Isabelle/HOLCF — Higher-Order Logic of Computable Functions

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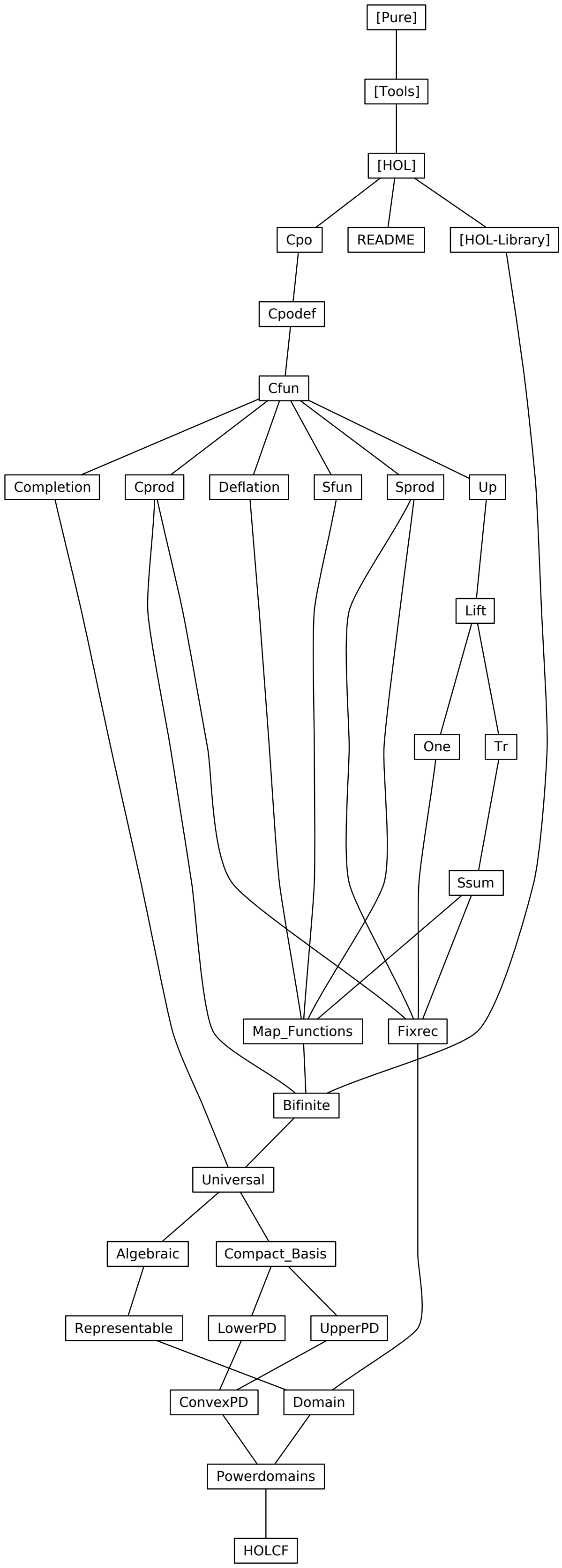
Contents


[Pure]
[Tools]
[HOL]
Cpo
README
[HOL-Library]
Cpodef
Cfun
Completion
Cprod
Deflation
Sfun
Sprod
Up
Lift
One
Tr
Ssum
Map_Functions
Fixrec
Bifinite
Universal
Algebraic
Compact_Basis
Representable
LowerPD
UpperPD
ConvexPD
Domain
Powerdomains
HOLCF

theory *Cpo*
imports *Main*
begin

1 Partial orders

declare [[*typedef-overloaded*]]

1.1 Type class for partial orders

class *below* =
fixes *below* :: $'a \Rightarrow 'a \Rightarrow bool$
begin

notation (*ASCII*)
below (**infix** ‹<<› *50*)

notation
below (**infix** ‹⊑› *50*)

abbreviation *not-below* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** ‹⋢› *50*)
where *not-below* $x\ y \equiv \neg$ *below* $x\ y$

notation (*ASCII*)
not-below (**infix** ‹~<<› *50*)

lemma *below-eq-trans*: $a \sqsubseteq b \Longrightarrow b = c \Longrightarrow a \sqsubseteq c$
⟨*proof*⟩

lemma *eq-below-trans*: $a = b \Longrightarrow b \sqsubseteq c \Longrightarrow a \sqsubseteq c$
⟨*proof*⟩

end

class *po* = *below* +
assumes *below-refl* [*iff*]: $x \sqsubseteq x$
assumes *below-trans*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq z \Longrightarrow x \sqsubseteq z$
assumes *below-antisym*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow x = y$
begin

lemma *eq-imp-below*: $x = y \Longrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *box-below*: $a \sqsubseteq b \Longrightarrow c \sqsubseteq a \Longrightarrow b \sqsubseteq d \Longrightarrow c \sqsubseteq d$
⟨*proof*⟩

lemma *po-eq-conv*: $x = y \longleftrightarrow x \sqsubseteq y \wedge y \sqsubseteq x$

⟨*proof*⟩

lemma *rev-below-trans*: $y \sqsubseteq z \Longrightarrow x \sqsubseteq y \Longrightarrow x \sqsubseteq z$
⟨*proof*⟩

lemma *not-below2not-eq*: $x \not\sqsubseteq y \Longrightarrow x \neq y$
⟨*proof*⟩

end

lemmas *HOLCF-trans-rules* [*trans*] =
below-trans
below-antisym
below-eq-trans
eq-below-trans

context *po*
begin

1.2 Upper bounds

definition *is-ub* :: *'a set* ⇒ *'a* ⇒ *bool* (**infix** ‹<|› *55*)
where $S <| x \longleftrightarrow (\forall y{\in}S.\ y \sqsubseteq x)$

lemma *is-ubI*: $(\bigwedge x.\ x \in S \Longrightarrow x \sqsubseteq u) \Longrightarrow S <| u$
⟨*proof*⟩

lemma *is-ubD*: $[\![S <| u; x \in S]\!] \Longrightarrow x \sqsubseteq u$
⟨*proof*⟩

lemma *ub-imageI*: $(\bigwedge x.\ x \in S \Longrightarrow f\ x \sqsubseteq u) \Longrightarrow (\lambda x.\ f\ x)$ ‘ $S <| u$
⟨*proof*⟩

lemma *ub-imageD*: $[\![f$ ‘ $S <| u; x \in S]\!] \Longrightarrow f\ x \sqsubseteq u$
⟨*proof*⟩

lemma *ub-rangeI*: $(\bigwedge i.\ S\ i \sqsubseteq x) \Longrightarrow$ *range* $S <| x$
⟨*proof*⟩

lemma *ub-rangeD*: *range* $S <| x \Longrightarrow S\ i \sqsubseteq x$
⟨*proof*⟩

lemma *is-ub-empty* [*simp*]: $\{\} <| u$
⟨*proof*⟩

lemma *is-ub-insert* [*simp*]: (*insert* $x\ A) <| y = (x \sqsubseteq y \wedge A <| y)$
⟨*proof*⟩

lemma *is-ub-upward*: $[\![S <| x; x \sqsubseteq y]\!] \Longrightarrow S <| y$

⟨proof⟩

1.3 Least upper bounds

definition *is-lub* :: $'a\ set \Rightarrow 'a \Rightarrow bool$ (**infix** ‹<<|› *55*)
  where $S <<| x \longleftrightarrow S <| x \wedge (\forall u.\ S <| u \longrightarrow x \sqsubseteq u)$

definition *lub* :: $'a\ set \Rightarrow 'a$
  where $lub\ S = (THE\ x.\ S <<| x)$

end

syntax (*ASCII*)
  -BLub :: [*pttrn*, 'a set, 'b] ⇒ 'b (‹(‹*indent=3 notation=*‹*binder LUB*››*LUB* -:-./ -)› [*0*,*0*, *10*] *10*)

syntax
  -BLub :: [*pttrn*, 'a set, 'b] ⇒ 'b (‹(‹*indent=3 notation=*‹*binder* ⨆››⨆-∈-./ -)› [*0*,*0*, *10*] *10*)

syntax-consts
  -BLub ⇌ *lub*

translations
  $LUB\ x{:}A.\ t \rightleftharpoons CONST\ lub\ ((\lambda x.\ t)\ `\ A)$

context *po*
begin

abbreviation *Lub* (**binder** ‹⨆› *10*)
  where $\bigsqcup n.\ t\ n \equiv lub\ (range\ t)$

notation (*ASCII*)
  Lub (**binder** ‹*LUB* › *10*)

access to some definition as inference rule

lemma *is-lubD1*: $S <<| x \Longrightarrow S <| x$
  ⟨proof⟩

lemma *is-lubD2*: $[\![S <<| x;\ S <| u]\!] \Longrightarrow x \sqsubseteq u$
  ⟨proof⟩

lemma *is-lubI*: $[\![S <| x;\ \bigwedge u.\ S <| u \Longrightarrow x \sqsubseteq u]\!] \Longrightarrow S <<| x$
  ⟨proof⟩

lemma *is-lub-below-iff*: $S <<| x \Longrightarrow x \sqsubseteq u \longleftrightarrow S <| u$
  ⟨proof⟩

lubs are unique

lemma *is-lub-unique*: $S <<| x \Longrightarrow S <<| y \Longrightarrow x = y$
⟨*proof*⟩

technical lemmas about *lub* and $(<<|)$

lemma *is-lub-lub*: $M <<| x \Longrightarrow M <<| lub\ M$
⟨*proof*⟩

lemma *lub-eqI*: $M <<| l \Longrightarrow lub\ M = l$
⟨*proof*⟩

lemma *is-lub-singleton* [*simp*]: $\{x\} <<| x$
⟨*proof*⟩

lemma *lub-singleton* [*simp*]: $lub\ \{x\} = x$
⟨*proof*⟩

lemma *is-lub-bin*: $x \sqsubseteq y \Longrightarrow \{x, y\} <<| y$
⟨*proof*⟩

lemma *lub-bin*: $x \sqsubseteq y \Longrightarrow lub\ \{x, y\} = y$
⟨*proof*⟩

lemma *is-lub-maximal*: $S <| x \Longrightarrow x \in S \Longrightarrow S <<| x$
⟨*proof*⟩

lemma *lub-maximal*: $S <| x \Longrightarrow x \in S \Longrightarrow lub\ S = x$
⟨*proof*⟩

1.4 Countable chains

definition *chain* :: $(nat \Rightarrow 'a) \Rightarrow bool$
where — Here we use countable chains and I prefer to code them as functions!
$chain\ Y = (\forall i.\ Y\ i \sqsubseteq Y\ (Suc\ i))$

lemma *chainI*: $(\bigwedge i.\ Y\ i \sqsubseteq Y\ (Suc\ i)) \Longrightarrow chain\ Y$
⟨*proof*⟩

lemma *chainE*: $chain\ Y \Longrightarrow Y\ i \sqsubseteq Y\ (Suc\ i)$
⟨*proof*⟩

chains are monotone functions

lemma *chain-mono-less*: $chain\ Y \Longrightarrow i < j \Longrightarrow Y\ i \sqsubseteq Y\ j$
⟨*proof*⟩

lemma *chain-mono*: $chain\ Y \Longrightarrow i \le j \Longrightarrow Y\ i \sqsubseteq Y\ j$
⟨*proof*⟩

lemma *chain-shift*: $chain\ Y \Longrightarrow chain\ (\lambda i.\ Y\ (i + j))$
⟨*proof*⟩

technical lemmas about (least) upper bounds of chains

lemma *is-lub-rangeD1*: *range S* <<| *x* $\Longrightarrow$ *S i* $\sqsubseteq$ *x*
⟨*proof*⟩

lemma *is-ub-range-shift*: *chain S* $\Longrightarrow$ *range* (λi. *S* (*i* + *j*)) <| *x* = *range S* <| *x*
⟨*proof*⟩

lemma *is-lub-range-shift*: *chain S* $\Longrightarrow$ *range* (λi. *S* (*i* + *j*)) <<| *x* = *range S* <<| *x*
⟨*proof*⟩

the lub of a constant chain is the constant

lemma *chain-const* [*simp*]: *chain* (λi. *c*)
⟨*proof*⟩

lemma *is-lub-const*: *range* (λx. *c*) <<| *c*
⟨*proof*⟩

lemma *lub-const* [*simp*]: ($\bigsqcup$ *i*. *c*) = *c*
⟨*proof*⟩

1.5 Finite chains

definition *max-in-chain* :: *nat* $\Rightarrow$ (*nat* $\Rightarrow$ *'a*) $\Rightarrow$ *bool*
where — finite chains, needed for monotony of continuous functions
max-in-chain i C $\longleftrightarrow$ ($\forall$ *j*. *i* $\le$ *j* $\longrightarrow$ *C i* = *C j*)

definition *finite-chain* :: (*nat* $\Rightarrow$ *'a*) $\Rightarrow$ *bool*
where *finite-chain C* = (*chain C* $\wedge$ ($\exists$ *i*. *max-in-chain i C*))

results about finite chains

lemma *max-in-chainI*: ($\bigwedge$*j*. *i* $\le$ *j* $\Longrightarrow$ *Y i* = *Y j*) $\Longrightarrow$ *max-in-chain i Y*
⟨*proof*⟩

lemma *max-in-chainD*: *max-in-chain i Y* $\Longrightarrow$ *i* $\le$ *j* $\Longrightarrow$ *Y i* = *Y j*
⟨*proof*⟩

lemma *finite-chainI*: *chain C* $\Longrightarrow$ *max-in-chain i C* $\Longrightarrow$ *finite-chain C*
⟨*proof*⟩

lemma *finite-chainE*: $\llbracket$*finite-chain C*; $\bigwedge$*i*. $\llbracket$*chain C*; *max-in-chain i C*$\rrbracket$ $\Longrightarrow$ *R*$\rrbracket$ $\Longrightarrow$ *R*
⟨*proof*⟩

lemma *lub-finch1*: *chain C* $\Longrightarrow$ *max-in-chain i C* $\Longrightarrow$ *range C* <<| *C i*
⟨*proof*⟩

lemma *lub-finch2*: *finite-chain C* $\Longrightarrow$ *range C* <<| *C* (*LEAST i*. *max-in-chain i C*)

