i.\ i \leq f\ i$ **and** *f2*: $\bigwedge i.\ P\ (Y\ (f\ i))$
using *LeastI-ex* [*OF P* [*rule-format*]] **by** (*simp-all add*: *f-def*)
have *lub-eq*: $(\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ Y\ (f\ i))$
apply (*rule below-antisym*)
apply (*rule lub-mono* [*OF chain chain'*])
apply (*rule chain-mono* [*OF chain f1*])
apply (*rule lub-range-mono* [*OF - chain chain'*])
apply *clarsimp*
done
show $P\ (\bigsqcup i.\ Y\ i)$
unfolding *lub-eq* **using** *adm chain' f2* **by** (*rule admD*)
qed

lemma *adm-disj-lemma2*: $\forall n::nat.\ P\ n \vee Q\ n \Longrightarrow (\forall i.\ \exists j \geq i.\ P\ j) \vee (\forall i.\ \exists j \geq i.\ Q\ j)$
apply (*erule contrapos-pp*)
apply (*clarsimp*, *rename-tac a b*)
apply (*rule-tac x=max a b* **in** *exI*)
apply *simp*
done

lemma *adm-disj* [*simp*]: $adm\ (\lambda x.\ P\ x) \Longrightarrow adm\ (\lambda x.\ Q\ x) \Longrightarrow adm\ (\lambda x.\ P\ x \vee Q\ x)$
apply (*rule admI*)
apply (*erule adm-disj-lemma2* [*THEN disjE*])
apply (*erule* (*2*) *adm-disj-lemma1* [*THEN disjI1*])
apply (*erule* (*2*) *adm-disj-lemma1* [*THEN disjI2*])
done

lemma *adm-imp* [*simp*]: $adm\ (\lambda x.\ \neg\ P\ x) \Longrightarrow adm\ (\lambda x.\ Q\ x) \Longrightarrow adm\ (\lambda x.\ P\ x \longrightarrow Q\ x)$
by (*subst imp-conv-disj*) (*rule adm-disj*)

lemma *adm-iff* [*simp*]: $adm\ (\lambda x.\ P\ x \longrightarrow Q\ x) \Longrightarrow adm\ (\lambda x.\ Q\ x \longrightarrow P\ x) \Longrightarrow adm\ (\lambda x.\ P\ x \longleftrightarrow Q\ x)$
by (*subst iff-conv-conj-imp*) (*rule adm-conj*)

end

admissibility and continuity

lemma *adm-below* [*simp*]: $cont\ (\lambda x.\ u\ x) \Longrightarrow cont\ (\lambda x.\ v\ x) \Longrightarrow adm\ (\lambda x.\ u\ x \sqsubseteq v\ x)$
by (*simp add*: *adm-def cont2contlubE lub-mono ch2ch-cont*)

lemma *adm-eq* [*simp*]: $cont\ (\lambda x.\ u\ x) \Longrightarrow cont\ (\lambda x.\ v\ x) \Longrightarrow adm\ (\lambda x.\ u\ x = v\ x)$
by (*simp add*: *po-eq-conv*)

lemma *adm-subst*: *cont* $(\lambda x.\ t\ x) \Longrightarrow$ *adm* $P \Longrightarrow$ *adm* $(\lambda x.\ P\ (t\ x))$
by (*simp add*: *adm-def cont2contlubE ch2ch-cont*)

lemma *adm-not-below* [*simp*]: *cont* $(\lambda x.\ t\ x) \Longrightarrow$ *adm* $(\lambda x.\ t\ x \not\sqsubseteq u)$
by (*rule admI*) (*simp add*: *cont2contlubE ch2ch-cont lub-below-iff*)

4.4 Compactness

context *cpo*
begin

definition *compact* :: $'a \Rightarrow$ *bool*
where *compact* $k =$ *adm* $(\lambda x.\ k \not\sqsubseteq x)$

lemma *compactI*: *adm* $(\lambda x.\ k \not\sqsubseteq x) \Longrightarrow$ *compact* k
unfolding *compact-def* **.**

lemma *compactD*: *compact* $k \Longrightarrow$ *adm* $(\lambda x.\ k \not\sqsubseteq x)$
unfolding *compact-def* **.**

lemma *compactI2*: $(\bigwedge Y.\ [\![chain\ Y;\ x \sqsubseteq (\bigsqcup i.\ Y\ i)]\!] \Longrightarrow \exists i.\ x \sqsubseteq Y\ i) \Longrightarrow$ *compact* x
unfolding *compact-def adm-def* **by** *fast*

lemma *compactD2*: *compact* $x \Longrightarrow$ *chain* $Y \Longrightarrow x \sqsubseteq (\bigsqcup i.\ Y\ i) \Longrightarrow \exists i.\ x \sqsubseteq Y\ i$
unfolding *compact-def adm-def* **by** *fast*

lemma *compact-below-lub-iff*: *compact* $x \Longrightarrow$ *chain* $Y \Longrightarrow x \sqsubseteq (\bigsqcup i.\ Y\ i) \longleftrightarrow (\exists i.\ x \sqsubseteq Y\ i)$
by (*fast intro*: *compactD2 elim*: *below-lub*)

end

lemma *compact-chfin* [*simp*]: *compact* x **for** x :: $'a$::*chfin*
by (*rule compactI* [*OF adm-chfin*])

lemma *compact-imp-max-in-chain*: *chain* $Y \Longrightarrow$ *compact* $(\bigsqcup i.\ Y\ i) \Longrightarrow \exists i.$ *max-in-chain* $i\ Y$
apply (*drule* (*1*) *compactD2*, *simp*)
apply (*erule exE*, *rule-tac x=i* **in** *exI*)
apply (*rule max-in-chainI*)
apply (*rule below-antisym*)
apply (*erule* (*1*) *chain-mono*)
apply (*erule* (*1*) *below-trans* [*OF is-ub-thelub*])
done

admissibility and compactness

lemma *adm-compact-not-below* [*simp*]:

compact k $\Longrightarrow$ *cont* ($\lambda x.$ *t x*) $\Longrightarrow$ *adm* ($\lambda x.$ *k* $\not\sqsubseteq$ *t x*)
unfolding *compact-def* **by** (*rule adm-subst*)

lemma *adm-neq-compact* [*simp*]: *compact k* $\Longrightarrow$ *cont* ($\lambda x.$ *t x*) $\Longrightarrow$ *adm* ($\lambda x.$ *t x* $\neq$ *k*)
by (*simp add*: *po-eq-conv*)

lemma *adm-compact-neq* [*simp*]: *compact k* $\Longrightarrow$ *cont* ($\lambda x.$ *t x*) $\Longrightarrow$ *adm* ($\lambda x.$ *k* $\neq$ *t x*)
by (*simp add*: *po-eq-conv*)

lemma *compact-bottom* [*simp*, *intro*]: *compact* $\bot$
by (*rule compactI*) *simp*

Any upward-closed predicate is admissible.

lemma *adm-upward*:
assumes *P*: $\bigwedge x\ y.$ $[\![P\ x;\ x \sqsubseteq y]\!] \Longrightarrow P\ y$
shows *adm P*
by (*rule admI*, *drule spec*, *erule P*, *erule is-ub-thelub*)

lemmas *adm-lemmas* =
adm-const adm-conj adm-all adm-ball adm-disj adm-imp adm-iff
adm-below adm-eq adm-not-below
adm-compact-not-below adm-compact-neq adm-neq-compact

5 Class instances for the full function space

5.1 Full function space is a partial order

instantiation *fun* :: (*type*, *below*) *below*
begin

definition *below-fun-def*: ($\sqsubseteq$) $\equiv$ ($\lambda f\ g.$ $\forall x.$ $f\ x \sqsubseteq g\ x$)

instance ..
end

instance *fun* :: (*type*, *po*) *po*
proof
fix $f\ g\ h$ :: $'a \Rightarrow\ 'b$
show $f \sqsubseteq f$
by (*simp add*: *below-fun-def*)
show $f \sqsubseteq g \Longrightarrow g \sqsubseteq f \Longrightarrow f = g$
by (*simp add*: *below-fun-def fun-eq-iff below-antisym*)
show $f \sqsubseteq g \Longrightarrow g \sqsubseteq h \Longrightarrow f \sqsubseteq h$
unfolding *below-fun-def* **by** (*fast elim*: *below-trans*)
qed

lemma *fun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall x.\ f\ x \sqsubseteq g\ x)$

by (*simp add*: *below-fun-def*)

lemma *fun-belowI*: $(\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow f \sqsubseteq g$
by (*simp add*: *below-fun-def*)

lemma *fun-belowD*: $f \sqsubseteq g \Longrightarrow f\ x \sqsubseteq g\ x$
by (*simp add*: *below-fun-def*)

5.2 Full function space is chain complete

Properties of chains of functions.

lemma *fun-chain-iff*: $\mathit{chain}\ S \longleftrightarrow (\forall x.\ \mathit{chain}\ (\lambda i.\ S\ i\ x))$
by (*auto simp*: *chain-def fun-below-iff*)

lemma *ch2ch-fun*: $\mathit{chain}\ S \Longrightarrow \mathit{chain}\ (\lambda i.\ S\ i\ x)$
by (*simp add*: *chain-def below-fun-def*)

lemma *ch2ch-lambda*: $(\bigwedge x.\ \mathit{chain}\ (\lambda i.\ S\ i\ x)) \Longrightarrow \mathit{chain}\ S$
by (*simp add*: *chain-def below-fun-def*)

Type $'a \Rightarrow\ 'b$ is chain complete

lemma *is-lub-lambda*: $(\bigwedge x.\ \mathit{range}\ (\lambda i.\ Y\ i\ x) <<|\ f\ x) \Longrightarrow \mathit{range}\ Y <<|\ f$
by (*simp add*: *is-lub-def is-ub-def below-fun-def*)

lemma *is-lub-fun*: $\mathit{chain}\ S \Longrightarrow \mathit{range}\ S <<|\ (\lambda x.\ \bigsqcup i.\ S\ i\ x)$
for $S :: \mathit{nat} \Rightarrow\ 'a::\mathit{type} \Rightarrow\ 'b$
apply (*rule is-lub-lambda*)
apply (*rule cpo-lubI*)
apply (*erule ch2ch-fun*)
done

lemma *lub-fun*: $\mathit{chain}\ S \Longrightarrow (\bigsqcup i.\ S\ i) = (\lambda x.\ \bigsqcup i.\ S\ i\ x)$
for $S :: \mathit{nat} \Rightarrow\ 'a::\mathit{type} \Rightarrow\ 'b$
by (*rule is-lub-fun* [*THEN lub-eqI*])

instance *fun* :: (*type*, *cpo*) *cpo*
by *intro-classes* (*rule exI*, *erule is-lub-fun*)

instance *fun* :: (*type*, *discrete-cpo*) *discrete-cpo*
proof
fix $f\ g :: 'a \Rightarrow\ 'b$
show $f \sqsubseteq g \longleftrightarrow f = g$
by (*simp add*: *fun-below-iff fun-eq-iff*)
qed

5.3 Full function space is pointed

lemma *minimal-fun*: $(\lambda x.\ \bot) \sqsubseteq f$
by (*simp add*: *below-fun-def*)

instance *fun* :: (*type*, *pcpo*) *pcpo*
by *standard* (*fast intro*: *minimal-fun*)

lemma *inst-fun-pcpo*: $\bot = (\lambda x.\ \bot)$
by (*rule minimal-fun* [*THEN bottomI*, *symmetric*])

lemma *app-strict* [*simp*]: $\bot\ x = \bot$
by (*simp add*: *inst-fun-pcpo*)

lemma *lambda-strict*: $(\lambda x.\ \bot) = \bot$
by (*rule bottomI*, *rule minimal-fun*)

5.4 Propagation of monotonicity and continuity

The lub of a chain of monotone functions is monotone.

lemma *adm-monofun*: *adm monofun*
by (*rule admI*) (*simp add*: *lub-fun fun-chain-iff monofun-def lub-mono*)

The lub of a chain of continuous functions is continuous.

lemma *adm-cont*: *adm cont*
by (*rule admI*) (*simp add*: *lub-fun fun-chain-iff*)

Function application preserves monotonicity and continuity.

lemma *mono2mono-fun*: *monofun f* $\Longrightarrow$ *monofun* $(\lambda x.\ f\ x\ y)$
by (*simp add*: *monofun-def fun-below-iff*)

lemma *cont2cont-fun*: *cont f* $\Longrightarrow$ *cont* $(\lambda x.\ f\ x\ y)$
apply (*rule contI2*)
apply (*erule cont2mono* [*THEN mono2mono-fun*])
apply (*simp add*: *cont2contlubE lub-fun ch2ch-cont*)
done

lemma *cont-fun*: *cont* $(\lambda f.\ f\ x)$
using *cont-id* **by** (*rule cont2cont-fun*)

Lambda abstraction preserves monotonicity and continuity. (Note $(\lambda x.\ \lambda y.\ f\ x\ y) = f$.)

lemma *mono2mono-lambda*: $(\bigwedge y.$ *monofun* $(\lambda x.\ f\ x\ y)) \Longrightarrow$ *monofun f*
by (*simp add*: *monofun-def fun-below-iff*)

lemma *cont2cont-lambda* [*simp*]:
assumes *f*: $\bigwedge y.$ *cont* $(\lambda x.\ f\ x\ y)$
shows *cont f*
by (*rule contI*, *rule is-lub-lambda*, *rule contE* [*OF f*])

What D.A.Schmidt calls continuity of abstraction; never used here

lemma *contlub-lambda*: $(\bigwedge x.\ chain\ (\lambda i.\ S\ i\ x)) \Longrightarrow (\lambda x.\ \bigsqcup i.\ S\ i\ x) = (\bigsqcup i.\ (\lambda x.\ S\ i\ x))$
  for $S :: nat \Rightarrow 'a::type \Rightarrow 'b$
  by (*simp add*: *lub-fun ch2ch-lambda*)

6 The cpo of cartesian products

6.1 Unit type is a pcpo

instantiation *unit* :: *discrete-cpo*
begin

definition *below-unit-def* [*simp*]: $x \sqsubseteq (y::unit) \longleftrightarrow True$

instance
  by *standard simp*

end

instance *unit* :: *pcpo*
  by *standard simp*

6.2 Product type is a partial order

instantiation *prod* :: (*below*, *below*) *below*
begin

definition *below-prod-def*: $(\sqsubseteq) \equiv \lambda p1\ p2.\ (fst\ p1 \sqsubseteq fst\ p2 \land snd\ p1 \sqsubseteq snd\ p2)$

instance ..

end

instance *prod* :: (*po*, *po*) *po*
proof
  fix $x\ y\ z :: 'a \times 'b$
  show $x \sqsubseteq x$
    by (*simp add*: *below-prod-def*)
  show $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow x = y$
    unfolding *below-prod-def prod-eq-iff*
    by (*fast intro*: *below-antisym*)
  show $x \sqsubseteq y \Longrightarrow y \sqsubseteq z \Longrightarrow x \sqsubseteq z$
    unfolding *below-prod-def*
    by (*fast intro*: *below-trans*)
qed

6.3 Monotonicity of *Pair*, *fst*, *snd*

lemma *prod-belowI*: $fst\ p \sqsubseteq fst\ q \Longrightarrow snd\ p \sqsubseteq snd\ q \Longrightarrow p \sqsubseteq q$

by (*simp add*: *below-prod-def*)

lemma *Pair-below-iff* [*simp*]: $(a, b) \sqsubseteq (c, d) \longleftrightarrow a \sqsubseteq c \wedge b \sqsubseteq d$
by (*simp add*: *below-prod-def*)

Pair (-,-) is monotone in both arguments

lemma *monofun-pair1*: *monofun* $(\lambda x. (x, y))$
by (*simp add*: *monofun-def*)

lemma *monofun-pair2*: *monofun* $(\lambda y. (x, y))$
by (*simp add*: *monofun-def*)

lemma *monofun-pair*: $x1 \sqsubseteq x2 \Longrightarrow y1 \sqsubseteq y2 \Longrightarrow (x1, y1) \sqsubseteq (x2, y2)$
by *simp*

lemma *ch2ch-Pair* [*simp*]: *chain X* $\Longrightarrow$ *chain Y* $\Longrightarrow$ *chain* $(\lambda i. (X\ i, Y\ i))$
by (*rule chainI*, *simp add*: *chainE*)

fst and *snd* are monotone

lemma *fst-monofun*: $x \sqsubseteq y \Longrightarrow fst\ x \sqsubseteq fst\ y$
by (*simp add*: *below-prod-def*)

lemma *snd-monofun*: $x \sqsubseteq y \Longrightarrow snd\ x \sqsubseteq snd\ y$
by (*simp add*: *below-prod-def*)

lemma *monofun-fst*: *monofun fst*
by (*simp add*: *monofun-def below-prod-def*)

lemma *monofun-snd*: *monofun snd*
by (*simp add*: *monofun-def below-prod-def*)

lemmas *ch2ch-fst* [*simp*] = *ch2ch-monofun* [*OF monofun-fst*]

lemmas *ch2ch-snd* [*simp*] = *ch2ch-monofun* [*OF monofun-snd*]

lemma *prod-chain-cases*:
assumes *chain*: *chain Y*
obtains *A B*
where *chain A* **and** *chain B* **and** $Y = (\lambda i. (A\ i, B\ i))$
proof
from *chain* **show** *chain* $(\lambda i. fst\ (Y\ i))$
by (*rule ch2ch-fst*)
from *chain* **show** *chain* $(\lambda i. snd\ (Y\ i))$
by (*rule ch2ch-snd*)
show $Y = (\lambda i. (fst\ (Y\ i), snd\ (Y\ i)))$
by *simp*
qed

6.4 Product type is a cpo

lemma *is-lub-Pair*: $\mathit{range}\ A <<|\ x \Longrightarrow \mathit{range}\ B <<|\ y \Longrightarrow \mathit{range}\ (\lambda i.\ (A\ i,\ B\ i))$ $<<|\ (x,\ y)$
  by (*simp add*: *is-lub-def is-ub-def below-prod-def*)

lemma *lub-Pair*: $\mathit{chain}\ A \Longrightarrow \mathit{chain}\ B \Longrightarrow (\bigsqcup i.\ (A\ i,\ B\ i)) = (\bigsqcup i.\ A\ i,\ \bigsqcup i.\ B\ i)$
  for $A :: \mathit{nat} \Rightarrow {}'a$ **and** $B :: \mathit{nat} \Rightarrow {}'b$
  by (*fast intro*: *lub-eqI is-lub-Pair elim*: *thelubE*)

lemma *is-lub-prod*:
  fixes $S :: \mathit{nat} \Rightarrow ({}'a \times {}'b)$
  assumes *chain S*
  shows $\mathit{range}\ S <<|\ (\bigsqcup i.\ \mathit{fst}\ (S\ i),\ \bigsqcup i.\ \mathit{snd}\ (S\ i))$
  using *assms* **by** (*auto elim*: *prod-chain-cases simp*: *is-lub-Pair cpo-lubI*)

lemma *lub-prod*: $\mathit{chain}\ S \Longrightarrow (\bigsqcup i.\ S\ i) = (\bigsqcup i.\ \mathit{fst}\ (S\ i),\ \bigsqcup i.\ \mathit{snd}\ (S\ i))$
  for $S :: \mathit{nat} \Rightarrow {}'a \times {}'b$
  by (*rule is-lub-prod* [*THEN lub-eqI*])

instance *prod* :: (*cpo*, *cpo*) *cpo*
proof
  fix $S :: \mathit{nat} \Rightarrow ({}'a \times {}'b)$
  assume *chain S*
  then have $\mathit{range}\ S <<|\ (\bigsqcup i.\ \mathit{fst}\ (S\ i),\ \bigsqcup i.\ \mathit{snd}\ (S\ i))$
    by (*rule is-lub-prod*)
  then show $\exists x.\ \mathit{range}\ S <<|\ x$ **..**
qed

instance *prod* :: (*discrete-cpo*, *discrete-cpo*) *discrete-cpo*
proof
  show $x \sqsubseteq y \longleftrightarrow x = y$ **for** $x\ y :: {}'a \times {}'b$
    by (*simp add*: *below-prod-def prod-eq-iff*)
qed

6.5 Product type is pointed

lemma *minimal-prod*: $(\bot,\ \bot) \sqsubseteq p$
  by (*simp add*: *below-prod-def*)

instance *prod* :: (*pcpo*, *pcpo*) *pcpo*
  by *intro-classes* (*fast intro*: *minimal-prod*)

lemma *inst-prod-pcpo*: $\bot = (\bot,\ \bot)$
  by (*rule minimal-prod* [*THEN bottomI*, *symmetric*])

lemma *Pair-bottom-iff* [*simp*]: $(x,\ y) = \bot \longleftrightarrow x = \bot \wedge y = \bot$
  by (*simp add*: *inst-prod-pcpo*)

lemma *fst-strict* [*simp*]: $\mathit{fst}\ \bot = \bot$

unfolding *inst-prod-pcpo* **by** (*rule fst-conv*)

lemma *snd-strict* [*simp*]: *snd* ⊥ = ⊥
unfolding *inst-prod-pcpo* **by** (*rule snd-conv*)

lemma *Pair-strict* [*simp*]: (⊥, ⊥) = ⊥
by *simp*

lemma *split-strict* [*simp*]: *case-prod f* ⊥ = *f* ⊥ ⊥
by (*simp add*: *split-def*)

6.6 Continuity of *Pair*, *fst*, *snd*

lemma *cont-pair1*: *cont* (λ*x*. (*x*, *y*))
apply (*rule contI*)
apply (*rule is-lub-Pair*)
apply (*erule cpo-lubI*)
apply (*rule is-lub-const*)
done

lemma *cont-pair2*: *cont* (λ*y*. (*x*, *y*))
apply (*rule contI*)
apply (*rule is-lub-Pair*)
apply (*rule is-lub-const*)
apply (*erule cpo-lubI*)
done

lemma *cont-fst*: *cont fst*
apply (*rule contI*)
apply (*simp add*: *lub-prod*)
apply (*erule cpo-lubI* [*OF ch2ch-fst*])
done

lemma *cont-snd*: *cont snd*
apply (*rule contI*)
apply (*simp add*: *lub-prod*)
apply (*erule cpo-lubI* [*OF ch2ch-snd*])
done

lemma *cont2cont-Pair* [*simp*, *cont2cont*]:
assumes *f*: *cont* (λ*x*. *f x*)
assumes *g*: *cont* (λ*x*. *g x*)
shows *cont* (λ*x*. (*f x*, *g x*))
apply (*rule cont-apply* [*OF f cont-pair1*])
apply (*rule cont-apply* [*OF g cont-pair2*])
apply (*rule cont-const*)
done

lemmas *cont2cont-fst* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-fst*]

```
lemmas cont2cont-snd [simp, cont2cont] = cont-compose [OF cont-snd]

lemma cont2cont-case-prod:
  assumes f1: ⋀a b. cont (λx. f x a b)
  assumes f2: ⋀x b. cont (λa. f x a b)
  assumes f3: ⋀x a. cont (λb. f x a b)
  assumes g: cont (λx. g x)
  shows cont (λx. case g x of (a, b) ⇒ f x a b)
  unfolding split-def
  apply (rule cont-apply [OF g])
   apply (rule cont-apply [OF cont-fst f2])
   apply (rule cont-apply [OF cont-snd f3])
   apply (rule cont-const)
  apply (rule f1)
  done

lemma prod-contI:
  assumes f1: ⋀y. cont (λx. f (x, y))
  assumes f2: ⋀x. cont (λy. f (x, y))
  shows cont f
proof –
  have cont (λ(x, y). f (x, y))
    by (intro cont2cont-case-prod f1 f2 cont2cont)
  then show cont f
    by (simp only: case-prod-eta)
qed

lemma prod-cont-iff: cont f ⟷ (∀ y. cont (λx. f (x, y))) ∧ (∀ x. cont (λy. f (x,
y)))
  apply safe
    apply (erule cont-compose [OF - cont-pair1])
   apply (erule cont-compose [OF - cont-pair2])
  apply (simp only: prod-contI)
  done

lemma cont2cont-case-prod′ [simp, cont2cont]:
  assumes f: cont (λp. f (fst p) (fst (snd p)) (snd (snd p)))
  assumes g: cont (λx. g x)
  shows cont (λx. case-prod (f x) (g x))
  using assms by (simp add: cont2cont-case-prod prod-cont-iff)
```

The simple version (due to Joachim Breitner) is needed if either element type of the pair is not a cpo.

```
lemma cont2cont-split-simple [simp, cont2cont]:
  assumes ⋀a b. cont (λx. f x a b)
  shows cont (λx. case p of (a, b) ⇒ f x a b)
  using assms by (cases p) auto
```

Admissibility of predicates on product types.

lemma *adm-case-prod* [*simp*]:
assumes *adm* (λ*x*. *P x* (*fst* (*f x*)) (*snd* (*f x*)))
shows *adm* (λ*x*. *case f x of* (*a*, *b*) ⇒ *P x a b*)
unfolding *case-prod-beta* **using** *assms* **.**

6.7 Compactness and chain-finiteness

lemma *fst-below-iff*: *fst x* ⊑ *y* ⟷ *x* ⊑ (*y*, *snd x*) **for** *x* :: *'a* × *'b*
by (*simp add*: *below-prod-def*)

lemma *snd-below-iff*: *snd x* ⊑ *y* ⟷ *x* ⊑ (*fst x*, *y*) **for** *x* :: *'a* × *'b*
by (*simp add*: *below-prod-def*)

lemma *compact-fst*: *compact x* ⟹ *compact* (*fst x*)
by (*rule compactI*) (*simp add*: *fst-below-iff*)

lemma *compact-snd*: *compact x* ⟹ *compact* (*snd x*)
by (*rule compactI*) (*simp add*: *snd-below-iff*)

lemma *compact-Pair*: *compact x* ⟹ *compact y* ⟹ *compact* (*x*, *y*)
by (*rule compactI*) (*simp add*: *below-prod-def*)

lemma *compact-Pair-iff* [*simp*]: *compact* (*x*, *y*) ⟷ *compact x* ∧ *compact y*
apply (*safe intro*!: *compact-Pair*)
apply (*drule compact-fst*, *simp*)
apply (*drule compact-snd*, *simp*)
done

instance *prod* :: (*chfin*, *chfin*) *chfin*
apply *intro-classes*
apply (*erule compact-imp-max-in-chain*)
apply (*case-tac* ⨆*i*. *Y i*, *simp*)
done

7 Discrete cpo types

datatype *'a discr* = *Discr 'a*::*type*

7.1 Discrete cpo class instance

instantiation *discr* :: (*type*) *discrete-cpo*
begin

definition ((⊑) :: *'a discr* ⇒ *'a discr* ⇒ *bool*) = (=)

instance
by *standard* (*simp add*: *below-discr-def*)

end

7.2 *undiscr*

definition *undiscr* :: *'a::type discr* ⇒ *'a*
where *undiscr x* = (*case x of Discr y* ⇒ *y*)

lemma *undiscr-Discr* [*simp*]: *undiscr* (*Discr x*) = *x*
by (*simp add*: *undiscr-def*)

lemma *Discr-undiscr* [*simp*]: *Discr* (*undiscr y*) = *y*
by (*induct y*) *simp*

end

8 Subtypes of pcpos

theory *Cpodef*
imports *Cpo*
keywords *pcpodef cpodef* :: *thy-goal-defn*
begin

8.1 Proving a subtype is a partial order

A subtype of a partial order is itself a partial order, if the ordering is defined in the standard way.

theorem (**in** *below*) *typedef-class-po*:
fixes *Abs* :: *'b::po* ⇒ *'a*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *class.po below*
apply (*rule class.po.intro*)
apply (*unfold below*)
apply (*rule below-refl*)
apply (*fact below-trans*)
apply (*rule type-definition.Rep-inject* [*OF type*, *THEN iffD1*])
apply (*fact below-antisym*)
done

lemmas *typedef-po-class* = *below.typedef-class-po* [*THEN po.intro-of-class*]

8.2 Proving a subtype is finite

lemma *typedef-finite-UNIV*:
fixes *Abs* :: *'a::type* ⇒ *'b::type*
assumes *type*: *type-definition Rep Abs A*
shows *finite A* ⟹ *finite* (*UNIV* :: *'b set*)
proof −

```
  assume finite A
  then have finite (Abs ‘ A)
    by (rule finite-imageI)
  then show finite (UNIV :: 'b set)
    by (simp only: type-definition.Abs-image [OF type])
qed
```

8.3 Proving a subtype is chain-finite

```
lemma ch2ch-Rep:
  assumes below: (⊑) ≡ λx y. Rep x ⊑ Rep y
  shows chain S ⟹ chain (λi. Rep (S i))
  unfolding chain-def below .

theorem typedef-chfin:
  fixes Abs :: 'a::chfin ⇒ 'b::po
  assumes type: type-definition Rep Abs A
    and below: (⊑) ≡ λx y. Rep x ⊑ Rep y
  shows OFCLASS('b, chfin-class)
  apply intro-classes
  apply (drule ch2ch-Rep [OF below])
  apply (drule chfin)
  apply (unfold max-in-chain-def)
  apply (simp add: type-definition.Rep-inject [OF type])
  done
```

8.4 Proving a subtype is complete

A subtype of a cpo is itself a cpo if the ordering is defined in the standard way, and the defining subset is closed with respect to limits of chains. A set is closed if and only if membership in the set is an admissible predicate.

```
lemma typedef-is-lubI:
  assumes below: (⊑) ≡ λx y. Rep x ⊑ Rep y
  shows range (λi. Rep (S i)) <<| Rep x ⟹ range S <<| x
  by (simp add: is-lub-def is-ub-def below)

lemma Abs-inverse-lub-Rep:
  fixes Abs :: 'a::cpo ⇒ 'b::po
  assumes type: type-definition Rep Abs A
    and below: (⊑) ≡ λx y. Rep x ⊑ Rep y
    and adm: adm (λx. x ∈ A)
  shows chain S ⟹ Rep (Abs (⨆ i. Rep (S i))) = (⨆ i. Rep (S i))
  apply (rule type-definition.Abs-inverse [OF type])
  apply (erule admD [OF adm ch2ch-Rep [OF below]])
  apply (rule type-definition.Rep [OF type])
  done

theorem typedef-is-lub:
  fixes Abs :: 'a::cpo ⇒ 'b::po
```

assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: $adm\ (\lambda x.\ x \in A)$
assumes *S*: *chain S*
shows $range\ S <<|\ Abs\ (\bigsqcup i.\ Rep\ (S\ i))$
proof −
from *S* **have** $chain\ (\lambda i.\ Rep\ (S\ i))$
by (*rule ch2ch-Rep* [*OF below*])
then have $range\ (\lambda i.\ Rep\ (S\ i)) <<|\ (\bigsqcup i.\ Rep\ (S\ i))$
by (*rule cpo-lubI*)
then have $range\ (\lambda i.\ Rep\ (S\ i)) <<|\ Rep\ (Abs\ (\bigsqcup i.\ Rep\ (S\ i)))$
by (*simp only*: *Abs-inverse-lub-Rep* [*OF type below adm S*])
then show $range\ S <<|\ Abs\ (\bigsqcup i.\ Rep\ (S\ i))$
by (*rule typedef-is-lubI* [*OF below*])
qed

lemmas *typedef-lub* = *typedef-is-lub* [*THEN lub-eqI*]

theorem *typedef-cpo*:
fixes *Abs* :: *'a::cpo* ⇒ *'b::po*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: $adm\ (\lambda x.\ x \in A)$
shows *OFCLASS*(*'b*, *cpo-class*)
proof
fix *S* :: *nat* ⇒ *'b*
assume *chain S*
then have $range\ S <<|\ Abs\ (\bigsqcup i.\ Rep\ (S\ i))$
by (*rule typedef-is-lub* [*OF type below adm*])
then show $\exists x.\ range\ S <<|\ x$ **..**
qed

8.4.1 Continuity of *Rep* and *Abs*

For any sub-cpo, the *Rep* function is continuous.

theorem *typedef-cont-Rep*:
fixes *Abs* :: *'a::cpo* ⇒ *'b::cpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: $adm\ (\lambda x.\ x \in A)$
shows $cont\ (\lambda x.\ f\ x) \Longrightarrow cont\ (\lambda x.\ Rep\ (f\ x))$
apply (*erule cont-apply* [*OF - - cont-const*])
apply (*rule contI*)
apply (*simp only*: *typedef-lub* [*OF type below adm*])
apply (*simp only*: *Abs-inverse-lub-Rep* [*OF type below adm*])
apply (*rule cpo-lubI*)
apply (*erule ch2ch-Rep* [*OF below*])
done

For a sub-cpo, we can make the *Abs* function continuous only if we restrict its domain to the defining subset by composing it with another continuous function.

theorem *typedef-cont-Abs*:
  fixes *Abs* :: *'a::cpo* ⇒ *'b::cpo*
  fixes *f* :: *'c::cpo* ⇒ *'a::cpo*
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *adm*: *adm* $(\lambda x.\ x \in A)$
    and *f-in-A*: $\bigwedge x.\ f\ x \in A$
  shows *cont f* $\Longrightarrow$ *cont* $(\lambda x.\ Abs\ (f\ x))$
  unfolding *cont-def is-lub-def is-ub-def ball-simps below*
  by (*simp add*: *type-definition.Abs-inverse* [*OF type f-in-A*])

8.5 Proving subtype elements are compact

theorem *typedef-compact*:
  fixes *Abs* :: *'a::cpo* ⇒ *'b::cpo*
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *adm*: *adm* $(\lambda x.\ x \in A)$
  shows *compact* (*Rep k*) $\Longrightarrow$ *compact k*
proof (*unfold compact-def*)
  have *cont-Rep*: *cont Rep*
    by (*rule typedef-cont-Rep* [*OF type below adm cont-id*])
  assume *adm* $(\lambda x.\ Rep\ k \not\sqsubseteq x)$
  with *cont-Rep* **have** *adm* $(\lambda x.\ Rep\ k \not\sqsubseteq Rep\ x)$ **by** (*rule adm-subst*)
  then show *adm* $(\lambda x.\ k \not\sqsubseteq x)$ **by** (*unfold below*)
qed

8.6 Proving a subtype is pointed

A subtype of a cpo has a least element if and only if the defining subset has a least element.

theorem *typedef-pcpo-generic*:
  fixes *Abs* :: *'a::cpo* ⇒ *'b::cpo*
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *z-in-A*: $z \in A$
    and *z-least*: $\bigwedge x.\ x \in A \Longrightarrow z \sqsubseteq x$
  shows *OFCLASS*(*'b*, *pcpo-class*)
  apply (*intro-classes*)
  apply (*rule-tac x=Abs z* **in** *exI*, *rule allI*)
  apply (*unfold below*)
  apply (*subst type-definition.Abs-inverse* [*OF type z-in-A*])
  apply (*rule z-least* [*OF type-definition.Rep* [*OF type*]])
  done

As a special case, a subtype of a pcpo has a least element if the defining subset contains ⊥.

theorem *typedef-pcpo*:
  fixes *Abs* :: *'a::pcpo ⇒ 'b::cpo*
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows *OFCLASS('b, pcpo-class)*
  by (*rule typedef-pcpo-generic* [*OF type below bottom-in-A*], *rule minimal*)

8.6.1 Strictness of *Rep* and *Abs*

For a sub-pcpo where ⊥ is a member of the defining subset, *Rep* and *Abs* are both strict.

theorem *typedef-Abs-strict*:
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows $Abs\ \bot = \bot$
  apply (*rule bottomI*, *unfold below*)
  apply (*simp add*: *type-definition.Abs-inverse* [*OF type bottom-in-A*])
  done

theorem *typedef-Rep-strict*:
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows $Rep\ \bot = \bot$
  apply (*rule typedef-Abs-strict* [*OF type below bottom-in-A*, *THEN subst*])
  apply (*rule type-definition.Abs-inverse* [*OF type bottom-in-A*])
  done

theorem *typedef-Abs-bottom-iff*:
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows $x \in A \Longrightarrow (Abs\ x = \bot) = (x = \bot)$
  apply (*rule typedef-Abs-strict* [*OF type below bottom-in-A*, *THEN subst*])
  apply (*simp add*: *type-definition.Abs-inject* [*OF type*] *bottom-in-A*)
  done

theorem *typedef-Rep-bottom-iff*:
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows $(Rep\ x = \bot) = (x = \bot)$
  apply (*rule typedef-Rep-strict* [*OF type below bottom-in-A*, *THEN subst*])
  apply (*simp add*: *type-definition.Rep-inject* [*OF type*])

done

8.7 Proving a subtype is flat

theorem *typedef-flat*:
fixes *Abs* :: *'a::flat ⇒ 'b::pcpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
and *bottom-in-A*: ⊥ ∈ *A*
shows *OFCLASS*(*'b*, *flat-class*)
apply (*intro-classes*)
apply (*unfold below*)
apply (*simp add*: *type-definition.Rep-inject* [*OF type*, *symmetric*])
apply (*simp add*: *typedef-Rep-strict* [*OF type below bottom-in-A*])
apply (*simp add*: *ax-flat*)
done

8.8 HOLCF type definition package

ML-file ‹*Tools/cpodef.ML*›

end

9 The type of continuous functions

theory *Cfun*
imports *Cpodef*
begin

9.1 Definition of continuous function type

definition *cfun* = {*f*::*'a* ⇒ *'b*. *cont f*}

cpodef (*'a*, *'b*) *cfun* (‹(‹*notation*=‹*infix* →››- →/ -)› [*1*, *0*] *0*) = *cfun* :: (*'a* ⇒ *'b*) *set*
by (*auto simp*: *cfun-def intro*: *cont-const adm-cont*)

type-notation (*ASCII*)
cfun (**infixr** ‹−>› *0*)

notation (*ASCII*)
Rep-cfun (‹(‹*notation*=‹*infix* $››-$/-)› [*999*,*1000*] *999*)

notation
Rep-cfun (‹(‹*notation*=‹*infix* ·››-·/-)› [*999*,*1000*] *999*)

9.2 Syntax for continuous lambda abstraction

syntax *-cabs* :: [*logic*, *logic*] ⇒ *logic*

```
parse-translation ‹
(* rewrite (-cabs x t) => (Abs-cfun (%x. t)) *)
  [Syntax-Trans.mk-binder-tr (syntax-const ‹-cabs›, const-syntax ‹Abs-cfun›)]
›

print-translation ‹
  [(const-syntax ‹Abs-cfun›, fn ctxt => fn [Abs abs] =>
      let val (x, t) = Syntax-Trans.atomic-abs-tr′ ctxt abs
      in Syntax.const syntax-const ‹-cabs› $ x $ t end)]
›  — To avoid eta-contraction of body
```

Syntax for nested abstractions

```
syntax (ASCII)
  -Lambda :: [cargs, logic] ⇒ logic  (‹(‹indent=3 notation=‹binder LAM››LAM -./
-)› [1000, 10] 10)

syntax
  -Lambda :: [cargs, logic] ⇒ logic (‹(‹indent=3 notation=‹binder Λ››Λ -./ -)›
[1000, 10] 10)

syntax-consts
  -Lambda ⇌ Abs-cfun

parse-ast-translation ‹
(* rewrite (LAM x y z. t) => (-cabs x (-cabs y (-cabs z t))) *)
(* cf. Syntax.lambda-ast-tr from src/Pure/Syntax/syn-trans.ML *)
  let
    fun Lambda-ast-tr [pats, body] =
          Ast.fold-ast-p syntax-const ‹-cabs›
            (Ast.unfold-ast syntax-const ‹-cargs› (Ast.strip-positions pats), body)
      | Lambda-ast-tr asts = raise Ast.AST (Lambda-ast-tr, asts);
  in [(syntax-const ‹-Lambda›, K Lambda-ast-tr)] end
›

print-ast-translation ‹
(* rewrite (-cabs x (-cabs y (-cabs z t))) => (LAM x y z. t) *)
(* cf. Syntax.abs-ast-tr′ from src/Pure/Syntax/syn-trans.ML *)
  let
    fun cabs-ast-tr′ asts =
      (case Ast.unfold-ast-p syntax-const ‹-cabs›
          (Ast.Appl (Ast.Constant syntax-const ‹-cabs› :: asts)) of
        ([], -) => raise Ast.AST (cabs-ast-tr′, asts)
      | (xs, body) => Ast.Appl
          [Ast.Constant syntax-const ‹-Lambda›,
           Ast.fold-ast syntax-const ‹-cargs› xs, body]);
  in [(syntax-const ‹-cabs›, K cabs-ast-tr′)] end
›
```

Dummy patterns for continuous abstraction

translations
Λ -. *t* ⇀ *CONST Abs-cfun* (λ-. *t*)

9.3 Continuous function space is pointed

lemma *bottom-cfun*: $\bot \in$ *cfun*
by (*simp add*: *cfun-def inst-fun-pcpo*)

instance *cfun* :: (*cpo*, *discrete-cpo*) *discrete-cpo*
by *intro-classes* (*simp add*: *below-cfun-def Rep-cfun-inject*)

instance *cfun* :: (*cpo*, *pcpo*) *pcpo*
by (*rule typedef-pcpo* [*OF type-definition-cfun below-cfun-def bottom-cfun*])

lemmas *Rep-cfun-strict* =
typedef-Rep-strict [*OF type-definition-cfun below-cfun-def bottom-cfun*]

lemmas *Abs-cfun-strict* =
typedef-Abs-strict [*OF type-definition-cfun below-cfun-def bottom-cfun*]

function application is strict in its first argument

lemma *Rep-cfun-strict1* [*simp*]: $\bot \cdot x = \bot$
by (*simp add*: *Rep-cfun-strict*)

lemma *LAM-strict* [*simp*]: $(\Lambda\ x.\ \bot) = \bot$
by (*simp add*: *inst-fun-pcpo* [*symmetric*] *Abs-cfun-strict*)

for compatibility with old HOLCF-Version

lemma *inst-cfun-pcpo*: $\bot = (\Lambda\ x.\ \bot)$
by *simp*

9.4 Basic properties of continuous functions

Beta-equality for continuous functions

lemma *Abs-cfun-inverse2*: *cont f* $\Longrightarrow$ *Rep-cfun* (*Abs-cfun f*) = *f*
by (*simp add*: *Abs-cfun-inverse cfun-def*)

lemma *beta-cfun*: *cont f* $\Longrightarrow (\Lambda\ x.\ f\ x) \cdot u = f\ u$
by (*simp add*: *Abs-cfun-inverse2*)

9.4.1 Beta-reduction simproc

Given the term $(\Lambda\ x.\ f\ x) \cdot y$, the procedure tries to construct the theorem $(\Lambda\ x.\ f\ x) \cdot y \equiv f\ y$. If this theorem cannot be completely solved by the cont2cont rules, then the procedure returns the ordinary conditional *beta-cfun* rule.

The simproc does not solve any more goals that would be solved by using *beta-cfun* as a simp rule. The advantage of the simproc is that it can avoid

deeply-nested calls to the simplifier that would otherwise be caused by large continuity side conditions.

Update: The simproc now uses rule *Abs-cfun-inverse2* instead of *beta-cfun*, to avoid problems with eta-contraction.

simproc-setup *beta-cfun-proc* (*Rep-cfun* (*Abs-cfun f*)) = ‹
  K (*fn ctxt* => *fn ct* =>
    let
      val f = *Thm.dest-arg* (*Thm.dest-arg ct*);
      val [*T*, *U*] = *Thm.dest-ctyp* (*Thm.ctyp-of-cterm f*);
     val tr = *Thm.instantiate′* [*SOME T*, *SOME U*] [*SOME f*] (*mk-meta-eq* @{*thm Abs-cfun-inverse2*});
      val rules = *Named-Theorems.get ctxt* ***named-theorems*** ‹*cont2cont*›;
      val tac = *SOLVED′* (*REPEAT-ALL-NEW* (*match-tac ctxt* (*rev rules*)));
    in SOME (*perhaps* (*SINGLE* (*tac 1*)) *tr*) *end*)
›

Eta-equality for continuous functions

lemma *eta-cfun*: $(\Lambda\ x.\ f\cdot x) = f$
  by (*rule Rep-cfun-inverse*)

Extensionality for continuous functions

lemma *cfun-eq-iff*: $f = g \longleftrightarrow (\forall x.\ f\cdot x = g\cdot x)$
  by (*simp add*: *Rep-cfun-inject* [*symmetric*] *fun-eq-iff*)

lemma *cfun-eqI*: $(\bigwedge x.\ f\cdot x = g\cdot x) \Longrightarrow f = g$
  by (*simp add*: *cfun-eq-iff*)

Extensionality wrt. ordering for continuous functions

lemma *cfun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall x.\ f\cdot x \sqsubseteq g\cdot x)$
  by (*simp add*: *below-cfun-def fun-below-iff*)

lemma *cfun-belowI*: $(\bigwedge x.\ f\cdot x \sqsubseteq g\cdot x) \Longrightarrow f \sqsubseteq g$
  by (*simp add*: *cfun-below-iff*)

Congruence for continuous function application

lemma *cfun-cong*: $f = g \Longrightarrow x = y \Longrightarrow f\cdot x = g\cdot y$
  by *simp*

lemma *cfun-fun-cong*: $f = g \Longrightarrow f\cdot x = g\cdot x$
  by *simp*

lemma *cfun-arg-cong*: $x = y \Longrightarrow f\cdot x = f\cdot y$
  by *simp*

9.5 Continuity of application

lemma *cont-Rep-cfun1*: *cont* $(\lambda f.\ f\cdot x)$

by (*rule cont-Rep-cfun* [*OF cont-id*, *THEN cont2cont-fun*])

lemma *cont-Rep-cfun2*: *cont* ($\lambda x.\ f\cdot x$)
using *Rep-cfun* [**where** $x = f$] **by** (*simp add*: *cfun-def*)

lemmas *monofun-Rep-cfun* = *cont-Rep-cfun* [*THEN cont2mono*]

lemmas *monofun-Rep-cfun1* = *cont-Rep-cfun1* [*THEN cont2mono*]
lemmas *monofun-Rep-cfun2* = *cont-Rep-cfun2* [*THEN cont2mono*]

contlub, cont properties of *Rep-cfun* in each argument

lemma *contlub-cfun-arg*: *chain* $Y \Longrightarrow f\cdot(\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ f\cdot(Y\ i))$
by (*rule cont-Rep-cfun2* [*THEN cont2contlubE*])

lemma *contlub-cfun-fun*: *chain* $F \Longrightarrow (\bigsqcup i.\ F\ i)\cdot x = (\bigsqcup i.\ F\ i\cdot x)$
by (*rule cont-Rep-cfun1* [*THEN cont2contlubE*])

monotonicity of application

lemma *monofun-cfun-fun*: $f \sqsubseteq g \Longrightarrow f\cdot x \sqsubseteq g\cdot x$
by (*simp add*: *cfun-below-iff*)

lemma *monofun-cfun-arg*: $x \sqsubseteq y \Longrightarrow f\cdot x \sqsubseteq f\cdot y$
by (*rule monofun-Rep-cfun2* [*THEN monofunE*])

lemma *monofun-cfun*: $f \sqsubseteq g \Longrightarrow x \sqsubseteq y \Longrightarrow f\cdot x \sqsubseteq g\cdot y$
by (*rule below-trans* [*OF monofun-cfun-fun monofun-cfun-arg*])

ch2ch - rules for the type $'a \rightarrow 'b$

lemma *chain-monofun*: *chain* $Y \Longrightarrow$ *chain* ($\lambda i.\ f\cdot(Y\ i)$)
by (*erule monofun-Rep-cfun2* [*THEN ch2ch-monofun*])

lemma *ch2ch-Rep-cfunR*: *chain* $Y \Longrightarrow$ *chain* ($\lambda i.\ f\cdot(Y\ i)$)
by (*rule monofun-Rep-cfun2* [*THEN ch2ch-monofun*])

lemma *ch2ch-Rep-cfunL*: *chain* $F \Longrightarrow$ *chain* ($\lambda i.\ (F\ i)\cdot x$)
by (*rule monofun-Rep-cfun1* [*THEN ch2ch-monofun*])

lemma *ch2ch-Rep-cfun* [*simp*]: *chain* $F \Longrightarrow$ *chain* $Y \Longrightarrow$ *chain* ($\lambda i.\ (F\ i)\cdot(Y\ i)$)
by (*simp add*: *chain-def monofun-cfun*)

lemma *ch2ch-LAM* [*simp*]:
($\bigwedge x.$ *chain* ($\lambda i.\ S\ i\ x$)) $\Longrightarrow$ ($\bigwedge i.$ *cont* ($\lambda x.\ S\ i\ x$)) $\Longrightarrow$ *chain* ($\lambda i.\ \Lambda\ x.\ S\ i\ x$)
by (*simp add*: *chain-def cfun-below-iff*)

contlub, cont properties of *Rep-cfun* in both arguments

lemma *lub-APP*: *chain* $F \Longrightarrow$ *chain* $Y \Longrightarrow (\bigsqcup i.\ F\ i\cdot(Y\ i)) = (\bigsqcup i.\ F\ i)\cdot(\bigsqcup i.\ Y\ i)$
by (*simp add*: *contlub-cfun-fun contlub-cfun-arg diag-lub*)

lemma *lub-LAM*:
  assumes $\bigwedge x.\ chain\ (\lambda i.\ F\ i\ x)$
    and $\bigwedge i.\ cont\ (\lambda x.\ F\ i\ x)$
  shows $(\bigsqcup i.\ \Lambda\ x.\ F\ i\ x) = (\Lambda\ x.\ \bigsqcup i.\ F\ i\ x)$
  using *assms* **by** (*simp add*: *lub-cfun lub-fun ch2ch-lambda*)

lemmas *lub-distribs* = *lub-APP lub-LAM*

strictness

lemma *strictI*: $f \cdot x = \bot \Longrightarrow f \cdot \bot = \bot$
  apply (*rule bottomI*)
  apply (*erule subst*)
  apply (*rule minimal* [*THEN monofun-cfun-arg*])
  done

type $'a \rightarrow 'b$ is chain complete

lemma *lub-cfun*: $chain\ F \Longrightarrow (\bigsqcup i.\ F\ i) = (\Lambda\ x.\ \bigsqcup i.\ F\ i \cdot x)$
  by (*simp add*: *lub-cfun lub-fun ch2ch-lambda*)

9.6 Continuity simplification procedure

cont2cont lemma for *Rep-cfun*

lemma *cont2cont-APP* [*simp*, *cont2cont*]:
  assumes *f*: $cont\ (\lambda x.\ f\ x)$
  assumes *t*: $cont\ (\lambda x.\ t\ x)$
  shows $cont\ (\lambda x.\ (f\ x) \cdot (t\ x))$
proof −
  from *cont-Rep-cfun1 f* **have** $cont\ (\lambda x.\ (f\ x) \cdot y)$ **for** *y*
    by (*rule cont-compose*)
  with *t cont-Rep-cfun2* **show** $cont\ (\lambda x.\ (f\ x) \cdot (t\ x))$
    by (*rule cont-apply*)
qed

Two specific lemmas for the combination of LCF and HOL terms. These lemmas are needed in theories that use types like $'a \rightarrow 'b \Rightarrow 'c$.

lemma *cont-APP-app* [*simp*]: $cont\ f \Longrightarrow cont\ g \Longrightarrow cont\ (\lambda x.\ ((f\ x) \cdot (g\ x))\ s)$
  by (*rule cont2cont-APP* [*THEN cont2cont-fun*])

lemma *cont-APP-app-app* [*simp*]: $cont\ f \Longrightarrow cont\ g \Longrightarrow cont\ (\lambda x.\ ((f\ x) \cdot (g\ x))\ s\ t)$
  by (*rule cont-APP-app* [*THEN cont2cont-fun*])

cont2mono Lemma for $\lambda x.\ \Lambda\ y.\ c1\ x\ y$

lemma *cont2mono-LAM*:
  $[\![\bigwedge x.\ cont\ (\lambda y.\ f\ x\ y);\ \bigwedge y.\ monofun\ (\lambda x.\ f\ x\ y)]\!]$
    $\Longrightarrow monofun\ (\lambda x.\ \Lambda\ y.\ f\ x\ y)$
  by (*simp add*: *monofun-def cfun-below-iff*)

cont2cont Lemma for $\lambda x.\ \Lambda\ y.\ f\ x\ y$

Not suitable as a cont2cont rule, because on nested lambdas it causes exponential blow-up in the number of subgoals.

```
lemma cont2cont-LAM:
  assumes f1: ⋀x. cont (λy. f x y)
  assumes f2: ⋀y. cont (λx. f x y)
  shows cont (λx. Λ y. f x y)
proof (rule cont-Abs-cfun)
  from f1 show f x ∈ cfun for x
    by (simp add: cfun-def)
  from f2 show cont f
    by (rule cont2cont-lambda)
qed
```

This version does work as a cont2cont rule, since it has only a single subgoal.

```
lemma cont2cont-LAM' [simp, cont2cont]:
  fixes f :: 'a::cpo ⇒ 'b::cpo ⇒ 'c::cpo
  assumes f: cont (λp. f (fst p) (snd p))
  shows cont (λx. Λ y. f x y)
  using assms by (simp add: cont2cont-LAM prod-cont-iff)

lemma cont2cont-LAM-discrete [simp, cont2cont]:
  (⋀y::'a::discrete-cpo. cont (λx. f x y)) ⟹ cont (λx. Λ y. f x y)
  by (simp add: cont2cont-LAM)
```

9.7 Miscellaneous

Monotonicity of *Abs-cfun*

```
lemma monofun-LAM: cont f ⟹ cont g ⟹ (⋀x. f x ⊑ g x) ⟹ (Λ x. f x) ⊑
(Λ x. g x)
  by (simp add: cfun-below-iff)
```

some lemmata for functions with flat/chfin domain/range types

```
lemma chfin-Rep-cfunR: chain Y ⟹ ∀ s. ∃ n. (LUB i. Y i)·s = Y n·s
  for Y :: nat ⇒ 'a::cpo → 'b::chfin
  apply (rule allI)
  apply (subst contlub-cfun-fun)
   apply assumption
  apply (fast intro!: lub-eqI chfin lub-finch2 chfin2finch ch2ch-Rep-cfunL)
  done

lemma adm-chfindom: adm (λ(u::'a::cpo → 'b::chfin). P(u·s))
  by (rule adm-subst, simp, rule adm-chfin)
```

9.8 Continuous injection-retraction pairs

Continuous retractions are strict.

lemma *retraction-strict*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow f\cdot\bot = \bot$
  apply (*rule bottomI*)
  apply (*drule-tac x*=⊥ **in** *spec*)
  apply (*erule subst*)
  apply (*rule monofun-cfun-arg*)
  apply (*rule minimal*)
  done

lemma *injection-eq*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow (g\cdot x = g\cdot y) = (x = y)$
  apply (*rule iffI*)
   apply (*drule-tac f*=*f* **in** *cfun-arg-cong*)
   apply *simp*
  apply *simp*
  done

lemma *injection-below*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow (g\cdot x \sqsubseteq g\cdot y) = (x \sqsubseteq y)$
  apply (*rule iffI*)
   apply (*drule-tac f*=*f* **in** *monofun-cfun-arg*)
   apply *simp*
  apply (*erule monofun-cfun-arg*)
  done

lemma *injection-defined-rev*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow g\cdot z = \bot \Longrightarrow z = \bot$
  apply (*drule-tac f*=*f* **in** *cfun-arg-cong*)
  apply (*simp add*: *retraction-strict*)
  done

lemma *injection-defined*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow z \neq \bot \Longrightarrow g\cdot z \neq \bot$
  by (*erule contrapos-nn*, *rule injection-defined-rev*)

a result about functions with flat codomain

lemma *flat-eqI*: $x \sqsubseteq y \Longrightarrow x \neq \bot \Longrightarrow x = y$
  for *x y* :: *'a*::*flat*
  by (*drule ax-flat*) *simp*

lemma *flat-codom*: $f\cdot x = c \Longrightarrow f\cdot\bot = \bot \vee (\forall z.\ f\cdot z = c)$
  for *c* :: *'b*::*flat*
  apply (*cases* $f\cdot x = \bot$)
   apply (*rule disjI1*)
   apply (*rule bottomI*)
   apply (*erule-tac t*=⊥ **in** *subst*)
   apply (*rule minimal* [*THEN monofun-cfun-arg*])
  apply *clarify*
  apply (*rule-tac a* = $f\cdot\bot$ **in** *refl* [*THEN box-equals*])
   apply (*erule minimal* [*THEN monofun-cfun-arg*, *THEN flat-eqI*])
  apply (*erule minimal* [*THEN monofun-cfun-arg*, *THEN flat-eqI*])
  done

9.9 Identity and composition

definition *ID* :: $'a \to 'a$
  where $ID = (\Lambda\ x.\ x)$

definition *cfcomp* :: $('b \to 'c) \to ('a \to 'b) \to 'a \to 'c$
  where *oo-def*: $cfcomp = (\Lambda\ f\ g\ x.\ f\cdot(g\cdot x))$

abbreviation *cfcomp-syn* :: $['b \to 'c, 'a \to 'b] \Rightarrow 'a \to 'c$ (**infixr** ‹*oo*› *100*)
  where $f\ oo\ g == cfcomp\cdot f\cdot g$

lemma *ID1* [*simp*]: $ID\cdot x = x$
  by (*simp add*: *ID-def*)

lemma *cfcomp1*: $(f\ oo\ g) = (\Lambda\ x.\ f\cdot(g\cdot x))$
  by (*simp add*: *oo-def*)

lemma *cfcomp2* [*simp*]: $(f\ oo\ g)\cdot x = f\cdot(g\cdot x)$
  by (*simp add*: *cfcomp1*)

lemma *cfcomp-LAM*: $cont\ g \Longrightarrow f\ oo\ (\Lambda\ x.\ g\ x) = (\Lambda\ x.\ f\cdot(g\ x))$
  by (*simp add*: *cfcomp1*)

lemma *cfcomp-strict* [*simp*]: $\bot\ oo\ f = \bot$
  by (*simp add*: *cfun-eq-iff*)

Show that interpretation of (pcpo, -→-) is a category.