⟨*proof*⟩

lemma *finch-imp-finite-range*: $finite\text{-}chain\ Y \Longrightarrow finite\ (range\ Y)$
⟨*proof*⟩

lemma *finite-range-has-max*:
fixes $f :: nat \Rightarrow 'a$
and $r :: 'a \Rightarrow 'a \Rightarrow bool$
assumes *mono*: $\bigwedge i\ j.\ i \leq j \Longrightarrow r\ (f\ i)\ (f\ j)$
assumes *finite-range*: $finite\ (range\ f)$
shows $\exists k.\ \forall i.\ r\ (f\ i)\ (f\ k)$
⟨*proof*⟩

lemma *finite-range-imp-finch*: $chain\ Y \Longrightarrow finite\ (range\ Y) \Longrightarrow finite\text{-}chain\ Y$
⟨*proof*⟩

lemma *bin-chain*: $x \sqsubseteq y \Longrightarrow chain\ (\lambda i.\ if\ i{=}0\ then\ x\ else\ y)$
⟨*proof*⟩

lemma *bin-chainmax*: $x \sqsubseteq y \Longrightarrow max\text{-}in\text{-}chain\ (Suc\ 0)\ (\lambda i.\ if\ i{=}0\ then\ x\ else\ y)$
⟨*proof*⟩

lemma *is-lub-bin-chain*: $x \sqsubseteq y \Longrightarrow range\ (\lambda i{::}nat.\ if\ i{=}0\ then\ x\ else\ y) <<|\ y$
⟨*proof*⟩

the maximal element in a chain is its lub

lemma *lub-chain-maxelem*: $Y\ i = c \Longrightarrow \forall i.\ Y\ i \sqsubseteq c \Longrightarrow lub\ (range\ Y) = c$
⟨*proof*⟩

end

2 Classes cpo and pcpo

2.1 Complete partial orders

The class cpo of chain complete partial orders

class $cpo = po\ +$
assumes *cpo*: $chain\ S \Longrightarrow \exists x.\ range\ S <<|\ x$

default-sort *cpo*

context *cpo*
begin

in cpo's everthing equal to THE lub has lub properties for every chain

lemma *cpo-lubI*: $chain\ S \Longrightarrow range\ S <<|\ (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

lemma *thelubE*: $[\![chain\ S;\ (\bigsqcup i.\ S\ i) = l]\!] \Longrightarrow range\ S <<|\ l$
⟨*proof*⟩

Properties of the lub

lemma *is-ub-thelub*: $chain\ S \Longrightarrow S\ x \sqsubseteq (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

lemma *is-lub-thelub*: $[\![chain\ S;\ range\ S <|\ x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
⟨*proof*⟩

lemma *lub-below-iff*: $chain\ S \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x \longleftrightarrow (\forall i.\ S\ i \sqsubseteq x)$
⟨*proof*⟩

lemma *lub-below*: $[\![chain\ S;\ \bigwedge i.\ S\ i \sqsubseteq x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
⟨*proof*⟩

lemma *below-lub*: $[\![chain\ S;\ x \sqsubseteq S\ i]\!] \Longrightarrow x \sqsubseteq (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

lemma *lub-range-mono*: $[\![range\ X \subseteq range\ Y;\ chain\ Y;\ chain\ X]\!] \Longrightarrow (\bigsqcup i.\ X\ i) \sqsubseteq (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *lub-range-shift*: $chain\ Y \Longrightarrow (\bigsqcup i.\ Y\ (i + j)) = (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *maxinch-is-thelub*: $chain\ Y \Longrightarrow max\text{-}in\text{-}chain\ i\ Y = ((\bigsqcup i.\ Y\ i) = Y\ i)$
⟨*proof*⟩

the $\sqsubseteq$ relation between two chains is preserved by their lubs

lemma *lub-mono*: $[\![chain\ X;\ chain\ Y;\ \bigwedge i.\ X\ i \sqsubseteq Y\ i]\!] \Longrightarrow (\bigsqcup i.\ X\ i) \sqsubseteq (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

the = relation between two chains is preserved by their lubs

lemma *lub-eq*: $(\bigwedge i.\ X\ i = Y\ i) \Longrightarrow (\bigsqcup i.\ X\ i) = (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *ch2ch-lub*:
  assumes *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
  assumes *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
  shows $chain\ (\lambda i.\ \bigsqcup j.\ Y\ i\ j)$
  ⟨*proof*⟩

lemma *diag-lub*:
  assumes *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
  assumes *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
  shows $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup i.\ Y\ i\ i)$
⟨*proof*⟩

lemma *ex-lub*:
  assumes *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
  assumes *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
  shows $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup j.\ \bigsqcup i.\ Y\ i\ j)$
  ⟨*proof*⟩

end

2.2 Pointed cpos

The class pcpo of pointed cpos

class *pcpo* = *cpo* +
  assumes *least*: $\exists x.\ \forall y.\ x \sqsubseteq y$
begin

definition *bottom* :: *'a* (‹⊥›)
  where $bottom = (THE\ x.\ \forall y.\ x \sqsubseteq y)$

lemma *minimal* [*iff*]: $\bot \sqsubseteq x$
  ⟨*proof*⟩

end

Old "UU" syntax:

abbreviation (*input*) $UU \equiv bottom$

Simproc to rewrite $\bot = x$ to $x = \bot$.

⟨*ML*⟩

useful lemmas about ⊥

lemma *below-bottom-iff* [*simp*]: $x \sqsubseteq \bot \longleftrightarrow x = \bot$
  ⟨*proof*⟩

lemma *eq-bottom-iff*: $x = \bot \longleftrightarrow x \sqsubseteq \bot$
  ⟨*proof*⟩

lemma *bottomI*: $x \sqsubseteq \bot \Longrightarrow x = \bot$
  ⟨*proof*⟩

lemma *lub-eq-bottom-iff*: $chain\ Y \Longrightarrow (\bigsqcup i.\ Y\ i) = \bot \longleftrightarrow (\forall i.\ Y\ i = \bot)$
  ⟨*proof*⟩

2.3 Chain-finite and flat cpos

further useful classes for HOLCF domains

class *chfin* = *po* +
  assumes *chfin*: $chain\ Y \Longrightarrow \exists n.\ max\text{-}in\text{-}chain\ n\ Y$
begin

subclass *cpo*
⟨*proof*⟩

lemma *chfin2finch*: $chain\ Y \Longrightarrow finite\text{-}chain\ Y$
⟨*proof*⟩

end

class *flat* = *pcpo* +
assumes *ax-flat*: $x \sqsubseteq y \Longrightarrow x = \bot \lor x = y$
begin

subclass *chfin*
⟨*proof*⟩

lemma *flat-below-iff*: $x \sqsubseteq y \longleftrightarrow x = \bot \lor x = y$
⟨*proof*⟩

lemma *flat-eq*: $a \neq \bot \Longrightarrow a \sqsubseteq b = (a = b)$
⟨*proof*⟩

end

2.4 Discrete cpos

class *discrete-cpo* = *below* +
assumes *discrete-cpo* [*simp*]: $x \sqsubseteq y \longleftrightarrow x = y$
begin

subclass *po*
⟨*proof*⟩

In a discrete cpo, every chain is constant

lemma *discrete-chain-const*:
assumes *S*: *chain S*
shows $\exists x.\ S = (\lambda i.\ x)$
⟨*proof*⟩

subclass *chfin*
⟨*proof*⟩

end

3 Continuity and monotonicity

3.1 Definitions

definition *monofun* :: $('a::po \Rightarrow 'b::po) \Rightarrow bool$ — monotonicity

where $monofun\ f \longleftrightarrow (\forall x\ y.\ x \sqsubseteq y \longrightarrow f\ x \sqsubseteq f\ y)$

definition $cont :: ('a \Rightarrow 'b) \Rightarrow bool$
where $cont\ f = (\forall Y.\ chain\ Y \longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i))$

lemma *contI*: $(\bigwedge Y.\ chain\ Y \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i)) \Longrightarrow cont\ f$
⟨*proof*⟩

lemma *contE*: $cont\ f \Longrightarrow chain\ Y \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *monofunI*: $(\bigwedge x\ y.\ x \sqsubseteq y \Longrightarrow f\ x \sqsubseteq f\ y) \Longrightarrow monofun\ f$
⟨*proof*⟩

lemma *monofunE*: $monofun\ f \Longrightarrow x \sqsubseteq y \Longrightarrow f\ x \sqsubseteq f\ y$
⟨*proof*⟩

3.2 Equivalence of alternate definition

monotone functions map chains to chains

lemma *ch2ch-monofun*: $monofun\ f \Longrightarrow chain\ Y \Longrightarrow chain\ (\lambda i.\ f\ (Y\ i))$
⟨*proof*⟩

monotone functions map upper bound to upper bounds

lemma *ub2ub-monofun*: $monofun\ f \Longrightarrow range\ Y <|\ u \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <|\ f\ u$
⟨*proof*⟩

a lemma about binary chains

lemma *binchain-cont*: $cont\ f \Longrightarrow x \sqsubseteq y \Longrightarrow range\ (\lambda i{::}nat.\ f\ (if\ i = 0\ then\ x\ else\ y)) <<|\ f\ y$
⟨*proof*⟩

continuity implies monotonicity

lemma *cont2mono*: $cont\ f \Longrightarrow monofun\ f$
⟨*proof*⟩

lemmas *cont2monofunE* = *cont2mono* [*THEN monofunE*]

lemmas *ch2ch-cont* = *cont2mono* [*THEN ch2ch-monofun*]

continuity implies preservation of lubs

lemma *cont2contlubE*: $cont\ f \Longrightarrow chain\ Y \Longrightarrow f\ (\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ f\ (Y\ i))$
⟨*proof*⟩

lemma *contI2*:
fixes $f :: 'a \Rightarrow 'b$
assumes *mono*: *monofun f*

assumes *below*: $\bigwedge Y.\ [\![chain\ Y;\ chain\ (\lambda i.\ f\ (Y\ i))]\!] \Longrightarrow f\ (\bigsqcup i.\ Y\ i) \sqsubseteq (\bigsqcup i.\ f\ (Y\ i))$
shows *cont f*
⟨*proof*⟩

3.3 Collection of continuity rules

named-theorems *cont2cont continuity intro rule*

3.4 Continuity of basic functions

The identity function is continuous

lemma *cont-id* [*simp*, *cont2cont*]: $cont\ (\lambda x.\ x)$
⟨*proof*⟩

constant functions are continuous

lemma *cont-const* [*simp*, *cont2cont*]: $cont\ (\lambda x.\ c)$
⟨*proof*⟩

application of functions is continuous

lemma *cont-apply*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c$ **and** $t :: 'a \Rightarrow 'b$
assumes *1*: $cont\ (\lambda x.\ t\ x)$
assumes *2*: $\bigwedge x.\ cont\ (\lambda y.\ f\ x\ y)$
assumes *3*: $\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)$
shows $cont\ (\lambda x.\ (f\ x)\ (t\ x))$
⟨*proof*⟩

lemma *cont-compose*: $cont\ c \Longrightarrow cont\ (\lambda x.\ f\ x) \Longrightarrow cont\ (\lambda x.\ c\ (f\ x))$
⟨*proof*⟩

Least upper bounds preserve continuity

lemma *cont2cont-lub* [*simp*]:
assumes *chain*: $\bigwedge x.\ chain\ (\lambda i.\ F\ i\ x)$
and *cont*: $\bigwedge i.\ cont\ (\lambda x.\ F\ i\ x)$
shows $cont\ (\lambda x.\ \bigsqcup i.\ F\ i\ x)$
⟨*proof*⟩

if-then-else is continuous

lemma *cont-if* [*simp*, *cont2cont*]: $cont\ f \Longrightarrow cont\ g \Longrightarrow cont\ (\lambda x.\ if\ b\ then\ f\ x\ else\ g\ x)$
⟨*proof*⟩

3.5 Finite chains and flat pcpos

Monotone functions map finite chains to finite chains.

lemma *monofun-finch2finch*: $monofun\ f \Longrightarrow finite\text{-}chain\ Y \Longrightarrow finite\text{-}chain\ (\lambda n.\ f\ (Y\ n))$

⟨*proof*⟩

The same holds for continuous functions.

lemma *cont-finch2finch*: *cont f* $\Longrightarrow$ *finite-chain* Y $\Longrightarrow$ *finite-chain* $(\lambda n.\ f\ (Y\ n))$
⟨*proof*⟩

All monotone functions with chain-finite domain are continuous.

lemma *chfindom-monofun2cont*: *monofun f* $\Longrightarrow$ *cont f*
for f :: $'a$::*chfin* $\Rightarrow$ $'b$
⟨*proof*⟩

All strict functions with flat domain are continuous.

lemma *flatdom-strict2mono*: $f\ \bot = \bot \Longrightarrow$ *monofun f*
for f :: $'a$::*flat* $\Rightarrow$ $'b$::*pcpo*
⟨*proof*⟩

lemma *flatdom-strict2cont*: $f\ \bot = \bot \Longrightarrow$ *cont f*
for f :: $'a$::*flat* $\Rightarrow$ $'b$::*pcpo*
⟨*proof*⟩

All functions with discrete domain are continuous.

lemma *cont-discrete-cpo* [*simp*, *cont2cont*]: *cont f*
for f :: $'a$::*discrete-cpo* $\Rightarrow$ $'b$
⟨*proof*⟩

4 Admissibility and compactness

4.1 Definitions

context *cpo*
begin

definition *adm* :: $('a \Rightarrow bool) \Rightarrow bool$
where *adm* $P \longleftrightarrow (\forall\, Y.\ chain\ Y \longrightarrow (\forall\, i.\ P\ (Y\ i)) \longrightarrow P\ (\bigsqcup i.\ Y\ i))$

lemma *admI*: $(\bigwedge Y.\ [\![chain\ Y;\ \forall\, i.\ P\ (Y\ i)]\!] \Longrightarrow P\ (\bigsqcup i.\ Y\ i)) \Longrightarrow adm\ P$
⟨*proof*⟩

lemma *admD*: $adm\ P \Longrightarrow chain\ Y \Longrightarrow (\bigwedge i.\ P\ (Y\ i)) \Longrightarrow P\ (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *admD2*: $adm\ (\lambda x.\ \neg\ P\ x) \Longrightarrow chain\ Y \Longrightarrow P\ (\bigsqcup i.\ Y\ i) \Longrightarrow \exists\, i.\ P\ (Y\ i)$
⟨*proof*⟩

lemma *triv-admI*: $\forall\, x.\ P\ x \Longrightarrow adm\ P$
⟨*proof*⟩

end

4.2 Admissibility on chain-finite types

For chain-finite (easy) types every formula is admissible.

lemma *adm-chfin* [*simp*]: *adm P* **for** *P* :: *'a::chfin* ⇒ *bool*
⟨*proof*⟩

4.3 Admissibility of special formulae and propagation

context *cpo*
begin

lemma *adm-const* [*simp*]: *adm* (λ*x*. *t*)
⟨*proof*⟩

lemma *adm-conj* [*simp*]: *adm* (λ*x*. *P x*) ⟹ *adm* (λ*x*. *Q x*) ⟹ *adm* (λ*x*. *P x* ∧ *Q x*)
⟨*proof*⟩

lemma *adm-all* [*simp*]: (⋀*y*. *adm* (λ*x*. *P x y*)) ⟹ *adm* (λ*x*. ∀ *y*. *P x y*)
⟨*proof*⟩

lemma *adm-ball* [*simp*]: (⋀*y*. *y* ∈ *A* ⟹ *adm* (λ*x*. *P x y*)) ⟹ *adm* (λ*x*. ∀ *y*∈*A*. *P x y*)
⟨*proof*⟩

Admissibility for disjunction is hard to prove. It requires 2 lemmas.

lemma *adm-disj-lemma1*:
assumes *adm*: *adm P*
assumes *chain*: *chain Y*
assumes *P*: ∀ *i*. ∃ *j*≥*i*. *P* (*Y j*)
shows *P* (⨆ *i*. *Y i*)
⟨*proof*⟩

lemma *adm-disj-lemma2*: ∀ *n*::*nat*. *P n* ∨ *Q n* ⟹ (∀ *i*. ∃ *j*≥*i*. *P j*) ∨ (∀ *i*. ∃ *j*≥*i*. *Q j*)
⟨*proof*⟩

lemma *adm-disj* [*simp*]: *adm* (λ*x*. *P x*) ⟹ *adm* (λ*x*. *Q x*) ⟹ *adm* (λ*x*. *P x* ∨ *Q x*)
⟨*proof*⟩

lemma *adm-imp* [*simp*]: *adm* (λ*x*. ¬ *P x*) ⟹ *adm* (λ*x*. *Q x*) ⟹ *adm* (λ*x*. *P x* ⟶ *Q x*)
⟨*proof*⟩

lemma *adm-iff* [*simp*]: *adm* (λ*x*. *P x* ⟶ *Q x*) ⟹ *adm* (λ*x*. *Q x* ⟶ *P x*) ⟹ *adm* (λ*x*. *P x* ⟷ *Q x*)
⟨*proof*⟩

end

admissibility and continuity

lemma *adm-below* [*simp*]: $cont\ (\lambda x.\ u\ x) \Longrightarrow cont\ (\lambda x.\ v\ x) \Longrightarrow adm\ (\lambda x.\ u\ x \sqsubseteq v\ x)$
⟨*proof*⟩

lemma *adm-eq* [*simp*]: $cont\ (\lambda x.\ u\ x) \Longrightarrow cont\ (\lambda x.\ v\ x) \Longrightarrow adm\ (\lambda x.\ u\ x = v\ x)$
⟨*proof*⟩

lemma *adm-subst*: $cont\ (\lambda x.\ t\ x) \Longrightarrow adm\ P \Longrightarrow adm\ (\lambda x.\ P\ (t\ x))$
⟨*proof*⟩

lemma *adm-not-below* [*simp*]: $cont\ (\lambda x.\ t\ x) \Longrightarrow adm\ (\lambda x.\ t\ x \not\sqsubseteq u)$
⟨*proof*⟩

4.4 Compactness

context *cpo*
begin

definition $compact :: 'a \Rightarrow bool$
where $compact\ k = adm\ (\lambda x.\ k \not\sqsubseteq x)$

lemma *compactI*: $adm\ (\lambda x.\ k \not\sqsubseteq x) \Longrightarrow compact\ k$
⟨*proof*⟩

lemma *compactD*: $compact\ k \Longrightarrow adm\ (\lambda x.\ k \not\sqsubseteq x)$
⟨*proof*⟩

lemma *compactI2*: $(\bigwedge Y.\ [\![chain\ Y;\ x \sqsubseteq (\bigsqcup i.\ Y\ i)]\!] \Longrightarrow \exists i.\ x \sqsubseteq Y\ i) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compactD2*: $compact\ x \Longrightarrow chain\ Y \Longrightarrow x \sqsubseteq (\bigsqcup i.\ Y\ i) \Longrightarrow \exists i.\ x \sqsubseteq Y\ i$
⟨*proof*⟩

lemma *compact-below-lub-iff*: $compact\ x \Longrightarrow chain\ Y \Longrightarrow x \sqsubseteq (\bigsqcup i.\ Y\ i) \longleftrightarrow (\exists i.\ x \sqsubseteq Y\ i)$
⟨*proof*⟩

end

lemma *compact-chfin* [*simp*]: *compact x* **for** $x :: 'a::chfin$
⟨*proof*⟩

lemma *compact-imp-max-in-chain*: $chain\ Y \Longrightarrow compact\ (\bigsqcup i.\ Y\ i) \Longrightarrow \exists i.\ max\text{-}in\text{-}chain\ i\ Y$
⟨*proof*⟩

admissibility and compactness

lemma *adm-compact-not-below* [*simp*]:
compact k $\Longrightarrow$ *cont* ($\lambda x.\ t\ x$) $\Longrightarrow$ *adm* ($\lambda x.\ k \not\sqsubseteq t\ x$)
⟨*proof*⟩

lemma *adm-neq-compact* [*simp*]: *compact k* $\Longrightarrow$ *cont* ($\lambda x.\ t\ x$) $\Longrightarrow$ *adm* ($\lambda x.\ t\ x \neq k$)
⟨*proof*⟩

lemma *adm-compact-neq* [*simp*]: *compact k* $\Longrightarrow$ *cont* ($\lambda x.\ t\ x$) $\Longrightarrow$ *adm* ($\lambda x.\ k \neq t\ x$)
⟨*proof*⟩

lemma *compact-bottom* [*simp*, *intro*]: *compact* $\bot$
⟨*proof*⟩

Any upward-closed predicate is admissible.

lemma *adm-upward*:
assumes *P*: $\bigwedge x\ y.\ [\![P\ x;\ x \sqsubseteq y]\!] \Longrightarrow P\ y$
shows *adm P*
⟨*proof*⟩

lemmas *adm-lemmas* =
adm-const adm-conj adm-all adm-ball adm-disj adm-imp adm-iff
adm-below adm-eq adm-not-below
adm-compact-not-below adm-compact-neq adm-neq-compact

5 Class instances for the full function space

5.1 Full function space is a partial order

instantiation *fun* :: (*type*, *below*) *below*
begin

definition *below-fun-def*: ($\sqsubseteq$) $\equiv$ ($\lambda f\ g.\ \forall x.\ f\ x \sqsubseteq g\ x$)

instance ⟨*proof*⟩
end

instance *fun* :: (*type*, *po*) *po*
⟨*proof*⟩

lemma *fun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall x.\ f\ x \sqsubseteq g\ x)$
⟨*proof*⟩

lemma *fun-belowI*: $(\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow f \sqsubseteq g$
⟨*proof*⟩

lemma *fun-belowD*: $f \sqsubseteq g \Longrightarrow f\ x \sqsubseteq g\ x$
⟨*proof*⟩

5.2 Full function space is chain complete

Properties of chains of functions.

lemma *fun-chain-iff*: $chain\ S \longleftrightarrow (\forall x.\ chain\ (\lambda i.\ S\ i\ x))$
⟨*proof*⟩

lemma *ch2ch-fun*: $chain\ S \Longrightarrow chain\ (\lambda i.\ S\ i\ x)$
⟨*proof*⟩

lemma *ch2ch-lambda*: $(\bigwedge x.\ chain\ (\lambda i.\ S\ i\ x)) \Longrightarrow chain\ S$
⟨*proof*⟩

Type $'a \Rightarrow 'b$ is chain complete

lemma *is-lub-lambda*: $(\bigwedge x.\ range\ (\lambda i.\ Y\ i\ x) <<|\ f\ x) \Longrightarrow range\ Y <<|\ f$
⟨*proof*⟩

lemma *is-lub-fun*: $chain\ S \Longrightarrow range\ S <<|\ (\lambda x.\ \bigsqcup i.\ S\ i\ x)$
for $S :: nat \Rightarrow 'a::type \Rightarrow 'b$
⟨*proof*⟩

lemma *lub-fun*: $chain\ S \Longrightarrow (\bigsqcup i.\ S\ i) = (\lambda x.\ \bigsqcup i.\ S\ i\ x)$
for $S :: nat \Rightarrow 'a::type \Rightarrow 'b$
⟨*proof*⟩

instance *fun* :: (*type*, *cpo*) *cpo*
⟨*proof*⟩

instance *fun* :: (*type*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

5.3 Full function space is pointed

lemma *minimal-fun*: $(\lambda x.\ \bot) \sqsubseteq f$
⟨*proof*⟩

instance *fun* :: (*type*, *pcpo*) *pcpo*
⟨*proof*⟩

lemma *inst-fun-pcpo*: $\bot = (\lambda x.\ \bot)$
⟨*proof*⟩

lemma *app-strict* [*simp*]: $\bot\ x = \bot$
⟨*proof*⟩

lemma *lambda-strict*: $(\lambda x.\ \bot) = \bot$
⟨*proof*⟩

5.4 Propagation of monotonicity and continuity

The lub of a chain of monotone functions is monotone.

lemma *adm-monofun*: *adm monofun*
⟨*proof*⟩

The lub of a chain of continuous functions is continuous.

lemma *adm-cont*: *adm cont*
⟨*proof*⟩

Function application preserves monotonicity and continuity.

lemma *mono2mono-fun*: $monofun\ f \Longrightarrow monofun\ (\lambda x.\ f\ x\ y)$
⟨*proof*⟩

lemma *cont2cont-fun*: $cont\ f \Longrightarrow cont\ (\lambda x.\ f\ x\ y)$
⟨*proof*⟩

lemma *cont-fun*: $cont\ (\lambda f.\ f\ x)$
⟨*proof*⟩

Lambda abstraction preserves monotonicity and continuity. (Note $(\lambda x.\ \lambda y.\ f\ x\ y) = f$.)

lemma *mono2mono-lambda*: $(\bigwedge y.\ monofun\ (\lambda x.\ f\ x\ y)) \Longrightarrow monofun\ f$
⟨*proof*⟩

lemma *cont2cont-lambda* [*simp*]:
assumes f: $\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)$
shows *cont f*
⟨*proof*⟩

What D.A.Schmidt calls continuity of abstraction; never used here

lemma *contlub-lambda*: $(\bigwedge x.\ chain\ (\lambda i.\ S\ i\ x)) \Longrightarrow (\lambda x.\ \bigsqcup i.\ S\ i\ x) = (\bigsqcup i.\ (\lambda x.\ S\ i\ x))$
for $S :: nat \Rightarrow 'a::type \Rightarrow 'b$
⟨*proof*⟩

6 The cpo of cartesian products

6.1 Unit type is a pcpo

instantiation *unit* :: *discrete-cpo*
begin

definition *below-unit-def* [*simp*]: $x \sqsubseteq (y::unit) \longleftrightarrow True$

instance
⟨*proof*⟩

end

instance *unit* :: *pcpo*
⟨*proof*⟩

6.2 Product type is a partial order

instantiation *prod* :: (*below*, *below*) *below*
begin

definition *below-prod-def*: $(\sqsubseteq) \equiv \lambda p1\ p2.\ (fst\ p1 \sqsubseteq fst\ p2 \wedge snd\ p1 \sqsubseteq snd\ p2)$

instance ⟨*proof*⟩

end

instance *prod* :: (*po*, *po*) *po*
⟨*proof*⟩

6.3 Monotonicity of *Pair*, *fst*, *snd*

lemma *prod-belowI*: $fst\ p \sqsubseteq fst\ q \Longrightarrow snd\ p \sqsubseteq snd\ q \Longrightarrow p \sqsubseteq q$
⟨*proof*⟩

lemma *Pair-below-iff* [*simp*]: $(a, b) \sqsubseteq (c, d) \longleftrightarrow a \sqsubseteq c \wedge b \sqsubseteq d$
⟨*proof*⟩

Pair (-,-) is monotone in both arguments

lemma *monofun-pair1*: $monofun\ (\lambda x.\ (x, y))$
⟨*proof*⟩

lemma *monofun-pair2*: $monofun\ (\lambda y.\ (x, y))$
⟨*proof*⟩

lemma *monofun-pair*: $x1 \sqsubseteq x2 \Longrightarrow y1 \sqsubseteq y2 \Longrightarrow (x1, y1) \sqsubseteq (x2, y2)$
⟨*proof*⟩

lemma *ch2ch-Pair* [*simp*]: $chain\ X \Longrightarrow chain\ Y \Longrightarrow chain\ (\lambda i.\ (X\ i, Y\ i))$
⟨*proof*⟩

fst and *snd* are monotone

lemma *fst-monofun*: $x \sqsubseteq y \Longrightarrow fst\ x \sqsubseteq fst\ y$
⟨*proof*⟩

lemma *snd-monofun*: $x \sqsubseteq y \Longrightarrow snd\ x \sqsubseteq snd\ y$
⟨*proof*⟩

lemma *monofun-fst*: *monofun fst*
⟨*proof*⟩

lemma *monofun-snd*: *monofun snd*
⟨*proof*⟩

lemmas *ch2ch-fst* [*simp*] = *ch2ch-monofun* [*OF monofun-fst*]

lemmas *ch2ch-snd* [*simp*] = *ch2ch-monofun* [*OF monofun-snd*]

lemma *prod-chain-cases*:
assumes *chain*: *chain Y*
obtains *A B*
where *chain A* **and** *chain B* **and** $Y = (\lambda i.\ (A\ i,\ B\ i))$
⟨*proof*⟩