- The class of objects is interpretation of syntactical class pcpo.
- The class of arrows between objects $'a$ and $'b$ is interpret. of $'a \to 'b$.
- The identity arrow is interpretation of *ID*.
- The composition of f and g is interpretation of *oo*.

lemma *ID2* [*simp*]: $f\ oo\ ID = f$
  by (*rule cfun-eqI*, *simp*)

lemma *ID3* [*simp*]: $ID\ oo\ f = f$
  by (*rule cfun-eqI*) *simp*

lemma *assoc-oo*: $f\ oo\ (g\ oo\ h) = (f\ oo\ g)\ oo\ h$
  by (*rule cfun-eqI*) *simp*

9.10 Strictified functions

definition *seq* :: $'a::pcpo \to 'b::pcpo \to 'b$
  where $seq = (\Lambda\ x.\ if\ x = \bot\ then\ \bot\ else\ ID)$

lemma *cont2cont-if-bottom* [*cont2cont*, *simp*]:
  assumes *f*: *cont* ($\lambda x.$ *f x*)
    and *g*: *cont* ($\lambda x.$ *g x*)
  shows *cont* ($\lambda x.$ *if f x* = ⊥ *then* ⊥ *else g x*)
proof (*rule cont-apply* [*OF f*])
  show *cont* ($\lambda y.$ *if y* = ⊥ *then* ⊥ *else g x*) **for** *x*
    unfolding *cont-def is-lub-def is-ub-def ball-simps*
    by (*simp add*: *lub-eq-bottom-iff*)
  show *cont* ($\lambda x.$ *if y* = ⊥ *then* ⊥ *else g x*) **for** *y*
    by (*simp add*: *g*)
qed

lemma *seq-conv-if*: *seq·x* = (*if x* = ⊥ *then* ⊥ *else ID*)
  by (*simp add*: *seq-def*)

lemma *seq-simps* [*simp*]:
  seq·⊥ = ⊥
  seq·x·⊥ = ⊥
  x ≠ ⊥ ⟹ *seq·x* = *ID*
  by (*simp-all add*: *seq-conv-if*)

definition *strictify* :: (*'a*::*pcpo* → *'b*::*pcpo*) → *'a* → *'b*
  where *strictify* = (Λ *f x*. *seq·x·*(*f·x*))

lemma *strictify-conv-if*: *strictify·f·x* = (*if x* = ⊥ *then* ⊥ *else f·x*)
  by (*simp add*: *strictify-def*)

lemma *strictify1* [*simp*]: *strictify·f*·⊥ = ⊥
  by (*simp add*: *strictify-conv-if*)

lemma *strictify2* [*simp*]: *x* ≠ ⊥ ⟹ *strictify·f·x* = *f·x*
  by (*simp add*: *strictify-conv-if*)

9.11 Continuity of let-bindings

lemma *cont2cont-Let*:
  assumes *f*: *cont* ($\lambda x.$ *f x*)
  assumes *g1*: ⋀*y*. *cont* ($\lambda x.$ *g x y*)
  assumes *g2*: ⋀*x*. *cont* ($\lambda y.$ *g x y*)
  shows *cont* ($\lambda x.$ *let y* = *f x in g x y*)
  unfolding *Let-def* **using** *f g2 g1* **by** (*rule cont-apply*)

lemma *cont2cont-Let'* [*simp*, *cont2cont*]:
  assumes *f*: *cont* ($\lambda x.$ *f x*)
  assumes *g*: *cont* ($\lambda p.$ *g* (*fst p*) (*snd p*))
  shows *cont* ($\lambda x.$ *let y* = *f x in g x y*)
  using *f*
proof (*rule cont2cont-Let*)
  from *g* **show** *cont* ($\lambda y.$ *g x y*) **for** *x*

```
    by (simp add: prod-cont-iff)
  from g show cont (λx. g x y) for y
    by (simp add: prod-cont-iff)
qed
```

The simple version (suggested by Joachim Breitner) is needed if the type of the defined term is not a cpo.

```
lemma cont2cont-Let-simple [simp, cont2cont]:
  assumes ⋀y. cont (λx. g x y)
  shows cont (λx. let y = t in g x y)
  unfolding Let-def using assms .

end
```

10 Continuous deflations and ep-pairs

```
theory Deflation
  imports Cfun
begin
```

10.1 Continuous deflations

```
locale deflation =
  fixes d :: 'a → 'a
  assumes idem: ⋀x. d·(d·x) = d·x
  assumes below: ⋀x. d·x ⊑ x
begin

lemma below-ID: d ⊑ ID
  by (rule cfun-belowI) (simp add: below)
```

The set of fixed points is the same as the range.

```
lemma fixes-eq-range: {x. d·x = x} = range (λx. d·x)
  by (auto simp add: eq-sym-conv idem)

lemma range-eq-fixes: range (λx. d·x) = {x. d·x = x}
  by (auto simp add: eq-sym-conv idem)
```

The pointwise ordering on deflation functions coincides with the subset ordering of their sets of fixed-points.

```
lemma belowI:
  assumes f: ⋀x. d·x = x ⟹ f·x = x
  shows d ⊑ f
proof (rule cfun-belowI)
  fix x
  from below have f·(d·x) ⊑ f·x
    by (rule monofun-cfun-arg)
  also from idem have f·(d·x) = d·x
```

```
    by (rule f)
  finally show d·x ⊑ f·x .
qed

lemma belowD: ⟦f ⊑ d; f·x = x⟧ ⟹ d·x = x
proof (rule below-antisym)
  from below show d·x ⊑ x .
  assume f ⊑ d
  then have f·x ⊑ d·x by (rule monofun-cfun-fun)
  also assume f·x = x
  finally show x ⊑ d·x .
qed

end

lemma deflation-strict: deflation d ⟹ d·⊥ = ⊥
  by (rule deflation.below [THEN bottomI])

lemma adm-deflation: adm (λd. deflation d)
  by (simp add: deflation-def)

lemma deflation-ID: deflation ID
  by (simp add: deflation.intro)

lemma deflation-bottom: deflation ⊥
  by (simp add: deflation.intro)

lemma deflation-below-iff: deflation p ⟹ deflation q ⟹ p ⊑ q ⟷ (∀x. p·x =
x ⟶ q·x = x)
  apply safe
   apply (simp add: deflation.belowD)
  apply (simp add: deflation.belowI)
  done
```

The composition of two deflations is equal to the lesser of the two (if they are comparable).

```
lemma deflation-below-comp1:
  assumes deflation f
  assumes deflation g
  shows f ⊑ g ⟹ f·(g·x) = f·x
proof (rule below-antisym)
  interpret g: deflation g by fact
  from g.below show f·(g·x) ⊑ f·x by (rule monofun-cfun-arg)
next
  interpret f: deflation f by fact
  assume f ⊑ g
  then have f·x ⊑ g·x by (rule monofun-cfun-fun)
  then have f·(f·x) ⊑ f·(g·x) by (rule monofun-cfun-arg)
  also have f·(f·x) = f·x by (rule f.idem)
```

finally show $f{\cdot}x \sqsubseteq f{\cdot}(g{\cdot}x)$ **.**
qed

lemma *deflation-below-comp2*: *deflation f* $\Longrightarrow$ *deflation g* $\Longrightarrow$ $f \sqsubseteq g \Longrightarrow g{\cdot}(f{\cdot}x) = f{\cdot}x$
by (*simp only*: *deflation.belowD deflation.idem*)

10.2 Deflations with finite range

lemma *finite-range-imp-finite-fixes*:
assumes *finite* (*range f*)
shows *finite* $\{x.\ f\ x = x\}$
proof −
have $\{x.\ f\ x = x\} \subseteq$ *range f*
by (*clarify*, *erule subst*, *rule rangeI*)
from *this assms* **show** *finite* $\{x.\ f\ x = x\}$
by (*rule finite-subset*)
qed

locale *finite-deflation* = *deflation* +
assumes *finite-fixes*: *finite* $\{x.\ d{\cdot}x = x\}$
begin

lemma *finite-range*: *finite* (*range* $(\lambda x.\ d{\cdot}x)$)
by (*simp add*: *range-eq-fixes finite-fixes*)

lemma *finite-image*: *finite* $((\lambda x.\ d{\cdot}x)$ ‘ $A)$
by (*rule finite-subset* [*OF image-mono* [*OF subset-UNIV*] *finite-range*])

lemma *compact*: *compact* $(d{\cdot}x)$
proof (*rule compactI2*)
fix $Y :: nat \Rightarrow 'a$
assume Y: *chain* Y
have *finite-chain* $(\lambda i.\ d{\cdot}(Y\ i))$
proof (*rule finite-range-imp-finch*)
from Y **show** *chain* $(\lambda i.\ d{\cdot}(Y\ i))$ **by** *simp*
have *range* $(\lambda i.\ d{\cdot}(Y\ i)) \subseteq$ *range* $(\lambda x.\ d{\cdot}x)$ **by** *auto*
then show *finite* (*range* $(\lambda i.\ d{\cdot}(Y\ i))$)
using *finite-range* **by** (*rule finite-subset*)
qed
then have $\exists j.\ (\bigsqcup i.\ d{\cdot}(Y\ i)) = d{\cdot}(Y\ j)$
by (*simp add*: *finite-chain-def maxinch-is-thelub* Y)
then obtain j **where** j: $(\bigsqcup i.\ d{\cdot}(Y\ i)) = d{\cdot}(Y\ j)$ **..**

assume $d{\cdot}x \sqsubseteq (\bigsqcup i.\ Y\ i)$
then have $d{\cdot}(d{\cdot}x) \sqsubseteq d{\cdot}(\bigsqcup i.\ Y\ i)$
by (*rule monofun-cfun-arg*)
then have $d{\cdot}x \sqsubseteq (\bigsqcup i.\ d{\cdot}(Y\ i))$
by (*simp add*: *contlub-cfun-arg* Y *idem*)

with j **have** $d{\cdot}x \sqsubseteq d{\cdot}(Y\ j)$ **by** *simp*
then have $d{\cdot}x \sqsubseteq Y\ j$
using *below* **by** (*rule below-trans*)
then show $\exists j.\ d{\cdot}x \sqsubseteq Y\ j$ **..**
qed

end

lemma *finite-deflation-intro*: *deflation d* $\Longrightarrow$ *finite* $\{x.\ d{\cdot}x = x\}$ $\Longrightarrow$ *finite-deflation d*
by (*intro finite-deflation.intro finite-deflation-axioms.intro*)

lemma *finite-deflation-imp-deflation*: *finite-deflation d* $\Longrightarrow$ *deflation d*
by (*simp add*: *finite-deflation-def*)

lemma *finite-deflation-bottom*: *finite-deflation* $\bot$
by *standard simp-all*

10.3 Continuous embedding-projection pairs

locale *ep-pair* =
fixes $e :: 'a \rightarrow 'b$ **and** $p :: 'b \rightarrow 'a$
assumes *e-inverse* [*simp*]: $\bigwedge x.\ p{\cdot}(e{\cdot}x) = x$
and *e-p-below*: $\bigwedge y.\ e{\cdot}(p{\cdot}y) \sqsubseteq y$
begin

lemma *e-below-iff* [*simp*]: $e{\cdot}x \sqsubseteq e{\cdot}y \longleftrightarrow x \sqsubseteq y$
proof
assume $e{\cdot}x \sqsubseteq e{\cdot}y$
then have $p{\cdot}(e{\cdot}x) \sqsubseteq p{\cdot}(e{\cdot}y)$ **by** (*rule monofun-cfun-arg*)
then show $x \sqsubseteq y$ **by** *simp*
next
assume $x \sqsubseteq y$
then show $e{\cdot}x \sqsubseteq e{\cdot}y$ **by** (*rule monofun-cfun-arg*)
qed

lemma *e-eq-iff* [*simp*]: $e{\cdot}x = e{\cdot}y \longleftrightarrow x = y$
unfolding *po-eq-conv e-below-iff* **..**

lemma *p-eq-iff*: $e{\cdot}(p{\cdot}x) = x \Longrightarrow e{\cdot}(p{\cdot}y) = y \Longrightarrow p{\cdot}x = p{\cdot}y \longleftrightarrow x = y$
by (*safe*, *erule subst*, *erule subst*, *simp*)

lemma *p-inverse*: $(\exists x.\ y = e{\cdot}x) \longleftrightarrow e{\cdot}(p{\cdot}y) = y$
by (*auto*, *rule exI*, *erule sym*)

lemma *e-below-iff-below-p*: $e{\cdot}x \sqsubseteq y \longleftrightarrow x \sqsubseteq p{\cdot}y$
proof
assume $e{\cdot}x \sqsubseteq y$
then have $p{\cdot}(e{\cdot}x) \sqsubseteq p{\cdot}y$ **by** (*rule monofun-cfun-arg*)

then show $x \sqsubseteq p{\cdot}y$ **by** *simp*
next
assume $x \sqsubseteq p{\cdot}y$
then have $e{\cdot}x \sqsubseteq e{\cdot}(p{\cdot}y)$ **by** (*rule monofun-cfun-arg*)
then show $e{\cdot}x \sqsubseteq y$ **using** *e-p-below* **by** (*rule below-trans*)
qed

lemma *compact-e-rev*: *compact* $(e{\cdot}x) \Longrightarrow$ *compact* x
proof –
assume *compact* $(e{\cdot}x)$
then have *adm* $(\lambda y.\ e{\cdot}x \not\sqsubseteq y)$ **by** (*rule compactD*)
then have *adm* $(\lambda y.\ e{\cdot}x \not\sqsubseteq e{\cdot}y)$ **by** (*rule adm-subst* [*OF cont-Rep-cfun2*])
then have *adm* $(\lambda y.\ x \not\sqsubseteq y)$ **by** *simp*
then show *compact* x **by** (*rule compactI*)
qed

lemma *compact-e*:
assumes *compact* x
shows *compact* $(e{\cdot}x)$
proof –
from *assms* **have** *adm* $(\lambda y.\ x \not\sqsubseteq y)$ **by** (*rule compactD*)
then have *adm* $(\lambda y.\ x \not\sqsubseteq p{\cdot}y)$ **by** (*rule adm-subst* [*OF cont-Rep-cfun2*])
then have *adm* $(\lambda y.\ e{\cdot}x \not\sqsubseteq y)$ **by** (*simp add*: *e-below-iff-below-p*)
then show *compact* $(e{\cdot}x)$ **by** (*rule compactI*)
qed

lemma *compact-e-iff*: *compact* $(e{\cdot}x) \longleftrightarrow$ *compact* x
by (*rule iffI* [*OF compact-e-rev compact-e*])

Deflations from ep-pairs

lemma *deflation-e-p*: *deflation* (*e oo p*)
by (*simp add*: *deflation.intro e-p-below*)

lemma *deflation-e-d-p*:
assumes *deflation d*
shows *deflation* (*e oo d oo p*)
proof
interpret *deflation d* **by** *fact*
fix x :: $'b$
show $(e\ oo\ d\ oo\ p){\cdot}((e\ oo\ d\ oo\ p){\cdot}x) = (e\ oo\ d\ oo\ p){\cdot}x$
by (*simp add*: *idem*)
show $(e\ oo\ d\ oo\ p){\cdot}x \sqsubseteq x$
by (*simp add*: *e-below-iff-below-p below*)
qed

lemma *finite-deflation-e-d-p*:
assumes *finite-deflation d*
shows *finite-deflation* (*e oo d oo p*)
proof

```
  interpret finite-deflation d by fact
  fix x :: 'b
  show (e oo d oo p)·((e oo d oo p)·x) = (e oo d oo p)·x
    by (simp add: idem)
  show (e oo d oo p)·x ⊑ x
    by (simp add: e-below-iff-below-p below)
  have finite ((λx. e·x) ‘ (λx. d·x) ‘ range (λx. p·x))
    by (simp add: finite-image)
  then have finite (range (λx. (e oo d oo p)·x))
    by (simp add: image-image)
  then show finite {x. (e oo d oo p)·x = x}
    by (rule finite-range-imp-finite-fixes)
qed

lemma deflation-p-d-e:
  assumes deflation d
  assumes d: ⋀x. d·x ⊑ e·(p·x)
  shows deflation (p oo d oo e)
proof −
  interpret d: deflation d by fact
  have p-d-e-below: (p oo d oo e)·x ⊑ x for x
  proof −
    have d·(e·x) ⊑ e·x
      by (rule d.below)
    then have p·(d·(e·x)) ⊑ p·(e·x)
      by (rule monofun-cfun-arg)
    then show ?thesis by simp
  qed
  show ?thesis
  proof
    show (p oo d oo e)·x ⊑ x for x
      by (rule p-d-e-below)
    show (p oo d oo e)·((p oo d oo e)·x) = (p oo d oo e)·x for x
    proof (rule below-antisym)
      show (p oo d oo e)·((p oo d oo e)·x) ⊑ (p oo d oo e)·x
        by (rule p-d-e-below)
      have p·(d·(d·(d·(e·x)))) ⊑ p·(d·(e·(p·(d·(e·x)))))
        by (intro monofun-cfun-arg d)
      then have p·(d·(e·x)) ⊑ p·(d·(e·(p·(d·(e·x)))))
        by (simp only: d.idem)
      then show (p oo d oo e)·x ⊑ (p oo d oo e)·((p oo d oo e)·x)
        by simp
    qed
  qed
qed

lemma finite-deflation-p-d-e:
  assumes finite-deflation d
  assumes d: ⋀x. d·x ⊑ e·(p·x)
```

```
  shows finite-deflation (p oo d oo e)
proof -
  interpret d: finite-deflation d by fact
  show ?thesis
  proof (rule finite-deflation-intro)
    have deflation d ..
    then show deflation (p oo d oo e)
      using d by (rule deflation-p-d-e)
  next
    have finite ((λx. d·x) ‘ range (λx. e·x))
      by (rule d.finite-image)
    then have finite ((λx. p·x) ‘ (λx. d·x) ‘ range (λx. e·x))
      by (rule finite-imageI)
    then have finite (range (λx. (p oo d oo e)·x))
      by (simp add: image-image)
    then show finite {x. (p oo d oo e)·x = x}
      by (rule finite-range-imp-finite-fixes)
  qed
qed

end
```

10.4 Uniqueness of ep-pairs

```
lemma ep-pair-unique-e-lemma:
  assumes 1: ep-pair e1 p
    and 2: ep-pair e2 p
  shows e1 ⊑ e2
proof (rule cfun-belowI)
  fix x
  have e1·(p·(e2·x)) ⊑ e2·x
    by (rule ep-pair.e-p-below [OF 1])
  then show e1·x ⊑ e2·x
    by (simp only: ep-pair.e-inverse [OF 2])
qed

lemma ep-pair-unique-e: ep-pair e1 p ⟹ ep-pair e2 p ⟹ e1 = e2
  by (fast intro: below-antisym elim: ep-pair-unique-e-lemma)

lemma ep-pair-unique-p-lemma:
  assumes 1: ep-pair e p1
    and 2: ep-pair e p2
  shows p1 ⊑ p2
proof (rule cfun-belowI)
  fix x
  have e·(p1·x) ⊑ x
    by (rule ep-pair.e-p-below [OF 1])
  then have p2·(e·(p1·x)) ⊑ p2·x
    by (rule monofun-cfun-arg)
```

```
  then show p1·x ⊑ p2·x
    by (simp only: ep-pair.e-inverse [OF 2])
qed

lemma ep-pair-unique-p: ep-pair e p1 ⟹ ep-pair e p2 ⟹ p1 = p2
  by (fast intro: below-antisym elim: ep-pair-unique-p-lemma)
```

10.5 Composing ep-pairs

```
lemma ep-pair-ID-ID: ep-pair ID ID
  by standard simp-all

lemma ep-pair-comp:
  assumes ep-pair e1 p1 and ep-pair e2 p2
  shows ep-pair (e2 oo e1) (p1 oo p2)
proof
  interpret ep1: ep-pair e1 p1 by fact
  interpret ep2: ep-pair e2 p2 by fact
  fix x y
  show (p1 oo p2)·((e2 oo e1)·x) = x
    by simp
  have e1·(p1·(p2·y)) ⊑ p2·y
    by (rule ep1.e-p-below)
  then have e2·(e1·(p1·(p2·y))) ⊑ e2·(p2·y)
    by (rule monofun-cfun-arg)
  also have e2·(p2·y) ⊑ y
    by (rule ep2.e-p-below)
  finally show (e2 oo e1)·((p1 oo p2)·y) ⊑ y
    by simp
qed

locale pcpo-ep-pair = ep-pair e p
  for e :: 'a::pcpo → 'b::pcpo
  and p :: 'b::pcpo → 'a::pcpo
begin

lemma e-strict [simp]: e·⊥ = ⊥
proof -
  have ⊥ ⊑ p·⊥ by (rule minimal)
  then have e·⊥ ⊑ e·(p·⊥) by (rule monofun-cfun-arg)
  also have e·(p·⊥) ⊑ ⊥ by (rule e-p-below)
  finally show e·⊥ = ⊥ by simp
qed

lemma e-bottom-iff [simp]: e·x = ⊥ ⟷ x = ⊥
  by (rule e-eq-iff [where y=⊥, unfolded e-strict])

lemma e-defined: x ≠ ⊥ ⟹ e·x ≠ ⊥
  by simp
```

lemma *p-strict* [*simp*]: *p*·⊥ = ⊥
  by (*rule e-inverse* [**where** *x*=⊥, *unfolded e-strict*])

lemmas *stricts* = *e-strict p-strict*

end

end

11 The type of strict products

theory *Sprod*
  imports *Cfun*
begin

11.1 Definition of strict product type

definition *sprod* = {*p*::′*a*::*pcpo* × ′*b*::*pcpo*. *p* = ⊥ ∨ (*fst p* ≠ ⊥ ∧ *snd p* ≠ ⊥)}

pcpodef (′*a*::*pcpo*, ′*b*::*pcpo*) *sprod* (‹(‹*notation*=‹*infix strict product*››- ⊗/ -)›
[*21*,*20*] *20*) =
  sprod :: (′*a* × ′*b*) *set*
  by (*simp-all add*: *sprod-def*)

instance *sprod* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
  by (*rule typedef-chfin* [*OF type-definition-sprod below-sprod-def*])

type-notation (*ASCII*)
  sprod (**infixr** ‹∗∗› *20*)

11.2 Definitions of constants

definition *sfst* :: (′*a*::*pcpo* ∗∗ ′*b*::*pcpo*) → ′*a*
  where *sfst* = (Λ *p*. *fst* (*Rep-sprod p*))

definition *ssnd* :: (′*a*::*pcpo* ∗∗ ′*b*::*pcpo*) → ′*b*
  where *ssnd* = (Λ *p*. *snd* (*Rep-sprod p*))

definition *spair* :: ′*a*::*pcpo* → ′*b*::*pcpo* → (′*a* ∗∗ ′*b*)
  where *spair* = (Λ *a b*. *Abs-sprod* (*seq*·*b*·*a*, *seq*·*a*·*b*))

definition *ssplit* :: (′*a*::*pcpo* → ′*b*::*pcpo* → ′*c*::*pcpo*) → (′*a* ∗∗ ′*b*) → ′*c*
  where *ssplit* = (Λ *f p*. *seq*·*p*·(*f*·(*sfst*·*p*)·(*ssnd*·*p*)))

syntax
  -stuple :: [*logic*, *args*] ⇒ *logic* (‹(‹*indent*=*1 notation*=‹*mixfix strict tuple*››′(:-,/
-:′))›)
syntax-consts

-stuple ⇌ *spair*
translations
(:x, y, z:) ⇌ (:x, (:y, z:):)
(:x, y:) ⇌ *CONST spair*·x·y

translations
Λ(*CONST spair*·x·y). t ⇌ *CONST ssplit*·(Λ x y. t)

11.3 Case analysis

lemma *spair-sprod*: (*seq*·b·a, *seq*·a·b) ∈ *sprod*
by (*simp add*: *sprod-def seq-conv-if*)

lemma *Rep-sprod-spair*: *Rep-sprod* (:a, b:) = (*seq*·b·a, *seq*·a·b)
by (*simp add*: *spair-def cont-Abs-sprod Abs-sprod-inverse spair-sprod*)

lemmas *Rep-sprod-simps* =
Rep-sprod-inject [*symmetric*] *below-sprod-def*
prod-eq-iff below-prod-def
Rep-sprod-strict Rep-sprod-spair

lemma *sprodE* [*case-names bottom spair*, *cases type*: *sprod*]:
obtains $p = \bot$ | x y **where** p = (:x, y:) **and** $x \neq \bot$ **and** $y \neq \bot$
using *Rep-sprod* [*of* p] **by** (*auto simp add*: *sprod-def Rep-sprod-simps*)

lemma *sprod-induct* [*case-names bottom spair*, *induct type*: *sprod*]:
⟦P ⊥; ⋀x y. ⟦$x \neq \bot$; $y \neq \bot$⟧ ⟹ P (:x, y:)⟧ ⟹ P x
by (*cases* x) *simp-all*

11.4 Properties of *spair*

lemma *spair-strict1* [*simp*]: (:⊥, y:) = ⊥
by (*simp add*: *Rep-sprod-simps*)

lemma *spair-strict2* [*simp*]: (:x, ⊥:) = ⊥
by (*simp add*: *Rep-sprod-simps*)

lemma *spair-bottom-iff* [*simp*]: (:x, y:) = ⊥ ⟷ $x = \bot \vee y = \bot$
by (*simp add*: *Rep-sprod-simps seq-conv-if*)

lemma *spair-below-iff*: (:a, b:) ⊑ (:c, d:) ⟷ $a = \bot \vee b = \bot \vee (a \sqsubseteq c \wedge b \sqsubseteq d)$
by (*simp add*: *Rep-sprod-simps seq-conv-if*)

lemma *spair-eq-iff*: (:a, b:) = (:c, d:) ⟷ $a = c \wedge b = d \vee (a = \bot \vee b = \bot) \wedge (c = \bot \vee d = \bot)$
by (*simp add*: *Rep-sprod-simps seq-conv-if*)

lemma *spair-strict*: $x = \bot \vee y = \bot$ ⟹ (:x, y:) = ⊥
by *simp*

lemma *spair-strict-rev*: $(:x,\ y:) \neq \bot \Longrightarrow x \neq \bot \wedge y \neq \bot$
by *simp*

lemma *spair-defined*: $[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow (:x,\ y:) \neq \bot$
by *simp*

lemma *spair-defined-rev*: $(:x,\ y:) = \bot \Longrightarrow x = \bot \vee y = \bot$
by *simp*

lemma *spair-below*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x,\ y:) \sqsubseteq (:a,\ b:) \longleftrightarrow x \sqsubseteq a \wedge y \sqsubseteq b$
by (*simp add*: *spair-below-iff*)

lemma *spair-eq*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x,\ y:) = (:a,\ b:) \longleftrightarrow x = a \wedge y = b$
by (*simp add*: *spair-eq-iff*)

lemma *spair-inject*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x,\ y:) = (:a,\ b:) \Longrightarrow x = a \wedge y = b$
by (*rule spair-eq* [*THEN iffD1*])

lemma *inst-sprod-pcpo2*: $\bot = (:\bot,\ \bot:)$
by *simp*

lemma *sprodE2*: $(\bigwedge x\ y.\ p = (:x,\ y:) \Longrightarrow Q) \Longrightarrow Q$
by (*cases p*) (*simp only*: *inst-sprod-pcpo2*, *simp*)

11.5 Properties of *sfst* and *ssnd*

lemma *sfst-strict* [*simp*]: $sfst\cdot\bot = \bot$
by (*simp add*: *sfst-def cont-Rep-sprod Rep-sprod-strict*)

lemma *ssnd-strict* [*simp*]: $ssnd\cdot\bot = \bot$
by (*simp add*: *ssnd-def cont-Rep-sprod Rep-sprod-strict*)

lemma *sfst-spair* [*simp*]: $y \neq \bot \Longrightarrow sfst\cdot(:x,\ y:) = x$
by (*simp add*: *sfst-def cont-Rep-sprod Rep-sprod-spair*)

lemma *ssnd-spair* [*simp*]: $x \neq \bot \Longrightarrow ssnd\cdot(:x,\ y:) = y$
by (*simp add*: *ssnd-def cont-Rep-sprod Rep-sprod-spair*)

lemma *sfst-bottom-iff* [*simp*]: $sfst\cdot p = \bot \longleftrightarrow p = \bot$
by (*cases p*) *simp-all*

lemma *ssnd-bottom-iff* [*simp*]: $ssnd\cdot p = \bot \longleftrightarrow p = \bot$
by (*cases p*) *simp-all*

lemma *sfst-defined*: $p \neq \bot \Longrightarrow sfst\cdot p \neq \bot$
by *simp*

lemma *ssnd-defined*: $p \neq \bot \Longrightarrow ssnd\cdot p \neq \bot$
by *simp*

lemma *spair-sfst-ssnd*: $(:sfst\cdot p,\ ssnd\cdot p:) = p$
by (*cases p*) *simp-all*

lemma *below-sprod*: $x \sqsubseteq y \longleftrightarrow sfst\cdot x \sqsubseteq sfst\cdot y \wedge ssnd\cdot x \sqsubseteq ssnd\cdot y$
by (*simp add*: *Rep-sprod-simps sfst-def ssnd-def cont-Rep-sprod*)

lemma *eq-sprod*: $x = y \longleftrightarrow sfst\cdot x = sfst\cdot y \wedge ssnd\cdot x = ssnd\cdot y$
by (*auto simp add*: *po-eq-conv below-sprod*)

lemma *sfst-below-iff*: $sfst\cdot x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:y,\ ssnd\cdot x:)$
by (*cases* $x = \bot$, *simp*, *cases* $y = \bot$, *simp*, *simp add*: *below-sprod*)

lemma *ssnd-below-iff*: $ssnd\cdot x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:sfst\cdot x,\ y:)$
by (*cases* $x = \bot$, *simp*, *cases* $y = \bot$, *simp*, *simp add*: *below-sprod*)

11.6 Compactness

lemma *compact-sfst*: *compact* $x \Longrightarrow$ *compact* $(sfst\cdot x)$
by (*rule compactI*) (*simp add*: *sfst-below-iff*)

lemma *compact-ssnd*: *compact* $x \Longrightarrow$ *compact* $(ssnd\cdot x)$
by (*rule compactI*) (*simp add*: *ssnd-below-iff*)

lemma *compact-spair*: *compact* $x \Longrightarrow$ *compact* $y \Longrightarrow$ *compact* $(:x,\ y:)$
by (*rule compact-sprod*) (*simp add*: *Rep-sprod-spair seq-conv-if*)

lemma *compact-spair-iff*: *compact* $(:x,\ y:) \longleftrightarrow x = \bot \vee y = \bot \vee$ (*compact* $x \wedge$ *compact* y)
apply (*safe elim*!: *compact-spair*)
apply (*drule compact-sfst*, *simp*)
apply (*drule compact-ssnd*, *simp*)
apply *simp*
apply *simp*
done

11.7 Properties of *ssplit*

lemma *ssplit1* [*simp*]: $ssplit\cdot f\cdot\bot = \bot$
by (*simp add*: *ssplit-def*)

lemma *ssplit2* [*simp*]: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow ssplit\cdot f\cdot(:x,\ y:) = f\cdot x\cdot y$
by (*simp add*: *ssplit-def*)

lemma *ssplit3* [*simp*]: $ssplit\cdot spair\cdot z = z$
by (*cases z*) *simp-all*

11.8 Strict product preserves flatness

instance *sprod* :: (*flat*, *flat*) *flat*

```
proof
  fix x y :: 'a ⊗ 'b
  assume x ⊑ y
  then show x = ⊥ ∨ x = y
    apply (induct x, simp)
    apply (induct y, simp)
    apply (simp add: spair-below-iff flat-below-iff)
    done
qed

end
```

12 The type of lifted values

```
theory Up
  imports Cfun
begin
```

12.1 Definition of new type for lifting

```
datatype 'a u (‹(‹notation=‹postfix lifting››-⊥)› [1000] 999) = Ibottom | Iup 'a

primrec Ifup :: ('a → 'b::pcpo) ⇒ 'a u ⇒ 'b
  where
    Ifup f Ibottom = ⊥
  | Ifup f (Iup x) = f·x
```

12.2 Ordering on lifted cpo

```
instantiation u :: (cpo) below
begin

definition below-up-def:
  (⊑) ≡
    (λx y.
      (case x of
        Ibottom ⇒ True
      | Iup a ⇒ (case y of Ibottom ⇒ False | Iup b ⇒ a ⊑ b)))

instance ..

end

lemma minimal-up [iff]: Ibottom ⊑ z
  by (simp add: below-up-def)

lemma not-Iup-below [iff]: Iup x ⋢ Ibottom
  by (simp add: below-up-def)
```

lemma *Iup-below* [*iff*]: $(Iup\ x \sqsubseteq Iup\ y) = (x \sqsubseteq y)$
  by (*simp add*: *below-up-def*)

12.3 Lifted cpo is a partial order

instance u :: (*cpo*) *po*
proof
  fix x :: $'a\ u$
  show $x \sqsubseteq x$
    by (*simp add*: *below-up-def split*: *u.split*)
next
  fix $x\ y$ :: $'a\ u$
  assume $x \sqsubseteq y$ $y \sqsubseteq x$
  then show $x = y$
    by (*auto simp*: *below-up-def split*: *u.split-asm intro*: *below-antisym*)
next
  fix $x\ y\ z$ :: $'a\ u$
  assume $x \sqsubseteq y$ $y \sqsubseteq z$
  then show $x \sqsubseteq z$
    by (*auto simp*: *below-up-def split*: *u.split-asm intro*: *below-trans*)
qed

12.4 Lifted cpo is a cpo

lemma *is-lub-Iup*: $range\ S <<|\ x \Longrightarrow range\ (\lambda i.\ Iup\ (S\ i)) <<|\ Iup\ x$
  by (*auto simp*: *is-lub-def is-ub-def ball-simps below-up-def split*: *u.split*)

lemma *up-chain-lemma*:
  assumes Y: *chain* Y
  obtains $\forall i.\ Y\ i = Ibottom$
  | $A\ k$ **where** $\forall i.\ Iup\ (A\ i) = Y\ (i + k)$ **and** *chain* A **and** $range\ Y <<|\ Iup$
$(\bigsqcup i.\ A\ i)$
proof (*cases* $\exists k.\ Y\ k \neq Ibottom$)
  case *True*
  then obtain k **where** k: $Y\ k \neq Ibottom$ **..**
  define A **where** $A\ i = (THE\ a.\ Iup\ a = Y\ (i + k))$ **for** i
  have *Iup-A*: $\forall i.\ Iup\ (A\ i) = Y\ (i + k)$
  proof
    fix i :: *nat*
    from Y *le-add2* **have** $Y\ k \sqsubseteq Y\ (i + k)$ **by** (*rule chain-mono*)
    with k **have** $Y\ (i + k) \neq Ibottom$ **by** (*cases* $Y\ k$) *auto*
    then show $Iup\ (A\ i) = Y\ (i + k)$
      by (*cases* $Y\ (i + k)$, *simp-all add*: *A-def*)
  qed
  from Y **have** *chain-A*: *chain* A
    by (*simp add*: *chain-def Iup-below* [*symmetric*] *Iup-A*)
  then have $range\ A <<|\ (\bigsqcup i.\ A\ i)$
    by (*rule cpo-lubI*)
  then have $range\ (\lambda i.\ Iup\ (A\ i)) <<|\ Iup\ (\bigsqcup i.\ A\ i)$
    by (*rule is-lub-Iup*)

```
  then have range (λi. Y (i + k)) <<| Iup (⨆i. A i)
    by (simp only: Iup-A)
  then have range (λi. Y i) <<| Iup (⨆i. A i)
    by (simp only: is-lub-range-shift [OF Y])
  with Iup-A chain-A show ?thesis ..
next
  case False
  then have ∀ i. Y i = Ibottom by simp
  then show ?thesis ..
qed

instance u :: (cpo) cpo
proof
  fix S :: nat ⇒ 'a u
  assume S: chain S
  then show ∃ x. range (λi. S i) <<| x
  proof (rule up-chain-lemma)
    assume ∀ i. S i = Ibottom
    then have range (λi. S i) <<| Ibottom
      by (simp add: is-lub-const)
    then show ?thesis ..
  next
    fix A :: nat ⇒ 'a
    assume range S <<| Iup (⨆i. A i)
    then show ?thesis ..
  qed
qed
```

12.5 Lifted cpo is pointed

```
instance u :: (cpo) pcpo
  by intro-classes fast
```

for compatibility with old HOLCF-Version

```
lemma inst-up-pcpo: ⊥ = Ibottom
  by (rule minimal-up [THEN bottomI, symmetric])
```

12.6 Continuity of *Iup* and *Ifup*

continuity for *Iup*

```
lemma cont-Iup: cont Iup
  apply (rule contI)
  apply (rule is-lub-Iup)
  apply (erule cpo-lubI)
  done
```

continuity for *Ifup*

```
lemma cont-Ifup1: cont (λf. Ifup f x)
```

```
  by (induct x) simp-all

lemma monofun-Ifup2: monofun (λx. Ifup f x)
  apply (rule monofunI)
  apply (case-tac x, simp)
  apply (case-tac y, simp)
  apply (simp add: monofun-cfun-arg)
  done

lemma cont-Ifup2: cont (λx. Ifup f x)
proof (rule contI2)
  fix Y
  assume Y: chain Y and Y': chain (λi. Ifup f (Y i))
  from Y show Ifup f (⨆ i. Y i) ⊑ (⨆ i. Ifup f (Y i))
  proof (rule up-chain-lemma)
    fix A and k
    assume A: ∀ i. Iup (A i) = Y (i + k)
    assume chain A and range Y <<| Iup (⨆ i. A i)
    then have Ifup f (⨆ i. Y i) = (⨆ i. Ifup f (Iup (A i)))
      by (simp add: lub-eqI contlub-cfun-arg)
    also have ... = (⨆ i. Ifup f (Y (i + k)))
      by (simp add: A)
    also have ... = (⨆ i. Ifup f (Y i))
      using Y' by (rule lub-range-shift)
    finally show ?thesis by simp
  qed simp
qed (rule monofun-Ifup2)
```

12.7 Continuous versions of constants

```
definition up :: 'a → 'a u
  where up = (Λ x. Iup x)

definition fup :: ('a → 'b::pcpo) → 'a u → 'b
  where fup = (Λ f p. Ifup f p)

translations
  case l of XCONST up·x ⇒ t ⇌ CONST fup·(Λ x. t)·l
  case l of (XCONST up :: 'a)·x ⇒ t ↽ CONST fup·(Λ x. t)·l
  Λ(XCONST up·x). t ⇌ CONST fup·(Λ x. t)
```

continuous versions of lemmas for $'a_\bot$

```
lemma Exh-Up: z = ⊥ ∨ (∃ x. z = up·x)
  by (induct z) (simp add: inst-up-pcpo, simp add: up-def cont-Iup)

lemma up-eq [simp]: (up·x = up·y) = (x = y)
  by (simp add: up-def cont-Iup)

lemma up-inject: up·x = up·y ⟹ x = y
```

by *simp*

lemma *up-defined* [*simp*]: $up{\cdot}x \neq \bot$
by (*simp add*: *up-def cont-Iup inst-up-pcpo*)

lemma *not-up-less-UU*: $up{\cdot}x \not\sqsubseteq \bot$
by *simp*

lemma *up-below* [*simp*]: $up{\cdot}x \sqsubseteq up{\cdot}y \longleftrightarrow x \sqsubseteq y$
by (*simp add*: *up-def cont-Iup*)

lemma *upE* [*case-names bottom up*, *cases type*: *u*]: $[\![p = \bot \Longrightarrow Q; \bigwedge x.\ p = up{\cdot}x \Longrightarrow Q]\!] \Longrightarrow Q$
by (*cases p*) (*simp add*: *inst-up-pcpo*, *simp add*: *up-def cont-Iup*)

lemma *up-induct* [*case-names bottom up*, *induct type*: *u*]: $P\ \bot \Longrightarrow (\bigwedge x.\ P\ (up{\cdot}x)) \Longrightarrow P\ x$
by (*cases x*) *simp-all*

lifting preserves chain-finiteness

lemma *up-chain-cases*:
assumes *Y*: *chain Y*
obtains $\forall i.\ Y\ i = \bot$
| *A k* **where** $\forall i.\ up{\cdot}(A\ i) = Y\ (i + k)$ **and** *chain A* **and** $(\bigsqcup i.\ Y\ i) = up{\cdot}(\bigsqcup i.\ A\ i)$
by (*rule up-chain-lemma* [*OF Y*]) (*simp-all add*: *inst-up-pcpo up-def cont-Iup lub-eqI*)

lemma *compact-up*: $compact\ x \Longrightarrow compact\ (up{\cdot}x)$
apply (*rule compactI2*)
apply (*erule up-chain-cases*)
apply *simp*
apply (*drule* (*1*) *compactD2*, *simp*)
apply (*erule exE*)
apply (*drule-tac f=up* **and** *x=x* **in** *monofun-cfun-arg*)
apply (*simp*, *erule exI*)
done

lemma *compact-upD*: $compact\ (up{\cdot}x) \Longrightarrow compact\ x$
unfolding *compact-def*
by (*drule adm-subst* [*OF cont-Rep-cfun2* [**where** *f=up*]], *simp*)

lemma *compact-up-iff* [*simp*]: $compact\ (up{\cdot}x) = compact\ x$
by (*safe elim*!: *compact-up compact-upD*)

instance *u* :: (*chfin*) *chfin*
apply *intro-classes*
apply (*erule compact-imp-max-in-chain*)
apply (*rule-tac* $p=\bigsqcup i.\ Y\ i$ **in** *upE*, *simp-all*)

done

properties of fup

lemma *fup1* [*simp*]: $fup \cdot f \cdot \bot = \bot$
by (*simp add*: *fup-def cont-Ifup1 cont-Ifup2 inst-up-pcpo cont2cont-LAM*)

lemma *fup2* [*simp*]: $fup \cdot f \cdot (up \cdot x) = f \cdot x$
by (*simp add*: *up-def fup-def cont-Iup cont-Ifup1 cont-Ifup2 cont2cont-LAM*)

lemma *fup3* [*simp*]: $fup \cdot up \cdot x = x$
by (*cases x*) *simp-all*

end

13 Lifting types of class type to flat pcpo's

theory *Lift*
imports *Up*
begin

pcpodef *'a::type lift* = *UNIV* :: *'a discr u set*
by *simp-all*

lemmas *inst-lift-pcpo* = *Abs-lift-strict* [*symmetric*]

definition
Def :: *'a::type* ⇒ *'a lift* **where**
Def x = *Abs-lift* ($up \cdot$(*Discr x*))

13.1 Lift as a datatype

lemma *lift-induct*: $[\![P\ \bot; \bigwedge x.\ P\ (Def\ x)]\!] \Longrightarrow P\ y$
apply (*induct y*)
apply (*rule-tac p=y* **in** *upE*)
apply (*simp add*: *Abs-lift-strict*)
apply (*case-tac x*)
apply (*simp add*: *Def-def*)
done

old-rep-datatype ⊥::*'a::type lift Def*
by (*erule lift-induct*) (*simp-all add*: *Def-def Abs-lift-inject inst-lift-pcpo*)

⊥ and *Def*

lemma *not-Undef-is-Def*: $(x \neq \bot) = (\exists y.\ x = Def\ y)$
by (*cases x*) *simp-all*

lemma *lift-definedE*: $[\![x \neq \bot; \bigwedge a.\ x = Def\ a \Longrightarrow R]\!] \Longrightarrow R$
by (*cases x*) *simp-all*

For $x \neq \bot$ in assumptions *defined* replaces x by *Def a* in conclusion.

method-setup *defined* = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD′
    (eresolve-tac ctxt @{thms lift-definedE} THEN′ asm-simp-tac ctxt))
›

lemma *DefE*: *Def x* = ⊥ ⟹ *R*
  by *simp*

lemma *DefE2*: ⟦*x* = *Def s*; *x* = ⊥⟧ ⟹ *R*
  by *simp*

lemma *Def-below-Def*: *Def x* ⊑ *Def y* ⟷ *x* = *y*
by (*simp add*: *below-lift-def Def-def Abs-lift-inverse*)

lemma *Def-below-iff* [*simp*]: *Def x* ⊑ *y* ⟷ *Def x* = *y*
by (*induct y*, *simp*, *simp add*: *Def-below-Def*)

13.2 Lift is flat

instance *lift* :: (*type*) *flat*
proof
  fix *x y* :: ′*a lift*
  assume *x* ⊑ *y* **thus** *x* = ⊥ ∨ *x* = *y*
    by (*induct x*) *auto*
qed

13.3 Continuity of *case-lift*

lemma *case-lift-eq*: *case-lift* ⊥ *f x* = *fup*·(Λ *y*. *f* (*undiscr y*))·(*Rep-lift x*)
apply (*induct x*, *unfold lift.case*)
apply (*simp add*: *Rep-lift-strict*)
apply (*simp add*: *Def-def Abs-lift-inverse*)
done

lemma *cont2cont-case-lift* [*simp*]:
  ⟦⋀*y*. *cont* (λ*x*. *f x y*); *cont g*⟧ ⟹ *cont* (λ*x*. *case-lift* ⊥ (*f x*) (*g x*))
unfolding *case-lift-eq* **by** (*simp add*: *cont-Rep-lift*)

13.4 Further operations

definition
  flift1 :: (′*a*::*type* ⇒ ′*b*::*pcpo*) ⇒ (′*a lift* → ′*b*) (**binder** ‹*FLIFT* › *10*) **where**
  flift1 = (λ*f*. (Λ *x*. *case-lift* ⊥ *f x*))

translations
  Λ(*XCONST Def x*). *t* => *CONST flift1* (λ*x*. *t*)
  Λ(*CONST Def x*). *FLIFT y*. *t* <= *FLIFT x y*. *t*
  Λ(*CONST Def x*). *t* <= *FLIFT x*. *t*

definition
flift2 :: (*'a*::*type* ⇒ *'b*::*type*) ⇒ (*'a lift* → *'b lift*) **where**
flift2 f = (*FLIFT x*. *Def* (*f x*))

lemma *flift1-Def* [*simp*]: *flift1 f*·(*Def x*) = (*f x*)
by (*simp add*: *flift1-def*)

lemma *flift2-Def* [*simp*]: *flift2 f*·(*Def x*) = *Def* (*f x*)
by (*simp add*: *flift2-def*)

lemma *flift1-strict* [*simp*]: *flift1 f*·⊥ = ⊥
by (*simp add*: *flift1-def*)

lemma *flift2-strict* [*simp*]: *flift2 f*·⊥ = ⊥
by (*simp add*: *flift2-def*)

lemma *flift2-defined* [*simp*]: *x* ≠ ⊥ ⟹ (*flift2 f*)·*x* ≠ ⊥
by (*erule lift-definedE*, *simp*)

lemma *flift2-bottom-iff* [*simp*]: (*flift2 f*·*x* = ⊥) = (*x* = ⊥)
by (*cases x*, *simp-all*)

lemma *FLIFT-mono*:
(⋀*x*. *f x* ⊑ *g x*) ⟹ (*FLIFT x*. *f x*) ⊑ (*FLIFT x*. *g x*)
by (*rule cfun-belowI*, *case-tac x*, *simp-all*)

lemma *cont2cont-flift1* [*simp*, *cont2cont*]:
⟦⋀*y*. *cont* (λ*x*. *f x y*)⟧ ⟹ *cont* (λ*x*. *FLIFT y*. *f x y*)
by (*simp add*: *flift1-def cont2cont-LAM*)

end

14 The type of lifted booleans

theory *Tr*
imports *Lift*
begin

14.1 Type definition and constructors

type-synonym *tr* = *bool lift*

translations
(*type*) *tr* ↽ (*type*) *bool lift*

definition *TT* :: *tr*
where *TT* = *Def True*

definition *FF* :: *tr*
  where *FF* = *Def False*

Exhaustion and Elimination for type *tr*

lemma *Exh-tr*: $t = \bot \vee t = TT \vee t = FF$
  by (*induct t*) (*auto simp*: *FF-def TT-def*)

lemma *trE* [*case-names bottom TT FF*, *cases type*: *tr*]:
  $[\![p = \bot \Longrightarrow Q; p = TT \Longrightarrow Q; p = FF \Longrightarrow Q]\!] \Longrightarrow Q$
  by (*induct p*) (*auto simp*: *FF-def TT-def*)

lemma *tr-induct* [*case-names bottom TT FF*, *induct type*: *tr*]:
  $P \bot \Longrightarrow P\ TT \Longrightarrow P\ FF \Longrightarrow P\ x$
  by (*cases x*) *simp-all*

distinctness for type *tr*

lemma *dist-below-tr* [*simp*]:
  $TT \not\sqsubseteq \bot\ FF \not\sqsubseteq \bot\ TT \not\sqsubseteq FF\ FF \not\sqsubseteq TT$
  by (*simp-all add*: *TT-def FF-def*)

lemma *dist-eq-tr* [*simp*]: $TT \neq \bot\ FF \neq \bot\ TT \neq FF\ \bot \neq TT\ \bot \neq FF\ FF \neq TT$
  by (*simp-all add*: *TT-def FF-def*)

lemma *TT-below-iff* [*simp*]: $TT \sqsubseteq x \longleftrightarrow x = TT$
  by (*induct x*) *simp-all*

lemma *FF-below-iff* [*simp*]: $FF \sqsubseteq x \longleftrightarrow x = FF$
  by (*induct x*) *simp-all*

lemma *not-below-TT-iff* [*simp*]: $x \not\sqsubseteq TT \longleftrightarrow x = FF$
  by (*induct x*) *simp-all*

lemma *not-below-FF-iff* [*simp*]: $x \not\sqsubseteq FF \longleftrightarrow x = TT$
  by (*induct x*) *simp-all*

14.2 Case analysis

definition *tr-case* :: $'a::pcpo \rightarrow 'a \rightarrow tr \rightarrow 'a$
  where *tr-case* = (Λ *t e* (*Def b*). *if b then t else e*)

abbreviation *cifte-syn* :: [*tr*, *'c::pcpo*, *'c*] ⇒ *'c* (‹(‹notation=‹mixfix If expression››If (-)/ then (-)/ else (-))› [*0*, *0*, *60*] *60*)
  where *If b then e1 else e2* ≡ *tr-case·e1·e2·b*

translations
  Λ (*XCONST TT*). *t* ⇌ *CONST tr-case·t·*⊥
  Λ (*XCONST FF*). *t* ⇌ *CONST tr-case·*⊥*·t*

lemma *ifte-thms* [*simp*]:

If ⊥ *then e1 else e2* = ⊥
If FF then e1 else e2 = *e2*
If TT then e1 else e2 = *e1*
by (*simp-all add*: *tr-case-def TT-def FF-def*)

14.3 Boolean connectives

definition *trand* :: *tr* → *tr* → *tr*
where *andalso-def*: *trand* = (Λ *x y*. *If x then y else FF*)

abbreviation *andalso-syn* :: *tr* ⇒ *tr* ⇒ *tr* (‹- *andalso* -› [*36*,*35*] *35*)
where *x andalso y* ≡ *trand*·*x*·*y*

definition *tror* :: *tr* → *tr* → *tr*
where *orelse-def*: *tror* = (Λ *x y*. *If x then TT else y*)

abbreviation *orelse-syn* :: *tr* ⇒ *tr* ⇒ *tr* (‹- *orelse* -› [*31*,*30*] *30*)
where *x orelse y* ≡ *tror*·*x*·*y*

definition *neg* :: *tr* → *tr*
where *neg* = *flift2 Not*

definition *If2* :: *tr* ⇒ ′*c*::*pcpo* ⇒ ′*c* ⇒ ′*c*
where *If2 Q x y* = (*If Q then x else y*)

tactic for tr-thms with case split

lemmas *tr-defs* = *andalso-def orelse-def neg-def tr-case-def TT-def FF-def*

lemmas about andalso, orelse, neg and if

lemma *andalso-thms* [*simp*]:
(*TT andalso y*) = *y*
(*FF andalso y*) = *FF*
(⊥ *andalso y*) = ⊥
(*y andalso TT*) = *y*
(*y andalso y*) = *y*
apply (*unfold andalso-def*, *simp-all*)
apply (*cases y*, *simp-all*)
apply (*cases y*, *simp-all*)
done

lemma *orelse-thms* [*simp*]:
(*TT orelse y*) = *TT*
(*FF orelse y*) = *y*
(⊥ *orelse y*) = ⊥
(*y orelse FF*) = *y*
(*y orelse y*) = *y*
apply (*unfold orelse-def*, *simp-all*)
apply (*cases y*, *simp-all*)
apply (*cases y*, *simp-all*)

done

lemma *neg-thms* [*simp*]:
$neg{\cdot}TT = FF$
$neg{\cdot}FF = TT$
$neg{\cdot}\bot = \bot$
by (*simp-all add*: *neg-def TT-def FF-def*)

split-tac for If via If2 because the constant has to be a constant

lemma *split-If2*: $P\ (If2\ Q\ x\ y) \longleftrightarrow ((Q = \bot \longrightarrow P\ \bot) \wedge (Q = TT \longrightarrow P\ x) \wedge (Q = FF \longrightarrow P\ y))$
by (*cases Q*) (*simp-all add*: *If2-def*)

ML ‹
```
fun split-If-tac ctxt =
  simp-tac (put-simpset HOL-basic-ss ctxt addsimps [@{thm If2-def} RS sym])
    THEN' (split-tac ctxt [@{thm split-If2}])
›
```

14.4 Rewriting of HOLCF operations to HOL functions

lemma *andalso-or*: $t \neq \bot \Longrightarrow (t\ andalso\ s) = FF \longleftrightarrow t = FF \vee s = FF$
by (*cases t*) *simp-all*

lemma *andalso-and*: $t \neq \bot \Longrightarrow ((t\ andalso\ s) \neq FF) \longleftrightarrow t \neq FF \wedge s \neq FF$
by (*cases t*) *simp-all*

lemma *Def-bool1* [*simp*]: $Def\ x \neq FF \longleftrightarrow x$
by (*simp add*: *FF-def*)

lemma *Def-bool2* [*simp*]: $Def\ x = FF \longleftrightarrow \neg\ x$
by (*simp add*: *FF-def*)

lemma *Def-bool3* [*simp*]: $Def\ x = TT \longleftrightarrow x$
by (*simp add*: *TT-def*)

lemma *Def-bool4* [*simp*]: $Def\ x \neq TT \longleftrightarrow \neg\ x$
by (*simp add*: *TT-def*)

lemma *If-and-if*: (*If Def P then A else B*) = (*if P then A else B*)
by (*cases Def P*) (*auto simp add*: *TT-def*[*symmetric*] *FF-def*[*symmetric*])

14.5 Compactness

lemma *compact-TT*: *compact TT*
by (*rule compact-chfin*)

lemma *compact-FF*: *compact FF*

by (*rule compact-chfin*)

end

15 The type of strict sums

theory *Ssum*
imports *Tr*
begin

15.1 Definition of strict sum type

definition *ssum* =
$\{p :: tr \times ('a::pcpo \times 'b::pcpo).\ p = \bot \lor$
$(fst\ p = TT \land fst\ (snd\ p) \neq \bot \land snd\ (snd\ p) = \bot) \lor$
$(fst\ p = FF \land fst\ (snd\ p) = \bot \land snd\ (snd\ p) \neq \bot)\}$

pcpodef (*'a::pcpo*, *'b::pcpo*) *ssum* (‹(‹*notation*=‹*infix strict sum*››- ⊕/ -)› [*21*, *20*] *20*) =
ssum :: (*tr* × *'a* × *'b*) *set*
by (*simp-all add*: *ssum-def*)

instance *ssum* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
by (*rule typedef-chfin* [*OF type-definition-ssum below-ssum-def*])

type-notation (*ASCII*)
ssum (**infixr** ‹++› *10*)

15.2 Definitions of constructors

definition *sinl* :: *'a::pcpo* → (*'a* ++ *'b::pcpo*)
where $sinl = (\Lambda\ a.\ Abs\text{-}ssum\ (seq{\cdot}a{\cdot}TT,\ a,\ \bot))$

definition *sinr* :: *'b::pcpo* → (*'a::pcpo* ++ *'b*)
where $sinr = (\Lambda\ b.\ Abs\text{-}ssum\ (seq{\cdot}b{\cdot}FF,\ \bot,\ b))$

lemma *sinl-ssum*: $(seq{\cdot}a{\cdot}TT,\ a,\ \bot) \in ssum$
by (*simp add*: *ssum-def seq-conv-if*)

lemma *sinr-ssum*: $(seq{\cdot}b{\cdot}FF,\ \bot,\ b) \in ssum$
by (*simp add*: *ssum-def seq-conv-if*)

lemma *Rep-ssum-sinl*: $Rep\text{-}ssum\ (sinl{\cdot}a) = (seq{\cdot}a{\cdot}TT,\ a,\ \bot)$
by (*simp add*: *sinl-def cont-Abs-ssum Abs-ssum-inverse sinl-ssum*)

lemma *Rep-ssum-sinr*: $Rep\text{-}ssum\ (sinr{\cdot}b) = (seq{\cdot}b{\cdot}FF,\ \bot,\ b)$
by (*simp add*: *sinr-def cont-Abs-ssum Abs-ssum-inverse sinr-ssum*)

lemmas *Rep-ssum-simps* =

Rep-ssum-inject [*symmetric*] *below-ssum-def*
prod-eq-iff below-prod-def
Rep-ssum-strict Rep-ssum-sinl Rep-ssum-sinr

15.3 Properties of *sinl* and *sinr*

Ordering

lemma *sinl-below* [*simp*]: $sinl \cdot x \sqsubseteq sinl \cdot y \longleftrightarrow x \sqsubseteq y$
by (*simp add*: *Rep-ssum-simps seq-conv-if*)

lemma *sinr-below* [*simp*]: $sinr \cdot x \sqsubseteq sinr \cdot y \longleftrightarrow x \sqsubseteq y$
by (*simp add*: *Rep-ssum-simps seq-conv-if*)

lemma *sinl-below-sinr* [*simp*]: $sinl \cdot x \sqsubseteq sinr \cdot y \longleftrightarrow x = \bot$
by (*simp add*: *Rep-ssum-simps seq-conv-if*)

lemma *sinr-below-sinl* [*simp*]: $sinr \cdot x \sqsubseteq sinl \cdot y \longleftrightarrow x = \bot$
by (*simp add*: *Rep-ssum-simps seq-conv-if*)

Equality

lemma *sinl-eq* [*simp*]: $sinl \cdot x = sinl \cdot y \longleftrightarrow x = y$
by (*simp add*: *po-eq-conv*)

lemma *sinr-eq* [*simp*]: $sinr \cdot x = sinr \cdot y \longleftrightarrow x = y$
by (*simp add*: *po-eq-conv*)

lemma *sinl-eq-sinr* [*simp*]: $sinl \cdot x = sinr \cdot y \longleftrightarrow x = \bot \land y = \bot$
by (*subst po-eq-conv*) *simp*

lemma *sinr-eq-sinl* [*simp*]: $sinr \cdot x = sinl \cdot y \longleftrightarrow x = \bot \land y = \bot$
by (*subst po-eq-conv*) *simp*

lemma *sinl-inject*: $sinl \cdot x = sinl \cdot y \Longrightarrow x = y$
by (*rule sinl-eq* [*THEN iffD1*])

lemma *sinr-inject*: $sinr \cdot x = sinr \cdot y \Longrightarrow x = y$
by (*rule sinr-eq* [*THEN iffD1*])

Strictness

lemma *sinl-strict* [*simp*]: $sinl \cdot \bot = \bot$
by (*simp add*: *Rep-ssum-simps*)

lemma *sinr-strict* [*simp*]: $sinr \cdot \bot = \bot$
by (*simp add*: *Rep-ssum-simps*)

lemma *sinl-bottom-iff* [*simp*]: $sinl \cdot x = \bot \longleftrightarrow x = \bot$
using *sinl-eq* [*of x* $\bot$] **by** *simp*

lemma *sinr-bottom-iff* [*simp*]: $sinr \cdot x = \bot \longleftrightarrow x = \bot$
using *sinr-eq* [*of x* ⊥] **by** *simp*

lemma *sinl-defined*: $x \neq \bot \Longrightarrow sinl \cdot x \neq \bot$
by *simp*

lemma *sinr-defined*: $x \neq \bot \Longrightarrow sinr \cdot x \neq \bot$
by *simp*

Compactness

lemma *compact-sinl*: *compact x* ⟹ *compact* (*sinl·x*)
by (*rule compact-ssum*) (*simp add*: *Rep-ssum-sinl*)

lemma *compact-sinr*: *compact x* ⟹ *compact* (*sinr·x*)
by (*rule compact-ssum*) (*simp add*: *Rep-ssum-sinr*)

lemma *compact-sinlD*: *compact* (*sinl·x*) ⟹ *compact x*
unfolding *compact-def*
by (*drule adm-subst* [*OF cont-Rep-cfun2* [**where** *f*=*sinl*]], *simp*)

lemma *compact-sinrD*: *compact* (*sinr·x*) ⟹ *compact x*
unfolding *compact-def*
by (*drule adm-subst* [*OF cont-Rep-cfun2* [**where** *f*=*sinr*]], *simp*)

lemma *compact-sinl-iff* [*simp*]: *compact* (*sinl·x*) = *compact x*
by (*safe elim*!: *compact-sinl compact-sinlD*)

lemma *compact-sinr-iff* [*simp*]: *compact* (*sinr·x*) = *compact x*
by (*safe elim*!: *compact-sinr compact-sinrD*)

15.4 Case analysis

lemma *ssumE* [*case-names bottom sinl sinr*, *cases type*: *ssum*]:
obtains $p = \bot$
| *x* **where** $p = sinl \cdot x$ **and** $x \neq \bot$
| *y* **where** $p = sinr \cdot y$ **and** $y \neq \bot$
using *Rep-ssum* [*of p*] **by** (*auto simp add*: *ssum-def Rep-ssum-simps*)

lemma *ssum-induct* [*case-names bottom sinl sinr*, *induct type*: *ssum*]:
$[\![P\ \bot;$
$\bigwedge x.\ x \neq \bot \Longrightarrow P\ (sinl \cdot x);$
$\bigwedge y.\ y \neq \bot \Longrightarrow P\ (sinr \cdot y)]\!] \Longrightarrow P\ x$
by (*cases x*) *simp-all*

lemma *ssumE2* [*case-names sinl sinr*]:
$[\![\bigwedge x.\ p = sinl \cdot x \Longrightarrow Q; \bigwedge y.\ p = sinr \cdot y \Longrightarrow Q]\!] \Longrightarrow Q$
by (*cases p*, *simp only*: *sinl-strict* [*symmetric*], *simp*, *simp*)

lemma *below-sinlD*: $p \sqsubseteq sinl \cdot x \Longrightarrow \exists\, y.\ p = sinl \cdot y \wedge y \sqsubseteq x$

by (*cases p*, *rule-tac x*=$\bot$ **in** *exI*, *simp-all*)

lemma *below-sinrD*: $p \sqsubseteq sinr\cdot x \Longrightarrow \exists\, y.\ p = sinr\cdot y \wedge y \sqsubseteq x$
by (*cases p*, *rule-tac x*=$\bot$ **in** *exI*, *simp-all*)

15.5 Case analysis combinator

definition *sscase* :: $('a::pcpo \rightarrow 'c::pcpo) \rightarrow ('b::pcpo \rightarrow 'c) \rightarrow ('a ++ 'b) \rightarrow 'c$
where $sscase = (\Lambda\ f\ g\ s.\ (\lambda(t, x, y).\ If\ t\ then\ f\cdot x\ else\ g\cdot y)\ (Rep\text{-}ssum\ s))$

translations
case s of XCONST sinl·x ⇒ *t1* | *XCONST sinr·y* ⇒ *t2* ⇌ *CONST sscase*·(Λ *x*. *t1*)·(Λ *y*. *t2*)·*s*
case s of (*XCONST sinl* :: *'a*)·*x* ⇒ *t1* | *XCONST sinr·y* ⇒ *t2* ⇀ *CONST sscase*·(Λ *x*. *t1*)·(Λ *y*. *t2*)·*s*

translations
Λ(*XCONST sinl·x*). *t* ⇌ *CONST sscase*·(Λ *x*. *t*)·⊥
Λ(*XCONST sinr·y*). *t* ⇌ *CONST sscase*·⊥·(Λ *y*. *t*)

lemma *beta-sscase*: $sscase\cdot f\cdot g\cdot s = (\lambda(t, x, y).\ If\ t\ then\ f\cdot x\ else\ g\cdot y)\ (Rep\text{-}ssum\ s)$
by (*simp add*: *sscase-def cont-Rep-ssum*)

lemma *sscase1* [*simp*]: $sscase\cdot f\cdot g\cdot\bot = \bot$
by (*simp add*: *beta-sscase Rep-ssum-strict*)

lemma *sscase2* [*simp*]: $x \neq \bot \Longrightarrow sscase\cdot f\cdot g\cdot(sinl\cdot x) = f\cdot x$
by (*simp add*: *beta-sscase Rep-ssum-sinl*)

lemma *sscase3* [*simp*]: $y \neq \bot \Longrightarrow sscase\cdot f\cdot g\cdot(sinr\cdot y) = g\cdot y$
by (*simp add*: *beta-sscase Rep-ssum-sinr*)

lemma *sscase4* [*simp*]: $sscase\cdot sinl\cdot sinr\cdot z = z$
by (*cases z*) *simp-all*

15.6 Strict sum preserves flatness

instance *ssum* :: (*flat*, *flat*) *flat*
apply (*intro-classes*, *clarify*)
apply (*case-tac x*, *simp*)
apply (*case-tac y*, *simp-all add*: *flat-below-iff*)
apply (*case-tac y*, *simp-all add*: *flat-below-iff*)
done

end

16 The Strict Function Type

theory *Sfun*

imports *Cfun*
begin

pcpodef ($'a$::*pcpo*, $'b$::*pcpo*) *sfun* (**infixr** ‹$\to$!› *0*) = $\{f :: 'a \to 'b.\ f\cdot\bot = \bot\}$
by *simp-all*

type-notation (*ASCII*)
sfun (**infixr** ‹−>!› *0*)

TODO: Define nice syntax for abstraction, application.

definition *sfun-abs* :: ($'a$::*pcpo* $\to$ $'b$::*pcpo*) $\to$ ($'a \to! 'b$)
where *sfun-abs* = (Λ *f*. *Abs-sfun* (*strictify*·*f*))

definition *sfun-rep* :: ($'a$::*pcpo* $\to$! $'b$::*pcpo*) $\to$ $'a \to 'b$
where *sfun-rep* = (Λ *f*. *Rep-sfun f*)

lemma *sfun-rep-beta*: *sfun-rep*·*f* = *Rep-sfun f*
by (*simp add*: *sfun-rep-def cont-Rep-sfun*)

lemma *sfun-rep-strict1* [*simp*]: *sfun-rep*·$\bot = \bot$
unfolding *sfun-rep-beta* **by** (*rule Rep-sfun-strict*)

lemma *sfun-rep-strict2* [*simp*]: *sfun-rep*·*f*·$\bot = \bot$
unfolding *sfun-rep-beta* **by** (*rule Rep-sfun* [*simplified*])

lemma *strictify-cancel*: $f\cdot\bot = \bot \Longrightarrow$ *strictify*·*f* = *f*
by (*simp add*: *cfun-eq-iff strictify-conv-if*)

lemma *sfun-abs-sfun-rep* [*simp*]: *sfun-abs*·(*sfun-rep*·*f*) = *f*
unfolding *sfun-abs-def sfun-rep-def*
apply (*simp add*: *cont-Abs-sfun cont-Rep-sfun*)
apply (*simp add*: *Rep-sfun-inject* [*symmetric*] *Abs-sfun-inverse*)
apply (*simp add*: *cfun-eq-iff strictify-conv-if*)
apply (*simp add*: *Rep-sfun* [*simplified*])
done

lemma *sfun-rep-sfun-abs* [*simp*]: *sfun-rep*·(*sfun-abs*·*f*) = *strictify*·*f*
unfolding *sfun-abs-def sfun-rep-def*
apply (*simp add*: *cont-Abs-sfun cont-Rep-sfun*)
apply (*simp add*: *Abs-sfun-inverse*)
done

lemma *sfun-eq-iff*: $f = g \longleftrightarrow$ *sfun-rep*·*f* = *sfun-rep*·*g*
by (*simp add*: *sfun-rep-def cont-Rep-sfun Rep-sfun-inject*)

lemma *sfun-below-iff*: $f \sqsubseteq g \longleftrightarrow$ *sfun-rep*·*f* $\sqsubseteq$ *sfun-rep*·*g*
by (*simp add*: *sfun-rep-def cont-Rep-sfun below-sfun-def*)

end

17 Map functions for various types

theory *Map-Functions*
imports *Deflation Sprod Ssum Sfun Up*
begin

17.1 Map operator for continuous function space

definition *cfun-map* :: $('b \to 'a) \to ('c \to 'd) \to ('a \to 'c) \to ('b \to 'd)$
where *cfun-map* = $(\Lambda\ a\ b\ f\ x.\ b\cdot(f\cdot(a\cdot x)))$

lemma *cfun-map-beta* [*simp*]: *cfun-map*$\cdot a\cdot b\cdot f\cdot x = b\cdot(f\cdot(a\cdot x))$
by (*simp add*: *cfun-map-def*)

lemma *cfun-map-ID*: *cfun-map*$\cdot ID\cdot ID = ID$
by (*simp add*: *cfun-eq-iff*)

lemma *cfun-map-map*: *cfun-map*$\cdot f1\cdot g1\cdot($*cfun-map*$\cdot f2\cdot g2\cdot p) =$ *cfun-map*$\cdot(\Lambda\ x.\ f2\cdot(f1\cdot x))\cdot(\Lambda\ x.\ g1\cdot(g2\cdot x))\cdot p$
by (*rule cfun-eqI*) *simp*

lemma *ep-pair-cfun-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*cfun-map*$\cdot p1\cdot e2$) (*cfun-map*$\cdot e1\cdot p2$)
proof
interpret *e1p1*: *ep-pair e1 p1* **by** *fact*
interpret *e2p2*: *ep-pair e2 p2* **by** *fact*
show *cfun-map*$\cdot e1\cdot p2\cdot($*cfun-map*$\cdot p1\cdot e2\cdot f) = f$ **for** *f*
by (*simp add*: *cfun-eq-iff*)
show *cfun-map*$\cdot p1\cdot e2\cdot($*cfun-map*$\cdot e1\cdot p2\cdot g) \sqsubseteq g$ **for** *g*
apply (*rule cfun-belowI*, *simp*)
apply (*rule below-trans* [*OF e2p2.e-p-below*])
apply (*rule monofun-cfun-arg*)
apply (*rule e1p1.e-p-below*)
done
qed

lemma *deflation-cfun-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* (*cfun-map*$\cdot d1\cdot d2$)
proof
interpret *d1*: *deflation d1* **by** *fact*
interpret *d2*: *deflation d2* **by** *fact*
fix *f*
show *cfun-map*$\cdot d1\cdot d2\cdot($*cfun-map*$\cdot d1\cdot d2\cdot f) =$ *cfun-map*$\cdot d1\cdot d2\cdot f$
by (*simp add*: *cfun-eq-iff d1.idem d2.idem*)
show *cfun-map*$\cdot d1\cdot d2\cdot f \sqsubseteq f$
apply (*rule cfun-belowI*, *simp*)
apply (*rule below-trans* [*OF d2.below*])
apply (*rule monofun-cfun-arg*)

```
    apply (rule d1.below)
    done
qed

lemma finite-range-cfun-map:
  assumes a: finite (range (λx. a·x))
  assumes b: finite (range (λy. b·y))
  shows finite (range (λf. cfun-map·a·b·f))  (is finite (range ?h))
proof (rule finite-imageD)
  let ?f = λg. range (λx. (a·x, g·x))
  show finite (?f ` range ?h)
  proof (rule finite-subset)
    let ?B = Pow (range (λx. a·x) × range (λy. b·y))
    show ?f ` range ?h ⊆ ?B
      by clarsimp
    show finite ?B
      by (simp add: a b)
  qed
  show inj-on ?f (range ?h)
  proof (rule inj-onI, rule cfun-eqI, clarsimp)
    fix x f g
    assume range (λx. (a·x, b·(f·(a·x)))) = range (λx. (a·x, b·(g·(a·x))))
    then have range (λx. (a·x, b·(f·(a·x)))) ⊆ range (λx. (a·x, b·(g·(a·x))))
      by (rule equalityD1)
    then have (a·x, b·(f·(a·x))) ∈ range (λx. (a·x, b·(g·(a·x))))
      by (simp add: subset-eq)
    then obtain y where (a·x, b·(f·(a·x))) = (a·y, b·(g·(a·y)))
      by (rule rangeE)
    then show b·(f·(a·x)) = b·(g·(a·x))
      by clarsimp
  qed
qed

lemma finite-deflation-cfun-map:
  assumes finite-deflation d1 and finite-deflation d2
  shows finite-deflation (cfun-map·d1·d2)
proof (rule finite-deflation-intro)
  interpret d1: finite-deflation d1 by fact
  interpret d2: finite-deflation d2 by fact
  from d1.deflation-axioms d2.deflation-axioms show deflation (cfun-map·d1·d2)
    by (rule deflation-cfun-map)
  have finite (range (λf. cfun-map·d1·d2·f))
    using d1.finite-range d2.finite-range
    by (rule finite-range-cfun-map)
  then show finite {f. cfun-map·d1·d2·f = f}
    by (rule finite-range-imp-finite-fixes)
qed
```

Finite deflations are compact elements of the function space

lemma *finite-deflation-imp-compact*: *finite-deflation d* $\Longrightarrow$ *compact d*
  apply (*frule finite-deflation-imp-deflation*)
  apply (*subgoal-tac compact* (*cfun-map*·*d*·*d*·*d*))
   apply (*simp add*: *cfun-map-def deflation.idem eta-cfun*)
  apply (*rule finite-deflation.compact*)
  apply (*simp only*: *finite-deflation-cfun-map*)
  done

17.2 Map operator for product type

definition *prod-map* :: $('a \to 'b) \to ('c \to 'd) \to 'a \times 'c \to 'b \times 'd$
  where *prod-map* = (Λ *f g p*. (*f*·(*fst p*), *g*·(*snd p*)))

lemma *prod-map-Pair* [*simp*]: *prod-map*·*f*·*g*·(*x*, *y*) = (*f*·*x*, *g*·*y*)
  by (*simp add*: *prod-map-def*)

lemma *prod-map-ID*: *prod-map*·*ID*·*ID* = *ID*
  by (*auto simp*: *cfun-eq-iff*)

lemma *prod-map-map*: *prod-map*·*f1*·*g1*·(*prod-map*·*f2*·*g2*·*p*) = *prod-map*·(Λ *x*. *f1*·(*f2*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
  by (*induct p*) *simp*

lemma *ep-pair-prod-map*:
  assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
  shows *ep-pair* (*prod-map*·*e1*·*e2*) (*prod-map*·*p1*·*p2*)
proof
  interpret *e1p1*: *ep-pair e1 p1* **by** *fact*
  interpret *e2p2*: *ep-pair e2 p2* **by** *fact*
  show *prod-map*·*p1*·*p2*·(*prod-map*·*e1*·*e2*·*x*) = *x* **for** *x*
    by (*induct x*) *simp*
  show *prod-map*·*e1*·*e2*·(*prod-map*·*p1*·*p2*·*y*) $\sqsubseteq$ *y* **for** *y*
    by (*induct y*) (*simp add*: *e1p1.e-p-below e2p2.e-p-below*)
qed

lemma *deflation-prod-map*:
  assumes *deflation d1* **and** *deflation d2*
  shows *deflation* (*prod-map*·*d1*·*d2*)
proof
  interpret *d1*: *deflation d1* **by** *fact*
  interpret *d2*: *deflation d2* **by** *fact*
  fix *x*
  show *prod-map*·*d1*·*d2*·(*prod-map*·*d1*·*d2*·*x*) = *prod-map*·*d1*·*d2*·*x*
    by (*induct x*) (*simp add*: *d1.idem d2.idem*)
  show *prod-map*·*d1*·*d2*·*x* $\sqsubseteq$ *x*
    by (*induct x*) (*simp add*: *d1.below d2.below*)
qed

lemma *finite-deflation-prod-map*:

assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* (*prod-map·d1·d2*)
proof (*rule finite-deflation-intro*)
interpret *d1*: *finite-deflation d1* **by** *fact*
interpret *d2*: *finite-deflation d2* **by** *fact*
from *d1.deflation-axioms d2.deflation-axioms* **show** *deflation* (*prod-map·d1·d2*)
by (*rule deflation-prod-map*)
have {*p. prod-map·d1·d2·p* = *p*} ⊆ {*x. d1·x* = *x*} × {*y. d2·y* = *y*}
by *auto*
then show *finite* {*p. prod-map·d1·d2·p* = *p*}
by (*rule finite-subset*, *simp add*: *d1.finite-fixes d2.finite-fixes*)
qed

17.3 Map function for lifted cpo

definition *u-map* :: (*'a* → *'b*) → *'a u* → *'b u*
where *u-map* = (Λ *f. fup·*(*up oo f*))

lemma *u-map-strict* [*simp*]: *u-map·f·*⊥ = ⊥
by (*simp add*: *u-map-def*)

lemma *u-map-up* [*simp*]: *u-map·f·*(*up·x*) = *up·*(*f·x*)
by (*simp add*: *u-map-def*)

lemma *u-map-ID*: *u-map·ID* = *ID*
by (*simp add*: *u-map-def cfun-eq-iff eta-cfun*)

lemma *u-map-map*: *u-map·f·*(*u-map·g·p*) = *u-map·*(Λ *x. f·*(*g·x*))*·p*
by (*induct p*) *simp-all*

lemma *u-map-oo*: *u-map·*(*f oo g*) = *u-map·f oo u-map·g*
by (*simp add*: *cfcomp1 u-map-map eta-cfun*)

lemma *ep-pair-u-map*: *ep-pair e p* ⟹ *ep-pair* (*u-map·e*) (*u-map·p*)
apply *standard*
subgoal for *x* **by** (*cases x*) (*simp-all add*: *ep-pair.e-inverse*)
subgoal for *y* **by** (*cases y*) (*simp-all add*: *ep-pair.e-p-below*)
done

lemma *deflation-u-map*: *deflation d* ⟹ *deflation* (*u-map·d*)
apply *standard*
subgoal for *x* **by** (*cases x*) (*simp-all add*: *deflation.idem*)
subgoal for *x* **by** (*cases x*) (*simp-all add*: *deflation.below*)
done

lemma *finite-deflation-u-map*:
assumes *finite-deflation d*
shows *finite-deflation* (*u-map·d*)
proof (*rule finite-deflation-intro*)

interpret *d*: *finite-deflation d* **by** *fact*
from *d.deflation-axioms* **show** *deflation* (*u-map*·*d*)
by (*rule deflation-u-map*)
have $\{x.\ u\text{-}map \cdot d \cdot x = x\} \subseteq insert\ \bot\ ((\lambda x.\ up \cdot x)\ `\ \{x.\ d \cdot x = x\})$
by (*rule subsetI*, *case-tac x*, *simp-all*)
then show *finite* $\{x.\ u\text{-}map \cdot d \cdot x = x\}$
by (*rule finite-subset*) (*simp add*: *d.finite-fixes*)
qed

17.4 Map function for strict products

definition *sprod-map* :: $('a::pcpo \to 'b::pcpo) \to ('c::pcpo \to 'd::pcpo) \to 'a \otimes 'c \to 'b \otimes 'd$
where $sprod\text{-}map = (\Lambda\ f\ g.\ ssplit \cdot (\Lambda\ x\ y.\ (:f \cdot x,\ g \cdot y:)))$

lemma *sprod-map-strict* [*simp*]: $sprod\text{-}map \cdot a \cdot b \cdot \bot = \bot$
by (*simp add*: *sprod-map-def*)

lemma *sprod-map-spair* [*simp*]: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow sprod\text{-}map \cdot f \cdot g \cdot (:x,\ y:) = (:f \cdot x,\ g \cdot y:)$
by (*simp add*: *sprod-map-def*)

lemma *sprod-map-spair'*: $f \cdot \bot = \bot \Longrightarrow g \cdot \bot = \bot \Longrightarrow sprod\text{-}map \cdot f \cdot g \cdot (:x,\ y:) = (:f \cdot x,\ g \cdot y:)$
by (*cases* $x = \bot \vee y = \bot$) *auto*

lemma *sprod-map-ID*: $sprod\text{-}map \cdot ID \cdot ID = ID$
by (*simp add*: *sprod-map-def cfun-eq-iff eta-cfun*)

lemma *sprod-map-map*:
$[\![f1 \cdot \bot = \bot;\ g1 \cdot \bot = \bot]\!] \Longrightarrow$
$sprod\text{-}map \cdot f1 \cdot g1 \cdot (sprod\text{-}map \cdot f2 \cdot g2 \cdot p) =$
$sprod\text{-}map \cdot (\Lambda\ x.\ f1 \cdot (f2 \cdot x)) \cdot (\Lambda\ x.\ g1 \cdot (g2 \cdot x)) \cdot p$
proof (*induct p*)
case *bottom*
then show *?case* **by** *simp*
next
case (*spair x y*)
then show *?case*
apply (*cases* $f2 \cdot x = \bot$, *simp*)
apply (*cases* $g2 \cdot y = \bot$, *simp*)
apply *simp*
done
qed

lemma *ep-pair-sprod-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*sprod-map*·*e1*·*e2*) (*sprod-map*·*p1*·*p2*)
proof

```
  interpret e1p1: pcpo-ep-pair e1 p1 unfolding pcpo-ep-pair-def by fact
  interpret e2p2: pcpo-ep-pair e2 p2 unfolding pcpo-ep-pair-def by fact
  show sprod-map·p1·p2·(sprod-map·e1·e2·x) = x for x
    by (induct x) simp-all
  show sprod-map·e1·e2·(sprod-map·p1·p2·y) ⊑ y for y
  proof (induct y)
    case bottom
    then show ?case by simp
  next
    case (spair x y)
    then show ?case
      apply simp
      apply (cases p1·x = ⊥, simp, cases p2·y = ⊥, simp)
      apply (simp add: monofun-cfun e1p1.e-p-below e2p2.e-p-below)
      done
  qed
qed

lemma deflation-sprod-map:
  assumes deflation d1 and deflation d2
  shows deflation (sprod-map·d1·d2)
proof
  interpret d1: deflation d1 by fact
  interpret d2: deflation d2 by fact
  fix x
  show sprod-map·d1·d2·(sprod-map·d1·d2·x) = sprod-map·d1·d2·x
  proof (induct x)
    case bottom
    then show ?case by simp
  next
    case (spair x y)
    then show ?case
      apply (cases d1·x = ⊥, simp, cases d2·y = ⊥, simp)
      apply (simp add: d1.idem d2.idem)
      done
  qed
  show sprod-map·d1·d2·x ⊑ x
  proof (induct x)
    case bottom
    then show ?case by simp
  next
    case spair
    then show ?case by (simp add: monofun-cfun d1.below d2.below)
  qed
qed

lemma finite-deflation-sprod-map:
  assumes finite-deflation d1 and finite-deflation d2
  shows finite-deflation (sprod-map·d1·d2)
```

proof (*rule finite-deflation-intro*)
  interpret *d1*: *finite-deflation d1* **by** *fact*
  interpret *d2*: *finite-deflation d2* **by** *fact*
  from *d1.deflation-axioms d2.deflation-axioms* **show** *deflation* (*sprod-map·d1·d2*)
    by (*rule deflation-sprod-map*)
  have $\{x.\ sprod\text{-}map \cdot d1 \cdot d2 \cdot x = x\} \subseteq$
      $insert\ \bot\ ((\lambda(x, y).\ (:x, y:))\ `\ (\{x.\ d1 \cdot x = x\} \times \{y.\ d2 \cdot y = y\}))$
    by (*rule subsetI*, *case-tac x*, *auto simp add*: *spair-eq-iff*)
  then show *finite* $\{x.\ sprod\text{-}map \cdot d1 \cdot d2 \cdot x = x\}$
    by (*rule finite-subset*) (*simp add*: *d1.finite-fixes d2.finite-fixes*)
qed

17.5 Map function for strict sums

definition *ssum-map* :: $('a::pcpo \to 'b::pcpo) \to ('c::pcpo \to 'd::pcpo) \to 'a \oplus 'c \to 'b \oplus 'd$
  where $ssum\text{-}map = (\Lambda\ f\ g.\ sscase \cdot (sinl\ oo\ f) \cdot (sinr\ oo\ g))$

lemma *ssum-map-strict* [*simp*]: $ssum\text{-}map \cdot f \cdot g \cdot \bot = \bot$
  by (*simp add*: *ssum-map-def*)

lemma *ssum-map-sinl* [*simp*]: $x \neq \bot \Longrightarrow ssum\text{-}map \cdot f \cdot g \cdot (sinl \cdot x) = sinl \cdot (f \cdot x)$
  by (*simp add*: *ssum-map-def*)

lemma *ssum-map-sinr* [*simp*]: $x \neq \bot \Longrightarrow ssum\text{-}map \cdot f \cdot g \cdot (sinr \cdot x) = sinr \cdot (g \cdot x)$
  by (*simp add*: *ssum-map-def*)

lemma *ssum-map-sinl'*: $f \cdot \bot = \bot \Longrightarrow ssum\text{-}map \cdot f \cdot g \cdot (sinl \cdot x) = sinl \cdot (f \cdot x)$
  by (*cases* $x = \bot$) *simp-all*

lemma *ssum-map-sinr'*: $g \cdot \bot = \bot \Longrightarrow ssum\text{-}map \cdot f \cdot g \cdot (sinr \cdot x) = sinr \cdot (g \cdot x)$
  by (*cases* $x = \bot$) *simp-all*

lemma *ssum-map-ID*: $ssum\text{-}map \cdot ID \cdot ID = ID$
  by (*simp add*: *ssum-map-def cfun-eq-iff eta-cfun*)

lemma *ssum-map-map*:
  $[\![f1 \cdot \bot = \bot;\ g1 \cdot \bot = \bot]\!] \Longrightarrow$
    $ssum\text{-}map \cdot f1 \cdot g1 \cdot (ssum\text{-}map \cdot f2 \cdot g2 \cdot p) =$
     $ssum\text{-}map \cdot (\Lambda\ x.\ f1 \cdot (f2 \cdot x)) \cdot (\Lambda\ x.\ g1 \cdot (g2 \cdot x)) \cdot p$
proof (*induct p*)
  case *bottom*
  then show *?case* **by** *simp*
next
  case (*sinl x*)
  then show *?case* **by** (*cases* $f2 \cdot x = \bot$) *simp-all*
next
  case (*sinr y*)
  then show *?case* **by** (*cases* $g2 \cdot y = \bot$) *simp-all*

```
qed

lemma ep-pair-ssum-map:
  assumes ep-pair e1 p1 and ep-pair e2 p2
  shows ep-pair (ssum-map·e1·e2) (ssum-map·p1·p2)
proof
  interpret e1p1: pcpo-ep-pair e1 p1 unfolding pcpo-ep-pair-def by fact
  interpret e2p2: pcpo-ep-pair e2 p2 unfolding pcpo-ep-pair-def by fact
  show ssum-map·p1·p2·(ssum-map·e1·e2·x) = x for x
    by (induct x) simp-all
  show ssum-map·e1·e2·(ssum-map·p1·p2·y) ⊑ y for y
  proof (induct y)
    case bottom
    then show ?case by simp
  next
    case (sinl x)
    then show ?case by (cases p1·x = ⊥) (simp-all add: e1p1.e-p-below)
  next
    case (sinr y)
    then show ?case by (cases p2·y = ⊥) (simp-all add: e2p2.e-p-below)
  qed
qed

lemma deflation-ssum-map:
  assumes deflation d1 and deflation d2
  shows deflation (ssum-map·d1·d2)
proof
  interpret d1: deflation d1 by fact
  interpret d2: deflation d2 by fact
  fix x
  show ssum-map·d1·d2·(ssum-map·d1·d2·x) = ssum-map·d1·d2·x
  proof (induct x)
    case bottom
    then show ?case by simp
  next
    case (sinl x)
    then show ?case by (cases d1·x = ⊥) (simp-all add: d1.idem)
  next
    case (sinr y)
    then show ?case by (cases d2·y = ⊥) (simp-all add: d2.idem)
  qed
  show ssum-map·d1·d2·x ⊑ x
  proof (induct x)
    case bottom
    then show ?case by simp
  next
    case (sinl x)
    then show ?case by (cases d1·x = ⊥) (simp-all add: d1.below)
  next
```

```
    case (sinr y)
    then show ?case by (cases d2·y = ⊥) (simp-all add: d2.below)
  qed
qed

lemma finite-deflation-ssum-map:
  assumes finite-deflation d1 and finite-deflation d2
  shows finite-deflation (ssum-map·d1·d2)
proof (rule finite-deflation-intro)
  interpret d1: finite-deflation d1 by fact
  interpret d2: finite-deflation d2 by fact
  from d1.deflation-axioms d2.deflation-axioms show deflation (ssum-map·d1·d2)
    by (rule deflation-ssum-map)
  have {x. ssum-map·d1·d2·x = x} ⊆
      (λx. sinl·x) ` {x. d1·x = x} ∪
      (λx. sinr·x) ` {x. d2·x = x} ∪ {⊥}
    by (rule subsetI, case-tac x, simp-all)
  then show finite {x. ssum-map·d1·d2·x = x}
    by (rule finite-subset, simp add: d1.finite-fixes d2.finite-fixes)
qed
```

17.6 Map operator for strict function space

```
definition sfun-map :: ('b::pcpo → 'a::pcpo) → ('c::pcpo → 'd::pcpo) → ('a →! 'c)
→ ('b →! 'd)
  where sfun-map = (Λ a b. sfun-abs oo cfun-map·a·b oo sfun-rep)

lemma sfun-map-ID: sfun-map·ID·ID = ID
  by (simp add: sfun-map-def cfun-map-ID cfun-eq-iff)

lemma sfun-map-map:
  assumes f2·⊥ = ⊥ and g2·⊥ = ⊥
  shows sfun-map·f1·g1·(sfun-map·f2·g2·p) =
    sfun-map·(Λ x. f2·(f1·x))·(Λ x. g1·(g2·x))·p
  by (simp add: sfun-map-def cfun-eq-iff strictify-cancel assms cfun-map-map)

lemma ep-pair-sfun-map:
  assumes 1: ep-pair e1 p1
  assumes 2: ep-pair e2 p2
  shows ep-pair (sfun-map·p1·e2) (sfun-map·e1·p2)
proof
  interpret e1p1: pcpo-ep-pair e1 p1
    unfolding pcpo-ep-pair-def by fact
  interpret e2p2: pcpo-ep-pair e2 p2
    unfolding pcpo-ep-pair-def by fact
  show sfun-map·e1·p2·(sfun-map·p1·e2·f) = f for f
    unfolding sfun-map-def
    apply (simp add: sfun-eq-iff strictify-cancel)
    apply (rule ep-pair.e-inverse)
```

```
    apply (rule ep-pair-cfun-map [OF 1 2])
    done
  show sfun-map·p1·e2·(sfun-map·e1·p2·g) ⊑ g for g
    unfolding sfun-map-def
    apply (simp add: sfun-below-iff strictify-cancel)
    apply (rule ep-pair.e-p-below)
    apply (rule ep-pair-cfun-map [OF 1 2])
    done
qed

lemma deflation-sfun-map:
  assumes 1: deflation d1
  assumes 2: deflation d2
  shows deflation (sfun-map·d1·d2)
  apply (simp add: sfun-map-def)
  apply (rule deflation.intro)
   apply simp
   apply (subst strictify-cancel)
    apply (simp add: cfun-map-def deflation-strict 1 2)
   apply (simp add: cfun-map-def deflation.idem 1 2)
  apply (simp add: sfun-below-iff)
  apply (subst strictify-cancel)
   apply (simp add: cfun-map-def deflation-strict 1 2)
  apply (rule deflation.below)
  apply (rule deflation-cfun-map [OF 1 2])
  done

lemma finite-deflation-sfun-map:
  assumes finite-deflation d1
    and finite-deflation d2
  shows finite-deflation (sfun-map·d1·d2)
proof (intro finite-deflation-intro)
  interpret d1: finite-deflation d1 by fact
  interpret d2: finite-deflation d2 by fact
  from d1.deflation-axioms d2.deflation-axioms show deflation (sfun-map·d1·d2)
    by (rule deflation-sfun-map)
  from assms have finite-deflation (cfun-map·d1·d2)
    by (rule finite-deflation-cfun-map)
  then have finite {f. cfun-map·d1·d2·f = f}
    by (rule finite-deflation.finite-fixes)
  moreover have inj (λf. sfun-rep·f)
    by (rule inj-onI) (simp add: sfun-eq-iff)
  ultimately have finite ((λf. sfun-rep·f) −` {f. cfun-map·d1·d2·f = f})
    by (rule finite-vimageI)
  with ‹deflation d1› ‹deflation d2› show finite {f. sfun-map·d1·d2·f = f}
    by (simp add: sfun-map-def sfun-eq-iff strictify-cancel deflation-strict)
qed

end
```

18 The cpo of cartesian products

theory *Cprod*
imports *Cfun*
begin

18.1 Continuous case function for unit type

definition *unit-when* :: $'a \rightarrow unit \rightarrow 'a$
where *unit-when* = (Λ *a* -. *a*)

translations
$\Lambda()$. *t* $\rightleftharpoons$ *CONST unit-when*$\cdot$*t*

lemma *unit-when* [*simp*]: *unit-when*$\cdot a \cdot u = a$
by (*simp add*: *unit-when-def*)

18.2 Continuous version of split function

definition *csplit* :: $('a \rightarrow 'b \rightarrow 'c) \rightarrow ('a \times 'b) \rightarrow 'c$
where *csplit* = (Λ *f p*. *f*$\cdot$(*fst p*)$\cdot$(*snd p*))

translations
Λ(*CONST Pair x y*). *t* $\rightleftharpoons$ *CONST csplit*$\cdot$(Λ *x y*. *t*)

abbreviation *cfst* :: $'a \times 'b \rightarrow 'a$
where *cfst* $\equiv$ *Abs-cfun fst*

abbreviation *csnd* :: $'a \times 'b \rightarrow 'b$
where *csnd* $\equiv$ *Abs-cfun snd*

18.3 Convert all lemmas to the continuous versions

lemma *csplit1* [*simp*]: *csplit*$\cdot f \cdot \bot = f \cdot \bot \cdot \bot$
by (*simp add*: *csplit-def*)

lemma *csplit-Pair* [*simp*]: *csplit*$\cdot f \cdot (x, y) = f \cdot x \cdot y$
by (*simp add*: *csplit-def*)

end

19 Profinite and bifinite cpos

theory *Bifinite*
imports *Map-Functions Cprod Sprod Sfun Up HOL−Library.Countable*
begin

19.1 Chains of finite deflations

locale *approx-chain* =

fixes *approx* :: *nat* ⇒ *'a* → *'a*
assumes *chain-approx* [*simp*]: *chain* (*λi. approx i*)
assumes *lub-approx* [*simp*]: (⨆ *i. approx i*) = *ID*
assumes *finite-deflation-approx* [*simp*]: ⋀*i. finite-deflation* (*approx i*)
begin

lemma *deflation-approx*: *deflation* (*approx i*)
using *finite-deflation-approx* **by** (*rule finite-deflation-imp-deflation*)

lemma *approx-idem*: *approx i*·(*approx i*·*x*) = *approx i*·*x*
using *deflation-approx* **by** (*rule deflation.idem*)

lemma *approx-below*: *approx i*·*x* ⊑ *x*
using *deflation-approx* **by** (*rule deflation.below*)

lemma *finite-range-approx*: *finite* (*range* (*λx. approx i*·*x*))
apply (*rule finite-deflation.finite-range*)
apply (*rule finite-deflation-approx*)
done

lemma *compact-approx* [*simp*]: *compact* (*approx n*·*x*)
apply (*rule finite-deflation.compact*)
apply (*rule finite-deflation-approx*)
done

lemma *compact-eq-approx*: *compact x* ⟹ ∃ *i. approx i*·*x* = *x*
by (*rule admD2*, *simp-all*)

end

19.2 Omega-profinite and bifinite domains

class *bifinite* = *pcpo* +
assumes *bifinite*: ∃ (*a*::*nat* ⇒ *'a* → *'a*). *approx-chain a*

class *profinite* = *cpo* +
assumes *profinite*: ∃ (*a*::*nat* ⇒ $'a_\bot$ → $'a_\bot$). *approx-chain a*

19.3 Building approx chains

lemma *approx-chain-iso*:
assumes *a*: *approx-chain a*
assumes [*simp*]: ⋀*x. f*·(*g*·*x*) = *x*
assumes [*simp*]: ⋀*y. g*·(*f*·*y*) = *y*
shows *approx-chain* (*λi. f oo a i oo g*)
proof −
have *1*: *f oo g* = *ID* **by** (*simp add*: *cfun-eqI*)
have *2*: *ep-pair f g* **by** (*simp add*: *ep-pair-def*)
from *1 2* **show** *?thesis*
using *a* **unfolding** *approx-chain-def*

by (*simp add*: *lub-APP ep-pair.finite-deflation-e-d-p*)
qed

lemma *approx-chain-u-map*:
assumes *approx-chain a*
shows *approx-chain* ($\lambda i.$ *u-map*·(*a i*))
using *assms* **unfolding** *approx-chain-def*
by (*simp add*: *lub-APP u-map-ID finite-deflation-u-map*)

lemma *approx-chain-sfun-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *sfun-map*·(*a i*)·(*b i*))
using *assms* **unfolding** *approx-chain-def*
by (*simp add*: *lub-APP sfun-map-ID finite-deflation-sfun-map*)

lemma *approx-chain-sprod-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *sprod-map*·(*a i*)·(*b i*))
using *assms* **unfolding** *approx-chain-def*
by (*simp add*: *lub-APP sprod-map-ID finite-deflation-sprod-map*)

lemma *approx-chain-ssum-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *ssum-map*·(*a i*)·(*b i*))
using *assms* **unfolding** *approx-chain-def*
by (*simp add*: *lub-APP ssum-map-ID finite-deflation-ssum-map*)

lemma *approx-chain-cfun-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *cfun-map*·(*a i*)·(*b i*))
using *assms* **unfolding** *approx-chain-def*
by (*simp add*: *lub-APP cfun-map-ID finite-deflation-cfun-map*)

lemma *approx-chain-prod-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *prod-map*·(*a i*)·(*b i*))
using *assms* **unfolding** *approx-chain-def*
by (*simp add*: *lub-APP prod-map-ID finite-deflation-prod-map*)

Approx chains for countable discrete types.