6.4 Product type is a cpo

lemma *is-lub-Pair*: *range A* <<| $x \Longrightarrow$ *range B* <<| $y \Longrightarrow$ *range* $(\lambda i.\ (A\ i,\ B\ i))$ <<| $(x,\ y)$
⟨*proof*⟩

lemma *lub-Pair*: *chain A* $\Longrightarrow$ *chain B* $\Longrightarrow (\bigsqcup i.\ (A\ i,\ B\ i)) = (\bigsqcup i.\ A\ i,\ \bigsqcup i.\ B\ i)$
for $A :: nat \Rightarrow {'a}$ **and** $B :: nat \Rightarrow {'b}$
⟨*proof*⟩

lemma *is-lub-prod*:
fixes $S :: nat \Rightarrow ({'a} \times {'b})$
assumes *chain S*
shows *range S* <<| $(\bigsqcup i.\ fst\ (S\ i),\ \bigsqcup i.\ snd\ (S\ i))$
⟨*proof*⟩

lemma *lub-prod*: *chain S* $\Longrightarrow (\bigsqcup i.\ S\ i) = (\bigsqcup i.\ fst\ (S\ i),\ \bigsqcup i.\ snd\ (S\ i))$
for $S :: nat \Rightarrow {'a} \times {'b}$
⟨*proof*⟩

instance *prod* :: (*cpo*, *cpo*) *cpo*
⟨*proof*⟩

instance *prod* :: (*discrete-cpo*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

6.5 Product type is pointed

lemma *minimal-prod*: $(\bot,\ \bot) \sqsubseteq p$
⟨*proof*⟩

instance *prod* :: (*pcpo*, *pcpo*) *pcpo*
⟨*proof*⟩

lemma *inst-prod-pcpo*: $\bot = (\bot,\ \bot)$
⟨*proof*⟩

lemma *Pair-bottom-iff* [*simp*]: $(x, y) = \bot \longleftrightarrow x = \bot \wedge y = \bot$
⟨*proof*⟩

lemma *fst-strict* [*simp*]: $fst\ \bot = \bot$
⟨*proof*⟩

lemma *snd-strict* [*simp*]: $snd\ \bot = \bot$
⟨*proof*⟩

lemma *Pair-strict* [*simp*]: $(\bot, \bot) = \bot$
⟨*proof*⟩

lemma *split-strict* [*simp*]: $case\text{-}prod\ f\ \bot = f\ \bot\ \bot$
⟨*proof*⟩

6.6 Continuity of *Pair*, *fst*, *snd*

lemma *cont-pair1*: $cont\ (\lambda x.\ (x, y))$
⟨*proof*⟩

lemma *cont-pair2*: $cont\ (\lambda y.\ (x, y))$
⟨*proof*⟩

lemma *cont-fst*: *cont fst*
⟨*proof*⟩

lemma *cont-snd*: *cont snd*
⟨*proof*⟩

lemma *cont2cont-Pair* [*simp*, *cont2cont*]:
assumes *f*: $cont\ (\lambda x.\ f\ x)$
assumes *g*: $cont\ (\lambda x.\ g\ x)$
shows $cont\ (\lambda x.\ (f\ x, g\ x))$
⟨*proof*⟩

lemmas *cont2cont-fst* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-fst*]

lemmas *cont2cont-snd* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-snd*]

lemma *cont2cont-case-prod*:
assumes *f1*: $\bigwedge a\ b.\ cont\ (\lambda x.\ f\ x\ a\ b)$
assumes *f2*: $\bigwedge x\ b.\ cont\ (\lambda a.\ f\ x\ a\ b)$
assumes *f3*: $\bigwedge x\ a.\ cont\ (\lambda b.\ f\ x\ a\ b)$
assumes *g*: $cont\ (\lambda x.\ g\ x)$
shows $cont\ (\lambda x.\ case\ g\ x\ of\ (a, b) \Rightarrow f\ x\ a\ b)$
⟨*proof*⟩

lemma *prod-contI*:

assumes *f1*: $\bigwedge y.$ *cont* ($\lambda x.$ *f* (x, y))
assumes *f2*: $\bigwedge x.$ *cont* ($\lambda y.$ *f* (x, y))
shows *cont f*
⟨*proof*⟩

lemma *prod-cont-iff*: *cont f* $\longleftrightarrow$ ($\forall$ *y*. *cont* ($\lambda x.$ *f* (x, y))) $\wedge$ ($\forall$ *x*. *cont* ($\lambda y.$ *f* (x, y)))
⟨*proof*⟩

lemma *cont2cont-case-prod'* [*simp*, *cont2cont*]:
assumes *f*: *cont* ($\lambda p.$ *f* (*fst p*) (*fst* (*snd p*)) (*snd* (*snd p*)))
assumes *g*: *cont* ($\lambda x.$ *g x*)
shows *cont* ($\lambda x.$ *case-prod* (*f x*) (*g x*))
⟨*proof*⟩

The simple version (due to Joachim Breitner) is needed if either element type of the pair is not a cpo.

lemma *cont2cont-split-simple* [*simp*, *cont2cont*]:
assumes $\bigwedge a\ b.$ *cont* ($\lambda x.$ *f x a b*)
shows *cont* ($\lambda x.$ *case p of* (a, b) $\Rightarrow$ *f x a b*)
⟨*proof*⟩

Admissibility of predicates on product types.

lemma *adm-case-prod* [*simp*]:
assumes *adm* ($\lambda x.$ *P x* (*fst* (*f x*)) (*snd* (*f x*)))
shows *adm* ($\lambda x.$ *case f x of* (a, b) $\Rightarrow$ *P x a b*)
⟨*proof*⟩

6.7 Compactness and chain-finiteness

lemma *fst-below-iff*: *fst x* $\sqsubseteq$ *y* $\longleftrightarrow$ *x* $\sqsubseteq$ (*y*, *snd x*) **for** *x* :: *'a* $\times$ *'b*
⟨*proof*⟩

lemma *snd-below-iff*: *snd x* $\sqsubseteq$ *y* $\longleftrightarrow$ *x* $\sqsubseteq$ (*fst x*, *y*) **for** *x* :: *'a* $\times$ *'b*
⟨*proof*⟩

lemma *compact-fst*: *compact x* $\Longrightarrow$ *compact* (*fst x*)
⟨*proof*⟩

lemma *compact-snd*: *compact x* $\Longrightarrow$ *compact* (*snd x*)
⟨*proof*⟩

lemma *compact-Pair*: *compact x* $\Longrightarrow$ *compact y* $\Longrightarrow$ *compact* (*x*, *y*)
⟨*proof*⟩

lemma *compact-Pair-iff* [*simp*]: *compact* (*x*, *y*) $\longleftrightarrow$ *compact x* $\wedge$ *compact y*
⟨*proof*⟩

instance *prod* :: (*chfin*, *chfin*) *chfin*

⟨*proof*⟩

7 Discrete cpo types

datatype *'a discr* = *Discr 'a::type*

7.1 Discrete cpo class instance

instantiation *discr* :: (*type*) *discrete-cpo*
begin

definition $((\sqsubseteq) :: 'a\ discr \Rightarrow 'a\ discr \Rightarrow bool) = (=)$

instance
⟨*proof*⟩

end

7.2 *undiscr*

definition *undiscr* :: *'a::type discr* ⇒ *'a*
where *undiscr x* = (*case x of Discr y* ⇒ *y*)

lemma *undiscr-Discr* [*simp*]: *undiscr* (*Discr x*) = *x*
⟨*proof*⟩

lemma *Discr-undiscr* [*simp*]: *Discr* (*undiscr y*) = *y*
⟨*proof*⟩

end

8 Subtypes of pcpos

theory *Cpodef*
imports *Cpo*
keywords *pcpodef cpodef* :: *thy-goal-defn*
begin

8.1 Proving a subtype is a partial order

A subtype of a partial order is itself a partial order, if the ordering is defined in the standard way.

theorem (**in** *below*) *typedef-class-po*:
fixes *Abs* :: *'b::po* ⇒ *'a*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
shows *class.po below*
⟨*proof*⟩

lemmas *typedef-po-class* = *below.typedef-class-po* [*THEN po.intro-of-class*]

8.2 Proving a subtype is finite

lemma *typedef-finite-UNIV*:
fixes *Abs* :: *'a::type* ⇒ *'b::type*
assumes *type*: *type-definition Rep Abs A*
shows *finite A* ⟹ *finite* (*UNIV* :: *'b set*)
⟨*proof*⟩

8.3 Proving a subtype is chain-finite

lemma *ch2ch-Rep*:
assumes *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *chain S* ⟹ *chain* (λ*i*. *Rep* (*S i*))
⟨*proof*⟩

theorem *typedef-chfin*:
fixes *Abs* :: *'a::chfin* ⇒ *'b::po*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *OFCLASS*(*'b*, *chfin-class*)
⟨*proof*⟩

8.4 Proving a subtype is complete

A subtype of a cpo is itself a cpo if the ordering is defined in the standard way, and the defining subset is closed with respect to limits of chains. A set is closed if and only if membership in the set is an admissible predicate.

lemma *typedef-is-lubI*:
assumes *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *range* (λ*i*. *Rep* (*S i*)) <<| *Rep x* ⟹ *range S* <<| *x*
⟨*proof*⟩

lemma *Abs-inverse-lub-Rep*:
fixes *Abs* :: *'a::cpo* ⇒ *'b::po*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x*. *x* ∈ *A*)
shows *chain S* ⟹ *Rep* (*Abs* (⨆*i*. *Rep* (*S i*))) = (⨆*i*. *Rep* (*S i*))
⟨*proof*⟩

theorem *typedef-is-lub*:
fixes *Abs* :: *'a::cpo* ⇒ *'b::po*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x*. *x* ∈ *A*)
assumes *S*: *chain S*

shows *range S <<| Abs* ($\bigsqcup$*i. Rep* (*S i*))
⟨*proof*⟩

lemmas *typedef-lub* = *typedef-is-lub* [*THEN lub-eqI*]

theorem *typedef-cpo*:
fixes *Abs* :: *'a::cpo* ⇒ *'b::po*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y. Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x. x* ∈ *A*)
shows *OFCLASS*(*'b*, *cpo-class*)
⟨*proof*⟩

8.4.1 Continuity of *Rep* and *Abs*

For any sub-cpo, the *Rep* function is continuous.

theorem *typedef-cont-Rep*:
fixes *Abs* :: *'a::cpo* ⇒ *'b::cpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y. Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x. x* ∈ *A*)
shows *cont* (λ*x. f x*) ⟹ *cont* (λ*x. Rep* (*f x*))
⟨*proof*⟩

For a sub-cpo, we can make the *Abs* function continuous only if we restrict its domain to the defining subset by composing it with another continuous function.

theorem *typedef-cont-Abs*:
fixes *Abs* :: *'a::cpo* ⇒ *'b::cpo*
fixes *f* :: *'c::cpo* ⇒ *'a::cpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y. Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x. x* ∈ *A*)
and *f-in-A*: ⋀*x. f x* ∈ *A*
shows *cont f* ⟹ *cont* (λ*x. Abs* (*f x*))
⟨*proof*⟩

8.5 Proving subtype elements are compact

theorem *typedef-compact*:
fixes *Abs* :: *'a::cpo* ⇒ *'b::cpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y. Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x. x* ∈ *A*)
shows *compact* (*Rep k*) ⟹ *compact k*
⟨*proof*⟩

8.6 Proving a subtype is pointed

A subtype of a cpo has a least element if and only if the defining subset has a least element.

theorem *typedef-pcpo-generic*:
  fixes $Abs :: 'a::cpo \Rightarrow 'b::cpo$
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *z-in-A*: $z \in A$
    and *z-least*: $\bigwedge x.\ x \in A \Longrightarrow z \sqsubseteq x$
  shows $OFCLASS('b,\ pcpo\text{-}class)$
  ⟨*proof*⟩

As a special case, a subtype of a pcpo has a least element if the defining subset contains ⊥.

theorem *typedef-pcpo*:
  fixes $Abs :: 'a::pcpo \Rightarrow 'b::cpo$
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows $OFCLASS('b,\ pcpo\text{-}class)$
  ⟨*proof*⟩

8.6.1 Strictness of *Rep* and *Abs*

For a sub-pcpo where ⊥ is a member of the defining subset, *Rep* and *Abs* are both strict.

theorem *typedef-Abs-strict*:
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows $Abs\ \bot = \bot$
  ⟨*proof*⟩

theorem *typedef-Rep-strict*:
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows $Rep\ \bot = \bot$
  ⟨*proof*⟩

theorem *typedef-Abs-bottom-iff*:
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows $x \in A \Longrightarrow (Abs\ x = \bot) = (x = \bot)$
  ⟨*proof*⟩

theorem *typedef-Rep-bottom-iff*:
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows $(Rep\ x = \bot) = (x = \bot)$
  ⟨*proof*⟩

8.7 Proving a subtype is flat

theorem *typedef-flat*:
  fixes *Abs* :: *'a*::*flat* ⇒ *'b*::*pcpo*
  assumes *type*: *type-definition Rep Abs A*
    and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
    and *bottom-in-A*: $\bot \in A$
  shows *OFCLASS*(*'b*, *flat-class*)
  ⟨*proof*⟩

8.8 HOLCF type definition package

⟨*ML*⟩

end

9 The type of continuous functions

theory *Cfun*
  imports *Cpodef*
begin

9.1 Definition of continuous function type

definition *cfun* = {*f*::*'a* ⇒ *'b*. *cont f*}

cpodef (*'a*, *'b*) *cfun* (‹(‹*notation*=‹*infix* →››- →/ -)› [*1*, *0*] *0*) = *cfun* :: (*'a* ⇒ *'b*) *set*
  ⟨*proof*⟩

type-notation (*ASCII*)
  cfun (**infixr** ‹–>› *0*)

notation (*ASCII*)
  Rep-cfun (‹(‹*notation*=‹*infix* $››-$/-)› [*999*,*1000*] *999*)

notation
  Rep-cfun (‹(‹*notation*=‹*infix* ·››-·/-)› [*999*,*1000*] *999*)