definition *discr-approx* :: *nat* ⇒ *'a::countable discr u* → *'a discr u*
where *discr-approx* = ($\lambda i.$ Λ(*up*·*x*). *if to-nat* (*undiscr x*) < *i then up*·*x else* ⊥)

lemma *chain-discr-approx* [*simp*]: *chain discr-approx*
unfolding *discr-approx-def*
by (*rule chainI*, *simp add*: *monofun-cfun monofun-LAM*)

lemma *lub-discr-approx* [*simp*]: ($\bigsqcup i.$ *discr-approx i*) = *ID*
apply (*rule cfun-eqI*)

```
  apply (simp add: contlub-cfun-fun)
  apply (simp add: discr-approx-def)
  subgoal for x
    apply (cases x)
     apply simp
    apply (rule lub-eqI)
    apply (rule is-lubI)
     apply (rule ub-rangeI, simp)
    apply (drule ub-rangeD)
    apply (erule rev-below-trans)
    apply simp
    apply (rule lessI)
    done
  done

lemma inj-on-undiscr [simp]: inj-on undiscr A
using Discr-undiscr by (rule inj-on-inverseI)

lemma finite-deflation-discr-approx: finite-deflation (discr-approx i)
proof
  fix x :: 'a discr u
  show discr-approx i·x ⊑ x
    unfolding discr-approx-def
    by (cases x, simp, simp)
  show discr-approx i·(discr-approx i·x) = discr-approx i·x
    unfolding discr-approx-def
    by (cases x, simp, simp)
  show finite {x::'a discr u. discr-approx i·x = x}
  proof (rule finite-subset)
    let ?S = insert (⊥::'a discr u) ((λx. up·x) ‘ undiscr −‘ to-nat −‘ {..<i})
    show {x::'a discr u. discr-approx i·x = x} ⊆ ?S
      unfolding discr-approx-def
      by (rule subsetI, case-tac x, simp, simp split: if-split-asm)
    show finite ?S
      by (simp add: finite-vimageI)
  qed
qed

lemma discr-approx: approx-chain discr-approx
using chain-discr-approx lub-discr-approx finite-deflation-discr-approx
by (rule approx-chain.intro)
```

19.4 Class instance proofs

```
instance bifinite ⊆ profinite
proof
  show ∃(a::nat ⇒ 'a⊥ → 'a⊥). approx-chain a
    using bifinite [where 'a='a]
    by (fast intro!: approx-chain-u-map)
```

qed

instance *u* :: (*profinite*) *bifinite*
  by *standard* (*rule profinite*)

Types $'a \to 'b$ and $'a_\perp \to! 'b$ are isomorphic.

definition *encode-cfun* = (Λ *f*. *sfun-abs*·(*fup*·*f*))

definition *decode-cfun* = (Λ *g x*. *sfun-rep*·*g*·(*up*·*x*))

lemma *decode-encode-cfun* [*simp*]: *decode-cfun*·(*encode-cfun*·*x*) = *x*
unfolding *encode-cfun-def decode-cfun-def*
by (*simp add*: *eta-cfun*)

lemma *encode-decode-cfun* [*simp*]: *encode-cfun*·(*decode-cfun*·*y*) = *y*
unfolding *encode-cfun-def decode-cfun-def*
apply (*simp add*: *sfun-eq-iff strictify-cancel*)
apply (*rule cfun-eqI*, *case-tac x*, *simp-all*)
done

instance *cfun* :: (*profinite*, *bifinite*) *bifinite*
proof
  obtain *a* :: *nat* ⇒ $'a_\perp \to 'a_\perp$ **where** *a*: *approx-chain a*
    using *profinite* **..**
  obtain *b* :: *nat* ⇒ $'b \to 'b$ **where** *b*: *approx-chain b*
    using *bifinite* **..**
  have *approx-chain* (λ*i*. *decode-cfun oo sfun-map*·(*a i*)·(*b i*) *oo encode-cfun*)
    using *a b* **by** (*simp add*: *approx-chain-iso approx-chain-sfun-map*)
  thus ∃ (*a*::*nat* ⇒ ($'a \to 'b$) → ($'a \to 'b$)). *approx-chain a*
    by − (*rule exI*)
qed

Types $('a \times 'b)_\perp$ and $'a_\perp \otimes 'b_\perp$ are isomorphic.

definition *encode-prod-u* = (Λ(*up*·(*x*, *y*)). (:*up*·*x*, *up*·*y*:))

definition *decode-prod-u* = (Λ(:*up*·*x*, *up*·*y*:). *up*·(*x*, *y*))

lemma *decode-encode-prod-u* [*simp*]: *decode-prod-u*·(*encode-prod-u*·*x*) = *x*
  unfolding *encode-prod-u-def decode-prod-u-def*
  apply (*cases x*)
   apply *simp*
  subgoal for *y* **by** (*cases y*) *simp*
  done

lemma *encode-decode-prod-u* [*simp*]: *encode-prod-u*·(*decode-prod-u*·*y*) = *y*
  unfolding *encode-prod-u-def decode-prod-u-def*
  apply (*cases y*)
   apply *simp*
  subgoal for *a b*

apply (*cases a*, *simp*)
apply (*cases b*, *simp*, *simp*)
done
done

instance *prod* :: (*profinite*, *profinite*) *profinite*
proof
obtain $a :: nat \Rightarrow 'a_\bot \to 'a_\bot$ **where** *a*: *approx-chain a*
using *profinite* **..**
obtain $b :: nat \Rightarrow 'b_\bot \to 'b_\bot$ **where** *b*: *approx-chain b*
using *profinite* **..**
have *approx-chain* ($\lambda i.$ *decode-prod-u oo sprod-map*$\cdot$(*a i*)$\cdot$(*b i*) *oo encode-prod-u*)
using *a b* **by** (*simp add*: *approx-chain-iso approx-chain-sprod-map*)
thus $\exists (a::nat \Rightarrow ('a \times 'b)_\bot \to ('a \times 'b)_\bot).$ *approx-chain a*
by − (*rule exI*)
qed

instance *prod* :: (*bifinite*, *bifinite*) *bifinite*
proof
show $\exists (a::nat \Rightarrow ('a \times 'b) \to ('a \times 'b)).$ *approx-chain a*
using *bifinite* [**where** $'a='a$] **and** *bifinite* [**where** $'a='b$]
by (*fast intro*!: *approx-chain-prod-map*)
qed

instance *sfun* :: (*bifinite*, *bifinite*) *bifinite*
proof
show $\exists (a::nat \Rightarrow ('a \to! 'b) \to ('a \to! 'b)).$ *approx-chain a*
using *bifinite* [**where** $'a='a$] **and** *bifinite* [**where** $'a='b$]
by (*fast intro*!: *approx-chain-sfun-map*)
qed

instance *sprod* :: (*bifinite*, *bifinite*) *bifinite*
proof
show $\exists (a::nat \Rightarrow ('a \otimes 'b) \to ('a \otimes 'b)).$ *approx-chain a*
using *bifinite* [**where** $'a='a$] **and** *bifinite* [**where** $'a='b$]
by (*fast intro*!: *approx-chain-sprod-map*)
qed

instance *ssum* :: (*bifinite*, *bifinite*) *bifinite*
proof
show $\exists (a::nat \Rightarrow ('a \oplus 'b) \to ('a \oplus 'b)).$ *approx-chain a*
using *bifinite* [**where** $'a='a$] **and** *bifinite* [**where** $'a='b$]
by (*fast intro*!: *approx-chain-ssum-map*)
qed

lemma *approx-chain-unit*: *approx-chain* ($\bot :: nat \Rightarrow unit \to unit$)
by (*simp add*: *approx-chain-def cfun-eq-iff finite-deflation-bottom*)

instance *unit* :: *bifinite*

by *standard* (*fast intro*!: *approx-chain-unit*)

instance *discr* :: (*countable*) *profinite*
by *standard* (*fast intro*!: *discr-approx*)

instance *lift* :: (*countable*) *bifinite*
proof
note [*simp*] = *cont-Abs-lift cont-Rep-lift Rep-lift-inverse Abs-lift-inverse*
obtain $a :: nat \Rightarrow ('a\ discr)_\bot \rightarrow ('a\ discr)_\bot$ **where** *a*: *approx-chain a*
using *profinite* **..**
hence *approx-chain* ($\lambda i.$ (Λ *y*. *Abs-lift y*) *oo a i oo* (Λ *x*. *Rep-lift x*))
by (*rule approx-chain-iso*) *simp-all*
thus $\exists (a::nat \Rightarrow 'a\ lift \rightarrow 'a\ lift).$ *approx-chain a*
by − (*rule exI*)
qed

end

20 Defining algebraic domains by ideal completion

theory *Completion*
imports *Cfun*
begin

20.1 Ideals over a preorder

locale *preorder* =
fixes *r* :: *'a*::*type* ⇒ *'a* ⇒ *bool* (**infix** ‹⪯› *50*)
assumes *r-refl*: $x \preceq x$
assumes *r-trans*: $[\![x \preceq y;\ y \preceq z]\!] \Longrightarrow x \preceq z$
begin

definition
ideal :: *'a set* ⇒ *bool* **where**
$ideal\ A = ((\exists x.\ x \in A) \land (\forall x{\in}A.\ \forall y{\in}A.\ \exists z{\in}A.\ x \preceq z \land y \preceq z) \land$
$(\forall x\ y.\ x \preceq y \longrightarrow y \in A \longrightarrow x \in A))$

lemma *idealI*:
assumes $\exists x.\ x \in A$
assumes $\bigwedge x\ y.\ [\![x \in A;\ y \in A]\!] \Longrightarrow \exists z{\in}A.\ x \preceq z \land y \preceq z$
assumes $\bigwedge x\ y.\ [\![x \preceq y;\ y \in A]\!] \Longrightarrow x \in A$
shows *ideal A*
unfolding *ideal-def* **using** *assms* **by** *fast*

lemma *idealD1*:
$ideal\ A \Longrightarrow \exists x.\ x \in A$
unfolding *ideal-def* **by** *fast*

lemma *idealD2*:

```
  ⟦ideal A; x ∈ A; y ∈ A⟧ ⟹ ∃z∈A. x ≼ z ∧ y ≼ z
unfolding ideal-def by fast

lemma idealD3:
  ⟦ideal A; x ≼ y; y ∈ A⟧ ⟹ x ∈ A
unfolding ideal-def by fast

lemma ideal-principal: ideal {x. x ≼ z}
  apply (rule idealI)
    apply (rule exI [where x = z])
    apply (fast intro: r-refl)
   apply (rule bexI [where x = z], fast)
   apply (fast intro: r-refl)
  apply (fast intro: r-trans)
  done

lemma ex-ideal: ∃A. A ∈ {A. ideal A}
by (fast intro: ideal-principal)
```

The set of ideals is a cpo

```
lemma ideal-UN:
  fixes A :: nat ⇒ 'a set
  assumes ideal-A: ⋀i. ideal (A i)
  assumes chain-A: ⋀i j. i ≤ j ⟹ A i ⊆ A j
  shows ideal (⋃i. A i)
  apply (rule idealI)
  using idealD1 [OF ideal-A] apply fast
   apply (clarify)
  subgoal for i j
    apply (drule subsetD [OF chain-A [OF max.cobounded1]])
    apply (drule subsetD [OF chain-A [OF max.cobounded2]])
    apply (drule (1) idealD2 [OF ideal-A])
    apply blast
    done
  apply clarify
  apply (drule (1) idealD3 [OF ideal-A])
  apply fast
  done

lemma typedef-ideal-po:
  fixes Abs :: 'a set ⇒ 'b::below
  assumes type: type-definition Rep Abs {S. ideal S}
  assumes below: ⋀x y. x ⊑ y ⟷ Rep x ⊆ Rep y
  shows OFCLASS('b, po-class)
 apply (intro-classes, unfold below)
   apply (rule subset-refl)
  apply (erule (1) subset-trans)
 apply (rule type-definition.Rep-inject [OF type, THEN iffD1])
 apply (erule (1) subset-antisym)
```

```
done

lemma
  fixes Abs :: 'a set ⇒ 'b::po
  assumes type: type-definition Rep Abs {S. ideal S}
  assumes below: ⋀x y. x ⊑ y ⟷ Rep x ⊆ Rep y
  assumes S: chain S
  shows typedef-ideal-lub: range S <<| Abs (⋃ i. Rep (S i))
    and typedef-ideal-rep-lub: Rep (⨆ i. S i) = (⋃ i. Rep (S i))
proof −
  have 1: ideal (⋃ i. Rep (S i))
    apply (rule ideal-UN)
     apply (rule type-definition.Rep [OF type, unfolded mem-Collect-eq])
    apply (subst below [symmetric])
    apply (erule chain-mono [OF S])
    done
  hence 2: Rep (Abs (⋃ i. Rep (S i))) = (⋃ i. Rep (S i))
    by (simp add: type-definition.Abs-inverse [OF type])
  show 3: range S <<| Abs (⋃ i. Rep (S i))
    apply (rule is-lubI)
     apply (rule is-ubI)
     apply (simp add: below 2, fast)
    apply (simp add: below 2 is-ub-def, fast)
    done
  hence 4: (⨆ i. S i) = Abs (⋃ i. Rep (S i))
    by (rule lub-eqI)
  show 5: Rep (⨆ i. S i) = (⋃ i. Rep (S i))
    by (simp add: 4 2)
qed

lemma typedef-ideal-cpo:
  fixes Abs :: 'a set ⇒ 'b::po
  assumes type: type-definition Rep Abs {S. ideal S}
  assumes below: ⋀x y. x ⊑ y ⟷ Rep x ⊆ Rep y
  shows OFCLASS('b, cpo-class)
  by standard (rule exI, erule typedef-ideal-lub [OF type below])

end

interpretation below: preorder below :: 'a::po ⇒ 'a ⇒ bool
apply unfold-locales
apply (rule below-refl)
apply (erule (1) below-trans)
done
```

20.2 Lemmas about least upper bounds

```
lemma is-ub-thelub-ex: ⟦∃ u. S <<| u; x ∈ S⟧ ⟹ x ⊑ lub S
apply (erule exE, drule is-lub-lub)
```

apply (*drule is-lubD1*)
apply (*erule* (*1*) *is-ubD*)
done

lemma *is-lub-thelub-ex*: $[\![\exists u.\ S <<|\ u;\ S <|\ x]\!] \Longrightarrow lub\ S \sqsubseteq x$
by (*erule exE*, *drule is-lub-lub*, *erule is-lubD2*)

20.3 Locale for ideal completion

hide-const (**open**) *Filter.principal*

locale *ideal-completion* = *preorder* +
  fixes *principal* :: $'a::type \Rightarrow 'b$
  fixes *rep* :: $'b \Rightarrow 'a::type\ set$
  assumes *ideal-rep*: $\bigwedge x.\ ideal\ (rep\ x)$
  assumes *rep-lub*: $\bigwedge Y.\ chain\ Y \Longrightarrow rep\ (\bigsqcup i.\ Y\ i) = (\bigcup i.\ rep\ (Y\ i))$
  assumes *rep-principal*: $\bigwedge a.\ rep\ (principal\ a) = \{b.\ b \preceq a\}$
  assumes *belowI*: $\bigwedge x\ y.\ rep\ x \subseteq rep\ y \Longrightarrow x \sqsubseteq y$
  assumes *countable*: $\exists f::'a \Rightarrow nat.\ inj\ f$
begin

lemma *rep-mono*: $x \sqsubseteq y \Longrightarrow rep\ x \subseteq rep\ y$
apply (*frule bin-chain*)
apply (*drule rep-lub*)
apply (*simp only*: *lub-eqI* [*OF is-lub-bin-chain*])
apply (*rule subsetI*, *rule UN-I* [**where** *a=0*], *simp-all*)
done

lemma *below-def*: $x \sqsubseteq y \longleftrightarrow rep\ x \subseteq rep\ y$
by (*rule iffI* [*OF rep-mono belowI*])

lemma *principal-below-iff-mem-rep*: $principal\ a \sqsubseteq x \longleftrightarrow a \in rep\ x$
unfolding *below-def rep-principal*
by (*auto intro*: *r-refl elim*: *idealD3* [*OF ideal-rep*])

lemma *principal-below-iff* [*simp*]: $principal\ a \sqsubseteq principal\ b \longleftrightarrow a \preceq b$
by (*simp add*: *principal-below-iff-mem-rep rep-principal*)

lemma *principal-eq-iff*: $principal\ a = principal\ b \longleftrightarrow a \preceq b \wedge b \preceq a$
unfolding *po-eq-conv* [**where** *'a='b*] *principal-below-iff* **..**

lemma *eq-iff*: $x = y \longleftrightarrow rep\ x = rep\ y$
unfolding *po-eq-conv below-def* **by** *auto*

lemma *principal-mono*: $a \preceq b \Longrightarrow principal\ a \sqsubseteq principal\ b$
by (*simp only*: *principal-below-iff*)

lemma *ch2ch-principal* [*simp*]:
  $\forall i.\ Y\ i \preceq Y\ (Suc\ i) \Longrightarrow chain\ (\lambda i.\ principal\ (Y\ i))$

by (*simp add*: *chainI principal-mono*)

20.3.1 Principal ideals approximate all elements

lemma *compact-principal* [*simp*]: *compact* (*principal a*)
by (*rule compactI2*, *simp add*: *principal-below-iff-mem-rep rep-lub*)

Construct a chain whose lub is the same as a given ideal

```
lemma obtain-principal-chain:
  obtains Y where ∀ i. Y i ⪯ Y (Suc i) and x = (⨆ i. principal (Y i))
proof −
  obtain count :: 'a ⇒ nat where inj: inj count
    using countable ..
  define enum where enum i = (THE a. count a = i) for i
  have enum-count [simp]: ⋀x. enum (count x) = x
    unfolding enum-def by (simp add: inj-eq [OF inj])
  define a where a = (LEAST i. enum i ∈ rep x)
  define b where b i = (LEAST j. enum j ∈ rep x ∧ ¬ enum j ⪯ enum i) for i
  define c where c i j = (LEAST k. enum k ∈ rep x ∧ enum i ⪯ enum k ∧ enum
j ⪯ enum k) for i j
  define P where P i ⟷ (∃ j. enum j ∈ rep x ∧ ¬ enum j ⪯ enum i) for i
  define X where X = rec-nat a (λn i. if P i then c i (b i) else i)
  have X-0: X 0 = a unfolding X-def by simp
  have X-Suc: ⋀n. X (Suc n) = (if P (X n) then c (X n) (b (X n)) else X n)
    unfolding X-def by simp
  have a-mem: enum a ∈ rep x
    unfolding a-def
    apply (rule LeastI-ex)
    apply (insert ideal-rep [of x])
    apply (drule idealD1)
    apply (clarify)
    subgoal for a by (rule exI [where x=count a]) simp
    done
  have b: ⋀i. P i ⟹ enum i ∈ rep x
    ⟹ enum (b i) ∈ rep x ∧ ¬ enum (b i) ⪯ enum i
    unfolding P-def b-def by (erule LeastI2-ex, simp)
  have c: ⋀i j. enum i ∈ rep x ⟹ enum j ∈ rep x
    ⟹ enum (c i j) ∈ rep x ∧ enum i ⪯ enum (c i j) ∧ enum j ⪯ enum (c i j)
    unfolding c-def
    apply (drule (1) idealD2 [OF ideal-rep], clarify)
    subgoal for ... z by (rule LeastI2 [where a=count z], simp, simp)
    done
  have X-mem: enum (X n) ∈ rep x for n
  proof (induct n)
    case 0
    then show ?case by (simp add: X-0 a-mem)
  next
    case (Suc n)
    with b c show ?case by (auto simp: X-Suc)
```

```
  qed
  have X-chain: ⋀n. enum (X n) ≼ enum (X (Suc n))
    apply (clarsimp simp add: X-Suc r-refl)
    apply (simp add: b c X-mem)
    done
  have less-b: ⋀n i. n < b i ⟹ enum n ∈ rep x ⟹ enum n ≼ enum i
    unfolding b-def by (drule not-less-Least, simp)
  have X-covers: ∀k≤n. enum k ∈ rep x ⟶ enum k ≼ enum (X n) for n
  proof (induct n)
    case 0
    then show ?case
      apply (clarsimp simp add: X-0 a-def)
      apply (drule Least-le [where k=0], simp add: r-refl)
      done
  next
    case (Suc n)
    then show ?case
      apply clarsimp
      apply (erule le-SucE)
       apply (rule r-trans [OF - X-chain], simp)
      apply (cases P (X n), simp add: X-Suc)
       apply (rule linorder-cases [where x=b (X n) and y=Suc n])
         apply (simp only: less-Suc-eq-le)
         apply (drule spec, drule (1) mp, simp add: b X-mem)
        apply (simp add: c X-mem)
       apply (drule (1) less-b)
       apply (erule r-trans)
       apply (simp add: b c X-mem)
      apply (simp add: X-Suc)
      apply (simp add: P-def)
      done
  qed
  have 1: ∀i. enum (X i) ≼ enum (X (Suc i))
    by (simp add: X-chain)
  have x = (⨆n. principal (enum (X n)))
    apply (simp add: eq-iff rep-lub 1 rep-principal)
    apply auto
    subgoal for a
      apply (subgoal-tac ∃i. a = enum i, erule exE)
       apply (rule-tac x=i in exI, simp add: X-covers)
      apply (rule-tac x=count a in exI, simp)
      done
    subgoal
      apply (erule idealD3 [OF ideal-rep])
      apply (rule X-mem)
      done
    done
  with 1 show ?thesis ..
qed
```

```
lemma principal-induct:
  assumes adm: adm P
  assumes P: ⋀a. P (principal a)
  shows P x
apply (rule obtain-principal-chain [of x])
apply (simp add: admD [OF adm] P)
done

lemma compact-imp-principal: compact x ⟹ ∃ a. x = principal a
apply (rule obtain-principal-chain [of x])
apply (drule adm-compact-neq [OF - cont-id])
apply (subgoal-tac chain (λi. principal (Y i)))
apply (drule (2) admD2, fast, simp)
done
```

20.4 Defining functions in terms of basis elements

```
definition
  extension :: ('a::type ⇒ 'c) ⇒ 'b → 'c where
  extension = (λf. (Λ x. lub (f ` rep x)))

lemma extension-lemma:
  fixes f :: 'a::type ⇒ 'c
  assumes f-mono: ⋀a b. a ≼ b ⟹ f a ⊑ f b
  shows ∃ u. f ` rep x <<| u
proof −
  obtain Y where Y: ∀ i. Y i ≼ Y (Suc i)
  and x: x = (⨆ i. principal (Y i))
    by (rule obtain-principal-chain [of x])
  have chain: chain (λi. f (Y i))
    by (rule chainI, simp add: f-mono Y)
  have rep-x: rep x = (⋃ n. {a. a ≼ Y n})
    by (simp add: x rep-lub Y rep-principal)
  have f ` rep x <<| (⨆ n. f (Y n))
    apply (rule is-lubI)
     apply (rule ub-imageI)
    subgoal for a
      apply (clarsimp simp add: rep-x)
      apply (drule f-mono)
      apply (erule below-lub [OF chain])
      done
    apply (rule lub-below [OF chain])
    subgoal for ... n
      apply (drule ub-imageD [where x=Y n])
       apply (simp add: rep-x, fast intro: r-refl)
      apply assumption
      done
    done
```

```
  then show ?thesis ..
qed

lemma extension-beta:
  fixes f :: 'a::type ⇒ 'c
  assumes f-mono: ⋀a b. a ≼ b ⟹ f a ⊑ f b
  shows extension f·x = lub (f ` rep x)
unfolding extension-def
proof (rule beta-cfun)
  have lub: ⋀x. ∃u. f ` rep x <<| u
    using f-mono by (rule extension-lemma)
  show cont: cont (λx. lub (f ` rep x))
    apply (rule contI2)
     apply (rule monofunI)
     apply (rule is-lub-thelub-ex [OF lub ub-imageI])
     apply (rule is-ub-thelub-ex [OF lub imageI])
     apply (erule (1) subsetD [OF rep-mono])
    apply (rule is-lub-thelub-ex [OF lub ub-imageI])
    apply (simp add: rep-lub, clarify)
    apply (erule rev-below-trans [OF is-ub-thelub])
    apply (erule is-ub-thelub-ex [OF lub imageI])
    done
qed

lemma extension-principal:
  fixes f :: 'a::type ⇒ 'c
  assumes f-mono: ⋀a b. a ≼ b ⟹ f a ⊑ f b
  shows extension f·(principal a) = f a
apply (subst extension-beta, erule f-mono)
apply (subst rep-principal)
apply (rule lub-eqI)
apply (rule is-lub-maximal)
apply (rule ub-imageI)
apply (simp add: f-mono)
apply (rule imageI)
apply (simp add: r-refl)
done

lemma extension-mono:
  assumes f-mono: ⋀a b. a ≼ b ⟹ f a ⊑ f b
  assumes g-mono: ⋀a b. a ≼ b ⟹ g a ⊑ g b
  assumes below: ⋀a. f a ⊑ g a
  shows extension f ⊑ extension g
  apply (rule cfun-belowI)
  apply (simp only: extension-beta f-mono g-mono)
  apply (rule is-lub-thelub-ex)
   apply (rule extension-lemma, erule f-mono)
  apply (rule ub-imageI)
  subgoal for x a
```

```
    apply (rule below-trans [OF below])
    apply (rule is-ub-thelub-ex)
     apply (rule extension-lemma, erule g-mono)
    apply (erule imageI)
    done
  done

lemma cont-extension:
  assumes f-mono: ⋀a b x. a ≼ b ⟹ f x a ⊑ f x b
  assumes f-cont: ⋀a. cont (λx. f x a)
  shows cont (λx. extension (λa. f x a))
 apply (rule contI2)
  apply (rule monofunI)
  apply (rule extension-mono, erule f-mono, erule f-mono)
  apply (erule cont2monofunE [OF f-cont])
 apply (rule cfun-belowI)
 apply (rule principal-induct, simp)
 apply (simp only: contlub-cfun-fun)
 apply (simp only: extension-principal f-mono)
 apply (simp add: cont2contlubE [OF f-cont])
done

end

lemma (in preorder) typedef-ideal-completion:
  fixes Abs :: 'a set ⇒ 'b
  assumes type: type-definition Rep Abs {S. ideal S}
  assumes below: ⋀x y. x ⊑ y ⟷ Rep x ⊆ Rep y
  assumes principal: ⋀a. principal a = Abs {b. b ≼ a}
  assumes countable: ∃f::'a ⇒ nat. inj f
  shows ideal-completion r principal Rep
proof
  interpret type-definition Rep Abs {S. ideal S} by fact
  fix a b :: 'a and x y :: 'b and Y :: nat ⇒ 'b
  show ideal (Rep x)
    using Rep [of x] by simp
  show chain Y ⟹ Rep (⨆i. Y i) = (⋃i. Rep (Y i))
    using type below by (rule typedef-ideal-rep-lub)
  show Rep (principal a) = {b. b ≼ a}
    by (simp add: principal Abs-inverse ideal-principal)
  show Rep x ⊆ Rep y ⟹ x ⊑ y
    by (simp only: below)
  show ∃f::'a ⇒ nat. inj f
    by (rule countable)
qed

end
```

21 A universal bifinite domain

theory *Universal*
imports *Bifinite Completion HOL−Library.Nat-Bijection*
begin

unbundle *no binomial-syntax*

21.1 Basis for universal domain

21.1.1 Basis datatype

type-synonym *ubasis* = *nat*

definition
node :: *nat* ⇒ *ubasis* ⇒ *ubasis set* ⇒ *ubasis*
where
node i a S = *Suc* (*prod-encode* (*i*, *prod-encode* (*a*, *set-encode S*)))

lemma *node-not-0* [*simp*]: *node i a S* ≠ *0*
unfolding *node-def* **by** *simp*

lemma *node-gt-0* [*simp*]: *0* < *node i a S*
unfolding *node-def* **by** *simp*

lemma *node-inject* [*simp*]:
⟦*finite S*; *finite T*⟧
⟹ *node i a S* = *node j b T* ⟷ *i* = *j* ∧ *a* = *b* ∧ *S* = *T*
unfolding *node-def* **by** (*simp add*: *prod-encode-eq set-encode-eq*)

lemma *node-gt0*: *i* < *node i a S*
unfolding *node-def less-Suc-eq-le*
by (*rule le-prod-encode-1*)

lemma *node-gt1*: *a* < *node i a S*
unfolding *node-def less-Suc-eq-le*
by (*rule order-trans* [*OF le-prod-encode-1 le-prod-encode-2*])

lemma *nat-less-power2*: *n* < *2^n*
by (*fact less-exp*)

lemma *node-gt2*: ⟦*finite S*; *b* ∈ *S*⟧ ⟹ *b* < *node i a S*
unfolding *node-def less-Suc-eq-le set-encode-def*
apply (*rule order-trans* [*OF - le-prod-encode-2*])
apply (*rule order-trans* [*OF - le-prod-encode-2*])
apply (*rule order-trans* [**where** *y*=*sum* ((^) *2*) {*b*}])
apply (*simp add*: *nat-less-power2* [*THEN order-less-imp-le*])
apply (*erule sum-mono2*, *simp*, *simp*)
done

lemma *eq-prod-encode-pairI*:
  $[\![fst\ (prod\text{-}decode\ x) = a;\ snd\ (prod\text{-}decode\ x) = b]\!] \Longrightarrow x = prod\text{-}encode\ (a,\ b)$
  by *auto*

lemma *node-cases*:
  assumes *1*: $x = 0 \Longrightarrow P$
  assumes *2*: $\bigwedge i\ a\ S.\ [\![finite\ S;\ x = node\ i\ a\ S]\!] \Longrightarrow P$
  shows *P*
 apply (*cases x*)
  apply (*erule 1*)
 apply (*rule 2*)
  apply (*rule finite-set-decode*)
 apply (*simp add*: *node-def*)
 apply (*rule eq-prod-encode-pairI* [*OF refl*])
 apply (*rule eq-prod-encode-pairI* [*OF refl refl*])
done

lemma *node-induct*:
  assumes *1*: *P 0*
  assumes *2*: $\bigwedge i\ a\ S.\ [\![P\ a;\ finite\ S;\ \forall\, b{\in}S.\ P\ b]\!] \Longrightarrow P\ (node\ i\ a\ S)$
  shows *P x*
 apply (*induct x rule*: *nat-less-induct*)
 apply (*case-tac n rule*: *node-cases*)
  apply (*simp add*: *1*)
 apply (*simp add*: *2 node-gt1 node-gt2*)
done

21.1.2 Basis ordering

inductive
  ubasis-le :: $nat \Rightarrow nat \Rightarrow bool$
where
  ubasis-le-refl: *ubasis-le a a*
| *ubasis-le-trans*:
    $[\![ubasis\text{-}le\ a\ b;\ ubasis\text{-}le\ b\ c]\!] \Longrightarrow ubasis\text{-}le\ a\ c$
| *ubasis-le-lower*:
    $finite\ S \Longrightarrow ubasis\text{-}le\ a\ (node\ i\ a\ S)$
| *ubasis-le-upper*:
    $[\![finite\ S;\ b \in S;\ ubasis\text{-}le\ a\ b]\!] \Longrightarrow ubasis\text{-}le\ (node\ i\ a\ S)\ b$

lemma *ubasis-le-minimal*: *ubasis-le 0 x*
apply (*induct x rule*: *node-induct*)
apply (*rule ubasis-le-refl*)
apply (*erule ubasis-le-trans*)
apply (*erule ubasis-le-lower*)
done

interpretation *udom*: *preorder ubasis-le*
apply *standard*

```
apply (rule ubasis-le-refl)
apply (erule (1) ubasis-le-trans)
done
```

21.1.3 Generic take function

```
function
  ubasis-until :: (ubasis ⇒ bool) ⇒ ubasis ⇒ ubasis
where
  ubasis-until P 0 = 0
| finite S ⟹ ubasis-until P (node i a S) =
    (if P (node i a S) then node i a S else ubasis-until P a)
   apply clarify
   apply (rule-tac x=b in node-cases)
    apply simp-all
done

termination ubasis-until
apply (relation measure snd)
apply (rule wf-measure)
apply (simp add: node-gt1)
done

lemma ubasis-until: P 0 ⟹ P (ubasis-until P x)
by (induct x rule: node-induct) simp-all

lemma ubasis-until': 0 < ubasis-until P x ⟹ P (ubasis-until P x)
by (induct x rule: node-induct) auto

lemma ubasis-until-same: P x ⟹ ubasis-until P x = x
by (induct x rule: node-induct) simp-all

lemma ubasis-until-idem:
  P 0 ⟹ ubasis-until P (ubasis-until P x) = ubasis-until P x
by (rule ubasis-until-same [OF ubasis-until])

lemma ubasis-until-0:
  ∀ x. x ≠ 0 ⟶ ¬ P x ⟹ ubasis-until P x = 0
by (induct x rule: node-induct) simp-all

lemma ubasis-until-less: ubasis-le (ubasis-until P x) x
apply (induct x rule: node-induct)
apply (simp add: ubasis-le-refl)
 by (metis ubasis-le.simps ubasis-until.simps(2))

lemma ubasis-until-chain:
  assumes PQ: ⋀x. P x ⟹ Q x
  shows ubasis-le (ubasis-until P x) (ubasis-until Q x)
apply (induct x rule: node-induct)
```

apply (*simp add*: *ubasis-le-refl*)
  by (*metis assms ubasis-until.simps*(*2*) *ubasis-until-less*)

lemma *ubasis-until-mono*:
  assumes $\bigwedge i\ a\ S\ b.\ [\![finite\ S;\ P\ (node\ i\ a\ S);\ b \in S;\ ubasis\text{-}le\ a\ b]\!] \Longrightarrow P\ b$
  shows $ubasis\text{-}le\ a\ b \Longrightarrow ubasis\text{-}le\ (ubasis\text{-}until\ P\ a)\ (ubasis\text{-}until\ P\ b)$
proof (*induct set*: *ubasis-le*)
  case (*ubasis-le-refl a*) **show** *?case* **by** (*rule ubasis-le.ubasis-le-refl*)
next
  case (*ubasis-le-trans a b c*) **thus** *?case* **by** − (*rule ubasis-le.ubasis-le-trans*)
next
  case (*ubasis-le-lower S a i*) **thus** *?case*
    by (*metis ubasis-le.simps ubasis-until.simps*(*2*) *ubasis-until-less*)
next
  case (*ubasis-le-upper S b a i*) **thus** *?case*
    by (*metis assms ubasis-le.simps ubasis-until.simps*(*2*) *ubasis-until-same*)
qed

lemma *finite-range-ubasis-until*:
  $finite\ \{x.\ P\ x\} \Longrightarrow finite\ (range\ (ubasis\text{-}until\ P))$
apply (*rule finite-subset* [**where** $B{=}insert\ 0\ \{x.\ P\ x\}$])
apply (*clarsimp simp add*: *ubasis-until′*)
apply *simp*
done

21.2 Defining the universal domain by ideal completion

typedef $udom = \{S.\ udom.ideal\ S\}$
by (*rule udom.ex-ideal*)

instantiation *udom* :: *below*
begin

definition
  $x \sqsubseteq y \longleftrightarrow Rep\text{-}udom\ x \subseteq Rep\text{-}udom\ y$

instance ..
end

instance *udom* :: *po*
using *type-definition-udom below-udom-def*
by (*rule udom.typedef-ideal-po*)

instance *udom* :: *cpo*
using *type-definition-udom below-udom-def*
by (*rule udom.typedef-ideal-cpo*)

definition
  udom-principal :: $nat \Rightarrow udom$ **where**

udom-principal t = *Abs-udom* $\{u.$ *ubasis-le* $u\ t\}$

lemma *ubasis-countable*: $\exists f$::*ubasis* $\Rightarrow$ *nat*. *inj* f
by (*rule exI*, *rule inj-on-id*)

interpretation *udom*:
ideal-completion ubasis-le udom-principal Rep-udom
using *type-definition-udom below-udom-def*
using *udom-principal-def ubasis-countable*
by (*rule udom.typedef-ideal-completion*)

Universal domain is pointed

lemma *udom-minimal*: *udom-principal 0* $\sqsubseteq x$
apply (*induct x rule*: *udom.principal-induct*)
apply (*simp*, *simp add*: *ubasis-le-minimal*)
done

instance *udom* :: *pcpo*
by *intro-classes* (*fast intro*: *udom-minimal*)

lemma *inst-udom-pcpo*: $\bot$ = *udom-principal 0*
by (*rule udom-minimal* [*THEN bottomI*, *symmetric*])

21.3 Compact bases of domains

typedef *'a compact-basis* = $\{x$::*'a*::*pcpo*. *compact* $x\}$
by *auto*

lemma *Rep-compact-basis'* [*simp*]: *compact* (*Rep-compact-basis a*)
by (*rule Rep-compact-basis* [*unfolded mem-Collect-eq*])

lemma *Abs-compact-basis-inverse'* [*simp*]:
compact x $\Longrightarrow$ *Rep-compact-basis* (*Abs-compact-basis x*) = x
by (*rule Abs-compact-basis-inverse* [*unfolded mem-Collect-eq*])

instantiation *compact-basis* :: (*pcpo*) *below*
begin

definition
compact-le-def:
($\sqsubseteq$) $\equiv$ ($\lambda x\ y$. *Rep-compact-basis* $x \sqsubseteq$ *Rep-compact-basis* y)

instance ..
end

instance *compact-basis* :: (*pcpo*) *po*
using *type-definition-compact-basis compact-le-def*
by (*rule typedef-po-class*)

definition
  approximants :: *'a::pcpo* ⇒ *'a compact-basis set* **where**
  approximants = (λx. {*a*. *Rep-compact-basis a* ⊑ *x*})

definition
  compact-bot :: *'a::pcpo compact-basis* **where**
  compact-bot = *Abs-compact-basis* ⊥

lemma *Rep-compact-bot* [*simp*]: *Rep-compact-basis compact-bot* = ⊥
unfolding *compact-bot-def* **by** *simp*

lemma *compact-bot-minimal* [*simp*]: *compact-bot* ⊑ *a*
unfolding *compact-le-def Rep-compact-bot* **by** *simp*

21.4 Universality of *udom*

We use a locale to parameterize the construction over a chain of approx functions on the type to be embedded.

locale *bifinite-approx-chain* =
  approx-chain approx **for** *approx* :: *nat* ⇒ *'a::bifinite* → *'a*
begin

21.4.1 Choosing a maximal element from a finite set

lemma *finite-has-maximal*:
  fixes *A* :: *'a compact-basis set*
  shows ⟦*finite A*; *A* ≠ {}⟧ ⟹ ∃ *x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*
proof (*induct rule*: *finite-ne-induct*)
  case (*singleton x*)
    show *?case* **by** *simp*
next
  case (*insert a A*)
  from ‹∃ *x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*›
  obtain *x* **where** *x*: *x* ∈ *A*
          and *x-eq*: ⋀*y*. ⟦*y* ∈ *A*; *x* ⊑ *y*⟧ ⟹ *x* = *y* **by** *fast*
  show *?case*
  proof (*intro bexI ballI impI*)
    fix *y*
    assume *y* ∈ *insert a A* **and** (*if x* ⊑ *a then a else x*) ⊑ *y*
    thus (*if x* ⊑ *a then a else x*) = *y*
      apply *auto*
      apply (*frule* (*1*) *below-trans*)
      apply (*frule* (*1*) *x-eq*)
      apply (*rule below-antisym*, *assumption*)
      apply *simp*
      apply (*erule* (*1*) *x-eq*)
      done
  next
    show (*if x* ⊑ *a then a else x*) ∈ *insert a A*

```
    by (simp add: x)
  qed
qed

definition
  choose :: 'a compact-basis set ⇒ 'a compact-basis
where
  choose A = (SOME x. x ∈ {x∈A. ∀ y∈A. x ⊑ y ⟶ x = y})

lemma choose-lemma:
  ⟦finite A; A ≠ {}⟧ ⟹ choose A ∈ {x∈A. ∀ y∈A. x ⊑ y ⟶ x = y}
unfolding choose-def
apply (rule someI-ex)
apply (frule (1) finite-has-maximal, fast)
done

lemma maximal-choose:
  ⟦finite A; y ∈ A; choose A ⊑ y⟧ ⟹ choose A = y
apply (cases A = {}, simp)
apply (frule (1) choose-lemma, simp)
done

lemma choose-in: ⟦finite A; A ≠ {}⟧ ⟹ choose A ∈ A
by (frule (1) choose-lemma, simp)

function
  choose-pos :: 'a compact-basis set ⇒ 'a compact-basis ⇒ nat
where
  choose-pos A x =
    (if finite A ∧ x ∈ A ∧ x ≠ choose A
      then Suc (choose-pos (A − {choose A}) x) else 0)
by auto

termination choose-pos
apply (relation measure (card ∘ fst), simp)
apply clarsimp
apply (rule card-Diff1-less)
apply assumption
apply (erule choose-in)
apply clarsimp
done

declare choose-pos.simps [simp del]

lemma choose-pos-choose: finite A ⟹ choose-pos A (choose A) = 0
by (simp add: choose-pos.simps)

lemma inj-on-choose-pos [OF refl]:
  ⟦card A = n; finite A⟧ ⟹ inj-on (choose-pos A) A
```

```
 apply (induct n arbitrary: A)
  apply simp
 apply (case-tac A = {}, simp)
 apply (frule (1) choose-in)
 apply (rule inj-onI)
 apply (drule-tac x=A − {choose A} in meta-spec, simp)
 apply (simp add: choose-pos.simps)
 apply (simp split: if-split-asm)
 apply (erule (1) inj-onD, simp, simp)
done

lemma choose-pos-bounded [OF refl]:
  ⟦card A = n; finite A; x ∈ A⟧ ⟹ choose-pos A x < n
apply (induct n arbitrary: A)
apply simp
 apply (case-tac A = {}, simp)
 apply (frule (1) choose-in)
apply (subst choose-pos.simps)
apply simp
done

lemma choose-pos-lessD:
  ⟦choose-pos A x < choose-pos A y; finite A; x ∈ A; y ∈ A⟧ ⟹ x ⋢ y
 apply (induct A x arbitrary: y rule: choose-pos.induct)
 apply simp
 apply (case-tac x = choose A)
  apply simp
  apply (rule notI)
  apply (frule (2) maximal-choose)
  apply simp
 apply (case-tac y = choose A)
  apply (simp add: choose-pos-choose)
 apply (drule-tac x=y in meta-spec)
 apply simp
 apply (erule meta-mp)
 apply (simp add: choose-pos.simps)
done
```

21.4.2 Compact basis take function

```
primrec
  cb-take :: nat ⇒ 'a compact-basis ⇒ 'a compact-basis where
  cb-take 0 = (λx. compact-bot)
| cb-take (Suc n) = (λa. Abs-compact-basis (approx n·(Rep-compact-basis a)))

declare cb-take.simps [simp del]

lemma cb-take-zero [simp]: cb-take 0 a = compact-bot
by (simp only: cb-take.simps)
```

lemma *Rep-cb-take*:
  Rep-compact-basis (*cb-take* (*Suc n*) *a*) = *approx n*·(*Rep-compact-basis a*)
by (*simp add*: *cb-take.simps*(*2*))

lemmas *approx-Rep-compact-basis* = *Rep-cb-take* [*symmetric*]

lemma *cb-take-covers*: ∃ *n*. *cb-take n x* = *x*
apply (*subgoal-tac* ∃ *n*. *cb-take* (*Suc n*) *x* = *x*, *fast*)
apply (*simp add*: *Rep-compact-basis-inject* [*symmetric*])
apply (*simp add*: *Rep-cb-take*)
apply (*rule compact-eq-approx*)
apply (*rule Rep-compact-basis′*)
done

lemma *cb-take-less*: *cb-take n x* ⊑ *x*
unfolding *compact-le-def*
by (*cases n*, *simp*, *simp add*: *Rep-cb-take approx-below*)

lemma *cb-take-idem*: *cb-take n* (*cb-take n x*) = *cb-take n x*
unfolding *Rep-compact-basis-inject* [*symmetric*]
by (*cases n*, *simp*, *simp add*: *Rep-cb-take approx-idem*)

lemma *cb-take-mono*: *x* ⊑ *y* ⟹ *cb-take n x* ⊑ *cb-take n y*
unfolding *compact-le-def*
by (*cases n*, *simp*, *simp add*: *Rep-cb-take monofun-cfun-arg*)

lemma *cb-take-chain-le*: *m* ≤ *n* ⟹ *cb-take m x* ⊑ *cb-take n x*
unfolding *compact-le-def*
apply (*cases m*, *simp*, *cases n*, *simp*)
apply (*simp add*: *Rep-cb-take*, *rule chain-mono*, *simp*, *simp*)
done

lemma *finite-range-cb-take*: *finite* (*range* (*cb-take n*))
apply (*cases n*)
apply (*subgoal-tac range* (*cb-take 0*) = {*compact-bot*}, *simp*, *force*)
apply (*rule finite-imageD* [**where** *f*=*Rep-compact-basis*])
apply (*rule finite-subset* [**where** *B*=*range* (λ*x*. *approx* (*n* − *1*)·*x*)])
apply (*clarsimp simp add*: *Rep-cb-take*)
apply (*rule finite-range-approx*)
apply (*rule inj-onI*, *simp add*: *Rep-compact-basis-inject*)
done

21.4.3 Rank of basis elements

definition
  rank :: ′*a compact-basis* ⇒ *nat*
where
  rank x = (*LEAST n*. *cb-take n x* = *x*)

lemma *compact-approx-rank*: *cb-take* (*rank x*) *x* = *x*
unfolding *rank-def*
apply (*rule LeastI-ex*)
apply (*rule cb-take-covers*)
done

lemma *rank-leD*: *rank x* ≤ *n* ⟹ *cb-take n x* = *x*
apply (*rule below-antisym* [*OF cb-take-less*])
apply (*subst compact-approx-rank* [*symmetric*])
apply (*erule cb-take-chain-le*)
done

lemma *rank-leI*: *cb-take n x* = *x* ⟹ *rank x* ≤ *n*
unfolding *rank-def* **by** (*rule Least-le*)

lemma *rank-le-iff*: *rank x* ≤ *n* ⟷ *cb-take n x* = *x*
by (*rule iffI* [*OF rank-leD rank-leI*])

lemma *rank-compact-bot* [*simp*]: *rank compact-bot* = *0*
using *rank-leI* [*of 0 compact-bot*] **by** *simp*

lemma *rank-eq-0-iff* [*simp*]: *rank x* = *0* ⟷ *x* = *compact-bot*
using *rank-le-iff* [*of x 0*] **by** *auto*

definition
  rank-le :: *'a compact-basis* ⇒ *'a compact-basis set*
where
  rank-le x = {*y*. *rank y* ≤ *rank x*}

definition
  rank-lt :: *'a compact-basis* ⇒ *'a compact-basis set*
where
  rank-lt x = {*y*. *rank y* < *rank x*}

definition
  rank-eq :: *'a compact-basis* ⇒ *'a compact-basis set*
where
  rank-eq x = {*y*. *rank y* = *rank x*}

lemma *rank-eq-cong*: *rank x* = *rank y* ⟹ *rank-eq x* = *rank-eq y*
unfolding *rank-eq-def* **by** *simp*

lemma *rank-lt-cong*: *rank x* = *rank y* ⟹ *rank-lt x* = *rank-lt y*
unfolding *rank-lt-def* **by** *simp*

lemma *rank-eq-subset*: *rank-eq x* ⊆ *rank-le x*
unfolding *rank-eq-def rank-le-def* **by** *auto*

lemma *rank-lt-subset*: *rank-lt x* ⊆ *rank-le x*
unfolding *rank-lt-def rank-le-def* **by** *auto*

lemma *finite-rank-le*: *finite* (*rank-le x*)
unfolding *rank-le-def*
apply (*rule finite-subset* [**where** *B*=*range* (*cb-take* (*rank x*))])
apply *clarify*
apply (*rule range-eqI*)
apply (*erule rank-leD* [*symmetric*])
apply (*rule finite-range-cb-take*)
done

lemma *finite-rank-eq*: *finite* (*rank-eq x*)
by (*rule finite-subset* [*OF rank-eq-subset finite-rank-le*])

lemma *finite-rank-lt*: *finite* (*rank-lt x*)
by (*rule finite-subset* [*OF rank-lt-subset finite-rank-le*])

lemma *rank-lt-Int-rank-eq*: *rank-lt x* ∩ *rank-eq x* = {}
unfolding *rank-lt-def rank-eq-def rank-le-def* **by** *auto*

lemma *rank-lt-Un-rank-eq*: *rank-lt x* ∪ *rank-eq x* = *rank-le x*
unfolding *rank-lt-def rank-eq-def rank-le-def* **by** *auto*

21.4.4 Sequencing basis elements

definition
  place :: *'a compact-basis* ⇒ *nat*
where
  place x = *card* (*rank-lt x*) + *choose-pos* (*rank-eq x*) *x*

lemma *place-bounded*: *place x* < *card* (*rank-le x*)
unfolding *place-def*
 apply (*rule ord-less-eq-trans*)
  apply (*rule add-strict-left-mono*)
  apply (*rule choose-pos-bounded*)
   apply (*rule finite-rank-eq*)
  apply (*simp add*: *rank-eq-def*)
 apply (*subst card-Un-disjoint* [*symmetric*])
    apply (*rule finite-rank-lt*)
   apply (*rule finite-rank-eq*)
  apply (*rule rank-lt-Int-rank-eq*)
 apply (*simp add*: *rank-lt-Un-rank-eq*)
done

lemma *place-ge*: *card* (*rank-lt x*) ≤ *place x*
unfolding *place-def* **by** *simp*

lemma *place-rank-mono*:

fixes *x y* :: *'a compact-basis*
shows *rank x* < *rank y* ⟹ *place x* < *place y*
apply (*rule less-le-trans* [*OF place-bounded*])
apply (*rule order-trans* [*OF - place-ge*])
apply (*rule card-mono*)
apply (*rule finite-rank-lt*)
apply (*simp add*: *rank-le-def rank-lt-def subset-eq*)
done

lemma *place-eqD*: *place x* = *place y* ⟹ *x* = *y*
apply (*rule linorder-cases* [**where** *x*=*rank x* **and** *y*=*rank y*])
apply (*drule place-rank-mono*, *simp*)
apply (*simp add*: *place-def*)
apply (*rule inj-on-choose-pos* [**where** *A*=*rank-eq x*, *THEN inj-onD*])
apply (*rule finite-rank-eq*)
apply (*simp cong*: *rank-lt-cong rank-eq-cong*)
apply (*simp add*: *rank-eq-def*)
apply (*simp add*: *rank-eq-def*)
apply (*drule place-rank-mono*, *simp*)
done

lemma *inj-place*: *inj place*
by (*rule inj-onI*, *erule place-eqD*)

21.4.5 Embedding and projection on basis elements

definition
sub :: *'a compact-basis* ⇒ *'a compact-basis*
where
sub x = (*case rank x of 0* ⇒ *compact-bot* | *Suc k* ⇒ *cb-take k x*)

lemma *rank-sub-less*: *x* ≠ *compact-bot* ⟹ *rank* (*sub x*) < *rank x*
unfolding *sub-def*
apply (*cases rank x*, *simp*)
apply (*simp add*: *less-Suc-eq-le*)
apply (*rule rank-leI*)
apply (*rule cb-take-idem*)
done

lemma *place-sub-less*: *x* ≠ *compact-bot* ⟹ *place* (*sub x*) < *place x*
apply (*rule place-rank-mono*)
apply (*erule rank-sub-less*)
done

lemma *sub-below*: *sub x* ⊑ *x*
unfolding *sub-def* **by** (*cases rank x*, *simp-all add*: *cb-take-less*)

lemma *rank-less-imp-below-sub*: ⟦*x* ⊑ *y*; *rank x* < *rank y*⟧ ⟹ *x* ⊑ *sub y*
unfolding *sub-def*

```
apply (cases rank y, simp)
apply (simp add: less-Suc-eq-le)
apply (subgoal-tac cb-take nat x ⊑ cb-take nat y)
apply (simp add: rank-leD)
apply (erule cb-take-mono)
done

function basis-emb :: 'a compact-basis ⇒ ubasis
  where basis-emb x = (if x = compact-bot then 0 else
    node (place x) (basis-emb (sub x))
      (basis-emb ‘ {y. place y < place x ∧ x ⊑ y}))
  by simp-all

termination basis-emb
  by (relation measure place) (simp-all add: place-sub-less)

declare basis-emb.simps [simp del]

lemma basis-emb-compact-bot [simp]:
  basis-emb compact-bot = 0
  using basis-emb.simps [of compact-bot] by simp

lemma basis-emb-rec:
  basis-emb x = node (place x) (basis-emb (sub x)) (basis-emb ‘ {y. place y < place
x ∧ x ⊑ y})
  if x ≠ compact-bot
  using that basis-emb.simps [of x] by simp

lemma basis-emb-eq-0-iff [simp]:
  basis-emb x = 0 ⟷ x = compact-bot
  by (cases x = compact-bot) (simp-all add: basis-emb-rec)

lemma fin1: finite {y. place y < place x ∧ x ⊑ y}
apply (subst Collect-conj-eq)
apply (rule finite-Int)
apply (rule disjI1)
apply (subgoal-tac finite (place −‘ {n. n < place x}), simp)
apply (rule finite-vimageI [OF - inj-place])
apply (simp add: lessThan-def [symmetric])
done

lemma fin2: finite (basis-emb ‘ {y. place y < place x ∧ x ⊑ y})
by (rule finite-imageI [OF fin1])

lemma rank-place-mono:
  ⟦place x < place y; x ⊑ y⟧ ⟹ rank x < rank y
apply (rule linorder-cases, assumption)
apply (simp add: place-def cong: rank-lt-cong rank-eq-cong)
apply (drule choose-pos-lessD)
```

```
apply (rule finite-rank-eq)
apply (simp add: rank-eq-def)
apply (simp add: rank-eq-def)
apply simp
apply (drule place-rank-mono, simp)
done

lemma basis-emb-mono:
  x ⊑ y ⟹ ubasis-le (basis-emb x) (basis-emb y)
proof (induct max (place x) (place y) arbitrary: x y rule: less-induct)
  case less
  show ?case proof (rule linorder-cases)
    assume place x < place y
    then have rank x < rank y
      using ‹x ⊑ y› by (rule rank-place-mono)
    with ‹place x < place y› show ?case
      apply (case-tac y = compact-bot, simp)
      apply (simp add: basis-emb.simps [of y])
      apply (rule ubasis-le-trans [OF - ubasis-le-lower [OF fin2]])
      apply (rule less)
       apply (simp add: less-max-iff-disj)
       apply (erule place-sub-less)
      apply (erule rank-less-imp-below-sub [OF ‹x ⊑ y›])
      done
  next
    assume place x = place y
    hence x = y by (rule place-eqD)
    thus ?case by (simp add: ubasis-le-refl)
  next
    assume place x > place y
    with ‹x ⊑ y› show ?case
      apply (case-tac x = compact-bot, simp add: ubasis-le-minimal)
      apply (simp add: basis-emb.simps [of x])
      apply (rule ubasis-le-upper [OF fin2], simp)
      apply (rule less)
       apply (simp add: less-max-iff-disj)
       apply (erule place-sub-less)
      apply (erule rev-below-trans)
      apply (rule sub-below)
      done
  qed
qed

lemma inj-basis-emb: inj basis-emb
proof (rule injI)
  fix x y
  assume basis-emb x = basis-emb y
  then show x = y
    by (cases x = compact-bot ∨ y = compact-bot) (auto simp add: basis-emb-rec
```

```
fin2 place-eqD)
qed

definition
  basis-prj :: ubasis ⇒ 'a compact-basis
where
  basis-prj x = inv basis-emb
    (ubasis-until (λx. x ∈ range (basis-emb :: 'a compact-basis ⇒ ubasis)) x)

lemma basis-prj-basis-emb: ⋀x. basis-prj (basis-emb x) = x
unfolding basis-prj-def
 apply (subst ubasis-until-same)
  apply (rule rangeI)
 apply (rule inv-f-f)
 apply (rule inj-basis-emb)
done

lemma basis-prj-node:
  ⟦finite S; node i a S ∉ range (basis-emb :: 'a compact-basis ⇒ nat)⟧
    ⟹ basis-prj (node i a S) = (basis-prj a :: 'a compact-basis)
unfolding basis-prj-def by simp

lemma basis-prj-0: basis-prj 0 = compact-bot
apply (subst basis-emb-compact-bot [symmetric])
apply (rule basis-prj-basis-emb)
done

lemma node-eq-basis-emb-iff:
  finite S ⟹ node i a S = basis-emb x ⟷
    x ≠ compact-bot ∧ i = place x ∧ a = basis-emb (sub x) ∧
        S = basis-emb ` {y. place y < place x ∧ x ⊑ y}
apply (cases x = compact-bot, simp)
apply (simp add: basis-emb.simps [of x])
apply (simp add: fin2)
done

lemma basis-prj-mono: ubasis-le a b ⟹ basis-prj a ⊑ basis-prj b
proof (induct a b rule: ubasis-le.induct)
  case (ubasis-le-refl a) show ?case by (rule below-refl)
next
  case (ubasis-le-trans a b c) thus ?case by − (rule below-trans)
next
  case (ubasis-le-lower S a i) thus ?case
    apply (cases node i a S ∈ range (basis-emb :: 'a compact-basis ⇒ nat))
     apply (erule rangeE, rename-tac x)
     apply (simp add: basis-prj-basis-emb)
     apply (simp add: node-eq-basis-emb-iff)
     apply (simp add: basis-prj-basis-emb)
     apply (rule sub-below)
```

```
    apply (simp add: basis-prj-node)
    done
next
  case (ubasis-le-upper S b a i) thus ?case
    apply (cases node i a S ∈ range (basis-emb :: 'a compact-basis ⇒ nat))
     apply (erule rangeE, rename-tac x)
     apply (simp add: basis-prj-basis-emb)
     apply (clarsimp simp add: node-eq-basis-emb-iff)
     apply (simp add: basis-prj-basis-emb)
    apply (simp add: basis-prj-node)
    done
qed

lemma basis-emb-prj-less: ubasis-le (basis-emb (basis-prj x)) x
unfolding basis-prj-def
 apply (subst f-inv-into-f [where f=basis-emb])
  apply (rule ubasis-until)
  apply (rule range-eqI [where x=compact-bot])
  apply simp
 apply (rule ubasis-until-less)
done

lemma ideal-completion:
  ideal-completion below Rep-compact-basis (approximants :: 'a ⇒ -)
proof
  fix w :: 'a
  show below.ideal (approximants w)
  proof (rule below.idealI)
    have Abs-compact-basis (approx 0·w) ∈ approximants w
      by (simp add: approximants-def approx-below)
    thus ∃x. x ∈ approximants w ..
  next
    fix x y :: 'a compact-basis
    assume x: x ∈ approximants w and y: y ∈ approximants w
    obtain i where i: approx i·(Rep-compact-basis x) = Rep-compact-basis x
      using compact-eq-approx Rep-compact-basis' by fast
    obtain j where j: approx j·(Rep-compact-basis y) = Rep-compact-basis y
      using compact-eq-approx Rep-compact-basis' by fast
    let ?z = Abs-compact-basis (approx (max i j)·w)
    have ?z ∈ approximants w
      by (simp add: approximants-def approx-below)
    moreover from x y have x ⊑ ?z ∧ y ⊑ ?z
      by (simp add: approximants-def compact-le-def)
       (metis i j monofun-cfun chain-mono chain-approx max.cobounded1 max.cobounded2)
    ultimately show ∃z ∈ approximants w. x ⊑ z ∧ y ⊑ z ..
  next
    fix x y :: 'a compact-basis
    assume x ⊑ y y ∈ approximants w thus x ∈ approximants w
      unfolding approximants-def compact-le-def
```

```
    by (auto elim: below-trans)
  qed
next
  fix Y :: nat ⇒ 'a
  assume chain Y
  thus approximants (⨆ i. Y i) = (⋃ i. approximants (Y i))
    unfolding approximants-def
    by (auto simp add: compact-below-lub-iff)
next
  fix a :: 'a compact-basis
  show approximants (Rep-compact-basis a) = {b. b ⊑ a}
    unfolding approximants-def compact-le-def ..
next
  fix x y :: 'a
  assume approximants x ⊆ approximants y
  hence ∀ z. compact z ⟶ z ⊑ x ⟶ z ⊑ y
    by (simp add: approximants-def subset-eq)
      (metis Abs-compact-basis-inverse')
  hence (⨆ i. approx i·x) ⊑ y
    by (simp add: lub-below approx-below)
  thus x ⊑ y
    by (simp add: lub-distribs)
next
  show ∃f::'a compact-basis ⇒ nat. inj f
    by (rule exI, rule inj-place)
qed

end

interpretation compact-basis:
  ideal-completion below Rep-compact-basis
    approximants :: 'a::bifinite ⇒ 'a compact-basis set
proof −
  obtain a :: nat ⇒ 'a → 'a where approx-chain a
    using bifinite ..
  hence bifinite-approx-chain a
    unfolding bifinite-approx-chain-def .
  thus ideal-completion below Rep-compact-basis (approximants :: 'a ⇒ -)
    by (rule bifinite-approx-chain.ideal-completion)
qed
```

21.4.6 EP-pair from any bifinite domain into *udom*

```
context bifinite-approx-chain begin

definition
  udom-emb :: 'a → udom
where
  udom-emb = compact-basis.extension (λx. udom-principal (basis-emb x))
```

definition
udom-prj :: *udom* → *'a*
where
udom-prj = *udom.extension* (*λx. Rep-compact-basis* (*basis-prj x*))

lemma *udom-emb-principal*:
udom-emb·(*Rep-compact-basis x*) = *udom-principal* (*basis-emb x*)
unfolding *udom-emb-def*
apply (*rule compact-basis.extension-principal*)
apply (*rule udom.principal-mono*)
apply (*erule basis-emb-mono*)
done

lemma *udom-prj-principal*:
udom-prj·(*udom-principal x*) = *Rep-compact-basis* (*basis-prj x*)
unfolding *udom-prj-def*
apply (*rule udom.extension-principal*)
apply (*rule compact-basis.principal-mono*)
apply (*erule basis-prj-mono*)
done

lemma *ep-pair-udom*: *ep-pair udom-emb udom-prj*
apply *standard*
apply (*rule compact-basis.principal-induct*, *simp*)
apply (*simp add*: *udom-emb-principal udom-prj-principal*)
apply (*simp add*: *basis-prj-basis-emb*)
apply (*rule udom.principal-induct*, *simp*)
apply (*simp add*: *udom-emb-principal udom-prj-principal*)
apply (*rule basis-emb-prj-less*)
done

end

abbreviation *udom-emb* ≡ *bifinite-approx-chain.udom-emb*
abbreviation *udom-prj* ≡ *bifinite-approx-chain.udom-prj*

lemmas *ep-pair-udom* =
bifinite-approx-chain.ep-pair-udom [*unfolded bifinite-approx-chain-def*]

21.5 Chain of approx functions for type *udom*

definition
udom-approx :: *nat* ⇒ *udom* → *udom*
where
udom-approx i =
udom.extension (*λx. udom-principal* (*ubasis-until* (*λy. y* ≤ *i*) *x*))

lemma *udom-approx-mono*:

```
  ubasis-le a b ⟹
    udom-principal (ubasis-until (λy. y ≤ i) a) ⊑
    udom-principal (ubasis-until (λy. y ≤ i) b)
apply (rule udom.principal-mono)
apply (rule ubasis-until-mono)
apply (frule (2) order-less-le-trans [OF node-gt2])
apply (erule order-less-imp-le)
apply assumption
done

lemma adm-mem-finite: ⟦cont f; finite S⟧ ⟹ adm (λx. f x ∈ S)
by (erule adm-subst, induct set: finite, simp-all)

lemma udom-approx-principal:
  udom-approx i·(udom-principal x) =
    udom-principal (ubasis-until (λy. y ≤ i) x)
unfolding udom-approx-def
apply (rule udom.extension-principal)
apply (erule udom-approx-mono)
done

lemma finite-deflation-udom-approx: finite-deflation (udom-approx i)
proof
  fix x show udom-approx i·(udom-approx i·x) = udom-approx i·x
    by (induct x rule: udom.principal-induct, simp)
      (simp add: udom-approx-principal ubasis-until-idem)
next
  fix x show udom-approx i·x ⊑ x
    by (induct x rule: udom.principal-induct, simp)
      (simp add: udom-approx-principal ubasis-until-less)
next
  have *: finite (range (λx. udom-principal (ubasis-until (λy. y ≤ i) x)))
    apply (subst range-composition [where f=udom-principal])
    apply (simp add: finite-range-ubasis-until)
    done
  show finite {x. udom-approx i·x = x}
    apply (rule finite-range-imp-finite-fixes)
    apply (rule rev-finite-subset [OF *])
    apply (clarsimp, rename-tac x)
    apply (induct-tac x rule: udom.principal-induct)
    apply (simp add: adm-mem-finite *)
    apply (simp add: udom-approx-principal)
    done
qed

interpretation udom-approx: finite-deflation udom-approx i
by (rule finite-deflation-udom-approx)

lemma chain-udom-approx [simp]: chain (λi. udom-approx i)
```

unfolding *udom-approx-def*
apply (*rule chainI*)
apply (*rule udom.extension-mono*)
apply (*erule udom-approx-mono*)
apply (*erule udom-approx-mono*)
apply (*rule udom.principal-mono*)
apply (*rule ubasis-until-chain*, *simp*)
done

lemma *lub-udom-approx* [*simp*]: ($\bigsqcup$ *i. udom-approx i*) = *ID*
apply (*rule cfun-eqI*, *simp add*: *contlub-cfun-fun*)
apply (*rule below-antisym*)
apply (*rule lub-below*)
apply (*simp*)
apply (*rule udom-approx.below*)
apply (*rule-tac x=x* **in** *udom.principal-induct*)
apply (*simp add*: *lub-distribs*)
apply (*rule-tac i=a* **in** *below-lub*)
apply *simp*
apply (*simp add*: *udom-approx-principal*)
apply (*simp add*: *ubasis-until-same ubasis-le-refl*)
done

lemma *udom-approx* [*simp*]: *approx-chain udom-approx*
proof
  show *chain* (*λi. udom-approx i*)
    by (*rule chain-udom-approx*)
  show ($\bigsqcup$ *i. udom-approx i*) = *ID*
    by (*rule lub-udom-approx*)
qed

instance *udom* :: *bifinite*
  by *standard* (*fast intro*: *udom-approx*)

hide-const (**open**) *node*

unbundle *binomial-syntax*

end

22 Algebraic deflations

theory *Algebraic*
imports *Universal Map-Functions*
begin

22.1 Type constructor for finite deflations

typedef *'a::bifinite fin-defl* = {*d::'a → 'a. finite-deflation d*}

by (*fast intro*: *finite-deflation-bottom*)

instantiation *fin-defl* :: (*bifinite*) *below*
begin

definition *below-fin-defl-def*:
  below $\equiv$ $\lambda x\ y$. *Rep-fin-defl* $x \sqsubseteq$ *Rep-fin-defl* y

instance ..
end

instance *fin-defl* :: (*bifinite*) *po*
using *type-definition-fin-defl below-fin-defl-def*
by (*rule typedef-po-class*)

lemma *finite-deflation-Rep-fin-defl*: *finite-deflation* (*Rep-fin-defl d*)
using *Rep-fin-defl* **by** *simp*

lemma *deflation-Rep-fin-defl*: *deflation* (*Rep-fin-defl d*)
using *finite-deflation-Rep-fin-defl*
by (*rule finite-deflation-imp-deflation*)

interpretation *Rep-fin-defl*: *finite-deflation Rep-fin-defl d*
by (*rule finite-deflation-Rep-fin-defl*)

lemma *fin-defl-belowI*:
  ($\bigwedge x$. *Rep-fin-defl* $a \cdot x = x \Longrightarrow$ *Rep-fin-defl* $b \cdot x = x$) $\Longrightarrow a \sqsubseteq b$
unfolding *below-fin-defl-def*
by (*rule Rep-fin-defl.belowI*)

lemma *fin-defl-belowD*:
  $[\![a \sqsubseteq b$; *Rep-fin-defl* $a \cdot x = x]\!] \Longrightarrow$ *Rep-fin-defl* $b \cdot x = x$
unfolding *below-fin-defl-def*
by (*rule Rep-fin-defl.belowD*)

lemma *fin-defl-eqI*:
  $a = b$ **if** ($\bigwedge x$. *Rep-fin-defl* $a \cdot x = x \longleftrightarrow$ *Rep-fin-defl* $b \cdot x = x$)
proof (*rule below-antisym*)
  show $a \sqsubseteq b$ **by** (*rule fin-defl-belowI*) (*simp add*: *that*)
  show $b \sqsubseteq a$ **by** (*rule fin-defl-belowI*) (*simp add*: *that*)
qed

lemma *Rep-fin-defl-mono*: $a \sqsubseteq b \Longrightarrow$ *Rep-fin-defl* $a \sqsubseteq$ *Rep-fin-defl* b
unfolding *below-fin-defl-def* **.**

lemma *Abs-fin-defl-mono*:
  $[\![$*finite-deflation* a; *finite-deflation* b; $a \sqsubseteq b]\!]$
    $\Longrightarrow$ *Abs-fin-defl* $a \sqsubseteq$ *Abs-fin-defl* b
unfolding *below-fin-defl-def*

by (*simp add*: *Abs-fin-defl-inverse*)

lemma (**in** *finite-deflation*) *compact-belowI*:
  $d \sqsubseteq f$ **if** $\bigwedge x.$ *compact* $x \Longrightarrow d{\cdot}x = x \Longrightarrow f{\cdot}x = x$
  by (*rule belowI*, *rule that*, *erule subst*, *rule compact*)

lemma *compact-Rep-fin-defl* [*simp*]: *compact* (*Rep-fin-defl a*)
using *finite-deflation-Rep-fin-defl*
by (*rule finite-deflation-imp-compact*)

22.2 Defining algebraic deflations by ideal completion

typedef *'a::bifinite defl* = {*S::'a fin-defl set. below.ideal S*}
by (*rule below.ex-ideal*)

instantiation *defl* :: (*bifinite*) *below*
begin

definition $x \sqsubseteq y \longleftrightarrow$ *Rep-defl x* $\subseteq$ *Rep-defl y*

instance ..

end

instance *defl* :: (*bifinite*) *po*
using *type-definition-defl below-defl-def*
by (*rule below.typedef-ideal-po*)

instance *defl* :: (*bifinite*) *cpo*
using *type-definition-defl below-defl-def*
by (*rule below.typedef-ideal-cpo*)

definition *defl-principal* :: *'a::bifinite fin-defl* $\Rightarrow$ *'a defl*
  where *defl-principal t* = *Abs-defl* {*u. u* $\sqsubseteq$ *t*}

lemma *fin-defl-countable*: $\exists$*f::'a::bifinite fin-defl* $\Rightarrow$ *nat. inj f*
proof −
  obtain *f* :: *'a compact-basis* $\Rightarrow$ *nat* **where** *inj-f*: *inj f*
    using *compact-basis.countable* **..**
  have $*$: $\bigwedge$*d. finite* (*f* ' *Rep-compact-basis* −' {*x. Rep-fin-defl d*·*x* = *x*})
    apply (*rule finite-imageI*)
    apply (*rule finite-vimageI*)
    apply (*rule Rep-fin-defl.finite-fixes*)
    apply (*simp add*: *inj-on-def Rep-compact-basis-inject*)
    done
  have *range-eq*: *range Rep-compact-basis* = {*x. compact x*}
    using *type-definition-compact-basis* **by** (*rule type-definition.Rep-range*)
  have *inj* (λ*d. set-encode*
    (*f* ' *Rep-compact-basis* −' {*x. Rep-fin-defl d*·*x* = *x*}))

```
  apply (rule inj-onI)
  apply (simp only: set-encode-eq ∗)
  apply (simp only: inj-image-eq-iff inj-f)
  apply (drule-tac f=image Rep-compact-basis in arg-cong)
  apply (simp del: vimage-Collect-eq add: range-eq set-eq-iff)
  apply (rule Rep-fin-defl-inject [THEN iffD1])
  apply (rule below-antisym)
  apply (rule Rep-fin-defl.compact-belowI, rename-tac z)
  apply (drule-tac x=z in spec, simp)
  apply (rule Rep-fin-defl.compact-belowI, rename-tac z)
  apply (drule-tac x=z in spec, simp)
  done
 thus ?thesis by − (rule exI)
qed

interpretation defl: ideal-completion below defl-principal Rep-defl
using type-definition-defl below-defl-def
using defl-principal-def fin-defl-countable
by (rule below.typedef-ideal-completion)
```

Algebraic deflations are pointed

```
lemma defl-minimal: defl-principal (Abs-fin-defl ⊥) ⊑ x
proof (induct x rule: defl.principal-induct)
 fix a :: 'a fin-defl
 have Abs-fin-defl ⊥ ⊑ a
  by (simp add: below-fin-defl-def Abs-fin-defl-inverse finite-deflation-bottom)
 then show defl-principal (Abs-fin-defl ⊥) ⊑ defl-principal a
  by (rule defl.principal-mono)
qed simp

instance defl :: (bifinite) pcpo
by intro-classes (fast intro: defl-minimal)

lemma inst-defl-pcpo: ⊥ = defl-principal (Abs-fin-defl ⊥)
by (rule defl-minimal [THEN bottomI, symmetric])
```

22.3 Applying algebraic deflations

```
definition cast :: 'a::bifinite defl → 'a → 'a
 where cast = defl.extension Rep-fin-defl

lemma cast-defl-principal: cast·(defl-principal a) = Rep-fin-defl a
 unfolding cast-def
 by (rule defl.extension-principal) (simp only: below-fin-defl-def)

lemma deflation-cast: deflation (cast·d)
apply (induct d rule: defl.principal-induct)
apply (rule adm-subst [OF - adm-deflation], simp)
apply (simp add: cast-defl-principal)
```

apply (*rule finite-deflation-imp-deflation*)
apply (*rule finite-deflation-Rep-fin-defl*)
done

lemma *finite-deflation-cast*: *compact d* $\Longrightarrow$ *finite-deflation* (*cast·d*)
  apply (*drule defl.compact-imp-principal*)
  apply *clarify*
  apply (*simp add*: *cast-defl-principal*)
  apply (*rule finite-deflation-Rep-fin-defl*)
  done

interpretation *cast*: *deflation cast·d*
by (*rule deflation-cast*)

declare *cast.idem* [*simp*]

lemma *compact-cast* [*simp*]: *compact* (*cast·d*) **if** *compact d*
  by (*rule finite-deflation-imp-compact*) (*use that* **in** ‹*rule finite-deflation-cast*›)

lemma *cast-below-cast*: *cast·A* $\sqsubseteq$ *cast·B* $\longleftrightarrow$ *A* $\sqsubseteq$ *B*
apply (*induct A rule*: *defl.principal-induct*, *simp*)
apply (*induct B rule*: *defl.principal-induct*, *simp*)
apply (*simp add*: *cast-defl-principal below-fin-defl-def*)
done

lemma *compact-cast-iff*: *compact* (*cast·d*) $\longleftrightarrow$ *compact d*
apply (*rule iffI*)
apply (*simp only*: *compact-def cast-below-cast* [*symmetric*])
apply (*erule adm-subst* [*OF cont-Rep-cfun2*])
apply (*erule compact-cast*)
done

lemma *cast-below-imp-below*: *cast·A* $\sqsubseteq$ *cast·B* $\Longrightarrow$ *A* $\sqsubseteq$ *B*
by (*simp only*: *cast-below-cast*)

lemma *cast-eq-imp-eq*: *cast·A* = *cast·B* $\Longrightarrow$ *A* = *B*
by (*simp add*: *below-antisym cast-below-imp-below*)

lemma *cast-strict1* [*simp*]: *cast·*$\bot$ = $\bot$
apply (*subst inst-defl-pcpo*)
apply (*subst cast-defl-principal*)
apply (*rule Abs-fin-defl-inverse*)
apply (*simp add*: *finite-deflation-bottom*)
done

lemma *cast-strict2* [*simp*]: *cast·A·*$\bot$ = $\bot$
by (*rule cast.below* [*THEN bottomI*])

22.4 Deflation combinators

definition
  defl-fun1 e p f =
    defl.extension (λ*a*.
      defl-principal (*Abs-fin-defl*
        (*e oo f*·(*Rep-fin-defl a*) *oo p*)))

definition
  defl-fun2 e p f =
    defl.extension (λ*a*.
      defl.extension (λ*b*.
        defl-principal (*Abs-fin-defl*
          (*e oo f*·(*Rep-fin-defl a*)·(*Rep-fin-defl b*) *oo p*))))

lemma *cast-defl-fun1*:
  assumes *ep*: *ep-pair e p*
  assumes *f*: ⋀*a*. *finite-deflation a* ⟹ *finite-deflation* (*f*·*a*)
  shows *cast*·(*defl-fun1 e p f*·*A*) = *e oo f*·(*cast*·*A*) *oo p*
proof −
  have *1*: *finite-deflation* (*e oo f*·(*Rep-fin-defl a*) *oo p*) **for** *a*
  proof −
    have *finite-deflation* (*f*·(*Rep-fin-defl a*))
      using *finite-deflation-Rep-fin-defl* **by** (*rule f*)
    with *ep* **show** *?thesis*
      by (*rule ep-pair.finite-deflation-e-d-p*)
  qed
  show *?thesis*
    by (*induct A rule*: *defl.principal-induct*, *simp*)
       (*simp only*: *defl-fun1-def*
                  defl.extension-principal
                  defl.extension-mono
                  defl.principal-mono
                  Abs-fin-defl-mono [*OF 1 1*]
                  monofun-cfun below-refl
                  Rep-fin-defl-mono
                  cast-defl-principal
                  Abs-fin-defl-inverse [*unfolded mem-Collect-eq*, *OF 1*])
qed

lemma *cast-defl-fun2*:
  assumes *ep*: *ep-pair e p*
  assumes *f*: ⋀*a b*. *finite-deflation a* ⟹ *finite-deflation b* ⟹
              finite-deflation (*f*·*a*·*b*)
  shows *cast*·(*defl-fun2 e p f*·*A*·*B*) = *e oo f*·(*cast*·*A*)·(*cast*·*B*) *oo p*
proof −
  have *1*: *finite-deflation* (*e oo f*·(*Rep-fin-defl a*)·(*Rep-fin-defl b*) *oo p*) **for** *a b*
  proof −
    have *finite-deflation* (*f*·(*Rep-fin-defl a*)·(*Rep-fin-defl b*))
      using *finite-deflation-Rep-fin-defl finite-deflation-Rep-fin-defl* **by** (*rule f*)

```
  with ep show ?thesis
    by (rule ep-pair.finite-deflation-e-d-p)
 qed
 show ?thesis
  apply (induct A rule: defl.principal-induct, simp)
  apply (induct B rule: defl.principal-induct, simp)
  by (simp only: defl-fun2-def
               defl.extension-principal
               defl.extension-mono
               defl.principal-mono
               Abs-fin-defl-mono [OF 1 1]
               monofun-cfun below-refl
               Rep-fin-defl-mono
               cast-defl-principal
               Abs-fin-defl-inverse [unfolded mem-Collect-eq, OF 1])
qed

end
```

23 Representable domains

```
theory Representable
imports Algebraic Map-Functions HOL−Library.Countable
begin
```

23.1 Class of representable domains

We define a "domain" as a pcpo that is isomorphic to some algebraic deflation over the universal domain; this is equivalent to being omega-bifinite.

A predomain is a cpo that, when lifted, becomes a domain. Predomains are represented by deflations over a lifted universal domain type.

```
class predomain-syn = cpo +
 fixes liftemb :: 'a⊥ → udom⊥
 fixes liftprj :: udom⊥ → 'a⊥
 fixes liftdefl :: 'a itself ⇒ udom u defl

class predomain = predomain-syn +
 assumes predomain-ep: ep-pair liftemb liftprj
 assumes cast-liftdefl: cast·(liftdefl TYPE('a)) = liftemb oo liftprj

syntax -LIFTDEFL :: type ⇒ logic  (‹(1LIFTDEFL/(1'(-')))›)
syntax-consts -LIFTDEFL ⇌ liftdefl
translations LIFTDEFL('t) ⇌ CONST liftdefl TYPE('t)

definition liftdefl-of :: udom defl → udom u defl
 where liftdefl-of = defl-fun1 ID ID u-map

lemma cast-liftdefl-of: cast·(liftdefl-of·t) = u-map·(cast·t)
```

by (*simp add*: *liftdefl-of-def cast-defl-fun1 ep-pair-def finite-deflation-u-map*)

class *domain* = *predomain-syn* + *pcpo* +
  fixes *emb* :: *'a* → *udom*
  fixes *prj* :: *udom* → *'a*
  fixes *defl* :: *'a itself* ⇒ *udom defl*
  assumes *ep-pair-emb-prj*: *ep-pair emb prj*
  assumes *cast-DEFL*: *cast·(defl TYPE('a))* = *emb oo prj*
  assumes *liftemb-eq*: *liftemb* = *u-map·emb*
  assumes *liftprj-eq*: *liftprj* = *u-map·prj*
  assumes *liftdefl-eq*: *liftdefl TYPE('a)* = *liftdefl-of·(defl TYPE('a))*

syntax *-DEFL* :: *type* ⇒ *logic* (‹(1DEFL/(1'(-')))›)
syntax-consts *-DEFL* ⇌ *defl*
translations *DEFL('t)* ⇌ *CONST defl TYPE('t)*

instance *domain* ⊆ *predomain*
proof
  show *ep-pair liftemb* (*liftprj*::*udom*$_\bot$ → *'a*$_\bot$)
    unfolding *liftemb-eq liftprj-eq*
    by (*intro ep-pair-u-map ep-pair-emb-prj*)
  show *cast·LIFTDEFL('a)* = *liftemb oo* (*liftprj*::*udom*$_\bot$ → *'a*$_\bot$)
    unfolding *liftemb-eq liftprj-eq liftdefl-eq*
    by (*simp add*: *cast-liftdefl-of cast-DEFL u-map-oo*)
qed

Constants *liftemb* and *liftprj* imply class predomain.

setup ‹
  fold Sign.add-const-constraint
  [(***const-name*** ‹*liftemb*›, *SOME* ***typ*** ‹*'a*::*predomain u* → *udom u*›),
   (***const-name*** ‹*liftprj*›, *SOME* ***typ*** ‹*udom u* → *'a*::*predomain u*›),
   (***const-name*** ‹*liftdefl*›, *SOME* ***typ*** ‹*'a*::*predomain itself* ⇒ *udom u defl*›)]
›

interpretation *predomain*: *pcpo-ep-pair liftemb liftprj*
  unfolding *pcpo-ep-pair-def* **by** (*rule predomain-ep*)

interpretation *domain*: *pcpo-ep-pair emb prj*
  unfolding *pcpo-ep-pair-def* **by** (*rule ep-pair-emb-prj*)

lemmas *emb-inverse* = *domain.e-inverse*
lemmas *emb-prj-below* = *domain.e-p-below*
lemmas *emb-eq-iff* = *domain.e-eq-iff*
lemmas *emb-strict* = *domain.e-strict*
lemmas *prj-strict* = *domain.p-strict*

23.2 Domains are bifinite

lemma *approx-chain-ep-cast*:

```
  assumes ep: ep-pair (e::'a::pcpo → 'b::bifinite) (p::'b → 'a)
  assumes cast-t: cast·t = e oo p
  shows ∃(a::nat ⇒ 'a::pcpo → 'a). approx-chain a
proof –
  interpret ep-pair e p by fact
  obtain Y where Y: ∀ i. Y i ⊑ Y (Suc i)
  and t: t = (⨆ i. defl-principal (Y i))
    by (rule defl.obtain-principal-chain)
  define approx where approx i = (p oo cast·(defl-principal (Y i)) oo e) for i
  have approx-chain approx
  proof (rule approx-chain.intro)
    show chain (λi. approx i)
      unfolding approx-def by (simp add: Y)
    show (⨆ i. approx i) = ID
      unfolding approx-def
      by (simp add: lub-distribs Y t [symmetric] cast-t cfun-eq-iff)
    show ⋀i. finite-deflation (approx i)
      unfolding approx-def
      apply (rule finite-deflation-p-d-e)
      apply (rule finite-deflation-cast)
      apply (rule defl.compact-principal)
      apply (rule below-trans [OF monofun-cfun-fun])
      apply (rule is-ub-thelub, simp add: Y)
      apply (simp add: lub-distribs Y t [symmetric] cast-t)
      done
  qed
  thus ∃(a::nat ⇒ 'a → 'a). approx-chain a by – (rule exI)
qed

instance domain ⊆ bifinite
by standard (rule approx-chain-ep-cast [OF ep-pair-emb-prj cast-DEFL])

instance predomain ⊆ profinite
by standard (rule approx-chain-ep-cast [OF predomain-ep cast-liftdefl])
```

23.3 Universal domain ep-pairs

```
definition u-emb = udom-emb (λi. u-map·(udom-approx i))
definition u-prj = udom-prj (λi. u-map·(udom-approx i))

definition prod-emb = udom-emb (λi. prod-map·(udom-approx i)·(udom-approx
i))
definition prod-prj = udom-prj (λi. prod-map·(udom-approx i)·(udom-approx i))

definition sprod-emb = udom-emb (λi. sprod-map·(udom-approx i)·(udom-approx
i))
definition sprod-prj = udom-prj (λi. sprod-map·(udom-approx i)·(udom-approx i))

definition ssum-emb = udom-emb (λi. ssum-map·(udom-approx i)·(udom-approx
```

i))
definition *ssum-prj* = *udom-prj* (λ*i*. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))

definition *sfun-emb* = *udom-emb* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))
definition *sfun-prj* = *udom-prj* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))

lemma *ep-pair-u*: *ep-pair u-emb u-prj*
unfolding *u-emb-def u-prj-def*
by (*simp add*: *ep-pair-udom approx-chain-u-map*)

lemma *ep-pair-prod*: *ep-pair prod-emb prod-prj*
unfolding *prod-emb-def prod-prj-def*
by (*simp add*: *ep-pair-udom approx-chain-prod-map*)

lemma *ep-pair-sprod*: *ep-pair sprod-emb sprod-prj*
unfolding *sprod-emb-def sprod-prj-def*
by (*simp add*: *ep-pair-udom approx-chain-sprod-map*)

lemma *ep-pair-ssum*: *ep-pair ssum-emb ssum-prj*
unfolding *ssum-emb-def ssum-prj-def*
by (*simp add*: *ep-pair-udom approx-chain-ssum-map*)

lemma *ep-pair-sfun*: *ep-pair sfun-emb sfun-prj*
unfolding *sfun-emb-def sfun-prj-def*
by (*simp add*: *ep-pair-udom approx-chain-sfun-map*)

23.4 Type combinators

definition *u-defl* :: *udom defl* → *udom defl*
where *u-defl* = *defl-fun1 u-emb u-prj u-map*

definition *prod-defl* :: *udom defl* → *udom defl* → *udom defl*
where *prod-defl* = *defl-fun2 prod-emb prod-prj prod-map*

definition *sprod-defl* :: *udom defl* → *udom defl* → *udom defl*
where *sprod-defl* = *defl-fun2 sprod-emb sprod-prj sprod-map*

definition *ssum-defl* :: *udom defl* → *udom defl* → *udom defl*
where *ssum-defl* = *defl-fun2 ssum-emb ssum-prj ssum-map*

definition *sfun-defl* :: *udom defl* → *udom defl* → *udom defl*
where *sfun-defl* = *defl-fun2 sfun-emb sfun-prj sfun-map*

lemma *cast-u-defl*:
cast·(*u-defl*·*A*) = *u-emb oo u-map*·(*cast*·*A*) *oo u-prj*
using *ep-pair-u finite-deflation-u-map*
unfolding *u-defl-def* **by** (*rule cast-defl-fun1*)

lemma *cast-prod-defl*:

cast·(*prod-defl*·*A*·*B*) =
prod-emb oo prod-map·(*cast*·*A*)·(*cast*·*B*) *oo prod-prj*
using *ep-pair-prod finite-deflation-prod-map*
unfolding *prod-defl-def* **by** (*rule cast-defl-fun2*)

lemma *cast-sprod-defl*:
cast·(*sprod-defl*·*A*·*B*) =
sprod-emb oo sprod-map·(*cast*·*A*)·(*cast*·*B*) *oo sprod-prj*
using *ep-pair-sprod finite-deflation-sprod-map*
unfolding *sprod-defl-def* **by** (*rule cast-defl-fun2*)

lemma *cast-ssum-defl*:
cast·(*ssum-defl*·*A*·*B*) =
ssum-emb oo ssum-map·(*cast*·*A*)·(*cast*·*B*) *oo ssum-prj*
using *ep-pair-ssum finite-deflation-ssum-map*
unfolding *ssum-defl-def* **by** (*rule cast-defl-fun2*)

lemma *cast-sfun-defl*:
cast·(*sfun-defl*·*A*·*B*) =
sfun-emb oo sfun-map·(*cast*·*A*)·(*cast*·*B*) *oo sfun-prj*
using *ep-pair-sfun finite-deflation-sfun-map*
unfolding *sfun-defl-def* **by** (*rule cast-defl-fun2*)

Special deflation combinator for unpointed types.

definition *u-liftdefl* :: *udom u defl* → *udom defl*
where *u-liftdefl* = *defl-fun1 u-emb u-prj ID*

lemma *cast-u-liftdefl*:
cast·(*u-liftdefl*·*A*) = *u-emb oo cast*·*A oo u-prj*
unfolding *u-liftdefl-def* **by** (*simp add*: *cast-defl-fun1 ep-pair-u*)

lemma *u-liftdefl-liftdefl-of*:
u-liftdefl·(*liftdefl-of*·*A*) = *u-defl*·*A*
by (*rule cast-eq-imp-eq*)
(*simp add*: *cast-u-liftdefl cast-liftdefl-of cast-u-defl*)

23.5 Class instance proofs

23.5.1 Universal domain

instantiation *udom* :: *domain*
begin

definition [*simp*]:
emb = (*ID* :: *udom* → *udom*)

definition [*simp*]:
prj = (*ID* :: *udom* → *udom*)

definition

defl (*t*::*udom itself*) = ($\bigsqcup$ *i*. *defl-principal* (*Abs-fin-defl* (*udom-approx i*)))

definition
(*liftemb* :: *udom u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → *udom u*) = *u-map*·*prj*

definition
liftdefl (*t*::*udom itself*) = *liftdefl-of*·*DEFL*(*udom*)

instance proof
show *ep-pair emb* (*prj* :: *udom* → *udom*)
by (*simp add*: *ep-pair.intro*)
show *cast*·*DEFL*(*udom*) = *emb oo* (*prj* :: *udom* → *udom*)
unfolding *defl-udom-def*
apply (*subst contlub-cfun-arg*)
apply (*rule chainI*)
apply (*rule defl.principal-mono*)
apply (*simp add*: *below-fin-defl-def*)
apply (*simp add*: *Abs-fin-defl-inverse finite-deflation-udom-approx*)
apply (*rule chainE*)
apply (*rule chain-udom-approx*)
apply (*subst cast-defl-principal*)
apply (*simp add*: *Abs-fin-defl-inverse finite-deflation-udom-approx*)
done
qed (*fact liftemb-udom-def liftprj-udom-def liftdefl-udom-def*)+

end

23.5.2 Lifted cpo

instantiation *u* :: (*predomain*) *domain*
begin

definition
emb = *u-emb oo liftemb*

definition
prj = *liftprj oo u-prj*

definition
defl (*t*::'*a u itself*) = *u-liftdefl*·*LIFTDEFL*('*a*)

definition
(*liftemb* :: '*a u u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a u u*) = *u-map*·*prj*

definition
liftdefl (*t*::′*a u itself*) = *liftdefl-of*·*DEFL*(′*a u*)

instance proof
show *ep-pair emb* (*prj* :: *udom* → ′*a u*)
unfolding *emb-u-def prj-u-def*
by (*intro ep-pair-comp ep-pair-u predomain-ep*)
show *cast*·*DEFL*(′*a u*) = *emb oo* (*prj* :: *udom* → ′*a u*)
unfolding *emb-u-def prj-u-def defl-u-def*
by (*simp add*: *cast-u-liftdefl cast-liftdefl assoc-oo*)
qed (*fact liftemb-u-def liftprj-u-def liftdefl-u-def*)+

end

lemma *DEFL-u*: *DEFL*(′*a*::*predomain u*) = *u-liftdefl*·*LIFTDEFL*(′*a*)
by (*rule defl-u-def*)

23.5.3 Strict function space

instantiation *sfun* :: (*domain*, *domain*) *domain*
begin

definition
emb = *sfun-emb oo sfun-map*·*prj*·*emb*

definition
prj = *sfun-map*·*emb*·*prj oo sfun-prj*

definition
defl (*t*::(′*a* →! ′*b*) *itself*) = *sfun-defl*·*DEFL*(′*a*)·*DEFL*(′*b*)

definition
(*liftemb* :: (′*a* →! ′*b*) *u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → (′*a* →! ′*b*) *u*) = *u-map*·*prj*

definition
liftdefl (*t*::(′*a* →! ′*b*) *itself*) = *liftdefl-of*·*DEFL*(′*a* →! ′*b*)

instance proof
show *ep-pair emb* (*prj* :: *udom* → ′*a* →! ′*b*)
unfolding *emb-sfun-def prj-sfun-def*
by (*intro ep-pair-comp ep-pair-sfun ep-pair-sfun-map ep-pair-emb-prj*)
show *cast*·*DEFL*(′*a* →! ′*b*) = *emb oo* (*prj* :: *udom* → ′*a* →! ′*b*)
unfolding *emb-sfun-def prj-sfun-def defl-sfun-def cast-sfun-defl*
by (*simp add*: *cast-DEFL oo-def sfun-eq-iff sfun-map-map*)
qed (*fact liftemb-sfun-def liftprj-sfun-def liftdefl-sfun-def*)+

end

lemma *DEFL-sfun*:
  DEFL('a::domain →! 'b::domain) = sfun-defl·DEFL('a)·DEFL('b)
by (*rule defl-sfun-def*)

23.5.4 Continuous function space

instantiation *cfun* :: (*predomain*, *domain*) *domain*
begin

definition
  emb = emb oo encode-cfun

definition
  prj = decode-cfun oo prj

definition
  defl (t::('a → 'b) itself) = DEFL('a u →! 'b)

definition
  (*liftemb* :: (*'a → 'b*) *u → udom u*) = *u-map·emb*

definition
  (*liftprj* :: *udom u →* (*'a → 'b*) *u*) = *u-map·prj*

definition
  liftdefl (t::('a → 'b) itself) = liftdefl-of·DEFL('a → 'b)

instance proof
  have *ep-pair encode-cfun decode-cfun*
    by (*rule ep-pair.intro*, *simp-all*)
  thus *ep-pair emb* (*prj* :: *udom → 'a → 'b*)
    unfolding *emb-cfun-def prj-cfun-def*
    using *ep-pair-emb-prj* **by** (*rule ep-pair-comp*)
  show *cast·DEFL('a → 'b) = emb oo* (*prj* :: *udom → 'a → 'b*)
    unfolding *emb-cfun-def prj-cfun-def defl-cfun-def*
    by (*simp add*: *cast-DEFL cfcomp1*)
qed (*fact liftemb-cfun-def liftprj-cfun-def liftdefl-cfun-def*)+

end

lemma *DEFL-cfun*:
  DEFL('a::predomain → 'b::domain) = DEFL('a u →! 'b)
by (*rule defl-cfun-def*)

23.5.5 Strict product

instantiation *sprod* :: (*domain*, *domain*) *domain*

begin

definition
emb = sprod-emb oo sprod-map·emb·emb

definition
prj = sprod-map·prj·prj oo sprod-prj

definition
defl (t::('a ⊗ 'b) itself) = sprod-defl·DEFL('a)·DEFL('b)

definition
(liftemb :: ('a ⊗ 'b) u → udom u) = u-map·emb

definition
(liftprj :: udom u → ('a ⊗ 'b) u) = u-map·prj

definition
liftdefl (t::('a ⊗ 'b) itself) = liftdefl-of·DEFL('a ⊗ 'b)

instance proof
show *ep-pair emb (prj :: udom → 'a ⊗ 'b)*
unfolding *emb-sprod-def prj-sprod-def*
by *(intro ep-pair-comp ep-pair-sprod ep-pair-sprod-map ep-pair-emb-prj)*
show *cast·DEFL('a ⊗ 'b) = emb oo (prj :: udom → 'a ⊗ 'b)*
unfolding *emb-sprod-def prj-sprod-def defl-sprod-def cast-sprod-defl*
by *(simp add: cast-DEFL oo-def cfun-eq-iff sprod-map-map)*
qed *(fact liftemb-sprod-def liftprj-sprod-def liftdefl-sprod-def)+*

end

lemma *DEFL-sprod*:
DEFL('a::domain ⊗ 'b::domain) = sprod-defl·DEFL('a)·DEFL('b)
by *(rule defl-sprod-def)*

23.5.6 Cartesian product

definition *prod-liftdefl :: udom u defl → udom u defl → udom u defl*
where *prod-liftdefl = defl-fun2 (u-map·prod-emb oo decode-prod-u)*
(encode-prod-u oo u-map·prod-prj) sprod-map

lemma *cast-prod-liftdefl*:
cast·(prod-liftdefl·a·b) =
(u-map·prod-emb oo decode-prod-u) oo sprod-map·(cast·a)·(cast·b) oo
(encode-prod-u oo u-map·prod-prj)
unfolding *prod-liftdefl-def*
apply *(rule cast-defl-fun2)*
apply *(intro ep-pair-comp ep-pair-u-map ep-pair-prod)*
apply *(simp add: ep-pair.intro)*

apply (*erule* (*1*) *finite-deflation-sprod-map*)
done

instantiation *prod* :: (*predomain*, *predomain*) *predomain*
begin

definition
liftemb = (*u-map·prod-emb oo decode-prod-u*) *oo*
(*sprod-map·liftemb·liftemb oo encode-prod-u*)

definition
liftprj = (*decode-prod-u oo sprod-map·liftprj·liftprj*) *oo*
(*encode-prod-u oo u-map·prod-prj*)

definition
liftdefl (*t*::(*'a* × *'b*) *itself*) = *prod-liftdefl·LIFTDEFL*(*'a*)·*LIFTDEFL*(*'b*)

instance proof
show *ep-pair liftemb* (*liftprj* :: *udom u* → (*'a* × *'b*) *u*)
unfolding *liftemb-prod-def liftprj-prod-def*
by (*intro ep-pair-comp ep-pair-sprod-map ep-pair-u-map*
ep-pair-prod predomain-ep, *simp-all add*: *ep-pair.intro*)
show *cast·LIFTDEFL*(*'a* × *'b*) = *liftemb oo* (*liftprj* :: *udom u* → (*'a* × *'b*) *u*)
unfolding *liftemb-prod-def liftprj-prod-def liftdefl-prod-def*
by (*simp add*: *cast-prod-liftdefl cast-liftdefl cfcomp1 sprod-map-map*)
qed

end

instantiation *prod* :: (*domain*, *domain*) *domain*
begin

definition
emb = *prod-emb oo prod-map·emb·emb*

definition
prj = *prod-map·prj·prj oo prod-prj*

definition
defl (*t*::(*'a* × *'b*) *itself*) = *prod-defl·DEFL*(*'a*)·*DEFL*(*'b*)

instance proof
show *1*: *ep-pair emb* (*prj* :: *udom* → *'a* × *'b*)
unfolding *emb-prod-def prj-prod-def*
by (*intro ep-pair-comp ep-pair-prod ep-pair-prod-map ep-pair-emb-prj*)
show *2*: *cast·DEFL*(*'a* × *'b*) = *emb oo* (*prj* :: *udom* → *'a* × *'b*)
unfolding *emb-prod-def prj-prod-def defl-prod-def cast-prod-defl*
by (*simp add*: *cast-DEFL oo-def cfun-eq-iff prod-map-map*)
show *3*: *liftemb* = *u-map*·(*emb* :: *'a* × *'b* → *udom*)

unfolding *emb-prod-def liftemb-prod-def liftemb-eq*
unfolding *encode-prod-u-def decode-prod-u-def*
by (*rule cfun-eqI*, *case-tac x*, *simp*, *clarsimp*)
show *4*: *liftprj* = *u-map*·(*prj* :: *udom* → *'a* × *'b*)
unfolding *prj-prod-def liftprj-prod-def liftprj-eq*
unfolding *encode-prod-u-def decode-prod-u-def*
apply (*rule cfun-eqI*, *case-tac x*, *simp*)
apply (*rename-tac y*, *case-tac prod-prj*·*y*, *simp*)
done
show *5*: *LIFTDEFL*(*'a* × *'b*) = *liftdefl-of*·*DEFL*(*'a* × *'b*)
by (*rule cast-eq-imp-eq*)
(*simp add*: *cast-liftdefl cast-liftdefl-of cast-DEFL 2 3 4 u-map-oo*)
qed

end

lemma *DEFL-prod*:
DEFL(*'a*::*domain* × *'b*::*domain*) = *prod-defl*·*DEFL*(*'a*)·*DEFL*(*'b*)
by (*rule defl-prod-def*)

lemma *LIFTDEFL-prod*:
LIFTDEFL(*'a*::*predomain* × *'b*::*predomain*) =
prod-liftdefl·*LIFTDEFL*(*'a*)·*LIFTDEFL*(*'b*)
by (*rule liftdefl-prod-def*)

23.5.7 Unit type

instantiation *unit* :: *domain*
begin

definition
emb = (⊥ :: *unit* → *udom*)

definition
prj = (⊥ :: *udom* → *unit*)

definition
defl (*t*::*unit itself*) = ⊥

definition
(*liftemb* :: *unit u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → *unit u*) = *u-map*·*prj*

definition
liftdefl (*t*::*unit itself*) = *liftdefl-of*·*DEFL*(*unit*)

instance proof

```
  show ep-pair emb (prj :: udom → unit)
    unfolding emb-unit-def prj-unit-def
    by (simp add: ep-pair.intro)
  show cast·DEFL(unit) = emb oo (prj :: udom → unit)
    unfolding emb-unit-def prj-unit-def defl-unit-def by simp
qed (fact liftemb-unit-def liftprj-unit-def liftdefl-unit-def)+

end
```

23.5.8 Discrete cpo

```
instantiation discr :: (countable) predomain
begin

definition
  (liftemb :: 'a discr u → udom u) = strictify·up oo udom-emb discr-approx

definition
  (liftprj :: udom u → 'a discr u) = udom-prj discr-approx oo fup·ID

definition
  liftdefl (t::'a discr itself) =
    (⨆ i. defl-principal (Abs-fin-defl (liftemb oo discr-approx i oo (liftprj::udom u
→ 'a discr u))))

instance proof
  show 1: ep-pair liftemb (liftprj :: udom u → 'a discr u)
    unfolding liftemb-discr-def liftprj-discr-def
    apply (intro ep-pair-comp ep-pair-udom [OF discr-approx])
    apply (rule ep-pair.intro)
    apply (simp add: strictify-conv-if)
    apply (case-tac y, simp, simp add: strictify-conv-if)
    done
  show cast·LIFTDEFL('a discr) = liftemb oo (liftprj :: udom u → 'a discr u)
    unfolding liftdefl-discr-def
    apply (subst contlub-cfun-arg)
    apply (rule chainI)
    apply (rule defl.principal-mono)
    apply (simp add: below-fin-defl-def)
    apply (simp add: Abs-fin-defl-inverse
        ep-pair.finite-deflation-e-d-p [OF 1]
        approx-chain.finite-deflation-approx [OF discr-approx])
    apply (intro monofun-cfun below-refl)
    apply (rule chainE)
    apply (rule chain-discr-approx)
    apply (subst cast-defl-principal)
    apply (simp add: Abs-fin-defl-inverse
        ep-pair.finite-deflation-e-d-p [OF 1]
        approx-chain.finite-deflation-approx [OF discr-approx])
```

```
  apply (simp add: lub-distribs)
  done
qed

end
```

23.5.9 Strict sum

```
instantiation ssum :: (domain, domain) domain
begin

definition
  emb = ssum-emb oo ssum-map·emb·emb

definition
  prj = ssum-map·prj·prj oo ssum-prj

definition
  defl (t::('a ⊕ 'b) itself) = ssum-defl·DEFL('a)·DEFL('b)

definition
  (liftemb :: ('a ⊕ 'b) u → udom u) = u-map·emb

definition
  (liftprj :: udom u → ('a ⊕ 'b) u) = u-map·prj

definition
  liftdefl (t::('a ⊕ 'b) itself) = liftdefl-of·DEFL('a ⊕ 'b)

instance proof
  show ep-pair emb (prj :: udom → 'a ⊕ 'b)
    unfolding emb-ssum-def prj-ssum-def
    by (intro ep-pair-comp ep-pair-ssum ep-pair-ssum-map ep-pair-emb-prj)
  show cast·DEFL('a ⊕ 'b) = emb oo (prj :: udom → 'a ⊕ 'b)
    unfolding emb-ssum-def prj-ssum-def defl-ssum-def cast-ssum-defl
    by (simp add: cast-DEFL oo-def cfun-eq-iff ssum-map-map)
qed (fact liftemb-ssum-def liftprj-ssum-def liftdefl-ssum-def)+

end

lemma DEFL-ssum:
  DEFL('a::domain ⊕ 'b::domain) = ssum-defl·DEFL('a)·DEFL('b)
by (rule defl-ssum-def)
```