9.2 Syntax for continuous lambda abstraction

syntax *-cabs* :: [*logic*, *logic*] ⇒ *logic*

⟨*ML*⟩

Syntax for nested abstractions

syntax (*ASCII*)
-Lambda :: [*cargs*, *logic*] ⇒ *logic* (‹(‹*indent=3 notation=*‹*binder LAM*››*LAM* -./ -)› [*1000*, *10*] *10*)

syntax
-Lambda :: [*cargs*, *logic*] ⇒ *logic* (‹(‹*indent=3 notation=*‹*binder* Λ››Λ -./ -)› [*1000*, *10*] *10*)

syntax-consts
-Lambda ⇌ *Abs-cfun*

⟨*ML*⟩

Dummy patterns for continuous abstraction

translations
Λ -. *t* ⇀ *CONST Abs-cfun* (λ-. *t*)

9.3 Continuous function space is pointed

lemma *bottom-cfun*: ⊥ ∈ *cfun*
⟨*proof*⟩

instance *cfun* :: (*cpo*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

instance *cfun* :: (*cpo*, *pcpo*) *pcpo*
⟨*proof*⟩

lemmas *Rep-cfun-strict* =
typedef-Rep-strict [*OF type-definition-cfun below-cfun-def bottom-cfun*]

lemmas *Abs-cfun-strict* =
typedef-Abs-strict [*OF type-definition-cfun below-cfun-def bottom-cfun*]

function application is strict in its first argument

lemma *Rep-cfun-strict1* [*simp*]: ⊥·*x* = ⊥
⟨*proof*⟩

lemma *LAM-strict* [*simp*]: (Λ *x*. ⊥) = ⊥
⟨*proof*⟩

for compatibility with old HOLCF-Version

lemma *inst-cfun-pcpo*: ⊥ = (Λ *x*. ⊥)
⟨*proof*⟩

9.4 Basic properties of continuous functions

Beta-equality for continuous functions

lemma *Abs-cfun-inverse2*: $cont\ f \Longrightarrow Rep\text{-}cfun\ (Abs\text{-}cfun\ f) = f$
⟨*proof*⟩

lemma *beta-cfun*: $cont\ f \Longrightarrow (\Lambda\ x.\ f\ x)\cdot u = f\ u$
⟨*proof*⟩

9.4.1 Beta-reduction simproc

Given the term $(\Lambda\ x.\ f\ x)\cdot y$, the procedure tries to construct the theorem $(\Lambda\ x.\ f\ x)\cdot y \equiv f\ y$. If this theorem cannot be completely solved by the cont2cont rules, then the procedure returns the ordinary conditional *beta-cfun* rule.

The simproc does not solve any more goals that would be solved by using *beta-cfun* as a simp rule. The advantage of the simproc is that it can avoid deeply-nested calls to the simplifier that would otherwise be caused by large continuity side conditions.

Update: The simproc now uses rule *Abs-cfun-inverse2* instead of *beta-cfun*, to avoid problems with eta-contraction.

⟨*ML*⟩

Eta-equality for continuous functions

lemma *eta-cfun*: $(\Lambda\ x.\ f\cdot x) = f$
⟨*proof*⟩

Extensionality for continuous functions

lemma *cfun-eq-iff*: $f = g \longleftrightarrow (\forall\, x.\ f\cdot x = g\cdot x)$
⟨*proof*⟩

lemma *cfun-eqI*: $(\bigwedge x.\ f\cdot x = g\cdot x) \Longrightarrow f = g$
⟨*proof*⟩

Extensionality wrt. ordering for continuous functions

lemma *cfun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall\, x.\ f\cdot x \sqsubseteq g\cdot x)$
⟨*proof*⟩

lemma *cfun-belowI*: $(\bigwedge x.\ f\cdot x \sqsubseteq g\cdot x) \Longrightarrow f \sqsubseteq g$
⟨*proof*⟩

Congruence for continuous function application

lemma *cfun-cong*: $f = g \Longrightarrow x = y \Longrightarrow f\cdot x = g\cdot y$
⟨*proof*⟩

lemma *cfun-fun-cong*: $f = g \Longrightarrow f\cdot x = g\cdot x$
⟨*proof*⟩

lemma *cfun-arg-cong*: $x = y \Longrightarrow f{\cdot}x = f{\cdot}y$
⟨*proof*⟩

9.5 Continuity of application

lemma *cont-Rep-cfun1*: *cont* $(\lambda f.\ f{\cdot}x)$
⟨*proof*⟩

lemma *cont-Rep-cfun2*: *cont* $(\lambda x.\ f{\cdot}x)$
⟨*proof*⟩

lemmas *monofun-Rep-cfun* = *cont-Rep-cfun* [*THEN cont2mono*]

lemmas *monofun-Rep-cfun1* = *cont-Rep-cfun1* [*THEN cont2mono*]
lemmas *monofun-Rep-cfun2* = *cont-Rep-cfun2* [*THEN cont2mono*]

contlub, cont properties of *Rep-cfun* in each argument

lemma *contlub-cfun-arg*: *chain* $Y \Longrightarrow f{\cdot}(\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

lemma *contlub-cfun-fun*: *chain* $F \Longrightarrow (\bigsqcup i.\ F\ i){\cdot}x = (\bigsqcup i.\ F\ i{\cdot}x)$
⟨*proof*⟩

monotonicity of application

lemma *monofun-cfun-fun*: $f \sqsubseteq g \Longrightarrow f{\cdot}x \sqsubseteq g{\cdot}x$
⟨*proof*⟩

lemma *monofun-cfun-arg*: $x \sqsubseteq y \Longrightarrow f{\cdot}x \sqsubseteq f{\cdot}y$
⟨*proof*⟩

lemma *monofun-cfun*: $f \sqsubseteq g \Longrightarrow x \sqsubseteq y \Longrightarrow f{\cdot}x \sqsubseteq g{\cdot}y$
⟨*proof*⟩

ch2ch - rules for the type $'a \to 'b$

lemma *chain-monofun*: *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

lemma *ch2ch-Rep-cfunR*: *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

lemma *ch2ch-Rep-cfunL*: *chain* $F \Longrightarrow$ *chain* $(\lambda i.\ (F\ i){\cdot}x)$
⟨*proof*⟩

lemma *ch2ch-Rep-cfun* [*simp*]: *chain* $F \Longrightarrow$ *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ (F\ i){\cdot}(Y\ i))$
⟨*proof*⟩

lemma *ch2ch-LAM* [*simp*]:
$(\bigwedge x.\ chain\ (\lambda i.\ S\ i\ x)) \Longrightarrow (\bigwedge i.\ cont\ (\lambda x.\ S\ i\ x)) \Longrightarrow chain\ (\lambda i.\ \Lambda\ x.\ S\ i\ x)$

⟨*proof*⟩

contlub, cont properties of *Rep-cfun* in both arguments

lemma *lub-APP*: *chain* $F \Longrightarrow$ *chain* $Y \Longrightarrow (\bigsqcup i.\ F\ i \cdot (Y\ i)) = (\bigsqcup i.\ F\ i) \cdot (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *lub-LAM*:
assumes $\bigwedge x.$ *chain* $(\lambda i.\ F\ i\ x)$
and $\bigwedge i.$ *cont* $(\lambda x.\ F\ i\ x)$
shows $(\bigsqcup i.\ \Lambda\ x.\ F\ i\ x) = (\Lambda\ x.\ \bigsqcup i.\ F\ i\ x)$
⟨*proof*⟩

lemmas *lub-distribs* = *lub-APP lub-LAM*

strictness

lemma *strictI*: $f \cdot x = \bot \Longrightarrow f \cdot \bot = \bot$
⟨*proof*⟩

type $'a \rightarrow 'b$ is chain complete

lemma *lub-cfun*: *chain* $F \Longrightarrow (\bigsqcup i.\ F\ i) = (\Lambda\ x.\ \bigsqcup i.\ F\ i \cdot x)$
⟨*proof*⟩

9.6 Continuity simplification procedure

cont2cont lemma for *Rep-cfun*

lemma *cont2cont-APP* [*simp*, *cont2cont*]:
assumes *f*: *cont* $(\lambda x.\ f\ x)$
assumes *t*: *cont* $(\lambda x.\ t\ x)$
shows *cont* $(\lambda x.\ (f\ x) \cdot (t\ x))$
⟨*proof*⟩

Two specific lemmas for the combination of LCF and HOL terms. These lemmas are needed in theories that use types like $'a \rightarrow 'b \Rightarrow 'c$.

lemma *cont-APP-app* [*simp*]: *cont* $f \Longrightarrow$ *cont* $g \Longrightarrow$ *cont* $(\lambda x.\ ((f\ x) \cdot (g\ x))\ s)$
⟨*proof*⟩

lemma *cont-APP-app-app* [*simp*]: *cont* $f \Longrightarrow$ *cont* $g \Longrightarrow$ *cont* $(\lambda x.\ ((f\ x) \cdot (g\ x))\ s\ t)$
⟨*proof*⟩

cont2mono Lemma for $\lambda x.\ \Lambda\ y.\ c1\ x\ y$

lemma *cont2mono-LAM*:
$[\![\bigwedge x.$ *cont* $(\lambda y.\ f\ x\ y)$; $\bigwedge y.$ *monofun* $(\lambda x.\ f\ x\ y)]\!]$
$\Longrightarrow$ *monofun* $(\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

cont2cont Lemma for $\lambda x.\ \Lambda\ y.\ f\ x\ y$

Not suitable as a cont2cont rule, because on nested lambdas it causes exponential blow-up in the number of subgoals.

lemma *cont2cont-LAM*:
assumes *f1*: $\bigwedge x.\ cont\ (\lambda y.\ f\ x\ y)$
assumes *f2*: $\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)$
shows $cont\ (\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

This version does work as a cont2cont rule, since it has only a single subgoal.

lemma *cont2cont-LAM′* [*simp*, *cont2cont*]:
fixes $f :: 'a::cpo \Rightarrow 'b::cpo \Rightarrow 'c::cpo$
assumes *f*: $cont\ (\lambda p.\ f\ (fst\ p)\ (snd\ p))$
shows $cont\ (\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

lemma *cont2cont-LAM-discrete* [*simp*, *cont2cont*]:
$(\bigwedge y::'a::discrete\text{-}cpo.\ cont\ (\lambda x.\ f\ x\ y)) \Longrightarrow cont\ (\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

9.7 Miscellaneous

Monotonicity of *Abs-cfun*

lemma *monofun-LAM*: $cont\ f \Longrightarrow cont\ g \Longrightarrow (\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow (\Lambda\ x.\ f\ x) \sqsubseteq (\Lambda\ x.\ g\ x)$
⟨*proof*⟩

some lemmata for functions with flat/chfin domain/range types

lemma *chfin-Rep-cfunR*: $chain\ Y \Longrightarrow \forall s.\ \exists n.\ (LUB\ i.\ Y\ i)\cdot s = Y\ n\cdot s$
for $Y :: nat \Rightarrow 'a::cpo \rightarrow 'b::chfin$
⟨*proof*⟩

lemma *adm-chfindom*: $adm\ (\lambda(u::'a::cpo \rightarrow 'b::chfin).\ P(u\cdot s))$
⟨*proof*⟩

9.8 Continuous injection-retraction pairs

Continuous retractions are strict.

lemma *retraction-strict*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow f\cdot\bot = \bot$
⟨*proof*⟩

lemma *injection-eq*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow (g\cdot x = g\cdot y) = (x = y)$
⟨*proof*⟩

lemma *injection-below*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow (g\cdot x \sqsubseteq g\cdot y) = (x \sqsubseteq y)$
⟨*proof*⟩

lemma *injection-defined-rev*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow g\cdot z = \bot \Longrightarrow z = \bot$

⟨*proof*⟩

lemma *injection-defined*: $\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow z \neq \bot \Longrightarrow g\cdot z \neq \bot$
⟨*proof*⟩

a result about functions with flat codomain

lemma *flat-eqI*: $x \sqsubseteq y \Longrightarrow x \neq \bot \Longrightarrow x = y$
for $x\ y$ :: *'a::flat*
⟨*proof*⟩

lemma *flat-codom*: $f\cdot x = c \Longrightarrow f\cdot\bot = \bot \vee (\forall z.\ f\cdot z = c)$
for c :: *'b::flat*
⟨*proof*⟩

9.9 Identity and composition

definition *ID* :: $'a \to 'a$
where $ID = (\Lambda\ x.\ x)$

definition *cfcomp* :: $('b \to 'c) \to ('a \to 'b) \to 'a \to 'c$
where *oo-def*: $cfcomp = (\Lambda\ f\ g\ x.\ f\cdot(g\cdot x))$

abbreviation *cfcomp-syn* :: $['b \to 'c, 'a \to 'b] \Rightarrow 'a \to 'c$ (**infixr** ‹*oo*› *100*)
where $f\ oo\ g == cfcomp\cdot f\cdot g$

lemma *ID1* [*simp*]: $ID\cdot x = x$
⟨*proof*⟩

lemma *cfcomp1*: $(f\ oo\ g) = (\Lambda\ x.\ f\cdot(g\cdot x))$
⟨*proof*⟩

lemma *cfcomp2* [*simp*]: $(f\ oo\ g)\cdot x = f\cdot(g\cdot x)$
⟨*proof*⟩

lemma *cfcomp-LAM*: $cont\ g \Longrightarrow f\ oo\ (\Lambda\ x.\ g\ x) = (\Lambda\ x.\ f\cdot(g\ x))$
⟨*proof*⟩

lemma *cfcomp-strict* [*simp*]: $\bot\ oo\ f = \bot$
⟨*proof*⟩

Show that interpretation of (pcpo, -→-) is a category.