23.5.10 Lifted HOL type

```
instantiation lift :: (countable) domain
begin

definition
```

emb = *emb oo* (Λ *x*. *Rep-lift x*)

definition
prj = (Λ *y*. *Abs-lift y*) *oo prj*

definition
defl (*t*::'*a lift itself*) = *DEFL*('*a discr u*)

definition
(*liftemb* :: '*a lift u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a lift u*) = *u-map*·*prj*

definition
liftdefl (*t*::'*a lift itself*) = *liftdefl-of*·*DEFL*('*a lift*)

instance proof
 note [*simp*] = *cont-Rep-lift cont-Abs-lift Rep-lift-inverse Abs-lift-inverse*
 have *ep-pair* (Λ(*x*::'*a lift*). *Rep-lift x*) (Λ *y*. *Abs-lift y*)
  by (*simp add*: *ep-pair-def*)
 thus *ep-pair emb* (*prj* :: *udom* → '*a lift*)
  unfolding *emb-lift-def prj-lift-def*
  using *ep-pair-emb-prj* **by** (*rule ep-pair-comp*)
 show *cast*·*DEFL*('*a lift*) = *emb oo* (*prj* :: *udom* → '*a lift*)
  unfolding *emb-lift-def prj-lift-def defl-lift-def cast-DEFL*
  by (*simp add*: *cfcomp1*)
qed (*fact liftemb-lift-def liftprj-lift-def liftdefl-lift-def*)+

end

end

24 The unit domain

theory *One*
 imports *Lift*
begin

type-synonym *one* = *unit lift*

translations
 (*type*) *one* ↽ (*type*) *unit lift*

definition *ONE* :: *one*
 where *ONE* ≡ *Def* ()

Exhaustion and Elimination for type *one*

lemma *Exh-one*: $t = \perp \lor t = ONE$

by (*induct t*) (*simp-all add*: *ONE-def*)

lemma *oneE* [*case-names bottom ONE*]: $[\![p = \bot \Longrightarrow Q;\ p = ONE \Longrightarrow Q]\!] \Longrightarrow Q$
by (*induct p*) (*simp-all add*: *ONE-def*)

lemma *one-induct* [*case-names bottom ONE*]: $P\ \bot \Longrightarrow P\ ONE \Longrightarrow P\ x$
by (*cases x rule*: *oneE*) *simp-all*

lemma *dist-below-one* [*simp*]: $ONE \not\sqsubseteq \bot$
by (*simp add*: *ONE-def*)

lemma *below-ONE* [*simp*]: $x \sqsubseteq ONE$
by (*induct x rule*: *one-induct*) *simp-all*

lemma *ONE-below-iff* [*simp*]: $ONE \sqsubseteq x \longleftrightarrow x = ONE$
by (*induct x rule*: *one-induct*) *simp-all*

lemma *ONE-defined* [*simp*]: $ONE \neq \bot$
by (*simp add*: *ONE-def*)

lemma *one-neq-iffs* [*simp*]:
$x \neq ONE \longleftrightarrow x = \bot$
$ONE \neq x \longleftrightarrow x = \bot$
$x \neq \bot \longleftrightarrow x = ONE$
$\bot \neq x \longleftrightarrow x = ONE$
by (*induct x rule*: *one-induct*) *simp-all*

lemma *compact-ONE*: *compact ONE*
by (*rule compact-chfin*)

Case analysis function for type *one*

definition *one-case* :: $'a::pcpo \rightarrow one \rightarrow 'a$
where *one-case* = $(\Lambda\ a\ x.\ seq\cdot x\cdot a)$

translations
$case\ x\ of\ XCONST\ ONE \Rightarrow t \rightleftharpoons CONST\ one\text{-}case\cdot t\cdot x$
$case\ x\ of\ XCONST\ ONE :: 'a \Rightarrow t \rightharpoonup CONST\ one\text{-}case\cdot t\cdot x$
$\Lambda\ (XCONST\ ONE).\ t \rightleftharpoons CONST\ one\text{-}case\cdot t$

lemma *one-case1* [*simp*]: $(case\ \bot\ of\ ONE \Rightarrow t) = \bot$
by (*simp add*: *one-case-def*)

lemma *one-case2* [*simp*]: $(case\ ONE\ of\ ONE \Rightarrow t) = t$
by (*simp add*: *one-case-def*)

lemma *one-case3* [*simp*]: $(case\ x\ of\ ONE \Rightarrow ONE) = x$
by (*induct x rule*: *one-induct*) *simp-all*

end

theory *Fixrec*
imports *Cprod Sprod Ssum Up One Tr Cfun*
keywords *fixrec* :: *thy-defn*
begin

25 Fixed point operator and admissibility

25.1 Iteration

primrec *iterate* :: *nat* ⇒ (*'a* → *'a*) → (*'a* → *'a*)
  where
    iterate 0 = (Λ *F x. x*)
  | *iterate* (*Suc n*) = (Λ *F x. F*·(*iterate n*·*F*·*x*))

Derive inductive properties of iterate from primitive recursion

lemma *iterate-0* [*simp*]: *iterate 0*·*F*·*x* = *x*
  by *simp*

lemma *iterate-Suc* [*simp*]: *iterate* (*Suc n*)·*F*·*x* = *F*·(*iterate n*·*F*·*x*)
  by *simp*

declare *iterate.simps* [*simp del*]

lemma *iterate-Suc2*: *iterate* (*Suc n*)·*F*·*x* = *iterate n*·*F*·(*F*·*x*)
  by (*induct n*) *simp-all*

lemma *iterate-iterate*: *iterate m*·*F*·(*iterate n*·*F*·*x*) = *iterate* (*m* + *n*)·*F*·*x*
  by (*induct m*) *simp-all*

The sequence of function iterations is a chain.

lemma *chain-iterate* [*simp*]: *chain* (λ*i. iterate i*·*F*·⊥)
  by (*rule chainI*, *unfold iterate-Suc2*, *rule monofun-cfun-arg*, *rule minimal*)

25.2 Least fixed point operator

definition *fix* :: (*'a*::*pcpo* → *'a*) → *'a*
  where *fix* = (Λ *F*. ⨆*i. iterate i*·*F*·⊥)

Binder syntax for *fix*

abbreviation *fix-syn* :: (*'a*::*pcpo* ⇒ *'a*) ⇒ *'a* (**binder** ‹μ › *10*)
  where *fix-syn* (λ*x. f x*) ≡ *fix*·(Λ *x. f x*)

notation (*ASCII*)
  fix-syn (**binder** ‹*FIX* › *10*)

Properties of *fix*

direct connection between *fix* and iteration

lemma *fix-def2*: $fix \cdot F = (\bigsqcup i.\ iterate\ i \cdot F \cdot \bot)$
  by (*simp add*: *fix-def*)

lemma *iterate-below-fix*: $iterate\ n \cdot f \cdot \bot \sqsubseteq fix \cdot f$
  unfolding *fix-def2*
  using *chain-iterate* **by** (*rule is-ub-thelub*)

Kleene's fixed point theorems for continuous functions in pointed omega cpo's

lemma *fix-eq*: $fix \cdot F = F \cdot (fix \cdot F)$
  apply (*simp add*: *fix-def2*)
  apply (*subst lub-range-shift* [*of - 1*, *symmetric*])
   apply (*rule chain-iterate*)
  apply (*subst contlub-cfun-arg*)
   apply (*rule chain-iterate*)
  apply *simp*
  done

lemma *fix-least-below*: $F \cdot x \sqsubseteq x \Longrightarrow fix \cdot F \sqsubseteq x$
  apply (*simp add*: *fix-def2*)
  apply (*rule lub-below*)
   apply (*rule chain-iterate*)
  apply (*induct-tac i*)
   apply *simp*
  apply *simp*
  apply (*erule rev-below-trans*)
  apply (*erule monofun-cfun-arg*)
  done

lemma *fix-least*: $F \cdot x = x \Longrightarrow fix \cdot F \sqsubseteq x$
  by (*rule fix-least-below*) *simp*

lemma *fix-eqI*:
  assumes *fixed*: $F \cdot x = x$
    and *least*: $\bigwedge z.\ F \cdot z = z \Longrightarrow x \sqsubseteq z$
  shows $fix \cdot F = x$
  apply (*rule below-antisym*)
   apply (*rule fix-least* [*OF fixed*])
  apply (*rule least* [*OF fix-eq* [*symmetric*]])
  done

lemma *fix-eq2*: $f \equiv fix \cdot F \Longrightarrow f = F \cdot f$
  by (*simp add*: *fix-eq* [*symmetric*])

lemma *fix-eq3*: $f \equiv fix \cdot F \Longrightarrow f \cdot x = F \cdot f \cdot x$
  by (*erule fix-eq2* [*THEN cfun-fun-cong*])

lemma *fix-eq4*: $f = fix \cdot F \Longrightarrow f = F \cdot f$

by (*erule ssubst*) (*rule fix-eq*)

lemma *fix-eq5*: $f = fix{\cdot}F \Longrightarrow f{\cdot}x = F{\cdot}f{\cdot}x$
by (*erule fix-eq4* [*THEN cfun-fun-cong*])

strictness of *fix*

lemma *fix-bottom-iff*: $fix{\cdot}F = \bot \longleftrightarrow F{\cdot}\bot = \bot$
apply (*rule iffI*)
apply (*erule subst*)
apply (*rule fix-eq* [*symmetric*])
apply (*erule fix-least* [*THEN bottomI*])
done

lemma *fix-strict*: $F{\cdot}\bot = \bot \Longrightarrow fix{\cdot}F = \bot$
by (*simp add*: *fix-bottom-iff*)

lemma *fix-defined*: $F{\cdot}\bot \neq \bot \Longrightarrow fix{\cdot}F \neq \bot$
by (*simp add*: *fix-bottom-iff*)

fix applied to identity and constant functions

lemma *fix-id*: $(\mu\ x.\ x) = \bot$
by (*simp add*: *fix-strict*)

lemma *fix-const*: $(\mu\ x.\ c) = c$
by (*subst fix-eq*) *simp*

25.3 Fixed point induction

lemma *fix-ind*: $adm\ P \Longrightarrow P\ \bot \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow P\ (F{\cdot}x)) \Longrightarrow P\ (fix{\cdot}F)$
unfolding *fix-def2*
apply (*erule admD*)
apply (*rule chain-iterate*)
apply (*rule nat-induct*, *simp-all*)
done

lemma *cont-fix-ind*: $cont\ F \Longrightarrow adm\ P \Longrightarrow P\ \bot \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow P\ (F\ x)) \Longrightarrow P\ (fix{\cdot}(Abs\text{-}cfun\ F))$
by (*simp add*: *fix-ind*)

lemma *def-fix-ind*: $[\![f \equiv fix{\cdot}F;\ adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F{\cdot}x)]\!] \Longrightarrow P\ f$
by (*simp add*: *fix-ind*)

lemma *fix-ind2*:
assumes *adm*: *adm P*
assumes *0*: $P\ \bot$ **and** *1*: $P\ (F{\cdot}\bot)$
assumes *step*: $\bigwedge x.\ [\![P\ x;\ P\ (F{\cdot}x)]\!] \Longrightarrow P\ (F{\cdot}(F{\cdot}x))$
shows $P\ (fix{\cdot}F)$
unfolding *fix-def2*
apply (*rule admD* [*OF adm chain-iterate*])

```
  apply (rule nat-less-induct)
  apply (case-tac n)
   apply (simp add: 0)
  apply (case-tac nat)
   apply (simp add: 1)
  apply (frule-tac x=nat in spec)
  apply (simp add: step)
  done

lemma parallel-fix-ind:
  assumes adm: adm (λx. P (fst x) (snd x))
  assumes base: P ⊥ ⊥
  assumes step: ⋀x y. P x y ⟹ P (F·x) (G·y)
  shows P (fix·F) (fix·G)
proof –
  from adm have adm': adm (case-prod P)
    unfolding split-def .
  have P (iterate i·F·⊥) (iterate i·G·⊥) for i
    by (induct i) (simp add: base, simp add: step)
  then have ⋀i. case-prod P (iterate i·F·⊥, iterate i·G·⊥)
    by simp
  then have case-prod P (⨆i. (iterate i·F·⊥, iterate i·G·⊥))
    by – (rule admD [OF adm'], simp, assumption)
  then have case-prod P (⨆i. iterate i·F·⊥, ⨆i. iterate i·G·⊥)
    by (simp add: lub-Pair)
  then have P (⨆i. iterate i·F·⊥) (⨆i. iterate i·G·⊥)
    by simp
  then show P (fix·F) (fix·G)
    by (simp add: fix-def2)
qed

lemma cont-parallel-fix-ind:
  assumes cont F and cont G
  assumes adm (λx. P (fst x) (snd x))
  assumes P ⊥ ⊥
  assumes ⋀x y. P x y ⟹ P (F x) (G y)
  shows P (fix·(Abs-cfun F)) (fix·(Abs-cfun G))
  by (rule parallel-fix-ind) (simp-all add: assms)
```

25.4 Fixed-points on product types

Bekic's Theorem: Simultaneous fixed points over pairs can be written in terms of separate fixed points.

```
lemma fix-cprod:
  fixes F :: 'a::pcpo × 'b::pcpo → 'a × 'b
  shows
    fix·F =
     (μ x. fst (F·(x, μ y. snd (F·(x, y)))),
      μ y. snd (F·(μ x. fst (F·(x, μ y. snd (F·(x, y)))), y)))
```

(**is** $fix \cdot F = (?x,\ ?y)$)
proof (*rule fix-eqI*)
have ∗: $fst\ (F \cdot (?x,\ ?y)) = ?x$
by (*rule trans* [*symmetric*, *OF fix-eq*], *simp*)
have $snd\ (F \cdot (?x,\ ?y)) = ?y$
by (*rule trans* [*symmetric*, *OF fix-eq*], *simp*)
with ∗ **show** $F \cdot (?x,\ ?y) = (?x,\ ?y)$
by (*simp add*: *prod-eq-iff*)
next
fix z
assume *F-z*: $F \cdot z = z$
obtain $x\ y$ **where** z: $z = (x,\ y)$ **by** (*rule prod.exhaust*)
from *F-z z* **have** *F-x*: $fst\ (F \cdot (x,\ y)) = x$ **by** *simp*
from *F-z z* **have** *F-y*: $snd\ (F \cdot (x,\ y)) = y$ **by** *simp*
let $?y1 = \mu\ y.\ snd\ (F \cdot (x,\ y))$
have $?y1 \sqsubseteq y$
by (*rule fix-least*) (*simp add*: *F-y*)
then have $fst\ (F \cdot (x,\ ?y1)) \sqsubseteq fst\ (F \cdot (x,\ y))$
by (*simp add*: *fst-monofun monofun-cfun*)
with *F-x* **have** $fst\ (F \cdot (x,\ ?y1)) \sqsubseteq x$
by *simp*
then have ∗: $?x \sqsubseteq x$
by (*simp add*: *fix-least-below*)
then have $snd\ (F \cdot (?x,\ y)) \sqsubseteq snd\ (F \cdot (x,\ y))$
by (*simp add*: *snd-monofun monofun-cfun*)
with *F-y* **have** $snd\ (F \cdot (?x,\ y)) \sqsubseteq y$
by *simp*
then have $?y \sqsubseteq y$
by (*simp add*: *fix-least-below*)
with z ∗ **show** $(?x,\ ?y) \sqsubseteq z$
by *simp*
qed

26 Package for defining recursive functions in HOLCF

26.1 Pattern-match monad

pcpodef $'a\ match = UNIV::(one\ {+}{+}\ 'a\ u)\ set$
by *simp-all*

definition
fail :: $'a\ match$ **where**
$fail = Abs\text{-}match\ (sinl \cdot ONE)$

definition
succeed :: $'a \to 'a\ match$ **where**
$succeed = (\Lambda\ x.\ Abs\text{-}match\ (sinr \cdot (up \cdot x)))$

lemma *matchE* [*case-names bottom fail succeed*, *cases type*: *match*]:

$[\![p = \bot \Longrightarrow Q;\ p = fail \Longrightarrow Q;\ \bigwedge x.\ p = succeed \cdot x \Longrightarrow Q]\!] \Longrightarrow Q$
unfolding *fail-def succeed-def*
apply (*cases p*, *rename-tac r*)
apply (*rule-tac p=r* **in** *ssumE*, *simp add*: *Abs-match-strict*)
apply (*rule-tac p=x* **in** *oneE*, *simp*, *simp*)
apply (*rule-tac p=y* **in** *upE*, *simp*, *simp add*: *cont-Abs-match*)
done

lemma *succeed-defined* [*simp*]: $succeed \cdot x \neq \bot$
by (*simp add*: *succeed-def cont-Abs-match Abs-match-bottom-iff*)

lemma *fail-defined* [*simp*]: $fail \neq \bot$
by (*simp add*: *fail-def Abs-match-bottom-iff*)

lemma *succeed-eq* [*simp*]: $(succeed \cdot x = succeed \cdot y) = (x = y)$
by (*simp add*: *succeed-def cont-Abs-match Abs-match-inject*)

lemma *succeed-neq-fail* [*simp*]:
$succeed \cdot x \neq fail$ $fail \neq succeed \cdot x$
by (*simp-all add*: *succeed-def fail-def cont-Abs-match Abs-match-inject*)

26.1.1 Run operator

definition
run :: *'a match* → *'a::pcpo* **where**
$run = (\Lambda\ m.\ sscase \cdot \bot \cdot (fup \cdot ID) \cdot (Rep\text{-}match\ m))$

rewrite rules for run

lemma *run-strict* [*simp*]: $run \cdot \bot = \bot$
unfolding *run-def*
by (*simp add*: *cont-Rep-match Rep-match-strict*)

lemma *run-fail* [*simp*]: $run \cdot fail = \bot$
unfolding *run-def fail-def*
by (*simp add*: *cont-Rep-match Abs-match-inverse*)

lemma *run-succeed* [*simp*]: $run \cdot (succeed \cdot x) = x$
unfolding *run-def succeed-def*
by (*simp add*: *cont-Rep-match cont-Abs-match Abs-match-inverse*)

26.1.2 Monad plus operator

definition
mplus :: *'a match* → *'a match* → *'a match* **where**
$mplus = (\Lambda\ m1\ m2.\ sscase \cdot (\Lambda\ \text{-}.\ m2) \cdot (\Lambda\ \text{-}.\ m1) \cdot (Rep\text{-}match\ m1))$

abbreviation
mplus-syn :: [*'a match*, *'a match*] ⇒ *'a match* (**infixr** ‹+++› *65*) **where**
$m1\ {+}{+}{+}\ m2 == mplus \cdot m1 \cdot m2$

rewrite rules for mplus

lemma *mplus-strict* [*simp*]: $\bot$ +++ *m* = $\bot$
unfolding *mplus-def*
by (*simp add*: *cont-Rep-match Rep-match-strict*)

lemma *mplus-fail* [*simp*]: *fail* +++ *m* = *m*
unfolding *mplus-def fail-def*
by (*simp add*: *cont-Rep-match Abs-match-inverse*)

lemma *mplus-succeed* [*simp*]: *succeed·x* +++ *m* = *succeed·x*
unfolding *mplus-def succeed-def*
by (*simp add*: *cont-Rep-match cont-Abs-match Abs-match-inverse*)

lemma *mplus-fail2* [*simp*]: *m* +++ *fail* = *m*
by (*cases m*, *simp-all*)

lemma *mplus-assoc*: (*x* +++ *y*) +++ *z* = *x* +++ (*y* +++ *z*)
by (*cases x*, *simp-all*)

26.2 Match functions for built-in types

definition
  match-bottom :: *'a*::*pcpo* → *'c match* → *'c match*
where
  match-bottom = (Λ *x k*. *seq·x·fail*)

definition
  match-Pair :: *'a* × *'b* → (*'a* → *'b* → *'c match*) → *'c match*
where
  match-Pair = (Λ *x k*. *csplit·k·x*)

definition
  match-spair :: *'a*::*pcpo* ⊗ *'b*::*pcpo* → (*'a* → *'b* → *'c match*) → *'c*::*pcpo match*
where
  match-spair = (Λ *x k*. *ssplit·k·x*)

definition
  match-sinl :: *'a*::*pcpo* ⊕ *'b*::*pcpo* → (*'a* → *'c*::*pcpo match*) → *'c match*
where
  match-sinl = (Λ *x k*. *sscase·k·*(Λ *b*. *fail*)*·x*)

definition
  match-sinr :: *'a*::*pcpo* ⊕ *'b*::*pcpo* → (*'b* → *'c*::*pcpo match*) → *'c match*
where
  match-sinr = (Λ *x k*. *sscase·*(Λ *a*. *fail*)*·k·x*)

definition
  match-up :: *'a u* → (*'a* → *'c*::*pcpo match*) → *'c match*
where

match-up = (Λ *x k*. *fup*·*k*·*x*)

definition
match-ONE :: *one* → ′*c*::*pcpo match* → ′*c match*
where
match-ONE = (Λ *ONE k*. *k*)

definition
match-TT :: *tr* → ′*c*::*pcpo match* → ′*c match*
where
match-TT = (Λ *x k*. *If x then k else fail*)

definition
match-FF :: *tr* → ′*c*::*pcpo match* → ′*c match*
where
match-FF = (Λ *x k*. *If x then fail else k*)

lemma *match-bottom-simps* [*simp*]:
match-bottom·*x*·*k* = (*if x* = ⊥ *then* ⊥ *else fail*)
by (*simp add*: *match-bottom-def*)

lemma *match-Pair-simps* [*simp*]:
match-Pair·(*x*, *y*)·*k* = *k*·*x*·*y*
by (*simp-all add*: *match-Pair-def*)

lemma *match-spair-simps* [*simp*]:
⟦*x* ≠ ⊥; *y* ≠ ⊥⟧ ⟹ *match-spair*·(:*x*, *y*:)·*k* = *k*·*x*·*y*
match-spair·⊥·*k* = ⊥
by (*simp-all add*: *match-spair-def*)

lemma *match-sinl-simps* [*simp*]:
x ≠ ⊥ ⟹ *match-sinl*·(*sinl*·*x*)·*k* = *k*·*x*
y ≠ ⊥ ⟹ *match-sinl*·(*sinr*·*y*)·*k* = *fail*
match-sinl·⊥·*k* = ⊥
by (*simp-all add*: *match-sinl-def*)

lemma *match-sinr-simps* [*simp*]:
x ≠ ⊥ ⟹ *match-sinr*·(*sinl*·*x*)·*k* = *fail*
y ≠ ⊥ ⟹ *match-sinr*·(*sinr*·*y*)·*k* = *k*·*y*
match-sinr·⊥·*k* = ⊥
by (*simp-all add*: *match-sinr-def*)

lemma *match-up-simps* [*simp*]:
match-up·(*up*·*x*)·*k* = *k*·*x*
match-up·⊥·*k* = ⊥
by (*simp-all add*: *match-up-def*)

lemma *match-ONE-simps* [*simp*]:
match-ONE·*ONE*·*k* = *k*

$match\text{-}ONE \cdot \bot \cdot k = \bot$
by (*simp-all add*: *match-ONE-def*)

lemma *match-TT-simps* [*simp*]:
$match\text{-}TT \cdot TT \cdot k = k$
$match\text{-}TT \cdot FF \cdot k = fail$
$match\text{-}TT \cdot \bot \cdot k = \bot$
by (*simp-all add*: *match-TT-def*)

lemma *match-FF-simps* [*simp*]:
$match\text{-}FF \cdot FF \cdot k = k$
$match\text{-}FF \cdot TT \cdot k = fail$
$match\text{-}FF \cdot \bot \cdot k = \bot$
by (*simp-all add*: *match-FF-def*)

26.3 Mutual recursion

The following rules are used to prove unfolding theorems from fixed-point definitions of mutually recursive functions.

lemma *Pair-equalI*: $[\![x \equiv fst\ p;\ y \equiv snd\ p]\!] \Longrightarrow (x,\ y) \equiv p$
by *simp*

lemma *Pair-eqD1*: $(x,\ y) = (x',\ y') \Longrightarrow x = x'$
by *simp*

lemma *Pair-eqD2*: $(x,\ y) = (x',\ y') \Longrightarrow y = y'$
by *simp*

lemma *def-cont-fix-eq*:
$[\![f \equiv fix \cdot (Abs\text{-}cfun\ F);\ cont\ F]\!] \Longrightarrow f = F\ f$
by (*simp*, *subst fix-eq*, *simp*)

lemma *def-cont-fix-ind*:
$[\![f \equiv fix \cdot (Abs\text{-}cfun\ F);\ cont\ F;\ adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F\ x)]\!] \Longrightarrow P\ f$
by (*simp add*: *fix-ind*)

lemma for proving rewrite rules

lemma *ssubst-lhs*: $[\![t = s;\ P\ s = Q]\!] \Longrightarrow P\ t = Q$
by *simp*

26.4 Initializing the fixrec package

ML-file ‹*Tools/holcf-library.ML*›
ML-file ‹*Tools/fixrec.ML*›

method-setup *fixrec-simp* = ‹
Scan.succeed (*SIMPLE-METHOD′ o Fixrec.fixrec-simp-tac*)
› *pattern prover for fixrec constants*

setup ‹
  Fixrec.add-matchers
    [(***const-name*** ‹*up*›, ***const-name*** ‹*match-up*›),
      (***const-name*** ‹*sinl*›, ***const-name*** ‹*match-sinl*›),
      (***const-name*** ‹*sinr*›, ***const-name*** ‹*match-sinr*›),
      (***const-name*** ‹*spair*›, ***const-name*** ‹*match-spair*›),
      (***const-name*** ‹*Pair*›, ***const-name*** ‹*match-Pair*›),
      (***const-name*** ‹*ONE*›, ***const-name*** ‹*match-ONE*›),
      (***const-name*** ‹*TT*›, ***const-name*** ‹*match-TT*›),
      (***const-name*** ‹*FF*›, ***const-name*** ‹*match-FF*›),
      (***const-name*** ‹*bottom*›, ***const-name*** ‹*match-bottom*›)]
›

hide-const (**open**) *succeed fail run*

end

27 Domain package

theory *Domain*
imports *Representable Map-Functions Fixrec*
keywords
  lazy unsafe **and**
  domaindef domain :: *thy-defn* **and**
  domain-isomorphism :: *thy-decl*
begin

27.1 Continuous isomorphisms

A locale for continuous isomorphisms

locale *iso* =
  fixes *abs* :: *'a::pcpo → 'b::pcpo*
  fixes *rep* :: *'b → 'a*
  assumes *abs-iso* [*simp*]: *rep·(abs·x) = x*
  assumes *rep-iso* [*simp*]: *abs·(rep·y) = y*
begin

lemma *swap*: *iso rep abs*
  by (*rule iso.intro* [*OF rep-iso abs-iso*])

lemma *abs-below*: $(abs{\cdot}x \sqsubseteq abs{\cdot}y) = (x \sqsubseteq y)$
proof
  assume $abs{\cdot}x \sqsubseteq abs{\cdot}y$
  then have $rep{\cdot}(abs{\cdot}x) \sqsubseteq rep{\cdot}(abs{\cdot}y)$ **by** (*rule monofun-cfun-arg*)
  then show $x \sqsubseteq y$ **by** *simp*
next
  assume $x \sqsubseteq y$
  then show $abs{\cdot}x \sqsubseteq abs{\cdot}y$ **by** (*rule monofun-cfun-arg*)

qed

lemma *rep-below*: $(rep{\cdot}x \sqsubseteq rep{\cdot}y) = (x \sqsubseteq y)$
  by (*rule iso.abs-below* [*OF swap*])

lemma *abs-eq*: $(abs{\cdot}x = abs{\cdot}y) = (x = y)$
  by (*simp add*: *po-eq-conv abs-below*)

lemma *rep-eq*: $(rep{\cdot}x = rep{\cdot}y) = (x = y)$
  by (*rule iso.abs-eq* [*OF swap*])

lemma *abs-strict*: $abs{\cdot}\bot = \bot$
proof −
  have $\bot \sqsubseteq rep{\cdot}\bot$ **..**
  then have $abs{\cdot}\bot \sqsubseteq abs{\cdot}(rep{\cdot}\bot)$ **by** (*rule monofun-cfun-arg*)
  then have $abs{\cdot}\bot \sqsubseteq \bot$ **by** *simp*
  then show *?thesis* **by** (*rule bottomI*)
qed

lemma *rep-strict*: $rep{\cdot}\bot = \bot$
  by (*rule iso.abs-strict* [*OF swap*])

lemma *abs-defin'*: $abs{\cdot}x = \bot \Longrightarrow x = \bot$
proof −
  have $x = rep{\cdot}(abs{\cdot}x)$ **by** *simp*
  also assume $abs{\cdot}x = \bot$
  also note *rep-strict*
  finally show $x = \bot$ **.**
qed

lemma *rep-defin'*: $rep{\cdot}z = \bot \Longrightarrow z = \bot$
  by (*rule iso.abs-defin'* [*OF swap*])

lemma *abs-defined*: $z \neq \bot \Longrightarrow abs{\cdot}z \neq \bot$
  by (*erule contrapos-nn*, *erule abs-defin'*)

lemma *rep-defined*: $z \neq \bot \Longrightarrow rep{\cdot}z \neq \bot$
  by (*rule iso.abs-defined* [*OF iso.swap*]) (*rule iso-axioms*)

lemma *abs-bottom-iff*: $(abs{\cdot}x = \bot) = (x = \bot)$
  by (*auto elim*: *abs-defin'* *intro*: *abs-strict*)

lemma *rep-bottom-iff*: $(rep{\cdot}x = \bot) = (x = \bot)$
  by (*rule iso.abs-bottom-iff* [*OF iso.swap*]) (*rule iso-axioms*)

lemma *casedist-rule*: $rep{\cdot}x = \bot \vee P \Longrightarrow x = \bot \vee P$
  by (*simp add*: *rep-bottom-iff*)

lemma *compact-abs-rev*: *compact* $(abs{\cdot}x) \Longrightarrow$ *compact* x

proof (*unfold compact-def*)
  assume *adm* ($\lambda y.$ *abs*$\cdot x \not\sqsubseteq y$)
  with *cont-Rep-cfun2*
  have *adm* ($\lambda y.$ *abs*$\cdot x \not\sqsubseteq$ *abs*$\cdot y$) **by** (*rule adm-subst*)
  then show *adm* ($\lambda y.\ x \not\sqsubseteq y$) **using** *abs-below* **by** *simp*
qed

lemma *compact-rep-rev*: *compact* (*rep*$\cdot x$) $\Longrightarrow$ *compact* x
  by (*rule iso.compact-abs-rev* [*OF iso.swap*]) (*rule iso-axioms*)

lemma *compact-abs*: *compact* $x \Longrightarrow$ *compact* (*abs*$\cdot x$)
  by (*rule compact-rep-rev*) *simp*

lemma *compact-rep*: *compact* $x \Longrightarrow$ *compact* (*rep*$\cdot x$)
  by (*rule iso.compact-abs* [*OF iso.swap*]) (*rule iso-axioms*)

lemma *iso-swap*: (x = *abs*$\cdot y$) = (*rep*$\cdot x$ = y)
proof
  assume x = *abs*$\cdot y$
  then have *rep*$\cdot x$ = *rep*$\cdot$(*abs*$\cdot y$) **by** *simp*
  then show *rep*$\cdot x$ = y **by** *simp*
next
  assume *rep*$\cdot x$ = y
  then have *abs*$\cdot$(*rep*$\cdot x$) = *abs*$\cdot y$ **by** *simp*
  then show x = *abs*$\cdot y$ **by** *simp*
qed

end

27.2 Proofs about take functions

This section contains lemmas that are used in a module that supports the domain isomorphism package; the module contains proofs related to take functions and the finiteness predicate.

lemma *deflation-abs-rep*:
  fixes *abs* **and** *rep* **and** d
  assumes *abs-iso*: $\bigwedge x.$ *rep*$\cdot$(*abs*$\cdot x$) = x
  assumes *rep-iso*: $\bigwedge y.$ *abs*$\cdot$(*rep*$\cdot y$) = y
  shows *deflation* $d \Longrightarrow$ *deflation* (*abs oo d oo rep*)
by (*rule ep-pair.deflation-e-d-p*) (*simp add*: *ep-pair.intro assms*)

lemma *deflation-chain-min*:
  assumes *chain*: *chain* d
  assumes *defl*: $\bigwedge n.$ *deflation* ($d\ n$)
  shows $d\ m\cdot(d\ n\cdot x) = d$ (*min* $m\ n$)$\cdot x$
proof (*rule linorder-le-cases*)
  assume $m \leq n$
  with *chain* **have** $d\ m \sqsubseteq d\ n$ **by** (*rule chain-mono*)
  then have $d\ m\cdot(d\ n\cdot x) = d\ m\cdot x$

by (*rule deflation-below-comp1* [*OF defl defl*])
moreover from ‹$m \le n$› **have** $min\ m\ n = m$ **by** *simp*
ultimately show *?thesis* **by** *simp*
next
assume $n \le m$
with *chain* **have** $d\ n \sqsubseteq d\ m$ **by** (*rule chain-mono*)
then have $d\ m \cdot (d\ n \cdot x) = d\ n \cdot x$
by (*rule deflation-below-comp2* [*OF defl defl*])
moreover from ‹$n \le m$› **have** $min\ m\ n = n$ **by** *simp*
ultimately show *?thesis* **by** *simp*
qed

lemma *lub-ID-take-lemma*:
assumes *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
assumes $\bigwedge n.\ t\ n \cdot x = t\ n \cdot y$ **shows** $x = y$
proof −
have $(\bigsqcup n.\ t\ n \cdot x) = (\bigsqcup n.\ t\ n \cdot y)$
using *assms*(*3*) **by** *simp*
then have $(\bigsqcup n.\ t\ n) \cdot x = (\bigsqcup n.\ t\ n) \cdot y$
using *assms*(*1*) **by** (*simp add*: *lub-distribs*)
then show $x = y$
using *assms*(*2*) **by** *simp*
qed

lemma *lub-ID-reach*:
assumes *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
shows $(\bigsqcup n.\ t\ n \cdot x) = x$
using *assms* **by** (*simp add*: *lub-distribs*)

lemma *lub-ID-take-induct*:
assumes *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
assumes *adm P* **and** $\bigwedge n.\ P\ (t\ n \cdot x)$ **shows** $P\ x$
proof −
from ‹*chain t*› **have** *chain* $(\lambda n.\ t\ n \cdot x)$ **by** *simp*
from ‹*adm P*› *this* ‹$\bigwedge n.\ P\ (t\ n \cdot x)$› **have** $P\ (\bigsqcup n.\ t\ n \cdot x)$ **by** (*rule admD*)
with ‹*chain t*› ‹$(\bigsqcup n.\ t\ n) = ID$› **show** $P\ x$ **by** (*simp add*: *lub-distribs*)
qed

27.3 Finiteness

Let a "decisive" function be a deflation that maps every input to either itself or bottom. Then if a domain's take functions are all decisive, then all values in the domain are finite.

definition
decisive :: $('a::pcpo \to 'a) \Rightarrow bool$
where
$decisive\ d \longleftrightarrow (\forall x.\ d \cdot x = x \lor d \cdot x = \bot)$

lemma *decisiveI*: $(\bigwedge x.\ d \cdot x = x \lor d \cdot x = \bot) \Longrightarrow decisive\ d$

```
  unfolding decisive-def by simp

lemma decisive-cases:
  assumes decisive d obtains d·x = x | d·x = ⊥
using assms unfolding decisive-def by auto

lemma decisive-bottom: decisive ⊥
  unfolding decisive-def by simp

lemma decisive-ID: decisive ID
  unfolding decisive-def by simp

lemma decisive-ssum-map:
  assumes f: decisive f
  assumes g: decisive g
  shows decisive (ssum-map·f·g)
  apply (rule decisiveI)
  subgoal for s
    apply (cases s, simp-all)
     apply (rule-tac x=x in decisive-cases [OF f], simp-all)
    apply (rule-tac x=y in decisive-cases [OF g], simp-all)
    done
  done

lemma decisive-sprod-map:
  assumes f: decisive f
  assumes g: decisive g
  shows decisive (sprod-map·f·g)
  apply (rule decisiveI)
  subgoal for s
    apply (cases s, simp)
    subgoal for x y
      apply (rule decisive-cases [OF f, where x = x], simp-all)
      apply (rule decisive-cases [OF g, where x = y], simp-all)
      done
    done
  done

lemma decisive-abs-rep:
  fixes abs rep
  assumes iso: iso abs rep
  assumes d: decisive d
  shows decisive (abs oo d oo rep)
  apply (rule decisiveI)
  subgoal for s
    apply (rule decisive-cases [OF d, where x=rep·s])
     apply (simp add: iso.rep-iso [OF iso])
    apply (simp add: iso.abs-strict [OF iso])
    done
```

done

lemma *lub-ID-finite*:
assumes *chain*: *chain d*
assumes *lub*: $(\bigsqcup n.\ d\ n) = ID$
assumes *decisive*: $\bigwedge n.\ decisive\ (d\ n)$
shows $\exists\, n.\ d\ n{\cdot}x = x$
proof −
have *1*: $chain\ (\lambda n.\ d\ n{\cdot}x)$ **using** *chain* **by** *simp*
have *2*: $(\bigsqcup n.\ d\ n{\cdot}x) = x$ **using** *chain lub* **by** (*rule lub-ID-reach*)
have $\forall\, n.\ d\ n{\cdot}x = x \lor d\ n{\cdot}x = \bot$
using *decisive* **unfolding** *decisive-def* **by** *simp*
hence $range\ (\lambda n.\ d\ n{\cdot}x) \subseteq \{x, \bot\}$
by *auto*
hence $finite\ (range\ (\lambda n.\ d\ n{\cdot}x))$
by (*rule finite-subset*, *simp*)
with *1* **have** $finite\text{-}chain\ (\lambda n.\ d\ n{\cdot}x)$
by (*rule finite-range-imp-finch*)
then have $\exists\, n.\ (\bigsqcup n.\ d\ n{\cdot}x) = d\ n{\cdot}x$
unfolding *finite-chain-def* **by** (*auto simp add*: *maxinch-is-thelub*)
with *2* **show** $\exists\, n.\ d\ n{\cdot}x = x$ **by** (*auto elim*: *sym*)
qed

lemma *lub-ID-finite-take-induct*:
assumes *chain d* **and** $(\bigsqcup n.\ d\ n) = ID$ **and** $\bigwedge n.\ decisive\ (d\ n)$
shows $(\bigwedge n.\ P\ (d\ n{\cdot}x)) \Longrightarrow P\ x$
using *lub-ID-finite* [*OF assms*] **by** *metis*

27.4 Proofs about constructor functions

Lemmas for proving nchotomy rule:

lemma *ex-one-bottom-iff*:
$(\exists\, x.\ P\ x \land x \neq \bot) = P\ ONE$
by *simp*

lemma *ex-up-bottom-iff*:
$(\exists\, x.\ P\ x \land x \neq \bot) = (\exists\, x.\ P\ (up{\cdot}x))$
by (*safe*, *case-tac x*, *auto*)

lemma *ex-sprod-bottom-iff*:
$(\exists\, y.\ P\ y \land y \neq \bot) =$
$(\exists\, x\ y.\ (P\ (:x,\ y:) \land x \neq \bot) \land y \neq \bot)$
by (*safe*, *case-tac y*, *auto*)

lemma *ex-sprod-up-bottom-iff*:
$(\exists\, y.\ P\ y \land y \neq \bot) =$
$(\exists\, x\ y.\ P\ (:up{\cdot}x,\ y:) \land y \neq \bot)$
by (*safe*, *case-tac y*, *simp*, *case-tac x*, *auto*)

lemma *ex-ssum-bottom-iff*:
$(\exists x.\ P\ x \wedge x \neq \bot) =$
$((\exists x.\ P\ (sinl \cdot x) \wedge x \neq \bot) \vee$
$(\exists x.\ P\ (sinr \cdot x) \wedge x \neq \bot))$
by (*safe*, *case-tac x*, *auto*)

lemma *exh-start*: $p = \bot \vee (\exists x.\ p = x \wedge x \neq \bot)$
by *auto*

lemmas *ex-bottom-iffs* =
ex-ssum-bottom-iff
ex-sprod-up-bottom-iff
ex-sprod-bottom-iff
ex-up-bottom-iff
ex-one-bottom-iff

Rules for turning nchotomy into exhaust:

lemma *exh-casedist0*: $[\![R;\ R \Longrightarrow P]\!] \Longrightarrow P$
by *auto*

lemma *exh-casedist1*: $((P \vee Q \Longrightarrow R) \Longrightarrow S) \equiv ([\![P \Longrightarrow R;\ Q \Longrightarrow R]\!] \Longrightarrow S)$
by *rule auto*

lemma *exh-casedist2*: $(\exists x.\ P\ x \Longrightarrow Q) \equiv (\bigwedge x.\ P\ x \Longrightarrow Q)$
by *rule auto*

lemma *exh-casedist3*: $(P \wedge Q \Longrightarrow R) \equiv (P \Longrightarrow Q \Longrightarrow R)$
by *rule auto*

lemmas *exh-casedists* = *exh-casedist1 exh-casedist2 exh-casedist3*

Rules for proving constructor properties

lemmas *con-strict-rules* =
sinl-strict sinr-strict spair-strict1 spair-strict2

lemmas *con-bottom-iff-rules* =
sinl-bottom-iff sinr-bottom-iff spair-bottom-iff up-defined ONE-defined

lemmas *con-below-iff-rules* =
sinl-below sinr-below sinl-below-sinr sinr-below-sinl con-bottom-iff-rules

lemmas *con-eq-iff-rules* =
sinl-eq sinr-eq sinl-eq-sinr sinr-eq-sinl con-bottom-iff-rules

lemmas *sel-strict-rules* =
cfcomp2 sscase1 sfst-strict ssnd-strict fup1

lemma *sel-app-extra-rules*:
$sscase \cdot ID \cdot \bot \cdot (sinr \cdot x) = \bot$

sscase·ID·⊥·(sinl·x) = x
sscase·⊥·ID·(sinl·x) = ⊥
sscase·⊥·ID·(sinr·x) = x
fup·ID·(up·x) = x
by (*cases x* = ⊥, *simp*, *simp*)+

lemmas *sel-app-rules* =
sel-strict-rules sel-app-extra-rules
ssnd-spair sfst-spair up-defined spair-defined

lemmas *sel-bottom-iff-rules* =
cfcomp2 sfst-bottom-iff ssnd-bottom-iff

lemmas *take-con-rules* =
ssum-map-sinl′ ssum-map-sinr′ sprod-map-spair′ u-map-up
deflation-strict deflation-ID ID1 cfcomp2

27.5 ML setup

named-theorems *domain-deflation theorems like deflation a ==> deflation* (*foo-map*$*a*)
and *domain-map-ID theorems like foo-map*$*ID* = *ID*

ML-file ‹*Tools/Domain/domain-take-proofs.ML*›
ML-file ‹*Tools/cont-consts.ML*›
ML-file ‹*Tools/cont-proc.ML*›
simproc-setup *cont* (*cont f*) = ‹*K ContProc.cont-proc*›

ML-file ‹*Tools/Domain/domain-constructors.ML*›
ML-file ‹*Tools/Domain/domain-induction.ML*›

27.6 Representations of types

lemma *emb-prj*: *emb*·((*prj*·*x*)::′*a*::*domain*) = *cast*·*DEFL*(′*a*)·*x*
by (*simp add*: *cast-DEFL*)

lemma *emb-prj-emb*:
fixes *x* :: ′*a*::*domain*
assumes *DEFL*(′*a*) ⊑ *DEFL*(′*b*)
shows *emb*·(*prj*·(*emb*·*x*) :: ′*b*::*domain*) = *emb*·*x*
unfolding *emb-prj*
apply (*rule cast.belowD*)
apply (*rule monofun-cfun-arg* [*OF assms*])
apply (*simp add*: *cast-DEFL*)
done

lemma *prj-emb-prj*:
assumes *DEFL*(′*a*::*domain*) ⊑ *DEFL*(′*b*::*domain*)
shows *prj*·(*emb*·(*prj*·*x* :: ′*b*)) = (*prj*·*x* :: ′*a*)
apply (*rule emb-eq-iff* [*THEN iffD1*])
apply (*simp only*: *emb-prj*)

apply (*rule deflation-below-comp1*)
apply (*rule deflation-cast*)
apply (*rule deflation-cast*)
apply (*rule monofun-cfun-arg* [*OF assms*])
done

Isomorphism lemmas used internally by the domain package:

lemma *domain-abs-iso*:
fixes *abs* **and** *rep*
assumes *DEFL*: *DEFL*('*b*::*domain*) = *DEFL*('*a*::*domain*)
assumes *abs-def*: (*abs* :: '*a* → '*b*) ≡ *prj oo emb*
assumes *rep-def*: (*rep* :: '*b* → '*a*) ≡ *prj oo emb*
shows *rep*·(*abs*·*x*) = *x*
unfolding *abs-def rep-def*
by (*simp add*: *emb-prj-emb DEFL*)

lemma *domain-rep-iso*:
fixes *abs* **and** *rep*
assumes *DEFL*: *DEFL*('*b*::*domain*) = *DEFL*('*a*::*domain*)
assumes *abs-def*: (*abs* :: '*a* → '*b*) ≡ *prj oo emb*
assumes *rep-def*: (*rep* :: '*b* → '*a*) ≡ *prj oo emb*
shows *abs*·(*rep*·*x*) = *x*
unfolding *abs-def rep-def*
by (*simp add*: *emb-prj-emb DEFL*)

27.7 Deflations as sets

definition *defl-set* :: '*a*::*bifinite defl* ⇒ '*a set*
where *defl-set A* = {*x*. *cast*·*A*·*x* = *x*}

lemma *adm-defl-set*: *adm* (λ*x*. *x* ∈ *defl-set A*)
unfolding *defl-set-def* **by** *simp*

lemma *defl-set-bottom*: ⊥ ∈ *defl-set A*
unfolding *defl-set-def* **by** *simp*

lemma *defl-set-cast* [*simp*]: *cast*·*A*·*x* ∈ *defl-set A*
unfolding *defl-set-def* **by** *simp*

lemma *defl-set-subset-iff*: *defl-set A* ⊆ *defl-set B* ⟷ *A* ⊑ *B*
apply (*simp add*: *defl-set-def subset-eq cast-below-cast* [*symmetric*])
apply (*auto simp add*: *cast.belowI cast.belowD*)
done

27.8 Proving a subtype is representable

Temporarily relax type constraints.

setup ‹
fold Sign.add-const-constraint

```
  [ (const-name ‹defl›, SOME typ ‹'a::pcpo itself ⇒ udom defl›)
  , (const-name ‹emb›, SOME typ ‹'a::pcpo → udom›)
  , (const-name ‹prj›, SOME typ ‹udom → 'a::pcpo›)
  , (const-name ‹liftdefl›, SOME typ ‹'a::pcpo itself ⇒ udom u defl›)
  , (const-name ‹liftemb›, SOME typ ‹'a::pcpo u → udom u›)
  , (const-name ‹liftprj›, SOME typ ‹udom u → 'a::pcpo u›) ]
›

lemma typedef-domain-class:
  fixes Rep :: 'a::pcpo ⇒ udom
  fixes Abs :: udom ⇒ 'a::pcpo
  fixes t :: udom defl
  assumes type: type-definition Rep Abs (defl-set t)
  assumes below: (⊑) ≡ λx y. Rep x ⊑ Rep y
  assumes emb: emb ≡ (Λ x. Rep x)
  assumes prj: prj ≡ (Λ x. Abs (cast·t·x))
  assumes defl: defl ≡ (λ a::'a itself. t)
  assumes liftemb: (liftemb :: 'a u → udom u) ≡ u-map·emb
  assumes liftprj: (liftprj :: udom u → 'a u) ≡ u-map·prj
  assumes liftdefl: (liftdefl :: 'a itself ⇒ -) ≡ (λt. liftdefl-of·DEFL('a))
  shows OFCLASS('a, domain-class)
proof
  have emb-beta: ⋀x. emb·x = Rep x
    unfolding emb
    apply (rule beta-cfun)
    apply (rule typedef-cont-Rep [OF type below adm-defl-set cont-id])
    done
  have prj-beta: ⋀y. prj·y = Abs (cast·t·y)
    unfolding prj
    apply (rule beta-cfun)
    apply (rule typedef-cont-Abs [OF type below adm-defl-set])
    apply simp-all
    done
  have prj-emb: ⋀x::'a. prj·(emb·x) = x
    using type-definition.Rep [OF type]
    unfolding prj-beta emb-beta defl-set-def
    by (simp add: type-definition.Rep-inverse [OF type])
  have emb-prj: ⋀y. emb·(prj·y :: 'a) = cast·t·y
    unfolding prj-beta emb-beta
    by (simp add: type-definition.Abs-inverse [OF type])
  show ep-pair (emb :: 'a → udom) prj
    apply standard
    apply (simp add: prj-emb)
    apply (simp add: emb-prj cast.below)
    done
  show cast·DEFL('a) = emb oo (prj :: udom → 'a)
    by (rule cfun-eqI, simp add: defl emb-prj)
qed (simp-all only: liftemb liftprj liftdefl)
```

lemma *typedef-DEFL*:
  assumes *defl* ≡ (λ*a*::'*a*::*pcpo itself*. *t*)
  shows *DEFL*('*a*::*pcpo*) = *t*
unfolding *assms* **..**

Restore original typing constraints.

setup ‹
  fold Sign.add-const-constraint
   [(***const-name*** ‹*defl*›, *SOME* ***typ*** ‹'*a*::*domain itself* ⇒ *udom defl*›),
    (***const-name*** ‹*emb*›, *SOME* ***typ*** ‹'*a*::*domain* → *udom*›),
    (***const-name*** ‹*prj*›, *SOME* ***typ*** ‹*udom* → '*a*::*domain*›),
    (***const-name*** ‹*liftdefl*›, *SOME* ***typ*** ‹'*a*::*predomain itself* ⇒ *udom u defl*›),
    (***const-name*** ‹*liftemb*›, *SOME* ***typ*** ‹'*a*::*predomain u* → *udom u*›),
    (***const-name*** ‹*liftprj*›, *SOME* ***typ*** ‹*udom u* → '*a*::*predomain u*›)]
›

ML-file ‹*Tools/domaindef.ML*›

27.9 Isomorphic deflations

definition *isodefl* :: ('*a*::*domain* → '*a*) ⇒ *udom defl* ⇒ *bool*
  where *isodefl d t* ⟷ *cast*·*t* = *emb oo d oo prj*

definition *isodefl'* :: ('*a*::*predomain* → '*a*) ⇒ *udom u defl* ⇒ *bool*
  where *isodefl' d t* ⟷ *cast*·*t* = *liftemb oo u-map*·*d oo liftprj*

lemma *isodeflI*: (⋀*x*. *cast*·*t*·*x* = *emb*·(*d*·(*prj*·*x*))) ⟹ *isodefl d t*
unfolding *isodefl-def* **by** (*simp add*: *cfun-eqI*)

lemma *cast-isodefl*: *isodefl d t* ⟹ *cast*·*t* = (Λ *x*. *emb*·(*d*·(*prj*·*x*)))
unfolding *isodefl-def* **by** (*simp add*: *cfun-eqI*)

lemma *isodefl-strict*: *isodefl d t* ⟹ *d*·⊥ = ⊥
unfolding *isodefl-def*
by (*drule cfun-fun-cong* [**where** *x*=⊥], *simp*)

lemma *isodefl-imp-deflation*:
  fixes *d* :: '*a*::*domain* → '*a*
  assumes *isodefl d t* **shows** *deflation d*
proof
  note *assms* [*unfolded isodefl-def*, *simp*]
  fix *x* :: '*a*
  show *d*·(*d*·*x*) = *d*·*x*
    using *cast.idem* [*of t emb*·*x*] **by** *simp*
  show *d*·*x* ⊑ *x*
    using *cast.below* [*of t emb*·*x*] **by** *simp*
qed

lemma *isodefl-ID-DEFL*: *isodefl* (*ID* :: '*a* → '*a*) *DEFL*('*a*::*domain*)

unfolding *isodefl-def* **by** (*simp add*: *cast-DEFL*)

lemma *isodefl-LIFTDEFL*:
isodefl′ (*ID* :: *′a* → *′a*) *LIFTDEFL*(*′a*::*predomain*)
unfolding *isodefl′-def* **by** (*simp add*: *cast-liftdefl u-map-ID*)

lemma *isodefl-DEFL-imp-ID*: *isodefl* (*d* :: *′a* → *′a*) *DEFL*(*′a*::*domain*) ⟹ *d* = *ID*
unfolding *isodefl-def*
apply (*simp add*: *cast-DEFL*)
apply (*simp add*: *cfun-eq-iff*)
apply (*rule allI*)
apply (*drule-tac x*=*emb*·*x* **in** *spec*)
apply *simp*
done

lemma *isodefl-bottom*: *isodefl* ⊥ ⊥
unfolding *isodefl-def* **by** (*simp add*: *cfun-eq-iff*)

lemma *adm-isodefl*:
cont f ⟹ *cont g* ⟹ *adm* (*λx*. *isodefl* (*f x*) (*g x*))
unfolding *isodefl-def* **by** *simp*

lemma *isodefl-lub*:
assumes *chain d* **and** *chain t*
assumes ⋀*i*. *isodefl* (*d i*) (*t i*)
shows *isodefl* (⨆*i*. *d i*) (⨆*i*. *t i*)
using *assms* **unfolding** *isodefl-def*
by (*simp add*: *contlub-cfun-arg contlub-cfun-fun*)

lemma *isodefl-fix*:
assumes ⋀*d t*. *isodefl d t* ⟹ *isodefl* (*f*·*d*) (*g*·*t*)
shows *isodefl* (*fix*·*f*) (*fix*·*g*)
unfolding *fix-def2*
apply (*rule isodefl-lub*, *simp*, *simp*)
apply (*induct-tac i*)
apply (*simp add*: *isodefl-bottom*)
apply (*simp add*: *assms*)
done

lemma *isodefl-abs-rep*:
fixes *abs* **and** *rep* **and** *d*
assumes *DEFL*: *DEFL*(*′b*::*domain*) = *DEFL*(*′a*::*domain*)
assumes *abs-def*: (*abs* :: *′a* → *′b*) ≡ *prj oo emb*
assumes *rep-def*: (*rep* :: *′b* → *′a*) ≡ *prj oo emb*
shows *isodefl d t* ⟹ *isodefl* (*abs oo d oo rep*) *t*
unfolding *isodefl-def*
by (*simp add*: *cfun-eq-iff assms prj-emb-prj emb-prj-emb*)

lemma *isodefl′-liftdefl-of*: *isodefl d t* ⟹ *isodefl′ d* (*liftdefl-of*·*t*)
unfolding *isodefl-def isodefl′-def*
by (*simp add*: *cast-liftdefl-of u-map-oo liftemb-eq liftprj-eq*)

lemma *isodefl-sfun*:
  isodefl d1 t1 ⟹ *isodefl d2 t2* ⟹
    isodefl (*sfun-map*·*d1*·*d2*) (*sfun-defl*·*t1*·*t2*)
apply (*rule isodeflI*)
apply (*simp add*: *cast-sfun-defl cast-isodefl*)
apply (*simp add*: *emb-sfun-def prj-sfun-def*)
apply (*simp add*: *sfun-map-map isodefl-strict*)
done

lemma *isodefl-ssum*:
  isodefl d1 t1 ⟹ *isodefl d2 t2* ⟹
    isodefl (*ssum-map*·*d1*·*d2*) (*ssum-defl*·*t1*·*t2*)
apply (*rule isodeflI*)
apply (*simp add*: *cast-ssum-defl cast-isodefl*)
apply (*simp add*: *emb-ssum-def prj-ssum-def*)
apply (*simp add*: *ssum-map-map isodefl-strict*)
done

lemma *isodefl-sprod*:
  isodefl d1 t1 ⟹ *isodefl d2 t2* ⟹
    isodefl (*sprod-map*·*d1*·*d2*) (*sprod-defl*·*t1*·*t2*)
apply (*rule isodeflI*)
apply (*simp add*: *cast-sprod-defl cast-isodefl*)
apply (*simp add*: *emb-sprod-def prj-sprod-def*)
apply (*simp add*: *sprod-map-map isodefl-strict*)
done

lemma *isodefl-prod*:
  isodefl d1 t1 ⟹ *isodefl d2 t2* ⟹
    isodefl (*prod-map*·*d1*·*d2*) (*prod-defl*·*t1*·*t2*)
apply (*rule isodeflI*)
apply (*simp add*: *cast-prod-defl cast-isodefl*)
apply (*simp add*: *emb-prod-def prj-prod-def*)
apply (*simp add*: *prod-map-map cfcomp1*)
done

lemma *isodefl-u*:
  isodefl d t ⟹ *isodefl* (*u-map*·*d*) (*u-defl*·*t*)
apply (*rule isodeflI*)
apply (*simp add*: *cast-u-defl cast-isodefl*)
apply (*simp add*: *emb-u-def prj-u-def liftemb-eq liftprj-eq u-map-map*)
done

lemma *isodefl-u-liftdefl*:
  isodefl′ d t ⟹ *isodefl* (*u-map*·*d*) (*u-liftdefl*·*t*)

apply (*rule isodeflI*)
apply (*simp add*: *cast-u-liftdefl isodefl′-def*)
apply (*simp add*: *emb-u-def prj-u-def liftemb-eq liftprj-eq*)
done

lemma *encode-prod-u-map*:
  encode-prod-u·(*u-map*·(*prod-map*·*f*·*g*)·(*decode-prod-u*·*x*))
    = *sprod-map*·(*u-map*·*f*)·(*u-map*·*g*)·*x*
unfolding *encode-prod-u-def decode-prod-u-def*
apply (*case-tac x*, *simp*, *rename-tac a b*)
apply (*case-tac a*, *simp*, *case-tac b*, *simp*, *simp*)
done

lemma *isodefl-prod-u*:
  assumes *isodefl′ d1 t1* **and** *isodefl′ d2 t2*
  shows *isodefl′* (*prod-map*·*d1*·*d2*) (*prod-liftdefl*·*t1*·*t2*)
using *assms* **unfolding** *isodefl′-def*
unfolding *liftemb-prod-def liftprj-prod-def*
by (*simp add*: *cast-prod-liftdefl cfcomp1 encode-prod-u-map sprod-map-map*)

lemma *encode-cfun-map*:
  encode-cfun·(*cfun-map*·*f*·*g*·(*decode-cfun*·*x*))
    = *sfun-map*·(*u-map*·*f*)·*g*·*x*
unfolding *encode-cfun-def decode-cfun-def*
apply (*simp add*: *sfun-eq-iff cfun-map-def sfun-map-def*)
apply (*rule cfun-eqI*, *rename-tac y*, *case-tac y*, *simp-all*)
done

lemma *isodefl-cfun*:
  assumes *isodefl* (*u-map*·*d1*) *t1* **and** *isodefl d2 t2*
  shows *isodefl* (*cfun-map*·*d1*·*d2*) (*sfun-defl*·*t1*·*t2*)
using *isodefl-sfun* [*OF assms*] **unfolding** *isodefl-def*
by (*simp add*: *emb-cfun-def prj-cfun-def cfcomp1 encode-cfun-map*)

27.10 Setting up the domain package

named-theorems *domain-defl-simps theorems like DEFL*(*′a t*) = *t-defl*$*DEFL*(*′a*)
 and *domain-isodefl theorems like isodefl d t ==> isodefl* (*foo-map*$*d*) (*foo-defl*$*t*)

ML-file ‹*Tools/Domain/domain-isomorphism.ML*›
ML-file ‹*Tools/Domain/domain-axioms.ML*›
ML-file ‹*Tools/Domain/domain.ML*›

lemmas [*domain-defl-simps*] =
  DEFL-cfun DEFL-sfun DEFL-ssum DEFL-sprod DEFL-prod DEFL-u
  liftdefl-eq LIFTDEFL-prod u-liftdefl-liftdefl-of

lemmas [*domain-map-ID*] =
  cfun-map-ID sfun-map-ID ssum-map-ID sprod-map-ID prod-map-ID u-map-ID

lemmas [*domain-isodefl*] =
isodefl-u isodefl-sfun isodefl-ssum isodefl-sprod
isodefl-cfun isodefl-prod isodefl-prod-u isodefl'-liftdefl-of
isodefl-u-liftdefl

lemmas [*domain-deflation*] =
deflation-cfun-map deflation-sfun-map deflation-ssum-map
deflation-sprod-map deflation-prod-map deflation-u-map

setup ‹
fold Domain-Take-Proofs.add-rec-type
[(***type-name*** ‹*cfun*›, [*true*, *true*]),
(***type-name*** ‹*sfun*›, [*true*, *true*]),
(***type-name*** ‹*ssum*›, [*true*, *true*]),
(***type-name*** ‹*sprod*›, [*true*, *true*]),
(***type-name*** ‹*prod*›, [*true*, *true*]),
(***type-name*** ‹*u*›, [*true*])]
›

end

28 A compact basis for powerdomains

theory *Compact-Basis*
imports *Universal*
begin

28.1 A compact basis for powerdomains

definition *pd-basis* = {*S*::'*a*::*bifinite compact-basis set*. *finite S* ∧ *S* ≠ {}}

typedef '*a*::*bifinite pd-basis* = *pd-basis* :: '*a compact-basis set set*
proof
show {*a*} ∈ *?pd-basis* **for** *a*
by (*simp add*: *pd-basis-def*)
qed

lemma *finite-Rep-pd-basis* [*simp*]: *finite* (*Rep-pd-basis u*)
using *Rep-pd-basis* [*of u*, *unfolded pd-basis-def*] **by** *simp*

lemma *Rep-pd-basis-nonempty* [*simp*]: *Rep-pd-basis u* ≠ {}
using *Rep-pd-basis* [*of u*, *unfolded pd-basis-def*] **by** *simp*

The powerdomain basis type is countable.

lemma *pd-basis-countable*: ∃*f*::'*a*::*bifinite pd-basis* ⇒ *nat*. *inj f* (**is** *Ex ?P*)
proof −
obtain *g* :: '*a compact-basis* ⇒ *nat* **where** *inj g*
using *compact-basis.countable* **..**

hence *image-g-eq*: g ' $A = g$ ' $B \longleftrightarrow A = B$ **for** A B
  by (*rule inj-image-eq-iff*)
have *inj* (λt. *set-encode* (g ' *Rep-pd-basis* t))
  by (*simp add*: *inj-on-def set-encode-eq image-g-eq Rep-pd-basis-inject*)
thus *?thesis* **by** (*rule exI* [*of ?P*])
qed

28.2 Unit and plus constructors

definition
  PDUnit :: '*a*::*bifinite compact-basis* ⇒ '*a pd-basis* **where**
  PDUnit = (λx. *Abs-pd-basis* $\{x\}$)

definition
  PDPlus :: '*a*::*bifinite pd-basis* ⇒ '*a pd-basis* ⇒ '*a pd-basis* **where**
  PDPlus t u = *Abs-pd-basis* (*Rep-pd-basis t* ∪ *Rep-pd-basis u*)

lemma *Rep-PDUnit*:
  Rep-pd-basis (*PDUnit x*) = $\{x\}$
unfolding *PDUnit-def* **by** (*rule Abs-pd-basis-inverse*) (*simp add*: *pd-basis-def*)

lemma *Rep-PDPlus*:
  Rep-pd-basis (*PDPlus u v*) = *Rep-pd-basis u* ∪ *Rep-pd-basis v*
unfolding *PDPlus-def* **by** (*rule Abs-pd-basis-inverse*) (*simp add*: *pd-basis-def*)

lemma *PDUnit-inject* [*simp*]: (*PDUnit a* = *PDUnit b*) = ($a = b$)
unfolding *Rep-pd-basis-inject* [*symmetric*] *Rep-PDUnit* **by** *simp*

lemma *PDPlus-assoc*: *PDPlus* (*PDPlus t u*) *v* = *PDPlus t* (*PDPlus u v*)
unfolding *Rep-pd-basis-inject* [*symmetric*] *Rep-PDPlus* **by** (*rule Un-assoc*)

lemma *PDPlus-commute*: *PDPlus t u* = *PDPlus u t*
unfolding *Rep-pd-basis-inject* [*symmetric*] *Rep-PDPlus* **by** (*rule Un-commute*)

lemma *PDPlus-absorb*: *PDPlus t t* = *t*
unfolding *Rep-pd-basis-inject* [*symmetric*] *Rep-PDPlus* **by** (*rule Un-absorb*)

lemma *pd-basis-induct1* [*case-names PDUnit PDPlus*]:
  assumes *PDUnit*: $\bigwedge a$. P (*PDUnit a*)
  assumes *PDPlus*: $\bigwedge a\ t$. $P\ t \Longrightarrow P$ (*PDPlus* (*PDUnit a*) *t*)
  shows $P\ x$
proof (*induct x*)
  case (*Abs-pd-basis y*)
  then have *finite y* **and** $y \neq \{\}$ **by** (*simp-all add*: *pd-basis-def*)
  then show *?case*
  proof (*induct rule*: *finite-ne-induct*)
    case (*singleton x*)
    show *?case* **by** (*rule PDUnit* [*unfolded PDUnit-def*])
  next

```
  case (insert x F)
  from insert(4) have P (PDPlus (PDUnit x) (Abs-pd-basis F)) by (rule PDPlus)
  with insert(1,2) show ?case
   by (simp add: PDUnit-def PDPlus-def Abs-pd-basis-inverse [unfolded pd-basis-def])
 qed
qed

lemma pd-basis-induct [case-names PDUnit PDPlus]:
  assumes PDUnit: ⋀a. P (PDUnit a)
  assumes PDPlus: ⋀t u. ⟦P t; P u⟧ ⟹ P (PDPlus t u)
  shows P x
  by (induct x rule: pd-basis-induct1) (fact PDUnit, fact PDPlus [OF PDUnit])
```

28.3 Fold operator

```
definition
  fold-pd ::
    ('a::bifinite compact-basis ⇒ 'b::type) ⇒ ('b ⇒ 'b ⇒ 'b) ⇒ 'a pd-basis ⇒ 'b
  where fold-pd g f t = semilattice-set.F f (g ` Rep-pd-basis t)

lemma fold-pd-PDUnit:
  assumes semilattice f
  shows fold-pd g f (PDUnit x) = g x
proof −
  from assms interpret semilattice-set f by (rule semilattice-set.intro)
  show ?thesis by (simp add: fold-pd-def Rep-PDUnit)
qed

lemma fold-pd-PDPlus:
  assumes semilattice f
  shows fold-pd g f (PDPlus t u) = f (fold-pd g f t) (fold-pd g f u)
proof −
  from assms interpret semilattice-set f by (rule semilattice-set.intro)
  show ?thesis by (simp add: image-Un fold-pd-def Rep-PDPlus union)
qed

end
```

29 Upper powerdomain

```
theory UpperPD
imports Compact-Basis
begin
```

29.1 Basis preorder

```
definition
  upper-le :: 'a::bifinite pd-basis ⇒ 'a pd-basis ⇒ bool (infix ‹≤♯› 50) where
  upper-le = (λu v. ∀ y∈Rep-pd-basis v. ∃ x∈Rep-pd-basis u. x ⊑ y)
```

lemma *upper-le-refl* [*simp*]: $t \leq\sharp\ t$
unfolding *upper-le-def* **by** *fast*

lemma *upper-le-trans*: $[\![t \leq\sharp\ u;\ u \leq\sharp\ v]\!] \Longrightarrow t \leq\sharp\ v$
unfolding *upper-le-def*
apply (*rule ballI*)
apply (*drule* (*1*) *bspec*, *erule bexE*)
apply (*drule* (*1*) *bspec*, *erule bexE*)
apply (*erule rev-bexI*)
apply (*erule* (*1*) *below-trans*)
done

interpretation *upper-le*: *preorder upper-le*
by (*rule preorder.intro*, *rule upper-le-refl*, *rule upper-le-trans*)

lemma *upper-le-minimal* [*simp*]: *PDUnit compact-bot* $\leq\sharp\ t$
unfolding *upper-le-def Rep-PDUnit* **by** *simp*

lemma *PDUnit-upper-mono*: $x \sqsubseteq y \Longrightarrow$ *PDUnit* $x \leq\sharp$ *PDUnit* y
unfolding *upper-le-def Rep-PDUnit* **by** *simp*

lemma *PDPlus-upper-mono*: $[\![s \leq\sharp\ t;\ u \leq\sharp\ v]\!] \Longrightarrow$ *PDPlus* $s\ u \leq\sharp$ *PDPlus* $t\ v$
unfolding *upper-le-def Rep-PDPlus* **by** *fast*

lemma *PDPlus-upper-le*: *PDPlus* $t\ u \leq\sharp\ t$
unfolding *upper-le-def Rep-PDPlus* **by** *fast*

lemma *upper-le-PDUnit-PDUnit-iff* [*simp*]:
  (*PDUnit* $a \leq\sharp$ *PDUnit* b) = ($a \sqsubseteq b$)
unfolding *upper-le-def Rep-PDUnit* **by** *fast*

lemma *upper-le-PDPlus-PDUnit-iff*:
  (*PDPlus* $t\ u \leq\sharp$ *PDUnit* a) = ($t \leq\sharp$ *PDUnit* $a \lor u \leq\sharp$ *PDUnit* a)
unfolding *upper-le-def Rep-PDPlus Rep-PDUnit* **by** *fast*

lemma *upper-le-PDPlus-iff*: ($t \leq\sharp$ *PDPlus* $u\ v$) = ($t \leq\sharp\ u \land t \leq\sharp\ v$)
unfolding *upper-le-def Rep-PDPlus* **by** *fast*

lemma *upper-le-induct* [*induct set*: *upper-le*]:
  assumes *le*: $t \leq\sharp\ u$
  assumes *1*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P$ (*PDUnit* a) (*PDUnit* b)
  assumes *2*: $\bigwedge t\ u\ a.\ P\ t$ (*PDUnit* a) $\Longrightarrow P$ (*PDPlus* $t\ u$) (*PDUnit* a)
  assumes *3*: $\bigwedge t\ u\ v.\ [\![P\ t\ u;\ P\ t\ v]\!] \Longrightarrow P\ t$ (*PDPlus* $u\ v$)
  shows $P\ t\ u$
  using *le*
proof (*induct u arbitrary*: *t rule*: *pd-basis-induct*)
  case (*PDUnit a*)
  then show *?case*

```
  proof (induct t rule: pd-basis-induct)
    case PDUnit
    then show ?case by (simp add: 1)
  next
    case (PDPlus t u)
    from PDPlus(3) consider (t) t ≤♯ PDUnit a | (u) u ≤♯ PDUnit a
      by (auto simp: upper-le-PDPlus-PDUnit-iff)
    then show ?case
    proof cases
      case t
      then have P t (PDUnit a) by (rule PDPlus(1))
      then show ?thesis by (rule 2)
    next
      case u
      then have P u (PDUnit a) by (rule PDPlus(2))
      then have P (PDPlus u t) (PDUnit a) by (rule 2)
      then show ?thesis by (simp only: PDPlus-commute)
    qed
  qed
next
  case (PDPlus t t' u)
  then show ?case by (simp add: upper-le-PDPlus-iff 3)
qed
```

29.2 Type definition

```
typedef 'a::bifinite upper-pd (‹(‹notation=‹postfix upper-pd››'(-')♯)›) =
  {S::'a pd-basis set. upper-le.ideal S}
by (rule upper-le.ex-ideal)

instantiation upper-pd :: (bifinite) below
begin

definition
  x ⊑ y ⟷ Rep-upper-pd x ⊆ Rep-upper-pd y

instance ..
end

instance upper-pd :: (bifinite) po
using type-definition-upper-pd below-upper-pd-def
by (rule upper-le.typedef-ideal-po)

instance upper-pd :: (bifinite) cpo
using type-definition-upper-pd below-upper-pd-def
by (rule upper-le.typedef-ideal-cpo)

definition
  upper-principal :: 'a::bifinite pd-basis ⇒ 'a upper-pd where
```

upper-principal t = *Abs-upper-pd* {*u*. *u* ≤♯ *t*}

interpretation *upper-pd*:
ideal-completion upper-le upper-principal Rep-upper-pd
using *type-definition-upper-pd below-upper-pd-def*
using *upper-principal-def pd-basis-countable*
by (*rule upper-le.typedef-ideal-completion*)

Upper powerdomain is pointed

lemma *upper-pd-minimal*: *upper-principal* (*PDUnit compact-bot*) ⊑ *ys*
by (*induct ys rule*: *upper-pd.principal-induct*, *simp*, *simp*)

instance *upper-pd* :: (*bifinite*) *pcpo*
by *intro-classes* (*fast intro*: *upper-pd-minimal*)

lemma *inst-upper-pd-pcpo*: ⊥ = *upper-principal* (*PDUnit compact-bot*)
by (*rule upper-pd-minimal* [*THEN bottomI*, *symmetric*])

29.3 Monadic unit and plus

definition
upper-unit :: *'a*::*bifinite* → *'a upper-pd* **where**
upper-unit = *compact-basis.extension* (λ*a*. *upper-principal* (*PDUnit a*))

definition
upper-plus :: *'a*::*bifinite upper-pd* → *'a upper-pd* → *'a upper-pd* **where**
upper-plus = *upper-pd.extension* (λ*t*. *upper-pd.extension* (λ*u*.
upper-principal (*PDPlus t u*)))

abbreviation
upper-add :: *'a*::*bifinite upper-pd* ⇒ *'a upper-pd* ⇒ *'a upper-pd*
(**infixl** ‹∪♯› *65*) **where**
xs ∪♯ *ys* == *upper-plus*·*xs*·*ys*

syntax
-upper-pd :: *args* ⇒ *logic* (‹(‹indent=1 notation=‹mixfix upper-pd enumeration››{-}♯)›)
translations
{*x*,*xs*}♯ == {*x*}♯ ∪♯ {*xs*}♯
{*x*}♯ == *CONST upper-unit*·*x*

lemma *upper-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♯ = *upper-principal* (*PDUnit a*)
unfolding *upper-unit-def*
by (*simp add*: *compact-basis.extension-principal PDUnit-upper-mono*)

lemma *upper-plus-principal* [*simp*]:
upper-principal t ∪♯ *upper-principal u* = *upper-principal* (*PDPlus t u*)
unfolding *upper-plus-def*

```
by (simp add: upper-pd.extension-principal
    upper-pd.extension-mono PDPlus-upper-mono)

interpretation upper-add: semilattice upper-add proof
  fix xs ys zs :: 'a upper-pd
  show (xs ∪♯ ys) ∪♯ zs = xs ∪♯ (ys ∪♯ zs)
    apply (induct xs rule: upper-pd.principal-induct, simp)
    apply (induct ys rule: upper-pd.principal-induct, simp)
    apply (induct zs rule: upper-pd.principal-induct, simp)
    apply (simp add: PDPlus-assoc)
    done
  show xs ∪♯ ys = ys ∪♯ xs
    apply (induct xs rule: upper-pd.principal-induct, simp)
    apply (induct ys rule: upper-pd.principal-induct, simp)
    apply (simp add: PDPlus-commute)
    done
  show xs ∪♯ xs = xs
    apply (induct xs rule: upper-pd.principal-induct, simp)
    apply (simp add: PDPlus-absorb)
    done
qed

lemmas upper-plus-assoc = upper-add.assoc
lemmas upper-plus-commute = upper-add.commute
lemmas upper-plus-absorb = upper-add.idem
lemmas upper-plus-left-commute = upper-add.left-commute
lemmas upper-plus-left-absorb = upper-add.left-idem
```

Useful for *simp add*: *upper-plus-ac*

```
lemmas upper-plus-ac =
  upper-plus-assoc upper-plus-commute upper-plus-left-commute
```

Useful for *simp only*: *upper-plus-aci*

```
lemmas upper-plus-aci =
  upper-plus-ac upper-plus-absorb upper-plus-left-absorb

lemma upper-plus-below1: xs ∪♯ ys ⊑ xs
apply (induct xs rule: upper-pd.principal-induct, simp)
apply (induct ys rule: upper-pd.principal-induct, simp)
apply (simp add: PDPlus-upper-le)
done

lemma upper-plus-below2: xs ∪♯ ys ⊑ ys
by (subst upper-plus-commute, rule upper-plus-below1)

lemma upper-plus-greatest: [[xs ⊑ ys; xs ⊑ zs]] ⟹ xs ⊑ ys ∪♯ zs
apply (subst upper-plus-absorb [of xs, symmetric])
apply (erule (1) monofun-cfun [OF monofun-cfun-arg])
done
```

lemma *upper-below-plus-iff* [*simp*]:
  $xs \sqsubseteq ys \cup\sharp\ zs \longleftrightarrow xs \sqsubseteq ys \wedge xs \sqsubseteq zs$
apply *safe*
apply (*erule below-trans* [*OF - upper-plus-below1*])
apply (*erule below-trans* [*OF - upper-plus-below2*])
apply (*erule* (*1*) *upper-plus-greatest*)
done

lemma *upper-plus-below-unit-iff* [*simp*]:
  $xs \cup\sharp\ ys \sqsubseteq \{z\}\sharp \longleftrightarrow xs \sqsubseteq \{z\}\sharp \vee ys \sqsubseteq \{z\}\sharp$
apply (*induct xs rule*: *upper-pd.principal-induct*, *simp*)
apply (*induct ys rule*: *upper-pd.principal-induct*, *simp*)
apply (*induct z rule*: *compact-basis.principal-induct*, *simp*)
apply (*simp add*: *upper-le-PDPlus-PDUnit-iff*)
done

lemma *upper-unit-below-iff* [*simp*]: $\{x\}\sharp \sqsubseteq \{y\}\sharp \longleftrightarrow x \sqsubseteq y$
apply (*induct x rule*: *compact-basis.principal-induct*, *simp*)
apply (*induct y rule*: *compact-basis.principal-induct*, *simp*)
apply *simp*
done

lemmas *upper-pd-below-simps* =
  upper-unit-below-iff
  upper-below-plus-iff
  upper-plus-below-unit-iff

lemma *upper-unit-eq-iff* [*simp*]: $\{x\}\sharp = \{y\}\sharp \longleftrightarrow x = y$
unfolding *po-eq-conv* **by** *simp*

lemma *upper-unit-strict* [*simp*]: $\{\bot\}\sharp = \bot$
using *upper-unit-Rep-compact-basis* [*of compact-bot*]
by (*simp add*: *inst-upper-pd-pcpo*)

lemma *upper-plus-strict1* [*simp*]: $\bot \cup\sharp\ ys = \bot$
by (*rule bottomI*, *rule upper-plus-below1*)

lemma *upper-plus-strict2* [*simp*]: $xs \cup\sharp\ \bot = \bot$
by (*rule bottomI*, *rule upper-plus-below2*)

lemma *upper-unit-bottom-iff* [*simp*]: $\{x\}\sharp = \bot \longleftrightarrow x = \bot$
unfolding *upper-unit-strict* [*symmetric*] **by** (*rule upper-unit-eq-iff*)

lemma *upper-plus-bottom-iff* [*simp*]:
  $xs \cup\sharp\ ys = \bot \longleftrightarrow xs = \bot \vee ys = \bot$
apply (*induct xs rule*: *upper-pd.principal-induct*, *simp*)
apply (*induct ys rule*: *upper-pd.principal-induct*, *simp*)
apply (*simp add*: *inst-upper-pd-pcpo upper-pd.principal-eq-iff*

upper-le-PDPlus-PDUnit-iff)
done

lemma *compact-upper-unit*: *compact x* ⟹ *compact* {*x*}♯
by (*auto dest*!: *compact-basis.compact-imp-principal*)

lemma *compact-upper-unit-iff* [*simp*]: *compact* {*x*}♯ ⟷ *compact x*
apply (*safe elim*!: *compact-upper-unit*)
apply (*simp only*: *compact-def upper-unit-below-iff* [*symmetric*])
apply (*erule adm-subst* [*OF cont-Rep-cfun2*])
done

lemma *compact-upper-plus* [*simp*]:
⟦*compact xs*; *compact ys*⟧ ⟹ *compact* (*xs* ∪♯ *ys*)
by (*auto dest*!: *upper-pd.compact-imp-principal*)

29.4 Induction rules

lemma *upper-pd-induct1*:
assumes *P*: *adm P*
assumes *unit*: ⋀*x*. *P* {*x*}♯
assumes *insert*: ⋀*x ys*. ⟦*P* {*x*}♯; *P ys*⟧ ⟹ *P* ({*x*}♯ ∪♯ *ys*)
shows *P* (*xs*::′*a*::*bifinite upper-pd*)
proof (*induct xs rule*: *upper-pd.principal-induct*)
have ∗: *P* {*Rep-compact-basis a*}♯ **for** *a*
by (*rule unit*)
show *P* (*upper-principal a*) **for** *a*
proof (*induct a rule*: *pd-basis-induct1*)
case (*PDUnit a*)
with ∗ **show** *?case*
by (*simp only*: *upper-unit-Rep-compact-basis* [*symmetric*])
next
case (*PDPlus a t*)
with ∗ **have** *P* ({*Rep-compact-basis a*}♯ ∪♯ *upper-principal t*)
by (*rule insert*)
then show *?case*
by (*simp only*: *upper-unit-Rep-compact-basis* [*symmetric*]
upper-plus-principal [*symmetric*])
qed
qed (*rule P*)

lemma *upper-pd-induct* [*case-names adm upper-unit upper-plus*, *induct type*: *upper-pd*]:
assumes *P*: *adm P*
assumes *unit*: ⋀*x*. *P* {*x*}♯
assumes *plus*: ⋀*xs ys*. ⟦*P xs*; *P ys*⟧ ⟹ *P* (*xs* ∪♯ *ys*)
shows *P* (*xs*::′*a*::*bifinite upper-pd*)
proof (*induct xs rule*: *upper-pd.principal-induct*)
show *P* (*upper-principal a*) **for** *a*

proof (*induct a rule*: *pd-basis-induct*)
  case *PDUnit*
  then show *?case*
    by (*simp only*: *upper-unit-Rep-compact-basis* [*symmetric*] *unit*)
next
  case *PDPlus*
  then show *?case*
    by (*simp only*: *upper-plus-principal* [*symmetric*] *plus*)
qed
qed (*rule P*)

29.5 Monadic bind

definition
  upper-bind-basis ::
  $'a$::*bifinite pd-basis* $\Rightarrow$ ($'a \rightarrow 'b$ *upper-pd*) $\rightarrow$ $'b$::*bifinite upper-pd* **where**
  upper-bind-basis = *fold-pd*
    (λa. Λ *f*. *f*·(*Rep-compact-basis a*))
    (λx *y*. Λ *f*. *x*·*f* $\cup\sharp$ *y*·*f*)

lemma *ACI-upper-bind*:
  semilattice (λx *y*. Λ *f*. *x*·*f* $\cup\sharp$ *y*·*f*)
apply *unfold-locales*
apply (*simp add*: *upper-plus-assoc*)
apply (*simp add*: *upper-plus-commute*)
apply (*simp add*: *eta-cfun*)
done

lemma *upper-bind-basis-simps* [*simp*]:
  upper-bind-basis (*PDUnit a*) =
    (Λ *f*. *f*·(*Rep-compact-basis a*))
  upper-bind-basis (*PDPlus t u*) =
    (Λ *f*. *upper-bind-basis t*·*f* $\cup\sharp$ *upper-bind-basis u*·*f*)
unfolding *upper-bind-basis-def*
apply −
apply (*rule fold-pd-PDUnit* [*OF ACI-upper-bind*])
apply (*rule fold-pd-PDPlus* [*OF ACI-upper-bind*])
done

lemma *upper-bind-basis-mono*:
  $t \leq\sharp u \Longrightarrow$ *upper-bind-basis t* $\sqsubseteq$ *upper-bind-basis u*
unfolding *cfun-below-iff*
apply (*erule upper-le-induct*, *safe*)
apply (*simp add*: *monofun-cfun*)
apply (*simp add*: *below-trans* [*OF upper-plus-below1*])
apply *simp*
done

definition

upper-bind :: *'a*::*bifinite upper-pd* → (*'a* → *'b upper-pd*) → *'b*::*bifinite upper-pd*
where
upper-bind = *upper-pd.extension upper-bind-basis*

syntax
-upper-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
(‹(‹*indent=3 notation=*‹*binder upper-bind*››⋃♯-∈-./ -)› [*0*, *0*, *10*] *10*)

translations
⋃♯*x*∈*xs*. *e* == *CONST upper-bind*·*xs*·(Λ *x*. *e*)

lemma *upper-bind-principal* [*simp*]:
upper-bind·(*upper-principal t*) = *upper-bind-basis t*
unfolding *upper-bind-def*
apply (*rule upper-pd.extension-principal*)
apply (*erule upper-bind-basis-mono*)
done

lemma *upper-bind-unit* [*simp*]:
upper-bind·{*x*}♯·*f* = *f*·*x*
by (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

lemma *upper-bind-plus* [*simp*]:
upper-bind·(*xs* ∪♯ *ys*)·*f* = *upper-bind*·*xs*·*f* ∪♯ *upper-bind*·*ys*·*f*
by (*induct xs rule*: *upper-pd.principal-induct*, *simp*,
induct ys rule: *upper-pd.principal-induct*, *simp*, *simp*)

lemma *upper-bind-strict* [*simp*]: *upper-bind*·⊥·*f* = *f*·⊥
unfolding *upper-unit-strict* [*symmetric*] **by** (*rule upper-bind-unit*)

lemma *upper-bind-bind*:
upper-bind·(*upper-bind*·*xs*·*f*)·*g* = *upper-bind*·*xs*·(Λ *x*. *upper-bind*·(*f*·*x*)·*g*)
by (*induct xs*, *simp-all*)

29.6 Map

definition
upper-map :: (*'a*::*bifinite* → *'b*::*bifinite*) → *'a upper-pd* → *'b upper-pd* **where**
upper-map = (Λ *f xs*. *upper-bind*·*xs*·(Λ *x*. {*f*·*x*}♯))

lemma *upper-map-unit* [*simp*]:
upper-map·*f*·{*x*}♯ = {*f*·*x*}♯
unfolding *upper-map-def* **by** *simp*

lemma *upper-map-plus* [*simp*]:
upper-map·*f*·(*xs* ∪♯ *ys*) = *upper-map*·*f*·*xs* ∪♯ *upper-map*·*f*·*ys*
unfolding *upper-map-def* **by** *simp*

lemma *upper-map-bottom* [*simp*]: *upper-map*·*f*·⊥ = {*f*·⊥}♯

```
unfolding upper-map-def by simp

lemma upper-map-ident: upper-map·(Λ x. x)·xs = xs
by (induct xs rule: upper-pd-induct, simp-all)

lemma upper-map-ID: upper-map·ID = ID
by (simp add: cfun-eq-iff ID-def upper-map-ident)

lemma upper-map-map:
  upper-map·f·(upper-map·g·xs) = upper-map·(Λ x. f·(g·x))·xs
by (induct xs rule: upper-pd-induct, simp-all)

lemma upper-bind-map:
  upper-bind·(upper-map·f·xs)·g = upper-bind·xs·(Λ x. g·(f·x))
by (simp add: upper-map-def upper-bind-bind)

lemma upper-map-bind:
  upper-map·f·(upper-bind·xs·g) = upper-bind·xs·(Λ x. upper-map·f·(g·x))
by (simp add: upper-map-def upper-bind-bind)

lemma ep-pair-upper-map: ep-pair e p ⟹ ep-pair (upper-map·e) (upper-map·p)
apply standard
apply (induct-tac x rule: upper-pd-induct, simp-all add: ep-pair.e-inverse)
apply (induct-tac y rule: upper-pd-induct)
apply (simp-all add: ep-pair.e-p-below monofun-cfun del: upper-below-plus-iff)
done

lemma deflation-upper-map: deflation d ⟹ deflation (upper-map·d)
apply standard
apply (induct-tac x rule: upper-pd-induct, simp-all add: deflation.idem)
apply (induct-tac x rule: upper-pd-induct)
apply (simp-all add: deflation.below monofun-cfun del: upper-below-plus-iff)
done


lemma finite-deflation-upper-map:
  assumes finite-deflation d shows finite-deflation (upper-map·d)
proof (rule finite-deflation-intro)
  interpret d: finite-deflation d by fact
  from d.deflation-axioms show deflation (upper-map·d)
    by (rule deflation-upper-map)
  have finite (range (λx. d·x)) by (rule d.finite-range)
  hence finite (Rep-compact-basis −‘ range (λx. d·x))
    by (rule finite-vimageI, simp add: inj-on-def Rep-compact-basis-inject)
  hence finite (Pow (Rep-compact-basis −‘ range (λx. d·x))) by simp
  hence finite (Rep-pd-basis −‘ (Pow (Rep-compact-basis −‘ range (λx. d·x))))
    by (rule finite-vimageI, simp add: inj-on-def Rep-pd-basis-inject)
  hence *: finite (upper-principal ‘ Rep-pd-basis −‘ (Pow (Rep-compact-basis −‘
range (λx. d·x)))) by simp
```

```
  hence finite (range (λxs. upper-map·d·xs))
    apply (rule rev-finite-subset)
    apply clarsimp
    apply (induct-tac xs rule: upper-pd.principal-induct)
    apply (simp add: adm-mem-finite ∗)
    apply (rename-tac t, induct-tac t rule: pd-basis-induct)
    apply (simp only: upper-unit-Rep-compact-basis [symmetric] upper-map-unit)
    apply simp
    apply (subgoal-tac ∃ b. d·(Rep-compact-basis a) = Rep-compact-basis b)
    apply clarsimp
    apply (rule imageI)
    apply (rule vimageI2)
    apply (simp add: Rep-PDUnit)
    apply (rule range-eqI)
    apply (erule sym)
    apply (rule exI)
    apply (rule Abs-compact-basis-inverse [symmetric])
    apply (simp add: d.compact)
    apply (simp only: upper-plus-principal [symmetric] upper-map-plus)
    apply clarsimp
    apply (rule imageI)
    apply (rule vimageI2)
    apply (simp add: Rep-PDPlus)
    done
  thus finite {xs. upper-map·d·xs = xs}
    by (rule finite-range-imp-finite-fixes)
qed
```

29.7 Upper powerdomain is bifinite

```
lemma approx-chain-upper-map:
  assumes approx-chain a
  shows approx-chain (λi. upper-map·(a i))
  using assms unfolding approx-chain-def
  by (simp add: lub-APP upper-map-ID finite-deflation-upper-map)

instance upper-pd :: (bifinite) bifinite
proof
  show ∃ (a::nat ⇒ 'a upper-pd → 'a upper-pd). approx-chain a
    using bifinite [where 'a='a]
    by (fast intro!: approx-chain-upper-map)
qed
```

29.8 Join

```
definition
  upper-join :: 'a::bifinite upper-pd upper-pd → 'a upper-pd where
  upper-join = (Λ xss. upper-bind·xss·(Λ xs. xs))

lemma upper-join-unit [simp]:
```

```
  upper-join·{xs}♯ = xs
unfolding upper-join-def by simp

lemma upper-join-plus [simp]:
  upper-join·(xss ∪♯ yss) = upper-join·xss ∪♯ upper-join·yss
unfolding upper-join-def by simp

lemma upper-join-bottom [simp]: upper-join·⊥ = ⊥
unfolding upper-join-def by simp

lemma upper-join-map-unit:
  upper-join·(upper-map·upper-unit·xs) = xs
by (induct xs rule: upper-pd-induct, simp-all)

lemma upper-join-map-join:
  upper-join·(upper-map·upper-join·xsss) = upper-join·(upper-join·xsss)
by (induct xsss rule: upper-pd-induct, simp-all)

lemma upper-join-map-map:
  upper-join·(upper-map·(upper-map·f)·xss) =
   upper-map·f·(upper-join·xss)
by (induct xss rule: upper-pd-induct, simp-all)

end
```

30 Lower powerdomain

```
theory LowerPD
imports Compact-Basis
begin
```

30.1 Basis preorder

```
definition
  lower-le :: 'a::bifinite pd-basis ⇒ 'a pd-basis ⇒ bool (infix ‹≤♭› 50) where
  lower-le = (λu v. ∀x∈Rep-pd-basis u. ∃y∈Rep-pd-basis v. x ⊑ y)

lemma lower-le-refl [simp]: t ≤♭ t
unfolding lower-le-def by fast

lemma lower-le-trans: ⟦t ≤♭ u; u ≤♭ v⟧ ⟹ t ≤♭ v
unfolding lower-le-def
apply (rule ballI)
apply (drule (1) bspec, erule bexE)
apply (drule (1) bspec, erule bexE)
apply (erule rev-bexI)
apply (erule (1) below-trans)
done
```

interpretation *lower-le*: *preorder lower-le*
by (*rule preorder.intro*, *rule lower-le-refl*, *rule lower-le-trans*)

lemma *lower-le-minimal* [*simp*]: *PDUnit compact-bot* $\leq\flat$ *t*
unfolding *lower-le-def Rep-PDUnit*
by (*simp*, *rule Rep-pd-basis-nonempty* [*folded ex-in-conv*])

lemma *PDUnit-lower-mono*: $x \sqsubseteq y \Longrightarrow$ *PDUnit x* $\leq\flat$ *PDUnit y*
unfolding *lower-le-def Rep-PDUnit* **by** *fast*

lemma *PDPlus-lower-mono*: $[\![s \leq\flat t;\ u \leq\flat v]\!] \Longrightarrow$ *PDPlus s u* $\leq\flat$ *PDPlus t v*
unfolding *lower-le-def Rep-PDPlus* **by** *fast*

lemma *PDPlus-lower-le*: *t* $\leq\flat$ *PDPlus t u*
unfolding *lower-le-def Rep-PDPlus* **by** *fast*

lemma *lower-le-PDUnit-PDUnit-iff* [*simp*]:
  (*PDUnit a* $\leq\flat$ *PDUnit b*) = ($a \sqsubseteq b$)
unfolding *lower-le-def Rep-PDUnit* **by** *fast*

lemma *lower-le-PDUnit-PDPlus-iff*:
  (*PDUnit a* $\leq\flat$ *PDPlus t u*) = (*PDUnit a* $\leq\flat$ *t* $\vee$ *PDUnit a* $\leq\flat$ *u*)
unfolding *lower-le-def Rep-PDPlus Rep-PDUnit* **by** *fast*

lemma *lower-le-PDPlus-iff*: (*PDPlus t u* $\leq\flat$ *v*) = (*t* $\leq\flat$ *v* $\wedge$ *u* $\leq\flat$ *v*)
unfolding *lower-le-def Rep-PDPlus* **by** *fast*

lemma *lower-le-induct* [*induct set*: *lower-le*]:
  assumes *le*: *t* $\leq\flat$ *u*
  assumes *1*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow$ *P* (*PDUnit a*) (*PDUnit b*)
  assumes *2*: $\bigwedge t\ u\ a.$ *P* (*PDUnit a*) *t* $\Longrightarrow$ *P* (*PDUnit a*) (*PDPlus t u*)
  assumes *3*: $\bigwedge t\ u\ v.\ [\![P\ t\ v;\ P\ u\ v]\!] \Longrightarrow$ *P* (*PDPlus t u*) *v*
  shows *P t u*
  using *le*
proof (*induct t arbitrary*: *u rule*: *pd-basis-induct*)
  case (*PDUnit a*)
  then show *?case*
  proof (*induct u rule*: *pd-basis-induct*)
    case *PDUnit*
    then show *?case* **by** (*simp add*: *1*)
  next
    case (*PDPlus t u*)
    from *PDPlus*(*3*) **consider** (*t*) *PDUnit a* $\leq\flat$ *t* | (*u*) *PDUnit a* $\leq\flat$ *u*
      by (*auto simp*: *lower-le-PDUnit-PDPlus-iff*)
    then show *?case*
    proof *cases*
      case *t*
      then have *P* (*PDUnit a*) *t* **by** (*rule PDPlus*(*1*))
      then show *?thesis* **by** (*rule 2*)

```
    next
      case u
      then have P (PDUnit a) u by (rule PDPlus(2))
      then have P (PDUnit a) (PDPlus u t) by (rule 2)
      then show ?thesis by (simp only: PDPlus-commute)
    qed
  qed
next
  case (PDPlus t t')
  then show ?case by (simp add: lower-le-PDPlus-iff 3)
qed
```

30.2 Type definition

```
typedef 'a::bifinite lower-pd (‹(‹notation=‹postfix lower-pd››'(-')♭)›) =
  {S::'a pd-basis set. lower-le.ideal S}
by (rule lower-le.ex-ideal)

instantiation lower-pd :: (bifinite) below
begin

definition
  x ⊑ y ⟷ Rep-lower-pd x ⊆ Rep-lower-pd y

instance ..
end

instance lower-pd :: (bifinite) po
using type-definition-lower-pd below-lower-pd-def
by (rule lower-le.typedef-ideal-po)

instance lower-pd :: (bifinite) cpo
using type-definition-lower-pd below-lower-pd-def
by (rule lower-le.typedef-ideal-cpo)

definition
  lower-principal :: 'a::bifinite pd-basis ⇒ 'a lower-pd where
  lower-principal t = Abs-lower-pd {u. u ≤♭ t}

interpretation lower-pd:
  ideal-completion lower-le lower-principal Rep-lower-pd
using type-definition-lower-pd below-lower-pd-def
using lower-principal-def pd-basis-countable
by (rule lower-le.typedef-ideal-completion)
```

Lower powerdomain is pointed

```
lemma lower-pd-minimal: lower-principal (PDUnit compact-bot) ⊑ ys
by (induct ys rule: lower-pd.principal-induct, simp, simp)
```

instance *lower-pd* :: (*bifinite*) *pcpo*
by *intro-classes* (*fast intro*: *lower-pd-minimal*)

lemma *inst-lower-pd-pcpo*: $\bot$ = *lower-principal* (*PDUnit compact-bot*)
by (*rule lower-pd-minimal* [*THEN bottomI*, *symmetric*])

30.3 Monadic unit and plus

definition
lower-unit :: *'a*::*bifinite* $\rightarrow$ *'a lower-pd* **where**
lower-unit = *compact-basis.extension* (λ*a*. *lower-principal* (*PDUnit a*))

definition
lower-plus :: *'a*::*bifinite lower-pd* $\rightarrow$ *'a lower-pd* $\rightarrow$ *'a lower-pd* **where**
lower-plus = *lower-pd.extension* (λ*t*. *lower-pd.extension* (λ*u*.
lower-principal (*PDPlus t u*)))

abbreviation
lower-add :: *'a*::*bifinite lower-pd* $\Rightarrow$ *'a lower-pd* $\Rightarrow$ *'a lower-pd*
(**infixl** ‹∪♭› *65*) **where**
xs ∪♭ *ys* == *lower-plus*·*xs*·*ys*

syntax
-lower-pd :: *args* $\Rightarrow$ *logic* (‹(‹*indent=1 notation=*‹*mixfix lower-pd enumeration*››{-}♭)›)
translations
{*x*,*xs*}♭ == {*x*}♭ ∪♭ {*xs*}♭
{*x*}♭ == *CONST lower-unit*·*x*

lemma *lower-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♭ = *lower-principal* (*PDUnit a*)
unfolding *lower-unit-def*
by (*simp add*: *compact-basis.extension-principal PDUnit-lower-mono*)

lemma *lower-plus-principal* [*simp*]:
lower-principal t ∪♭ *lower-principal u* = *lower-principal* (*PDPlus t u*)
unfolding *lower-plus-def*
by (*simp add*: *lower-pd.extension-principal*
lower-pd.extension-mono PDPlus-lower-mono)

interpretation *lower-add*: *semilattice lower-add* **proof**
fix *xs ys zs* :: *'a*::*bifinite lower-pd*
show (*xs* ∪♭ *ys*) ∪♭ *zs* = *xs* ∪♭ (*ys* ∪♭ *zs*)
apply (*induct xs rule*: *lower-pd.principal-induct*, *simp*)
apply (*induct ys rule*: *lower-pd.principal-induct*, *simp*)
apply (*induct zs rule*: *lower-pd.principal-induct*, *simp*)
apply (*simp add*: *PDPlus-assoc*)
done
show *xs* ∪♭ *ys* = *ys* ∪♭ *xs*

apply (*induct xs rule*: *lower-pd.principal-induct*, *simp*)
apply (*induct ys rule*: *lower-pd.principal-induct*, *simp*)
apply (*simp add*: *PDPlus-commute*)
done
show $xs \cup\flat\ xs = xs$
apply (*induct xs rule*: *lower-pd.principal-induct*, *simp*)
apply (*simp add*: *PDPlus-absorb*)
done
qed

lemmas *lower-plus-assoc* = *lower-add.assoc*
lemmas *lower-plus-commute* = *lower-add.commute*
lemmas *lower-plus-absorb* = *lower-add.idem*
lemmas *lower-plus-left-commute* = *lower-add.left-commute*
lemmas *lower-plus-left-absorb* = *lower-add.left-idem*

Useful for *simp add*: *lower-plus-ac*

lemmas *lower-plus-ac* =
lower-plus-assoc lower-plus-commute lower-plus-left-commute

Useful for *simp only*: *lower-plus-aci*

lemmas *lower-plus-aci* =
lower-plus-ac lower-plus-absorb lower-plus-left-absorb

lemma *lower-plus-below1*: $xs \sqsubseteq xs \cup\flat\ ys$
apply (*induct xs rule*: *lower-pd.principal-induct*, *simp*)
apply (*induct ys rule*: *lower-pd.principal-induct*, *simp*)
apply (*simp add*: *PDPlus-lower-le*)
done

lemma *lower-plus-below2*: $ys \sqsubseteq xs \cup\flat\ ys$
by (*subst lower-plus-commute*, *rule lower-plus-below1*)

lemma *lower-plus-least*: $[\![xs \sqsubseteq zs;\ ys \sqsubseteq zs]\!] \Longrightarrow xs \cup\flat\ ys \sqsubseteq zs$
apply (*subst lower-plus-absorb* [*of zs*, *symmetric*])
apply (*erule* (*1*) *monofun-cfun* [*OF monofun-cfun-arg*])
done

lemma *lower-plus-below-iff* [*simp*]:
$xs \cup\flat\ ys \sqsubseteq zs \longleftrightarrow xs \sqsubseteq zs \wedge ys \sqsubseteq zs$
apply *safe*
apply (*erule below-trans* [*OF lower-plus-below1*])
apply (*erule below-trans* [*OF lower-plus-below2*])
apply (*erule* (*1*) *lower-plus-least*)
done

lemma *lower-unit-below-plus-iff* [*simp*]:
$\{x\}\flat \sqsubseteq ys \cup\flat\ zs \longleftrightarrow \{x\}\flat \sqsubseteq ys \vee \{x\}\flat \sqsubseteq zs$
apply (*induct x rule*: *compact-basis.principal-induct*, *simp*)

apply (*induct ys rule*: *lower-pd.principal-induct*, *simp*)
apply (*induct zs rule*: *lower-pd.principal-induct*, *simp*)
apply (*simp add*: *lower-le-PDUnit-PDPlus-iff*)
done

lemma *lower-unit-below-iff* [*simp*]: $\{x\}\flat \sqsubseteq \{y\}\flat \longleftrightarrow x \sqsubseteq y$
apply (*induct x rule*: *compact-basis.principal-induct*, *simp*)
apply (*induct y rule*: *compact-basis.principal-induct*, *simp*)
apply *simp*
done

lemmas *lower-pd-below-simps* =
lower-unit-below-iff
lower-plus-below-iff
lower-unit-below-plus-iff

lemma *lower-unit-eq-iff* [*simp*]: $\{x\}\flat = \{y\}\flat \longleftrightarrow x = y$
by (*simp add*: *po-eq-conv*)

lemma *lower-unit-strict* [*simp*]: $\{\bot\}\flat = \bot$
using *lower-unit-Rep-compact-basis* [*of compact-bot*]
by (*simp add*: *inst-lower-pd-pcpo*)

lemma *lower-unit-bottom-iff* [*simp*]: $\{x\}\flat = \bot \longleftrightarrow x = \bot$
unfolding *lower-unit-strict* [*symmetric*] **by** (*rule lower-unit-eq-iff*)

lemma *lower-plus-bottom-iff* [*simp*]:
$xs \cup\flat\ ys = \bot \longleftrightarrow xs = \bot \wedge ys = \bot$
apply *safe*
apply (*rule bottomI*, *erule subst*, *rule lower-plus-below1*)
apply (*rule bottomI*, *erule subst*, *rule lower-plus-below2*)
apply (*rule lower-plus-absorb*)
done

lemma *lower-plus-strict1* [*simp*]: $\bot \cup\flat\ ys = ys$
apply (*rule below-antisym* [*OF - lower-plus-below2*])
apply (*simp add*: *lower-plus-least*)
done

lemma *lower-plus-strict2* [*simp*]: $xs \cup\flat \bot = xs$
apply (*rule below-antisym* [*OF - lower-plus-below1*])
apply (*simp add*: *lower-plus-least*)
done

lemma *compact-lower-unit*: *compact x* $\Longrightarrow$ *compact* $\{x\}\flat$
by (*auto dest*!: *compact-basis.compact-imp-principal*)

lemma *compact-lower-unit-iff* [*simp*]: *compact* $\{x\}\flat \longleftrightarrow$ *compact x*
apply (*safe elim*!: *compact-lower-unit*)

apply (*simp only*: *compact-def lower-unit-below-iff* [*symmetric*])
apply (*erule adm-subst* [*OF cont-Rep-cfun2*])
done

lemma *compact-lower-plus* [*simp*]:
  ⟦*compact xs*; *compact ys*⟧ ⟹ *compact* (*xs* ∪♭ *ys*)
by (*auto dest*!: *lower-pd.compact-imp-principal*)