- The class of objects is interpretation of syntactical class pcpo.
- The class of arrows between objects $'a$ and $'b$ is interpret. of $'a \to 'b$.
- The identity arrow is interpretation of *ID*.
- The composition of f and g is interpretation of *oo*.

lemma *ID2* [*simp*]: $f\ oo\ ID = f$
⟨*proof*⟩

lemma *ID3* [*simp*]: $ID\ oo\ f = f$
⟨*proof*⟩

lemma *assoc-oo*: $f\ oo\ (g\ oo\ h) = (f\ oo\ g)\ oo\ h$
⟨*proof*⟩

9.10 Strictified functions

definition *seq* :: $'a::pcpo \to 'b::pcpo \to 'b$
where $seq = (\Lambda\ x.\ if\ x = \bot\ then\ \bot\ else\ ID)$

lemma *cont2cont-if-bottom* [*cont2cont*, *simp*]:
assumes *f*: $cont\ (\lambda x.\ f\ x)$
and *g*: $cont\ (\lambda x.\ g\ x)$
shows $cont\ (\lambda x.\ if\ f\ x = \bot\ then\ \bot\ else\ g\ x)$
⟨*proof*⟩

lemma *seq-conv-if*: $seq{\cdot}x = (if\ x = \bot\ then\ \bot\ else\ ID)$
⟨*proof*⟩

lemma *seq-simps* [*simp*]:
$seq{\cdot}\bot = \bot$
$seq{\cdot}x{\cdot}\bot = \bot$
$x \neq \bot \Longrightarrow seq{\cdot}x = ID$
⟨*proof*⟩

definition *strictify* :: $('a::pcpo \to 'b::pcpo) \to 'a \to 'b$
where $strictify = (\Lambda\ f\ x.\ seq{\cdot}x{\cdot}(f{\cdot}x))$

lemma *strictify-conv-if*: $strictify{\cdot}f{\cdot}x = (if\ x = \bot\ then\ \bot\ else\ f{\cdot}x)$
⟨*proof*⟩

lemma *strictify1* [*simp*]: $strictify{\cdot}f{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *strictify2* [*simp*]: $x \neq \bot \Longrightarrow strictify{\cdot}f{\cdot}x = f{\cdot}x$
⟨*proof*⟩

9.11 Continuity of let-bindings

lemma *cont2cont-Let*:
assumes *f*: $cont\ (\lambda x.\ f\ x)$
assumes *g1*: $\bigwedge y.\ cont\ (\lambda x.\ g\ x\ y)$
assumes *g2*: $\bigwedge x.\ cont\ (\lambda y.\ g\ x\ y)$
shows $cont\ (\lambda x.\ let\ y = f\ x\ in\ g\ x\ y)$
⟨*proof*⟩

lemma *cont2cont-Let′* [*simp*, *cont2cont*]:
assumes *f*: *cont* (λ*x*. *f x*)
assumes *g*: *cont* (λ*p*. *g* (*fst p*) (*snd p*))
shows *cont* (λ*x*. *let y* = *f x in g x y*)
⟨*proof*⟩

The simple version (suggested by Joachim Breitner) is needed if the type of the defined term is not a cpo.

lemma *cont2cont-Let-simple* [*simp*, *cont2cont*]:
assumes ⋀*y*. *cont* (λ*x*. *g x y*)
shows *cont* (λ*x*. *let y* = *t in g x y*)
⟨*proof*⟩

end

10 Continuous deflations and ep-pairs

theory *Deflation*
imports *Cfun*
begin

10.1 Continuous deflations

locale *deflation* =
fixes *d* :: ′*a* → ′*a*
assumes *idem*: ⋀*x*. *d*·(*d*·*x*) = *d*·*x*
assumes *below*: ⋀*x*. *d*·*x* ⊑ *x*
begin

lemma *below-ID*: *d* ⊑ *ID*
⟨*proof*⟩

The set of fixed points is the same as the range.

lemma *fixes-eq-range*: {*x*. *d*·*x* = *x*} = *range* (λ*x*. *d*·*x*)
⟨*proof*⟩

lemma *range-eq-fixes*: *range* (λ*x*. *d*·*x*) = {*x*. *d*·*x* = *x*}
⟨*proof*⟩

The pointwise ordering on deflation functions coincides with the subset ordering of their sets of fixed-points.

lemma *belowI*:
assumes *f*: ⋀*x*. *d*·*x* = *x* ⟹ *f*·*x* = *x*
shows *d* ⊑ *f*
⟨*proof*⟩

lemma *belowD*: ⟦*f* ⊑ *d*; *f*·*x* = *x*⟧ ⟹ *d*·*x* = *x*
⟨*proof*⟩

end

lemma *deflation-strict*: *deflation* $d \Longrightarrow d{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *adm-deflation*: *adm* $(\lambda d.$ *deflation* $d)$
⟨*proof*⟩

lemma *deflation-ID*: *deflation ID*
⟨*proof*⟩

lemma *deflation-bottom*: *deflation* $\bot$
⟨*proof*⟩

lemma *deflation-below-iff*: *deflation* $p \Longrightarrow$ *deflation* $q \Longrightarrow p \sqsubseteq q \longleftrightarrow (\forall x.\ p{\cdot}x = x \longrightarrow q{\cdot}x = x)$
⟨*proof*⟩

The composition of two deflations is equal to the lesser of the two (if they are comparable).

lemma *deflation-below-comp1*:
assumes *deflation f*
assumes *deflation g*
shows $f \sqsubseteq g \Longrightarrow f{\cdot}(g{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

lemma *deflation-below-comp2*: *deflation* $f \Longrightarrow$ *deflation* $g \Longrightarrow f \sqsubseteq g \Longrightarrow g{\cdot}(f{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

10.2 Deflations with finite range

lemma *finite-range-imp-finite-fixes*:
assumes *finite* (*range f*)
shows *finite* $\{x.\ f\ x = x\}$
⟨*proof*⟩

locale *finite-deflation* = *deflation* +
assumes *finite-fixes*: *finite* $\{x.\ d{\cdot}x = x\}$
begin

lemma *finite-range*: *finite* (*range* $(\lambda x.\ d{\cdot}x)$)
⟨*proof*⟩

lemma *finite-image*: *finite* $((\lambda x.\ d{\cdot}x)$ ‘ $A)$
⟨*proof*⟩

lemma *compact*: *compact* $(d{\cdot}x)$

⟨*proof*⟩

end

lemma *finite-deflation-intro*: *deflation d* $\Longrightarrow$ *finite* $\{x.\ d\cdot x = x\}$ $\Longrightarrow$ *finite-deflation d*
⟨*proof*⟩

lemma *finite-deflation-imp-deflation*: *finite-deflation d* $\Longrightarrow$ *deflation d*
⟨*proof*⟩

lemma *finite-deflation-bottom*: *finite-deflation* $\bot$
⟨*proof*⟩

10.3 Continuous embedding-projection pairs

locale *ep-pair* =
fixes $e :: {'a} \rightarrow {'b}$ **and** $p :: {'b} \rightarrow {'a}$
assumes *e-inverse* [*simp*]: $\bigwedge x.\ p\cdot(e\cdot x) = x$
and *e-p-below*: $\bigwedge y.\ e\cdot(p\cdot y) \sqsubseteq y$
begin

lemma *e-below-iff* [*simp*]: $e\cdot x \sqsubseteq e\cdot y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *e-eq-iff* [*simp*]: $e\cdot x = e\cdot y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *p-eq-iff*: $e\cdot(p\cdot x) = x \Longrightarrow e\cdot(p\cdot y) = y \Longrightarrow p\cdot x = p\cdot y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *p-inverse*: $(\exists\, x.\ y = e\cdot x) \longleftrightarrow e\cdot(p\cdot y) = y$
⟨*proof*⟩

lemma *e-below-iff-below-p*: $e\cdot x \sqsubseteq y \longleftrightarrow x \sqsubseteq p\cdot y$
⟨*proof*⟩

lemma *compact-e-rev*: *compact* $(e\cdot x) \Longrightarrow$ *compact x*
⟨*proof*⟩

lemma *compact-e*:
assumes *compact x*
shows *compact* $(e\cdot x)$
⟨*proof*⟩

lemma *compact-e-iff*: *compact* $(e\cdot x) \longleftrightarrow$ *compact x*
⟨*proof*⟩

Deflations from ep-pairs

lemma *deflation-e-p*: *deflation* (*e oo p*)
⟨*proof*⟩

lemma *deflation-e-d-p*:
assumes *deflation d*
shows *deflation* (*e oo d oo p*)
⟨*proof*⟩

lemma *finite-deflation-e-d-p*:
assumes *finite-deflation d*
shows *finite-deflation* (*e oo d oo p*)
⟨*proof*⟩

lemma *deflation-p-d-e*:
assumes *deflation d*
assumes *d*: $\bigwedge x.\ d\cdot x \sqsubseteq e\cdot(p\cdot x)$
shows *deflation* (*p oo d oo e*)
⟨*proof*⟩

lemma *finite-deflation-p-d-e*:
assumes *finite-deflation d*
assumes *d*: $\bigwedge x.\ d\cdot x \sqsubseteq e\cdot(p\cdot x)$
shows *finite-deflation* (*p oo d oo e*)
⟨*proof*⟩

end

10.4 Uniqueness of ep-pairs

lemma *ep-pair-unique-e-lemma*:
assumes *1*: *ep-pair e1 p*
and *2*: *ep-pair e2 p*
shows $e1 \sqsubseteq e2$
⟨*proof*⟩

lemma *ep-pair-unique-e*: *ep-pair e1 p* $\Longrightarrow$ *ep-pair e2 p* $\Longrightarrow$ $e1 = e2$
⟨*proof*⟩

lemma *ep-pair-unique-p-lemma*:
assumes *1*: *ep-pair e p1*
and *2*: *ep-pair e p2*
shows $p1 \sqsubseteq p2$
⟨*proof*⟩

lemma *ep-pair-unique-p*: *ep-pair e p1* $\Longrightarrow$ *ep-pair e p2* $\Longrightarrow$ $p1 = p2$
⟨*proof*⟩

10.5 Composing ep-pairs

lemma *ep-pair-ID-ID*: *ep-pair ID ID*

⟨*proof*⟩

lemma *ep-pair-comp*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*e2 oo e1*) (*p1 oo p2*)
⟨*proof*⟩

locale *pcpo-ep-pair* = *ep-pair e p*
for *e* :: *'a::pcpo* → *'b::pcpo*
and *p* :: *'b::pcpo* → *'a::pcpo*
begin

lemma *e-strict* [*simp*]: $e \cdot \bot = \bot$
⟨*proof*⟩

lemma *e-bottom-iff* [*simp*]: $e \cdot x = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *e-defined*: $x \neq \bot \Longrightarrow e \cdot x \neq \bot$
⟨*proof*⟩

lemma *p-strict* [*simp*]: $p \cdot \bot = \bot$
⟨*proof*⟩

lemmas *stricts* = *e-strict p-strict*

end

end

11 The type of strict products

theory *Sprod*
imports *Cfun*
begin

11.1 Definition of strict product type

definition *sprod* = $\{p::'a::pcpo \times 'b::pcpo.\ p = \bot \lor (fst\ p \neq \bot \land snd\ p \neq \bot)\}$

pcpodef (*'a::pcpo*, *'b::pcpo*) *sprod* (‹(‹*notation*=‹*infix strict product*››- ⊗/ -)› [*21*,*20*] *20*) =
sprod :: (*'a* × *'b*) *set*
⟨*proof*⟩

instance *sprod* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
⟨*proof*⟩

type-notation (*ASCII*)

sprod (**infixr** ‹**› *20*)

11.2 Definitions of constants

definition *sfst* :: ('*a*::*pcpo* ** '*b*::*pcpo*) → '*a*
where *sfst* = (Λ *p*. *fst* (*Rep-sprod p*))

definition *ssnd* :: ('*a*::*pcpo* ** '*b*::*pcpo*) → '*b*
where *ssnd* = (Λ *p*. *snd* (*Rep-sprod p*))

definition *spair* :: '*a*::*pcpo* → '*b*::*pcpo* → ('*a* ** '*b*)
where *spair* = (Λ *a b*. *Abs-sprod* (*seq*·*b*·*a*, *seq*·*a*·*b*))

definition *ssplit* :: ('*a*::*pcpo* → '*b*::*pcpo* → '*c*::*pcpo*) → ('*a* ** '*b*) → '*c*
where *ssplit* = (Λ *f p*. *seq*·*p*·(*f*·(*sfst*·*p*)·(*ssnd*·*p*)))

syntax
-stuple :: [*logic*, *args*] ⇒ *logic* (‹(‹*indent=1 notation=*‹*mixfix strict tuple*››'(:-,/ -:'))›)
syntax-consts
-stuple ⇌ *spair*
translations
(:*x*, *y*, *z*:) ⇌ (:*x*, (:*y*, *z*:):)
(:*x*, *y*:) ⇌ *CONST spair*·*x*·*y*

translations
Λ(*CONST spair*·*x*·*y*). *t* ⇌ *CONST ssplit*·(Λ *x y*. *t*)

11.3 Case analysis

lemma *spair-sprod*: (*seq*·*b*·*a*, *seq*·*a*·*b*) ∈ *sprod*
⟨*proof*⟩

lemma *Rep-sprod-spair*: *Rep-sprod* (:*a*, *b*:) = (*seq*·*b*·*a*, *seq*·*a*·*b*)
⟨*proof*⟩

lemmas *Rep-sprod-simps* =
Rep-sprod-inject [*symmetric*] *below-sprod-def*
prod-eq-iff below-prod-def
Rep-sprod-strict Rep-sprod-spair

lemma *sprodE* [*case-names bottom spair*, *cases type*: *sprod*]:
obtains *p* = ⊥ | *x y* **where** *p* = (:*x*, *y*:) **and** *x* ≠ ⊥ **and** *y* ≠ ⊥
⟨*proof*⟩

lemma *sprod-induct* [*case-names bottom spair*, *induct type*: *sprod*]:
⟦*P* ⊥; ⋀*x y*. ⟦*x* ≠ ⊥; *y* ≠ ⊥⟧ ⟹ *P* (:*x*, *y*:)⟧ ⟹ *P x*
⟨*proof*⟩

11.4 Properties of *spair*

lemma *spair-strict1* [*simp*]: $(:\bot,\ y:) = \bot$
⟨*proof*⟩

lemma *spair-strict2* [*simp*]: $(:x,\ \bot:) = \bot$
⟨*proof*⟩

lemma *spair-bottom-iff* [*simp*]: $(:x,\ y:) = \bot \longleftrightarrow x = \bot \lor y = \bot$
⟨*proof*⟩

lemma *spair-below-iff*: $(:a,\ b:) \sqsubseteq (:c,\ d:) \longleftrightarrow a = \bot \lor b = \bot \lor (a \sqsubseteq c \land b \sqsubseteq d)$
⟨*proof*⟩

lemma *spair-eq-iff*: $(:a,\ b:) = (:c,\ d:) \longleftrightarrow a = c \land b = d \lor (a = \bot \lor b = \bot) \land (c = \bot \lor d = \bot)$
⟨*proof*⟩

lemma *spair-strict*: $x = \bot \lor y = \bot \Longrightarrow (:x,\ y:) = \bot$
⟨*proof*⟩

lemma *spair-strict-rev*: $(:x,\ y:) \neq \bot \Longrightarrow x \neq \bot \land y \neq \bot$
⟨*proof*⟩

lemma *spair-defined*: $[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow (:x,\ y:) \neq \bot$
⟨*proof*⟩

lemma *spair-defined-rev*: $(:x,\ y:) = \bot \Longrightarrow x = \bot \lor y = \bot$
⟨*proof*⟩

lemma *spair-below*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x,\ y:) \sqsubseteq (:a,\ b:) \longleftrightarrow x \sqsubseteq a \land y \sqsubseteq b$
⟨*proof*⟩

lemma *spair-eq*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x,\ y:) = (:a,\ b:) \longleftrightarrow x = a \land y = b$
⟨*proof*⟩

lemma *spair-inject*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x,\ y:) = (:a,\ b:) \Longrightarrow x = a \land y = b$
⟨*proof*⟩

lemma *inst-sprod-pcpo2*: $\bot = (:\bot,\ \bot:)$
⟨*proof*⟩

lemma *sprodE2*: $(\bigwedge x\ y.\ p = (:x,\ y:) \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

11.5 Properties of *sfst* and *ssnd*

lemma *sfst-strict* [*simp*]: $sfst \cdot \bot = \bot$
⟨*proof*⟩

lemma *ssnd-strict* [*simp*]: $ssnd\cdot\bot = \bot$
⟨*proof*⟩

lemma *sfst-spair* [*simp*]: $y \neq \bot \Longrightarrow sfst\cdot(:x,\ y:) = x$
⟨*proof*⟩

lemma *ssnd-spair* [*simp*]: $x \neq \bot \Longrightarrow ssnd\cdot(:x,\ y:) = y$
⟨*proof*⟩

lemma *sfst-bottom-iff* [*simp*]: $sfst\cdot p = \bot \longleftrightarrow p = \bot$
⟨*proof*⟩

lemma *ssnd-bottom-iff* [*simp*]: $ssnd\cdot p = \bot \longleftrightarrow p = \bot$
⟨*proof*⟩

lemma *sfst-defined*: $p \neq \bot \Longrightarrow sfst\cdot p \neq \bot$
⟨*proof*⟩

lemma *ssnd-defined*: $p \neq \bot \Longrightarrow ssnd\cdot p \neq \bot$
⟨*proof*⟩

lemma *spair-sfst-ssnd*: $(:sfst\cdot p,\ ssnd\cdot p:) = p$
⟨*proof*⟩

lemma *below-sprod*: $x \sqsubseteq y \longleftrightarrow sfst\cdot x \sqsubseteq sfst\cdot y \wedge ssnd\cdot x \sqsubseteq ssnd\cdot y$
⟨*proof*⟩

lemma *eq-sprod*: $x = y \longleftrightarrow sfst\cdot x = sfst\cdot y \wedge ssnd\cdot x = ssnd\cdot y$
⟨*proof*⟩

lemma *sfst-below-iff*: $sfst\cdot x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:y,\ ssnd\cdot x:)$
⟨*proof*⟩

lemma *ssnd-below-iff*: $ssnd\cdot x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:sfst\cdot x,\ y:)$
⟨*proof*⟩

11.6 Compactness

lemma *compact-sfst*: $compact\ x \Longrightarrow compact\ (sfst\cdot x)$
⟨*proof*⟩

lemma *compact-ssnd*: $compact\ x \Longrightarrow compact\ (ssnd\cdot x)$
⟨*proof*⟩

lemma *compact-spair*: $compact\ x \Longrightarrow compact\ y \Longrightarrow compact\ (:x,\ y:)$
⟨*proof*⟩

lemma *compact-spair-iff*: $compact\ (:x,\ y:) \longleftrightarrow x = \bot \vee y = \bot \vee (compact\ x \wedge compact\ y)$

⟨*proof*⟩

11.7 Properties of *ssplit*

lemma *ssplit1* [*simp*]: *ssplit·f·*⊥ = ⊥
⟨*proof*⟩

lemma *ssplit2* [*simp*]: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow$ *ssplit·f·*(:*x*, *y*:) = *f·x·y*
⟨*proof*⟩

lemma *ssplit3* [*simp*]: *ssplit·spair·z* = *z*
⟨*proof*⟩

11.8 Strict product preserves flatness

instance *sprod* :: (*flat*, *flat*) *flat*
⟨*proof*⟩

end

12 The type of lifted values

theory *Up*
imports *Cfun*
begin

12.1 Definition of new type for lifting

datatype *′a u* (‹(‹*notation*=‹*postfix lifting*››-$_\bot$)› [*1000*] *999*) = *Ibottom* | *Iup ′a*

primrec *Ifup* :: (*′a* → *′b*::*pcpo*) ⇒ *′a u* ⇒ *′b*
where
Ifup f Ibottom = ⊥
| *Ifup f* (*Iup x*) = *f·x*

12.2 Ordering on lifted cpo

instantiation *u* :: (*cpo*) *below*
begin

definition *below-up-def*:
(⊑) ≡
(λ*x y*.
(*case x of*
Ibottom ⇒ *True*
| *Iup a* ⇒ (*case y of Ibottom* ⇒ *False* | *Iup b* ⇒ *a* ⊑ *b*)))

instance ⟨*proof*⟩

end

lemma *minimal-up* [*iff*]: *Ibottom* ⊑ *z*
⟨*proof*⟩

lemma *not-Iup-below* [*iff*]: *Iup x* ⋢ *Ibottom*
⟨*proof*⟩

lemma *Iup-below* [*iff*]: (*Iup x* ⊑ *Iup y*) = (*x* ⊑ *y*)
⟨*proof*⟩

12.3 Lifted cpo is a partial order

instance *u* :: (*cpo*) *po*
⟨*proof*⟩

12.4 Lifted cpo is a cpo

lemma *is-lub-Iup*: *range S* <<| *x* ⟹ *range* (λ*i*. *Iup* (*S i*)) <<| *Iup x*
⟨*proof*⟩

lemma *up-chain-lemma*:
assumes *Y*: *chain Y*
obtains ∀ *i*. *Y i* = *Ibottom*
| *A k* **where** ∀ *i*. *Iup* (*A i*) = *Y* (*i* + *k*) **and** *chain A* **and** *range Y* <<| *Iup* (⨆ *i*. *A i*)
⟨*proof*⟩

instance *u* :: (*cpo*) *cpo*
⟨*proof*⟩

12.5 Lifted cpo is pointed

instance *u* :: (*cpo*) *pcpo*
⟨*proof*⟩

for compatibility with old HOLCF-Version

lemma *inst-up-pcpo*: ⊥ = *Ibottom*
⟨*proof*⟩

12.6 Continuity of *Iup* and *Ifup*

continuity for *Iup*

lemma *cont-Iup*: *cont Iup*
⟨*proof*⟩

continuity for *Ifup*

lemma *cont-Ifup1*: *cont* (λ*f*. *Ifup f x*)
⟨*proof*⟩

lemma *monofun-Ifup2*: *monofun* $(\lambda x.\ Ifup\ f\ x)$
⟨*proof*⟩

lemma *cont-Ifup2*: *cont* $(\lambda x.\ Ifup\ f\ x)$
⟨*proof*⟩

12.7 Continuous versions of constants

definition *up* :: $'a \rightarrow 'a\ u$
where $up = (\Lambda\ x.\ Iup\ x)$

definition *fup* :: $('a \rightarrow 'b::pcpo) \rightarrow 'a\ u \rightarrow 'b$
where $fup = (\Lambda\ f\ p.\ Ifup\ f\ p)$

translations
$case\ l\ of\ XCONST\ up \cdot x \Rightarrow t \rightleftharpoons CONST\ fup \cdot (\Lambda\ x.\ t) \cdot l$
$case\ l\ of\ (XCONST\ up :: 'a) \cdot x \Rightarrow t \rightharpoonup CONST\ fup \cdot (\Lambda\ x.\ t) \cdot l$
$\Lambda (XCONST\ up \cdot x).\ t \rightleftharpoons CONST\ fup \cdot (\Lambda\ x.\ t)$

continuous versions of lemmas for $'a_\bot$

lemma *Exh-Up*: $z = \bot \vee (\exists\, x.\ z = up \cdot x)$
⟨*proof*⟩

lemma *up-eq* [*simp*]: $(up \cdot x = up \cdot y) = (x = y)$
⟨*proof*⟩

lemma *up-inject*: $up \cdot x = up \cdot y \Longrightarrow x = y$
⟨*proof*⟩

lemma *up-defined* [*simp*]: $up \cdot x \neq \bot$
⟨*proof*⟩

lemma *not-up-less-UU*: $up \cdot x \not\sqsubseteq \bot$
⟨*proof*⟩

lemma *up-below* [*simp*]: $up \cdot x \sqsubseteq up \cdot y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *upE* [*case-names bottom up*, *cases type*: *u*]: $[\![p = \bot \Longrightarrow Q;\ \bigwedge x.\ p = up \cdot x \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *up-induct* [*case-names bottom up*, *induct type*: *u*]: $P\ \bot \Longrightarrow (\bigwedge x.\ P\ (up \cdot x)) \Longrightarrow P\ x$
⟨*proof*⟩

lifting preserves chain-finiteness

lemma *up-chain-cases*:

assumes Y: *chain* Y
obtains $\forall\, i.\ Y\ i = \bot$
| $A\ k$ **where** $\forall\, i.\ up\cdot(A\ i) = Y\ (i + k)$ **and** *chain* A **and** $(\bigsqcup i.\ Y\ i) = up\cdot(\bigsqcup i.\ A\ i)$
⟨*proof*⟩

lemma *compact-up*: *compact* $x \Longrightarrow$ *compact* $(up\cdot x)$
⟨*proof*⟩

lemma *compact-upD*: *compact* $(up\cdot x) \Longrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-up-iff* [*simp*]: *compact* $(up\cdot x)$ = *compact* x
⟨*proof*⟩

instance u :: (*chfin*) *chfin*
⟨*proof*⟩

properties of fup

lemma *fup1* [*simp*]: $fup\cdot f\cdot\bot = \bot$
⟨*proof*⟩

lemma *fup2* [*simp*]: $fup\cdot f\cdot(up\cdot x) = f\cdot x$
⟨*proof*⟩

lemma *fup3* [*simp*]: $fup\cdot up\cdot x = x$
⟨*proof*⟩

end

13 Lifting types of class type to flat pcpo's

theory *Lift*
imports *Up*
begin

pcpodef *'a::type lift* = *UNIV* :: *'a discr u set*
⟨*proof*⟩

lemmas *inst-lift-pcpo* = *Abs-lift-strict* [*symmetric*]

definition
Def :: *'a::type* ⇒ *'a lift* **where**
Def x = *Abs-lift* $(up\cdot(Discr\ x))$

13.1 Lift as a datatype

lemma *lift-induct*: $[\![P\ \bot;\ \bigwedge x.\ P\ (Def\ x)]\!] \Longrightarrow P\ y$
⟨*proof*⟩

old-rep-datatype $\bot$::$'a$::*type lift Def*
⟨*proof*⟩

$\bot$ and *Def*

lemma *not-Undef-is-Def*: $(x \neq \bot) = (\exists\, y.\ x = Def\ y)$
⟨*proof*⟩

lemma *lift-definedE*: $[\![x \neq \bot; \bigwedge a.\ x = Def\ a \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

For $x \neq \bot$ in assumptions *defined* replaces x by *Def* a in conclusion.