30.4 Induction rules

lemma *lower-pd-induct1*:
  assumes *P*: *adm P*
  assumes *unit*: ⋀*x*. *P* {*x*}♭
  assumes *insert*: ⋀*x ys*. ⟦*P* {*x*}♭; *P ys*⟧ ⟹ *P* ({*x*}♭ ∪♭ *ys*)
  shows *P* (*xs*::'*a*::*bifinite lower-pd*)
proof (*induct xs rule*: *lower-pd.principal-induct*)
  have ∗: *P* {*Rep-compact-basis a*}♭ **for** *a*
    by (*rule unit*)
  show *P* (*lower-principal a*) **for** *a*
  proof (*induct a rule*: *pd-basis-induct1*)
    case *PDUnit*
    from ∗ **show** *?case*
      by (*simp only*: *lower-unit-Rep-compact-basis* [*symmetric*])
  next
    case (*PDPlus a t*)
    with ∗ **have** *P* ({*Rep-compact-basis a*}♭ ∪♭ *lower-principal t*)
      by (*rule insert*)
    then show *?case*
      by (*simp only*: *lower-unit-Rep-compact-basis* [*symmetric*] *lower-plus-principal* [*symmetric*])
  qed
qed (*rule P*)

lemma *lower-pd-induct* [*case-names adm lower-unit lower-plus*, *induct type*: *lower-pd*]:
  assumes *P*: *adm P*
  assumes *unit*: ⋀*x*. *P* {*x*}♭
  assumes *plus*: ⋀*xs ys*. ⟦*P xs*; *P ys*⟧ ⟹ *P* (*xs* ∪♭ *ys*)
  shows *P* (*xs*::'*a*::*bifinite lower-pd*)
proof (*induct xs rule*: *lower-pd.principal-induct*)
  show *P* (*lower-principal a*) **for** *a*
  proof (*induct a rule*: *pd-basis-induct*)
    case *PDUnit*
    then show *?case*
      by (*simp only*: *lower-unit-Rep-compact-basis* [*symmetric*] *unit*)
  next
    case *PDPlus*
    then show *?case*
      by (*simp only*: *lower-plus-principal* [*symmetric*] *plus*)
  qed

qed (*rule P*)

30.5 Monadic bind

definition
  lower-bind-basis ::
  'a::bifinite pd-basis ⇒ (*'a* → *'b lower-pd*) → *'b::bifinite lower-pd* **where**
  lower-bind-basis = *fold-pd*
    (λ*a*. Λ *f*. *f*·(*Rep-compact-basis a*))
    (λ*x y*. Λ *f*. *x*·*f* ∪♭ *y*·*f*)

lemma *ACI-lower-bind*:
  semilattice (λ*x y*. Λ *f*. *x*·*f* ∪♭ *y*·*f*)
apply *unfold-locales*
apply (*simp add*: *lower-plus-assoc*)
apply (*simp add*: *lower-plus-commute*)
apply (*simp add*: *eta-cfun*)
done

lemma *lower-bind-basis-simps* [*simp*]:
  lower-bind-basis (*PDUnit a*) =
    (Λ *f*. *f*·(*Rep-compact-basis a*))
  lower-bind-basis (*PDPlus t u*) =
    (Λ *f*. *lower-bind-basis t*·*f* ∪♭ *lower-bind-basis u*·*f*)
unfolding *lower-bind-basis-def*
apply −
apply (*rule fold-pd-PDUnit* [*OF ACI-lower-bind*])
apply (*rule fold-pd-PDPlus* [*OF ACI-lower-bind*])
done

lemma *lower-bind-basis-mono*:
  t ≤♭ *u* ⟹ *lower-bind-basis t* ⊑ *lower-bind-basis u*
unfolding *cfun-below-iff*
apply (*erule lower-le-induct*, *safe*)
apply (*simp add*: *monofun-cfun*)
apply (*simp add*: *rev-below-trans* [*OF lower-plus-below1*])
apply *simp*
done

definition
  lower-bind :: *'a::bifinite lower-pd* → (*'a* → *'b lower-pd*) → *'b::bifinite lower-pd*
where
  lower-bind = *lower-pd.extension lower-bind-basis*

syntax
  -lower-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
    (‹(‹*indent=3 notation*=‹*binder lower-bind*››⋃♭-∈-./ -)› [*0*, *0*, *10*] *10*)

translations

$\bigcup\flat x{\in}xs.\ e$ == *CONST lower-bind*·*xs*·(Λ *x*. *e*)

lemma *lower-bind-principal* [*simp*]:
lower-bind·(*lower-principal t*) = *lower-bind-basis t*
unfolding *lower-bind-def*
apply (*rule lower-pd.extension-principal*)
apply (*erule lower-bind-basis-mono*)
done

lemma *lower-bind-unit* [*simp*]:
lower-bind·{*x*}♭·*f* = *f*·*x*
by (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

lemma *lower-bind-plus* [*simp*]:
lower-bind·(*xs* ∪♭ *ys*)·*f* = *lower-bind*·*xs*·*f* ∪♭ *lower-bind*·*ys*·*f*
by (*induct xs rule*: *lower-pd.principal-induct*, *simp*,
induct ys rule: *lower-pd.principal-induct*, *simp*, *simp*)

lemma *lower-bind-strict* [*simp*]: *lower-bind*·⊥·*f* = *f*·⊥
unfolding *lower-unit-strict* [*symmetric*] **by** (*rule lower-bind-unit*)

lemma *lower-bind-bind*:
lower-bind·(*lower-bind*·*xs*·*f*)·*g* = *lower-bind*·*xs*·(Λ *x*. *lower-bind*·(*f*·*x*)·*g*)
by (*induct xs*, *simp-all*)

30.6 Map

definition
lower-map :: (*'a*::*bifinite* → *'b*::*bifinite*) → *'a lower-pd* → *'b lower-pd* **where**
lower-map = (Λ *f xs*. *lower-bind*·*xs*·(Λ *x*. {*f*·*x*}♭))

lemma *lower-map-unit* [*simp*]:
lower-map·*f*·{*x*}♭ = {*f*·*x*}♭
unfolding *lower-map-def* **by** *simp*

lemma *lower-map-plus* [*simp*]:
lower-map·*f*·(*xs* ∪♭ *ys*) = *lower-map*·*f*·*xs* ∪♭ *lower-map*·*f*·*ys*
unfolding *lower-map-def* **by** *simp*

lemma *lower-map-bottom* [*simp*]: *lower-map*·*f*·⊥ = {*f*·⊥}♭
unfolding *lower-map-def* **by** *simp*

lemma *lower-map-ident*: *lower-map*·(Λ *x*. *x*)·*xs* = *xs*
by (*induct xs rule*: *lower-pd-induct*, *simp-all*)

lemma *lower-map-ID*: *lower-map*·*ID* = *ID*
by (*simp add*: *cfun-eq-iff ID-def lower-map-ident*)

lemma *lower-map-map*:

lower-map·f·(lower-map·g·xs) = lower-map·(Λ x. f·(g·x))·xs
by (*induct xs rule*: *lower-pd-induct*, *simp-all*)

lemma *lower-bind-map*:
lower-bind·(lower-map·f·xs)·g = lower-bind·xs·(Λ x. g·(f·x))
by (*simp add*: *lower-map-def lower-bind-bind*)

lemma *lower-map-bind*:
lower-map·f·(lower-bind·xs·g) = lower-bind·xs·(Λ x. lower-map·f·(g·x))
by (*simp add*: *lower-map-def lower-bind-bind*)

lemma *ep-pair-lower-map*: *ep-pair e p* ⟹ *ep-pair* (*lower-map·e*) (*lower-map·p*)
apply *standard*
apply (*induct-tac x rule*: *lower-pd-induct*, *simp-all add*: *ep-pair.e-inverse*)
apply (*induct-tac y rule*: *lower-pd-induct*)
apply (*simp-all add*: *ep-pair.e-p-below monofun-cfun del*: *lower-plus-below-iff*)
done

lemma *deflation-lower-map*: *deflation d* ⟹ *deflation* (*lower-map·d*)
apply *standard*
apply (*induct-tac x rule*: *lower-pd-induct*, *simp-all add*: *deflation.idem*)
apply (*induct-tac x rule*: *lower-pd-induct*)
apply (*simp-all add*: *deflation.below monofun-cfun del*: *lower-plus-below-iff*)
done

lemma *finite-deflation-lower-map*:
assumes *finite-deflation d* **shows** *finite-deflation* (*lower-map·d*)
proof (*rule finite-deflation-intro*)
interpret *d*: *finite-deflation d* **by** *fact*
from *d.deflation-axioms* **show** *deflation* (*lower-map·d*)
by (*rule deflation-lower-map*)
have *finite* (*range* (*λx. d·x*)) **by** (*rule d.finite-range*)
hence *finite* (*Rep-compact-basis −‘ range* (*λx. d·x*))
by (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-compact-basis-inject*)
hence *finite* (*Pow* (*Rep-compact-basis −‘ range* (*λx. d·x*))) **by** *simp*
hence *finite* (*Rep-pd-basis −‘* (*Pow* (*Rep-compact-basis −‘ range* (*λx. d·x*))))
by (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-pd-basis-inject*)
hence ∗: *finite* (*lower-principal ‘ Rep-pd-basis −‘* (*Pow* (*Rep-compact-basis −‘ range* (*λx. d·x*)))) **by** *simp*
hence *finite* (*range* (*λxs. lower-map·d·xs*))
apply (*rule rev-finite-subset*)
apply *clarsimp*
apply (*induct-tac xs rule*: *lower-pd.principal-induct*)
apply (*simp add*: *adm-mem-finite* ∗)
apply (*rename-tac t*, *induct-tac t rule*: *pd-basis-induct*)
apply (*simp only*: *lower-unit-Rep-compact-basis* [*symmetric*] *lower-map-unit*)
apply *simp*
apply (*subgoal-tac* ∃ *b. d·*(*Rep-compact-basis a*) = *Rep-compact-basis b*)

apply *clarsimp*
apply (*rule imageI*)
apply (*rule vimageI2*)
apply (*simp add*: *Rep-PDUnit*)
apply (*rule range-eqI*)
apply (*erule sym*)
apply (*rule exI*)
apply (*rule Abs-compact-basis-inverse* [*symmetric*])
apply (*simp add*: *d.compact*)
apply (*simp only*: *lower-plus-principal* [*symmetric*] *lower-map-plus*)
apply *clarsimp*
apply (*rule imageI*)
apply (*rule vimageI2*)
apply (*simp add*: *Rep-PDPlus*)
done
thus *finite* {*xs. lower-map·d·xs* = *xs*}
by (*rule finite-range-imp-finite-fixes*)
qed

30.7 Lower powerdomain is bifinite

lemma *approx-chain-lower-map*:
assumes *approx-chain a*
shows *approx-chain* (*λi. lower-map·*(*a i*))
using *assms* **unfolding** *approx-chain-def*
by (*simp add*: *lub-APP lower-map-ID finite-deflation-lower-map*)

instance *lower-pd* :: (*bifinite*) *bifinite*
proof
show ∃(*a*::*nat* ⇒ *'a lower-pd* → *'a lower-pd*). *approx-chain a*
using *bifinite* [**where** *'a*=*'a*]
by (*fast intro*!: *approx-chain-lower-map*)
qed

30.8 Join

definition
lower-join :: *'a*::*bifinite lower-pd lower-pd* → *'a lower-pd* **where**
lower-join = (Λ *xss. lower-bind·xss·*(Λ *xs. xs*))

lemma *lower-join-unit* [*simp*]:
lower-join·{*xs*}♭ = *xs*
unfolding *lower-join-def* **by** *simp*

lemma *lower-join-plus* [*simp*]:
lower-join·(*xss* ∪♭ *yss*) = *lower-join·xss* ∪♭ *lower-join·yss*
unfolding *lower-join-def* **by** *simp*

lemma *lower-join-bottom* [*simp*]: *lower-join·*⊥ = ⊥
unfolding *lower-join-def* **by** *simp*

lemma *lower-join-map-unit*:
lower-join·(lower-map·lower-unit·xs) = xs
by (*induct xs rule*: *lower-pd-induct*, *simp-all*)

lemma *lower-join-map-join*:
lower-join·(lower-map·lower-join·xsss) = lower-join·(lower-join·xsss)
by (*induct xsss rule*: *lower-pd-induct*, *simp-all*)

lemma *lower-join-map-map*:
lower-join·(lower-map·(lower-map·f)·xss) =
lower-map·f·(lower-join·xss)
by (*induct xss rule*: *lower-pd-induct*, *simp-all*)

end

31 Convex powerdomain

theory *ConvexPD*
imports *UpperPD LowerPD*
begin

31.1 Basis preorder

definition
convex-le :: *'a::bifinite pd-basis ⇒ 'a pd-basis ⇒ bool* (**infix** ‹≤♮› *50*) **where**
convex-le = (λu v. u ≤♯ v ∧ u ≤♭ v)

lemma *convex-le-refl* [*simp*]: *t ≤♮ t*
unfolding *convex-le-def* **by** (*fast intro*: *upper-le-refl lower-le-refl*)

lemma *convex-le-trans*: ⟦*t ≤♮ u*; *u ≤♮ v*⟧ ⟹ *t ≤♮ v*
unfolding *convex-le-def* **by** (*fast intro*: *upper-le-trans lower-le-trans*)

interpretation *convex-le*: *preorder convex-le*
by (*rule preorder.intro*, *rule convex-le-refl*, *rule convex-le-trans*)

lemma *upper-le-minimal* [*simp*]: *PDUnit compact-bot ≤♮ t*
unfolding *convex-le-def Rep-PDUnit* **by** *simp*

lemma *PDUnit-convex-mono*: *x ⊑ y* ⟹ *PDUnit x ≤♮ PDUnit y*
unfolding *convex-le-def* **by** (*fast intro*: *PDUnit-upper-mono PDUnit-lower-mono*)

lemma *PDPlus-convex-mono*: ⟦*s ≤♮ t*; *u ≤♮ v*⟧ ⟹ *PDPlus s u ≤♮ PDPlus t v*
unfolding *convex-le-def* **by** (*fast intro*: *PDPlus-upper-mono PDPlus-lower-mono*)

lemma *convex-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a ≤♮ PDUnit b*) = (*a ⊑ b*)
unfolding *convex-le-def upper-le-def lower-le-def Rep-PDUnit* **by** *fast*

```
lemma convex-le-PDUnit-lemma1:
  (PDUnit a ≤♮ t) = (∀ b∈Rep-pd-basis t. a ⊑ b)
unfolding convex-le-def upper-le-def lower-le-def Rep-PDUnit
using Rep-pd-basis-nonempty [of t, folded ex-in-conv] by fast

lemma convex-le-PDUnit-PDPlus-iff [simp]:
  (PDUnit a ≤♮ PDPlus t u) = (PDUnit a ≤♮ t ∧ PDUnit a ≤♮ u)
unfolding convex-le-PDUnit-lemma1 Rep-PDPlus by fast

lemma convex-le-PDUnit-lemma2:
  (t ≤♮ PDUnit b) = (∀ a∈Rep-pd-basis t. a ⊑ b)
unfolding convex-le-def upper-le-def lower-le-def Rep-PDUnit
using Rep-pd-basis-nonempty [of t, folded ex-in-conv] by fast

lemma convex-le-PDPlus-PDUnit-iff [simp]:
  (PDPlus t u ≤♮ PDUnit a) = (t ≤♮ PDUnit a ∧ u ≤♮ PDUnit a)
unfolding convex-le-PDUnit-lemma2 Rep-PDPlus by fast

lemma convex-le-PDPlus-lemma:
  assumes z: PDPlus t u ≤♮ z
  shows ∃ v w. z = PDPlus v w ∧ t ≤♮ v ∧ u ≤♮ w
proof (intro exI conjI)
  let ?A = {b∈Rep-pd-basis z. ∃ a∈Rep-pd-basis t. a ⊑ b}
  let ?B = {b∈Rep-pd-basis z. ∃ a∈Rep-pd-basis u. a ⊑ b}
  let ?v = Abs-pd-basis ?A
  let ?w = Abs-pd-basis ?B
  have Rep-v: Rep-pd-basis ?v = ?A
    apply (rule Abs-pd-basis-inverse)
    apply (rule Rep-pd-basis-nonempty [of t, folded ex-in-conv, THEN exE])
    apply (cut-tac z, simp only: convex-le-def lower-le-def, clarify)
    apply (drule-tac x=x in bspec, simp add: Rep-PDPlus, erule bexE)
    apply (simp add: pd-basis-def)
    apply fast
    done
  have Rep-w: Rep-pd-basis ?w = ?B
    apply (rule Abs-pd-basis-inverse)
    apply (rule Rep-pd-basis-nonempty [of u, folded ex-in-conv, THEN exE])
    apply (cut-tac z, simp only: convex-le-def lower-le-def, clarify)
    apply (drule-tac x=x in bspec, simp add: Rep-PDPlus, erule bexE)
    apply (simp add: pd-basis-def)
    apply fast
    done
  show z = PDPlus ?v ?w
    apply (insert z)
    apply (simp add: convex-le-def, erule conjE)
    apply (simp add: Rep-pd-basis-inject [symmetric] Rep-PDPlus)
    apply (simp add: Rep-v Rep-w)
    apply (rule equalityI)
```

```
      apply (rule subsetI)
      apply (simp only: upper-le-def)
      apply (drule (1) bspec, erule bexE)
      apply (simp add: Rep-PDPlus)
      apply fast
     apply fast
    done
  show t ≤♮ ?v u ≤♮ ?w
     using z by (simp-all add: convex-le-def upper-le-def lower-le-def Rep-PDPlus
Rep-v Rep-w) fast+
qed

lemma convex-le-induct [induct set: convex-le]:
  assumes le: t ≤♮ u
  assumes 2: ⋀t u v. ⟦P t u; P u v⟧ ⟹ P t v
  assumes 3: ⋀a b. a ⊑ b ⟹ P (PDUnit a) (PDUnit b)
  assumes 4: ⋀t u v w. ⟦P t v; P u w⟧ ⟹ P (PDPlus t u) (PDPlus v w)
  shows P t u
  using le
proof (induct t arbitrary: u rule: pd-basis-induct)
  case (PDUnit a)
  then show ?case
  proof (induct u rule: pd-basis-induct1)
    case (PDUnit b)
    then show ?case by (simp add: 3)
  next
    case (PDPlus b t)
    have P (PDPlus (PDUnit a) (PDUnit a)) (PDPlus (PDUnit b) t)
      by (rule 4 [OF 3]) (use PDPlus in simp-all)
    then show ?case by (simp add: PDPlus-absorb)
  qed
next
  case PDPlus
  from PDPlus(1,2) show ?case
    using convex-le-PDPlus-lemma [OF PDPlus(3)] by (auto simp add: 4)
qed
```

31.2 Type definition

```
typedef 'a::bifinite convex-pd (‹(‹notation=‹postfix convex-pd››'(-')♮)›) =
  {S::'a pd-basis set. convex-le.ideal S}
by (rule convex-le.ex-ideal)

instantiation convex-pd :: (bifinite) below
begin

definition
  x ⊑ y ⟷ Rep-convex-pd x ⊆ Rep-convex-pd y
```

instance ..
end

instance *convex-pd* :: (*bifinite*) *po*
using *type-definition-convex-pd below-convex-pd-def*
by (*rule convex-le.typedef-ideal-po*)

instance *convex-pd* :: (*bifinite*) *cpo*
using *type-definition-convex-pd below-convex-pd-def*
by (*rule convex-le.typedef-ideal-cpo*)

definition
convex-principal :: *'a::bifinite pd-basis* ⇒ *'a convex-pd* **where**
convex-principal t = *Abs-convex-pd* {*u. u* ≤♮ *t*}

interpretation *convex-pd*:
ideal-completion convex-le convex-principal Rep-convex-pd
using *type-definition-convex-pd below-convex-pd-def*
using *convex-principal-def pd-basis-countable*
by (*rule convex-le.typedef-ideal-completion*)

Convex powerdomain is pointed

lemma *convex-pd-minimal*: *convex-principal* (*PDUnit compact-bot*) ⊑ *ys*
by (*induct ys rule*: *convex-pd.principal-induct*, *simp*, *simp*)

instance *convex-pd* :: (*bifinite*) *pcpo*
by *intro-classes* (*fast intro*: *convex-pd-minimal*)

lemma *inst-convex-pd-pcpo*: ⊥ = *convex-principal* (*PDUnit compact-bot*)
by (*rule convex-pd-minimal* [*THEN bottomI*, *symmetric*])

31.3 Monadic unit and plus

definition
convex-unit :: *'a::bifinite* → *'a convex-pd* **where**
convex-unit = *compact-basis.extension* (λ*a. convex-principal* (*PDUnit a*))

definition
convex-plus :: *'a::bifinite convex-pd* → *'a convex-pd* → *'a convex-pd* **where**
convex-plus = *convex-pd.extension* (λ*t. convex-pd.extension* (λ*u.*
convex-principal (*PDPlus t u*)))

abbreviation
convex-add :: *'a::bifinite convex-pd* ⇒ *'a convex-pd* ⇒ *'a convex-pd*
(**infixl** ‹∪♮› *65*) **where**
xs ∪♮ *ys* == *convex-plus·xs·ys*

syntax
-convex-pd :: *args* ⇒ *logic* (‹(‹*indent=1 notation=*‹*mixfix convex-pd enumera-*

tion››{-}♮)›)
translations
  {*x*,*xs*}♮ == {*x*}♮ ∪♮ {*xs*}♮
  {*x*}♮ == *CONST convex-unit·x*

lemma *convex-unit-Rep-compact-basis* [*simp*]:
  {*Rep-compact-basis a*}♮ = *convex-principal* (*PDUnit a*)
unfolding *convex-unit-def*
by (*simp add*: *compact-basis.extension-principal PDUnit-convex-mono*)

lemma *convex-plus-principal* [*simp*]:
  convex-principal t ∪♮ *convex-principal u* = *convex-principal* (*PDPlus t u*)
unfolding *convex-plus-def*
by (*simp add*: *convex-pd.extension-principal*
    convex-pd.extension-mono PDPlus-convex-mono)

interpretation *convex-add*: *semilattice convex-add* **proof**
  fix *xs ys zs* :: *'a convex-pd*
  show (*xs* ∪♮ *ys*) ∪♮ *zs* = *xs* ∪♮ (*ys* ∪♮ *zs*)
    apply (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
    apply (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
    apply (*induct zs rule*: *convex-pd.principal-induct*, *simp*)
    apply (*simp add*: *PDPlus-assoc*)
    done
  show *xs* ∪♮ *ys* = *ys* ∪♮ *xs*
    apply (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
    apply (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
    apply (*simp add*: *PDPlus-commute*)
    done
  show *xs* ∪♮ *xs* = *xs*
    apply (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
    apply (*simp add*: *PDPlus-absorb*)
    done
qed

lemmas *convex-plus-assoc* = *convex-add.assoc*
lemmas *convex-plus-commute* = *convex-add.commute*
lemmas *convex-plus-absorb* = *convex-add.idem*
lemmas *convex-plus-left-commute* = *convex-add.left-commute*
lemmas *convex-plus-left-absorb* = *convex-add.left-idem*

Useful for *simp add*: *convex-plus-ac*

lemmas *convex-plus-ac* =
  convex-plus-assoc convex-plus-commute convex-plus-left-commute

Useful for *simp only*: *convex-plus-aci*

lemmas *convex-plus-aci* =
  convex-plus-ac convex-plus-absorb convex-plus-left-absorb

lemma *convex-unit-below-plus-iff* [*simp*]:
$\{x\}\natural \sqsubseteq ys \cup\natural\ zs \longleftrightarrow \{x\}\natural \sqsubseteq ys \wedge \{x\}\natural \sqsubseteq zs$
apply (*induct x rule*: *compact-basis.principal-induct*, *simp*)
apply (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
apply (*induct zs rule*: *convex-pd.principal-induct*, *simp*)
apply *simp*
done

lemma *convex-plus-below-unit-iff* [*simp*]:
$xs \cup\natural\ ys \sqsubseteq \{z\}\natural \longleftrightarrow xs \sqsubseteq \{z\}\natural \wedge ys \sqsubseteq \{z\}\natural$
apply (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
apply (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
apply (*induct z rule*: *compact-basis.principal-induct*, *simp*)
apply *simp*
done

lemma *convex-unit-below-iff* [*simp*]: $\{x\}\natural \sqsubseteq \{y\}\natural \longleftrightarrow x \sqsubseteq y$
apply (*induct x rule*: *compact-basis.principal-induct*, *simp*)
apply (*induct y rule*: *compact-basis.principal-induct*, *simp*)
apply *simp*
done

lemma *convex-unit-eq-iff* [*simp*]: $\{x\}\natural = \{y\}\natural \longleftrightarrow x = y$
unfolding *po-eq-conv* **by** *simp*

lemma *convex-unit-strict* [*simp*]: $\{\bot\}\natural = \bot$
using *convex-unit-Rep-compact-basis* [*of compact-bot*]
by (*simp add*: *inst-convex-pd-pcpo*)

lemma *convex-unit-bottom-iff* [*simp*]: $\{x\}\natural = \bot \longleftrightarrow x = \bot$
unfolding *convex-unit-strict* [*symmetric*] **by** (*rule convex-unit-eq-iff*)

lemma *compact-convex-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\natural$
by (*auto dest*!: *compact-basis.compact-imp-principal*)

lemma *compact-convex-unit-iff* [*simp*]: *compact* $\{x\}\natural \longleftrightarrow$ *compact* x
apply (*safe elim*!: *compact-convex-unit*)
apply (*simp only*: *compact-def convex-unit-below-iff* [*symmetric*])
apply (*erule adm-subst* [*OF cont-Rep-cfun2*])
done

lemma *compact-convex-plus* [*simp*]:
$[\![$*compact xs*; *compact ys*$]\!] \Longrightarrow$ *compact* $(xs \cup\natural\ ys)$
by (*auto dest*!: *convex-pd.compact-imp-principal*)

31.4 Induction rules

lemma *convex-pd-induct1*:
assumes *P*: *adm P*

assumes *unit*: $\bigwedge x.\ P\ \{x\}\natural$
assumes *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\natural;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\natural \cup\natural\ ys)$
shows $P\ (xs::'a::bifinite\ convex\text{-}pd)$
proof (*induct xs rule*: *convex-pd.principal-induct*)
show *P* (*convex-principal a*) **for** *a*
proof (*induct a rule*: *pd-basis-induct1*)
case *PDUnit*
show *?case* **by** (*simp only*: *convex-unit-Rep-compact-basis* [*symmetric*]) (*rule unit*)
next
case *PDPlus*
show *?case*
by (*simp only*: *convex-unit-Rep-compact-basis* [*symmetric*] *convex-plus-principal* [*symmetric*])
(*rule insert* [*OF unit PDPlus*])
qed
qed (*rule P*)

lemma *convex-pd-induct* [*case-names adm convex-unit convex-plus*, *induct type*: *convex-pd*]:
assumes *P*: *adm P*
assumes *unit*: $\bigwedge x.\ P\ \{x\}\natural$
assumes *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\natural\ ys)$
shows $P\ (xs::'a::bifinite\ convex\text{-}pd)$
proof (*induct xs rule*: *convex-pd.principal-induct*)
show *P* (*convex-principal a*) **for** *a*
proof (*induct a rule*: *pd-basis-induct*)
case *PDUnit*
then show *?case* **by** (*simp only*: *convex-unit-Rep-compact-basis* [*symmetric*] *unit*)
next
case *PDPlus*
then show *?case* **by** (*simp only*: *convex-plus-principal* [*symmetric*] *plus*)
qed
qed (*rule P*)

31.5 Monadic bind

definition
convex-bind-basis ::
$'a::bifinite\ pd\text{-}basis \Rightarrow ('a \rightarrow 'b\ convex\text{-}pd) \rightarrow 'b::bifinite\ convex\text{-}pd$ **where**
convex-bind-basis = *fold-pd*
$(\lambda a.\ \Lambda\ f.\ f\cdot(Rep\text{-}compact\text{-}basis\ a))$
$(\lambda x\ y.\ \Lambda\ f.\ x\cdot f \cup\natural\ y\cdot f)$

lemma *ACI-convex-bind*:
semilattice $(\lambda x\ y.\ \Lambda\ f.\ x\cdot f \cup\natural\ y\cdot f)$
apply *unfold-locales*
apply (*simp add*: *convex-plus-assoc*)

apply (*simp add*: *convex-plus-commute*)
apply (*simp add*: *eta-cfun*)
done

lemma *convex-bind-basis-simps* [*simp*]:
convex-bind-basis (*PDUnit a*) =
(Λ *f*. *f*·(*Rep-compact-basis a*))
convex-bind-basis (*PDPlus t u*) =
(Λ *f*. *convex-bind-basis t*·*f* ∪♮ *convex-bind-basis u*·*f*)
unfolding *convex-bind-basis-def*
apply −
apply (*rule fold-pd-PDUnit* [*OF ACI-convex-bind*])
apply (*rule fold-pd-PDPlus* [*OF ACI-convex-bind*])
done

lemma *convex-bind-basis-mono*:
t ≤♮ *u* ⟹ *convex-bind-basis t* ⊑ *convex-bind-basis u*
apply (*erule convex-le-induct*)
apply (*erule* (*1*) *below-trans*)
apply (*simp add*: *monofun-LAM monofun-cfun*)
apply (*simp add*: *monofun-LAM monofun-cfun*)
done

definition
convex-bind :: ′*a*::*bifinite convex-pd* → (′*a* → ′*b convex-pd*) → ′*b*::*bifinite convex-pd*
where
convex-bind = *convex-pd.extension convex-bind-basis*

syntax
-convex-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
(‹(‹*indent=3 notation=*‹*binder convex-bind*››⋃♮-∈-./ -)› [*0*, *0*, *10*] *10*)

translations
⋃♮*x*∈*xs*. *e* == *CONST convex-bind*·*xs*·(Λ *x*. *e*)

lemma *convex-bind-principal* [*simp*]:
convex-bind·(*convex-principal t*) = *convex-bind-basis t*
unfolding *convex-bind-def*
apply (*rule convex-pd.extension-principal*)
apply (*erule convex-bind-basis-mono*)
done

lemma *convex-bind-unit* [*simp*]:
convex-bind·{*x*}♮·*f* = *f*·*x*
by (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

lemma *convex-bind-plus* [*simp*]:
convex-bind·(*xs* ∪♮ *ys*)·*f* = *convex-bind*·*xs*·*f* ∪♮ *convex-bind*·*ys*·*f*
by (*induct xs rule*: *convex-pd.principal-induct*, *simp*,

induct ys rule: *convex-pd.principal-induct*, *simp*, *simp*)

lemma *convex-bind-strict* [*simp*]: *convex-bind*$\cdot\bot\cdot$*f* = *f*$\cdot\bot$
unfolding *convex-unit-strict* [*symmetric*] **by** (*rule convex-bind-unit*)

lemma *convex-bind-bind*:
convex-bind$\cdot$(*convex-bind*$\cdot$*xs*$\cdot$*f*)$\cdot$*g* =
convex-bind$\cdot$*xs*$\cdot$(Λ *x*. *convex-bind*$\cdot$(*f*$\cdot$*x*)$\cdot$*g*)
by (*induct xs*, *simp-all*)

31.6 Map

definition
convex-map :: ('*a*::*bifinite* $\rightarrow$ '*b*) $\rightarrow$ '*a convex-pd* $\rightarrow$ '*b*::*bifinite convex-pd* **where**
convex-map = (Λ *f xs*. *convex-bind*$\cdot$*xs*$\cdot$(Λ *x*. {*f*$\cdot$*x*}$\natural$))

lemma *convex-map-unit* [*simp*]:
convex-map$\cdot$*f*$\cdot${*x*}$\natural$ = {*f*$\cdot$*x*}$\natural$
unfolding *convex-map-def* **by** *simp*

lemma *convex-map-plus* [*simp*]:
convex-map$\cdot$*f*$\cdot$(*xs* $\cup\natural$ *ys*) = *convex-map*$\cdot$*f*$\cdot$*xs* $\cup\natural$ *convex-map*$\cdot$*f*$\cdot$*ys*
unfolding *convex-map-def* **by** *simp*

lemma *convex-map-bottom* [*simp*]: *convex-map*$\cdot$*f*$\cdot\bot$ = {*f*$\cdot\bot$}$\natural$
unfolding *convex-map-def* **by** *simp*

lemma *convex-map-ident*: *convex-map*$\cdot$(Λ *x*. *x*)$\cdot$*xs* = *xs*
by (*induct xs rule*: *convex-pd-induct*, *simp-all*)

lemma *convex-map-ID*: *convex-map*$\cdot$*ID* = *ID*
by (*simp add*: *cfun-eq-iff ID-def convex-map-ident*)

lemma *convex-map-map*:
convex-map$\cdot$*f*$\cdot$(*convex-map*$\cdot$*g*$\cdot$*xs*) = *convex-map*$\cdot$(Λ *x*. *f*$\cdot$(*g*$\cdot$*x*))$\cdot$*xs*
by (*induct xs rule*: *convex-pd-induct*, *simp-all*)

lemma *convex-bind-map*:
convex-bind$\cdot$(*convex-map*$\cdot$*f*$\cdot$*xs*)$\cdot$*g* = *convex-bind*$\cdot$*xs*$\cdot$(Λ *x*. *g*$\cdot$(*f*$\cdot$*x*))
by (*simp add*: *convex-map-def convex-bind-bind*)

lemma *convex-map-bind*:
convex-map$\cdot$*f*$\cdot$(*convex-bind*$\cdot$*xs*$\cdot$*g*) = *convex-bind*$\cdot$*xs*$\cdot$(Λ *x*. *convex-map*$\cdot$*f*$\cdot$(*g*$\cdot$*x*))
by (*simp add*: *convex-map-def convex-bind-bind*)

lemma *ep-pair-convex-map*: *ep-pair e p* $\Longrightarrow$ *ep-pair* (*convex-map*$\cdot$*e*) (*convex-map*$\cdot$*p*)
apply *standard*
apply (*induct-tac x rule*: *convex-pd-induct*, *simp-all add*: *ep-pair.e-inverse*)
apply (*induct-tac y rule*: *convex-pd-induct*)

apply (*simp-all add*: *ep-pair.e-p-below monofun-cfun*)
done

lemma *deflation-convex-map*: *deflation d* $\Longrightarrow$ *deflation* (*convex-map*·*d*)
apply *standard*
apply (*induct-tac x rule*: *convex-pd-induct*, *simp-all add*: *deflation.idem*)
apply (*induct-tac x rule*: *convex-pd-induct*)
apply (*simp-all add*: *deflation.below monofun-cfun*)
done

lemma *finite-deflation-convex-map*:
  assumes *finite-deflation d* **shows** *finite-deflation* (*convex-map*·*d*)
proof (*rule finite-deflation-intro*)
  interpret *d*: *finite-deflation d* **by** *fact*
  from *d.deflation-axioms* **show** *deflation* (*convex-map*·*d*)
    by (*rule deflation-convex-map*)
  have *finite* (*range* ($\lambda x.$ *d*·*x*)) **by** (*rule d.finite-range*)
  hence *finite* (*Rep-compact-basis* −' *range* ($\lambda x.$ *d*·*x*))
    by (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-compact-basis-inject*)
  hence *finite* (*Pow* (*Rep-compact-basis* −' *range* ($\lambda x.$ *d*·*x*))) **by** *simp*
  hence *finite* (*Rep-pd-basis* −' (*Pow* (*Rep-compact-basis* −' *range* ($\lambda x.$ *d*·*x*))))
    by (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-pd-basis-inject*)
  hence ∗: *finite* (*convex-principal* ' *Rep-pd-basis* −' (*Pow* (*Rep-compact-basis* −'
range ($\lambda x.$ *d*·*x*)))) **by** *simp*
  hence *finite* (*range* ($\lambda xs.$ *convex-map*·*d*·*xs*))
    apply (*rule rev-finite-subset*)
    apply *clarsimp*
    apply (*induct-tac xs rule*: *convex-pd.principal-induct*)
    apply (*simp add*: *adm-mem-finite* ∗)
    apply (*rename-tac t*, *induct-tac t rule*: *pd-basis-induct*)
    apply (*simp only*: *convex-unit-Rep-compact-basis* [*symmetric*] *convex-map-unit*)
    apply *simp*
    apply (*subgoal-tac* ∃ *b*. *d*·(*Rep-compact-basis a*) = *Rep-compact-basis b*)
    apply *clarsimp*
    apply (*rule imageI*)
    apply (*rule vimageI2*)
    apply (*simp add*: *Rep-PDUnit*)
    apply (*rule range-eqI*)
    apply (*erule sym*)
    apply (*rule exI*)
    apply (*rule Abs-compact-basis-inverse* [*symmetric*])
    apply (*simp add*: *d.compact*)
    apply (*simp only*: *convex-plus-principal* [*symmetric*] *convex-map-plus*)
    apply *clarsimp*
    apply (*rule imageI*)
    apply (*rule vimageI2*)
    apply (*simp add*: *Rep-PDPlus*)
    done

thus *finite* {*xs*. *convex-map·d·xs* = *xs*}
  by (*rule finite-range-imp-finite-fixes*)
qed

31.7 Convex powerdomain is bifinite

lemma *approx-chain-convex-map*:
  assumes *approx-chain a*
  shows *approx-chain* (λ*i*. *convex-map*·(*a i*))
  using *assms* **unfolding** *approx-chain-def*
  by (*simp add*: *lub-APP convex-map-ID finite-deflation-convex-map*)

instance *convex-pd* :: (*bifinite*) *bifinite*
proof
  show ∃(*a*::*nat* ⇒ ′*a convex-pd* → ′*a convex-pd*). *approx-chain a*
    using *bifinite* [**where** ′*a*=′*a*]
    by (*fast intro*!: *approx-chain-convex-map*)
qed

31.8 Join

definition
  convex-join :: ′*a*::*bifinite convex-pd convex-pd* → ′*a convex-pd* **where**
  convex-join = (Λ *xss*. *convex-bind·xss*·(Λ *xs*. *xs*))

lemma *convex-join-unit* [*simp*]:
  convex-join·{*xs*}♮ = *xs*
unfolding *convex-join-def* **by** *simp*

lemma *convex-join-plus* [*simp*]:
  convex-join·(*xss* ∪♮ *yss*) = *convex-join·xss* ∪♮ *convex-join·yss*
unfolding *convex-join-def* **by** *simp*

lemma *convex-join-bottom* [*simp*]: *convex-join*·⊥ = ⊥
unfolding *convex-join-def* **by** *simp*

lemma *convex-join-map-unit*:
  convex-join·(*convex-map·convex-unit·xs*) = *xs*
by (*induct xs rule*: *convex-pd-induct*, *simp-all*)

lemma *convex-join-map-join*:
  convex-join·(*convex-map·convex-join·xsss*) = *convex-join*·(*convex-join·xsss*)
by (*induct xsss rule*: *convex-pd-induct*, *simp-all*)

lemma *convex-join-map-map*:
  convex-join·(*convex-map*·(*convex-map·f*)·*xss*) =
   convex-map·f·(*convex-join·xss*)
by (*induct xss rule*: *convex-pd-induct*, *simp-all*)

31.9 Conversions to other powerdomains

Convex to upper

lemma *convex-le-imp-upper-le*: $t \leq\natural\ u \Longrightarrow t \leq\sharp\ u$
unfolding *convex-le-def* **by** *simp*

definition
  convex-to-upper :: *'a::bifinite convex-pd* → *'a upper-pd* **where**
  convex-to-upper = *convex-pd.extension upper-principal*

lemma *convex-to-upper-principal* [*simp*]:
  convex-to-upper·(*convex-principal t*) = *upper-principal t*
unfolding *convex-to-upper-def*
apply (*rule convex-pd.extension-principal*)
apply (*rule upper-pd.principal-mono*)
apply (*erule convex-le-imp-upper-le*)
done

lemma *convex-to-upper-unit* [*simp*]:
  convex-to-upper·{*x*}♮ = {*x*}♯
by (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

lemma *convex-to-upper-plus* [*simp*]:
  convex-to-upper·(*xs* ∪♮ *ys*) = *convex-to-upper·xs* ∪♯ *convex-to-upper·ys*
by (*induct xs rule*: *convex-pd.principal-induct*, *simp*,
    induct ys rule: *convex-pd.principal-induct*, *simp*, *simp*)

lemma *convex-to-upper-bind* [*simp*]:
  convex-to-upper·(*convex-bind·xs·f*) =
    upper-bind·(*convex-to-upper·xs*)·(*convex-to-upper oo f*)
by (*induct xs rule*: *convex-pd-induct*, *simp*, *simp*, *simp*)

lemma *convex-to-upper-map* [*simp*]:
  convex-to-upper·(*convex-map·f·xs*) = *upper-map·f·*(*convex-to-upper·xs*)
by (*simp add*: *convex-map-def upper-map-def cfcomp-LAM*)

lemma *convex-to-upper-join* [*simp*]:
  convex-to-upper·(*convex-join·xss*) =
    upper-bind·(*convex-to-upper·xss*)*·convex-to-upper*
by (*simp add*: *convex-join-def upper-join-def cfcomp-LAM eta-cfun*)

Convex to lower

lemma *convex-le-imp-lower-le*: $t \leq\natural\ u \Longrightarrow t \leq\flat\ u$
unfolding *convex-le-def* **by** *simp*

definition
  convex-to-lower :: *'a::bifinite convex-pd* → *'a lower-pd* **where**
  convex-to-lower = *convex-pd.extension lower-principal*

lemma *convex-to-lower-principal* [*simp*]:
  convex-to-lower·(*convex-principal t*) = *lower-principal t*
unfolding *convex-to-lower-def*
apply (*rule convex-pd.extension-principal*)
apply (*rule lower-pd.principal-mono*)
apply (*erule convex-le-imp-lower-le*)
done

lemma *convex-to-lower-unit* [*simp*]:
  convex-to-lower·{*x*}♮ = {*x*}♭
by (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

lemma *convex-to-lower-plus* [*simp*]:
  convex-to-lower·(*xs* ∪♮ *ys*) = *convex-to-lower*·*xs* ∪♭ *convex-to-lower*·*ys*
by (*induct xs rule*: *convex-pd.principal-induct*, *simp*,
    induct ys rule: *convex-pd.principal-induct*, *simp*, *simp*)

lemma *convex-to-lower-bind* [*simp*]:
  convex-to-lower·(*convex-bind*·*xs*·*f*) =
    lower-bind·(*convex-to-lower*·*xs*)·(*convex-to-lower oo f*)
by (*induct xs rule*: *convex-pd-induct*, *simp*, *simp*, *simp*)

lemma *convex-to-lower-map* [*simp*]:
  convex-to-lower·(*convex-map*·*f*·*xs*) = *lower-map*·*f*·(*convex-to-lower*·*xs*)
by (*simp add*: *convex-map-def lower-map-def cfcomp-LAM*)

lemma *convex-to-lower-join* [*simp*]:
  convex-to-lower·(*convex-join*·*xss*) =
    lower-bind·(*convex-to-lower*·*xss*)·*convex-to-lower*
by (*simp add*: *convex-join-def lower-join-def cfcomp-LAM eta-cfun*)

Ordering property

lemma *convex-pd-below-iff*:
  (*xs* ⊑ *ys*) =
    (*convex-to-upper*·*xs* ⊑ *convex-to-upper*·*ys* ∧
     convex-to-lower·*xs* ⊑ *convex-to-lower*·*ys*)
apply (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
apply (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
apply (*simp add*: *convex-le-def*)
done

lemmas *convex-plus-below-plus-iff* =
  convex-pd-below-iff [**where** *xs*=*xs* ∪♮ *ys* **and** *ys*=*zs* ∪♮ *ws*]
  for *xs ys zs ws*

lemmas *convex-pd-below-simps* =
  convex-unit-below-plus-iff
  convex-plus-below-unit-iff
  convex-plus-below-plus-iff

convex-unit-below-iff
convex-to-upper-unit
convex-to-upper-plus
convex-to-lower-unit
convex-to-lower-plus
upper-pd-below-simps
lower-pd-below-simps

end

32 Powerdomains

theory *Powerdomains*
imports *ConvexPD Domain*
begin

32.1 Universal domain embeddings

definition *upper-emb* = *udom-emb* ($\lambda i.$ *upper-map*·(*udom-approx i*))
definition *upper-prj* = *udom-prj* ($\lambda i.$ *upper-map*·(*udom-approx i*))

definition *lower-emb* = *udom-emb* ($\lambda i.$ *lower-map*·(*udom-approx i*))
definition *lower-prj* = *udom-prj* ($\lambda i.$ *lower-map*·(*udom-approx i*))

definition *convex-emb* = *udom-emb* ($\lambda i.$ *convex-map*·(*udom-approx i*))
definition *convex-prj* = *udom-prj* ($\lambda i.$ *convex-map*·(*udom-approx i*))

lemma *ep-pair-upper*: *ep-pair upper-emb upper-prj*
unfolding *upper-emb-def upper-prj-def*
by (*simp add*: *ep-pair-udom approx-chain-upper-map*)

lemma *ep-pair-lower*: *ep-pair lower-emb lower-prj*
unfolding *lower-emb-def lower-prj-def*
by (*simp add*: *ep-pair-udom approx-chain-lower-map*)

lemma *ep-pair-convex*: *ep-pair convex-emb convex-prj*
unfolding *convex-emb-def convex-prj-def*
by (*simp add*: *ep-pair-udom approx-chain-convex-map*)

32.2 Deflation combinators

definition *upper-defl* :: *udom defl* $\to$ *udom defl*
where *upper-defl* = *defl-fun1 upper-emb upper-prj upper-map*

definition *lower-defl* :: *udom defl* $\to$ *udom defl*
where *lower-defl* = *defl-fun1 lower-emb lower-prj lower-map*

definition *convex-defl* :: *udom defl* $\to$ *udom defl*
where *convex-defl* = *defl-fun1 convex-emb convex-prj convex-map*

lemma *cast-upper-defl*:
cast·(upper-defl·A) = upper-emb oo upper-map·(cast·A) oo upper-prj
using *ep-pair-upper finite-deflation-upper-map*
unfolding *upper-defl-def* **by** (*rule cast-defl-fun1*)

lemma *cast-lower-defl*:
cast·(lower-defl·A) = lower-emb oo lower-map·(cast·A) oo lower-prj
using *ep-pair-lower finite-deflation-lower-map*
unfolding *lower-defl-def* **by** (*rule cast-defl-fun1*)

lemma *cast-convex-defl*:
cast·(convex-defl·A) = convex-emb oo convex-map·(cast·A) oo convex-prj
using *ep-pair-convex finite-deflation-convex-map*
unfolding *convex-defl-def* **by** (*rule cast-defl-fun1*)

32.3 Domain class instances

instantiation *upper-pd* :: (*domain*) *domain*
begin

definition
emb = upper-emb oo upper-map·emb

definition
prj = upper-map·prj oo upper-prj

definition
defl (t::'a upper-pd itself) = upper-defl·DEFL('a)

definition
(*liftemb* :: *'a upper-pd u → udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u → 'a upper-pd u*) = *u-map·prj*

definition
liftdefl (t::'a upper-pd itself) = liftdefl-of·DEFL('a upper-pd)

instance proof
show *ep-pair emb (prj :: udom → 'a upper-pd)*
unfolding *emb-upper-pd-def prj-upper-pd-def*
by (*intro ep-pair-comp ep-pair-upper ep-pair-upper-map ep-pair-emb-prj*)
next
show *cast·DEFL('a upper-pd) = emb oo (prj :: udom → 'a upper-pd)*
unfolding *emb-upper-pd-def prj-upper-pd-def defl-upper-pd-def cast-upper-defl*
by (*simp add: cast-DEFL oo-def cfun-eq-iff upper-map-map*)
qed (*fact liftemb-upper-pd-def liftprj-upper-pd-def liftdefl-upper-pd-def*)+

end

instantiation *lower-pd* :: (*domain*) *domain*
begin

definition
emb = *lower-emb oo lower-map·emb*

definition
prj = *lower-map·prj oo lower-prj*

definition
defl (*t*::′*a lower-pd itself*) = *lower-defl·DEFL*(′*a*)

definition
(*liftemb* :: ′*a lower-pd u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → ′*a lower-pd u*) = *u-map·prj*

definition
liftdefl (*t*::′*a lower-pd itself*) = *liftdefl-of·DEFL*(′*a lower-pd*)

instance proof
show *ep-pair emb* (*prj* :: *udom* → ′*a lower-pd*)
unfolding *emb-lower-pd-def prj-lower-pd-def*
by (*intro ep-pair-comp ep-pair-lower ep-pair-lower-map ep-pair-emb-prj*)
next
show *cast·DEFL*(′*a lower-pd*) = *emb oo* (*prj* :: *udom* → ′*a lower-pd*)
unfolding *emb-lower-pd-def prj-lower-pd-def defl-lower-pd-def cast-lower-defl*
by (*simp add*: *cast-DEFL oo-def cfun-eq-iff lower-map-map*)
qed (*fact liftemb-lower-pd-def liftprj-lower-pd-def liftdefl-lower-pd-def*)+

end

instantiation *convex-pd* :: (*domain*) *domain*
begin

definition
emb = *convex-emb oo convex-map·emb*

definition
prj = *convex-map·prj oo convex-prj*

definition
defl (*t*::′*a convex-pd itself*) = *convex-defl·DEFL*(′*a*)

definition
(*liftemb* :: ′*a convex-pd u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → *'a convex-pd u*) = *u-map·prj*

definition
liftdefl (*t*::*'a convex-pd itself*) = *liftdefl-of·DEFL*(*'a convex-pd*)

instance proof
show *ep-pair emb* (*prj* :: *udom* → *'a convex-pd*)
unfolding *emb-convex-pd-def prj-convex-pd-def*
by (*intro ep-pair-comp ep-pair-convex ep-pair-convex-map ep-pair-emb-prj*)
next
show *cast·DEFL*(*'a convex-pd*) = *emb oo* (*prj* :: *udom* → *'a convex-pd*)
unfolding *emb-convex-pd-def prj-convex-pd-def defl-convex-pd-def cast-convex-defl*
by (*simp add*: *cast-DEFL oo-def cfun-eq-iff convex-map-map*)
qed (*fact liftemb-convex-pd-def liftprj-convex-pd-def liftdefl-convex-pd-def*)+

end

lemma *DEFL-upper*: *DEFL*(*'a*::*domain upper-pd*) = *upper-defl·DEFL*(*'a*)
by (*rule defl-upper-pd-def*)

lemma *DEFL-lower*: *DEFL*(*'a*::*domain lower-pd*) = *lower-defl·DEFL*(*'a*)
by (*rule defl-lower-pd-def*)

lemma *DEFL-convex*: *DEFL*(*'a*::*domain convex-pd*) = *convex-defl·DEFL*(*'a*)
by (*rule defl-convex-pd-def*)

32.4 Isomorphic deflations

lemma *isodefl-upper*:
isodefl d t ⟹ *isodefl* (*upper-map·d*) (*upper-defl·t*)
apply (*rule isodeflI*)
apply (*simp add*: *cast-upper-defl cast-isodefl*)
apply (*simp add*: *emb-upper-pd-def prj-upper-pd-def*)
apply (*simp add*: *upper-map-map*)
done

lemma *isodefl-lower*:
isodefl d t ⟹ *isodefl* (*lower-map·d*) (*lower-defl·t*)
apply (*rule isodeflI*)
apply (*simp add*: *cast-lower-defl cast-isodefl*)
apply (*simp add*: *emb-lower-pd-def prj-lower-pd-def*)
apply (*simp add*: *lower-map-map*)
done

lemma *isodefl-convex*:
isodefl d t ⟹ *isodefl* (*convex-map·d*) (*convex-defl·t*)
apply (*rule isodeflI*)

apply (*simp add*: *cast-convex-defl cast-isodefl*)
apply (*simp add*: *emb-convex-pd-def prj-convex-pd-def*)
apply (*simp add*: *convex-map-map*)
done

32.5 Domain package setup for powerdomains

lemmas [*domain-defl-simps*] = *DEFL-upper DEFL-lower DEFL-convex*
lemmas [*domain-map-ID*] = *upper-map-ID lower-map-ID convex-map-ID*
lemmas [*domain-isodefl*] = *isodefl-upper isodefl-lower isodefl-convex*

lemmas [*domain-deflation*] =
deflation-upper-map deflation-lower-map deflation-convex-map

setup ‹
fold Domain-Take-Proofs.add-rec-type
[(***type-name*** ‹*upper-pd*›, [*true*]),
(***type-name*** ‹*lower-pd*›, [*true*]),
(***type-name*** ‹*convex-pd*›, [*true*])]
›

end

theory *HOLCF*
imports
Main
Domain
Powerdomains
begin

default-sort *domain*

end