⟨*ML*⟩

lemma *DefE*: $Def\ x = \bot \Longrightarrow R$
⟨*proof*⟩

lemma *DefE2*: $[\![x = Def\ s;\ x = \bot]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *Def-below-Def*: $Def\ x \sqsubseteq Def\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *Def-below-iff* [*simp*]: $Def\ x \sqsubseteq y \longleftrightarrow Def\ x = y$
⟨*proof*⟩

13.2 Lift is flat

instance *lift* :: (*type*) *flat*
⟨*proof*⟩

13.3 Continuity of *case-lift*

lemma *case-lift-eq*: $case\text{-}lift\ \bot\ f\ x = fup\cdot(\Lambda\ y.\ f\ (undiscr\ y))\cdot(Rep\text{-}lift\ x)$
⟨*proof*⟩

lemma *cont2cont-case-lift* [*simp*]:
$[\![\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y);\ cont\ g]\!] \Longrightarrow cont\ (\lambda x.\ case\text{-}lift\ \bot\ (f\ x)\ (g\ x))$
⟨*proof*⟩

13.4 Further operations

definition
flift1 :: $('a::type \Rightarrow 'b::pcpo) \Rightarrow ('a\ lift \rightarrow 'b)$ (**binder** ‹*FLIFT* › *10*) **where**
$flift1 = (\lambda f.\ (\Lambda\ x.\ case\text{-}lift\ \bot\ f\ x))$

translations
Λ(*XCONST Def x*). *t* => *CONST flift1* (λx. *t*)
Λ(*CONST Def x*). *FLIFT y*. *t* <= *FLIFT x y*. *t*

$\Lambda(CONST\ Def\ x).\ t$ <= $FLIFT\ x.\ t$

definition
flift2 :: ('*a*::*type* ⇒ '*b*::*type*) ⇒ ('*a lift* → '*b lift*) **where**
flift2 f = (*FLIFT x. Def* (*f x*))

lemma *flift1-Def* [*simp*]: *flift1 f*·(*Def x*) = (*f x*)
⟨*proof*⟩

lemma *flift2-Def* [*simp*]: *flift2 f*·(*Def x*) = *Def* (*f x*)
⟨*proof*⟩

lemma *flift1-strict* [*simp*]: *flift1 f*·⊥ = ⊥
⟨*proof*⟩

lemma *flift2-strict* [*simp*]: *flift2 f*·⊥ = ⊥
⟨*proof*⟩

lemma *flift2-defined* [*simp*]: $x \neq \bot \Longrightarrow (flift2\ f) \cdot x \neq \bot$
⟨*proof*⟩

lemma *flift2-bottom-iff* [*simp*]: $(flift2\ f \cdot x = \bot) = (x = \bot)$
⟨*proof*⟩

lemma *FLIFT-mono*:
$(\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow (FLIFT\ x.\ f\ x) \sqsubseteq (FLIFT\ x.\ g\ x)$
⟨*proof*⟩

lemma *cont2cont-flift1* [*simp*, *cont2cont*]:
$[\![\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)]\!] \Longrightarrow cont\ (\lambda x.\ FLIFT\ y.\ f\ x\ y)$
⟨*proof*⟩

end

14 The type of lifted booleans

theory *Tr*
imports *Lift*
begin

14.1 Type definition and constructors

type-synonym *tr* = *bool lift*

translations
(*type*) *tr* ↼ (*type*) *bool lift*

definition *TT* :: *tr*
where *TT* = *Def True*

definition *FF* :: *tr*
  where $FF = Def\ False$

Exhaustion and Elimination for type *tr*

lemma *Exh-tr*: $t = \bot \lor t = TT \lor t = FF$
  ⟨*proof*⟩

lemma *trE* [*case-names bottom TT FF*, *cases type*: *tr*]:
  $[\![p = \bot \Longrightarrow Q; p = TT \Longrightarrow Q; p = FF \Longrightarrow Q]\!] \Longrightarrow Q$
  ⟨*proof*⟩

lemma *tr-induct* [*case-names bottom TT FF*, *induct type*: *tr*]:
  $P\ \bot \Longrightarrow P\ TT \Longrightarrow P\ FF \Longrightarrow P\ x$
  ⟨*proof*⟩

distinctness for type *tr*

lemma *dist-below-tr* [*simp*]:
  $TT \not\sqsubseteq \bot\ FF \not\sqsubseteq \bot\ TT \not\sqsubseteq FF\ FF \not\sqsubseteq TT$
  ⟨*proof*⟩

lemma *dist-eq-tr* [*simp*]: $TT \neq \bot\ FF \neq \bot\ TT \neq FF\ \bot \neq TT\ \bot \neq FF\ FF \neq TT$
  ⟨*proof*⟩

lemma *TT-below-iff* [*simp*]: $TT \sqsubseteq x \longleftrightarrow x = TT$
  ⟨*proof*⟩

lemma *FF-below-iff* [*simp*]: $FF \sqsubseteq x \longleftrightarrow x = FF$
  ⟨*proof*⟩

lemma *not-below-TT-iff* [*simp*]: $x \not\sqsubseteq TT \longleftrightarrow x = FF$
  ⟨*proof*⟩

lemma *not-below-FF-iff* [*simp*]: $x \not\sqsubseteq FF \longleftrightarrow x = TT$
  ⟨*proof*⟩

14.2 Case analysis

definition *tr-case* :: $'a::pcpo \rightarrow 'a \rightarrow tr \rightarrow 'a$
  where $tr\text{-}case = (\Lambda\ t\ e\ (Def\ b).\ if\ b\ then\ t\ else\ e)$

abbreviation *cifte-syn* :: $[tr, 'c::pcpo, 'c] \Rightarrow 'c$ (‹(‹notation=‹mixfix If expression››If (-)/ then (-)/ else (-))› [*0*, *0*, *60*] *60*)
  where $If\ b\ then\ e1\ else\ e2 \equiv tr\text{-}case{\cdot}e1{\cdot}e2{\cdot}b$

translations
  $\Lambda\ (XCONST\ TT).\ t \rightleftharpoons CONST\ tr\text{-}case{\cdot}t{\cdot}\bot$
  $\Lambda\ (XCONST\ FF).\ t \rightleftharpoons CONST\ tr\text{-}case{\cdot}\bot{\cdot}t$

lemma *ifte-thms* [*simp*]:
If ⊥ *then e1 else e2* = ⊥
If FF then e1 else e2 = *e2*
If TT then e1 else e2 = *e1*
⟨*proof*⟩

14.3 Boolean connectives

definition *trand* :: *tr* → *tr* → *tr*
where *andalso-def*: *trand* = (Λ *x y*. *If x then y else FF*)

abbreviation *andalso-syn* :: *tr* ⇒ *tr* ⇒ *tr* (‹- *andalso* -› [*36*,*35*] *35*)
where *x andalso y* ≡ *trand·x·y*

definition *tror* :: *tr* → *tr* → *tr*
where *orelse-def*: *tror* = (Λ *x y*. *If x then TT else y*)

abbreviation *orelse-syn* :: *tr* ⇒ *tr* ⇒ *tr* (‹- *orelse* -› [*31*,*30*] *30*)
where *x orelse y* ≡ *tror·x·y*

definition *neg* :: *tr* → *tr*
where *neg* = *flift2 Not*

definition *If2* :: *tr* ⇒ ′*c*::*pcpo* ⇒ ′*c* ⇒ ′*c*
where *If2 Q x y* = (*If Q then x else y*)

tactic for tr-thms with case split

lemmas *tr-defs* = *andalso-def orelse-def neg-def tr-case-def TT-def FF-def*

lemmas about andalso, orelse, neg and if

lemma *andalso-thms* [*simp*]:
(*TT andalso y*) = *y*
(*FF andalso y*) = *FF*
(⊥ *andalso y*) = ⊥
(*y andalso TT*) = *y*
(*y andalso y*) = *y*
⟨*proof*⟩

lemma *orelse-thms* [*simp*]:
(*TT orelse y*) = *TT*
(*FF orelse y*) = *y*
(⊥ *orelse y*) = ⊥
(*y orelse FF*) = *y*
(*y orelse y*) = *y*
⟨*proof*⟩

lemma *neg-thms* [*simp*]:
neg·TT = *FF*
neg·FF = *TT*

$neg \cdot \bot = \bot$
⟨*proof*⟩

split-tac for If via If2 because the constant has to be a constant

lemma *split-If2*: $P\ (If2\ Q\ x\ y) \longleftrightarrow ((Q = \bot \longrightarrow P\ \bot) \wedge (Q = TT \longrightarrow P\ x) \wedge (Q = FF \longrightarrow P\ y))$
⟨*proof*⟩

⟨*ML*⟩

14.4 Rewriting of HOLCF operations to HOL functions

lemma *andalso-or*: $t \neq \bot \Longrightarrow (t\ andalso\ s) = FF \longleftrightarrow t = FF \vee s = FF$
⟨*proof*⟩

lemma *andalso-and*: $t \neq \bot \Longrightarrow ((t\ andalso\ s) \neq FF) \longleftrightarrow t \neq FF \wedge s \neq FF$
⟨*proof*⟩

lemma *Def-bool1* [*simp*]: $Def\ x \neq FF \longleftrightarrow x$
⟨*proof*⟩

lemma *Def-bool2* [*simp*]: $Def\ x = FF \longleftrightarrow \neg\ x$
⟨*proof*⟩

lemma *Def-bool3* [*simp*]: $Def\ x = TT \longleftrightarrow x$
⟨*proof*⟩

lemma *Def-bool4* [*simp*]: $Def\ x \neq TT \longleftrightarrow \neg\ x$
⟨*proof*⟩

lemma *If-and-if*: $(If\ Def\ P\ then\ A\ else\ B) = (if\ P\ then\ A\ else\ B)$
⟨*proof*⟩

14.5 Compactness

lemma *compact-TT*: *compact TT*
⟨*proof*⟩

lemma *compact-FF*: *compact FF*
⟨*proof*⟩

end

15 The type of strict sums

theory *Ssum*
imports *Tr*
begin

15.1 Definition of strict sum type

definition *ssum* =
{*p* :: *tr* × ('*a*::*pcpo* × '*b*::*pcpo*). *p* = ⊥ ∨
(*fst p* = *TT* ∧ *fst* (*snd p*) ≠ ⊥ ∧ *snd* (*snd p*) = ⊥) ∨
(*fst p* = *FF* ∧ *fst* (*snd p*) = ⊥ ∧ *snd* (*snd p*) ≠ ⊥)}

pcpodef ('*a*::*pcpo*, '*b*::*pcpo*) *ssum* (‹(‹notation=‹infix strict sum››- ⊕/ -)› [*21*, *20*] *20*) =
ssum :: (*tr* × '*a* × '*b*) *set*
⟨*proof*⟩

instance *ssum* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
⟨*proof*⟩

type-notation (*ASCII*)
ssum (**infixr** ‹++› *10*)

15.2 Definitions of constructors

definition *sinl* :: '*a*::*pcpo* → ('*a* ++ '*b*::*pcpo*)
where *sinl* = (Λ *a*. *Abs-ssum* (*seq*·*a*·*TT*, *a*, ⊥))

definition *sinr* :: '*b*::*pcpo* → ('*a*::*pcpo* ++ '*b*)
where *sinr* = (Λ *b*. *Abs-ssum* (*seq*·*b*·*FF*, ⊥, *b*))

lemma *sinl-ssum*: (*seq*·*a*·*TT*, *a*, ⊥) ∈ *ssum*
⟨*proof*⟩

lemma *sinr-ssum*: (*seq*·*b*·*FF*, ⊥, *b*) ∈ *ssum*
⟨*proof*⟩

lemma *Rep-ssum-sinl*: *Rep-ssum* (*sinl*·*a*) = (*seq*·*a*·*TT*, *a*, ⊥)
⟨*proof*⟩

lemma *Rep-ssum-sinr*: *Rep-ssum* (*sinr*·*b*) = (*seq*·*b*·*FF*, ⊥, *b*)
⟨*proof*⟩

lemmas *Rep-ssum-simps* =
Rep-ssum-inject [*symmetric*] *below-ssum-def*
prod-eq-iff below-prod-def
Rep-ssum-strict Rep-ssum-sinl Rep-ssum-sinr

15.3 Properties of *sinl* and *sinr*

Ordering

lemma *sinl-below* [*simp*]: *sinl*·*x* ⊑ *sinl*·*y* ⟷ *x* ⊑ *y*
⟨*proof*⟩

lemma *sinr-below* [*simp*]: $sinr{\cdot}x \sqsubseteq sinr{\cdot}y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *sinl-below-sinr* [*simp*]: $sinl{\cdot}x \sqsubseteq sinr{\cdot}y \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *sinr-below-sinl* [*simp*]: $sinr{\cdot}x \sqsubseteq sinl{\cdot}y \longleftrightarrow x = \bot$
⟨*proof*⟩

Equality

lemma *sinl-eq* [*simp*]: $sinl{\cdot}x = sinl{\cdot}y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *sinr-eq* [*simp*]: $sinr{\cdot}x = sinr{\cdot}y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *sinl-eq-sinr* [*simp*]: $sinl{\cdot}x = sinr{\cdot}y \longleftrightarrow x = \bot \land y = \bot$
⟨*proof*⟩

lemma *sinr-eq-sinl* [*simp*]: $sinr{\cdot}x = sinl{\cdot}y \longleftrightarrow x = \bot \land y = \bot$
⟨*proof*⟩

lemma *sinl-inject*: $sinl{\cdot}x = sinl{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

lemma *sinr-inject*: $sinr{\cdot}x = sinr{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

Strictness

lemma *sinl-strict* [*simp*]: $sinl{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *sinr-strict* [*simp*]: $sinr{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *sinl-bottom-iff* [*simp*]: $sinl{\cdot}x = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *sinr-bottom-iff* [*simp*]: $sinr{\cdot}x = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *sinl-defined*: $x \neq \bot \Longrightarrow sinl{\cdot}x \neq \bot$
⟨*proof*⟩

lemma *sinr-defined*: $x \neq \bot \Longrightarrow sinr{\cdot}x \neq \bot$
⟨*proof*⟩

Compactness

lemma *compact-sinl*: $compact\ x \Longrightarrow compact\ (sinl{\cdot}x)$

⟨*proof*⟩

lemma *compact-sinr*: *compact* $x \Longrightarrow$ *compact* (*sinr*·*x*)
⟨*proof*⟩

lemma *compact-sinlD*: *compact* (*sinl*·*x*) $\Longrightarrow$ *compact* *x*
⟨*proof*⟩

lemma *compact-sinrD*: *compact* (*sinr*·*x*) $\Longrightarrow$ *compact* *x*
⟨*proof*⟩

lemma *compact-sinl-iff* [*simp*]: *compact* (*sinl*·*x*) = *compact* *x*
⟨*proof*⟩

lemma *compact-sinr-iff* [*simp*]: *compact* (*sinr*·*x*) = *compact* *x*
⟨*proof*⟩

15.4 Case analysis

lemma *ssumE* [*case-names bottom sinl sinr*, *cases type*: *ssum*]:
obtains $p = \bot$
| *x* **where** *p* = *sinl*·*x* **and** $x \neq \bot$
| *y* **where** *p* = *sinr*·*y* **and** $y \neq \bot$
⟨*proof*⟩

lemma *ssum-induct* [*case-names bottom sinl sinr*, *induct type*: *ssum*]:
⟦*P* ⊥;
⋀*x*. $x \neq \bot \Longrightarrow$ *P* (*sinl*·*x*);
⋀*y*. $y \neq \bot \Longrightarrow$ *P* (*sinr*·*y*)⟧ $\Longrightarrow$ *P x*
⟨*proof*⟩

lemma *ssumE2* [*case-names sinl sinr*]:
⟦⋀*x*. *p* = *sinl*·*x* $\Longrightarrow$ *Q*; ⋀*y*. *p* = *sinr*·*y* $\Longrightarrow$ *Q*⟧ $\Longrightarrow$ *Q*
⟨*proof*⟩

lemma *below-sinlD*: *p* ⊑ *sinl*·*x* $\Longrightarrow$ ∃ *y*. *p* = *sinl*·*y* ∧ *y* ⊑ *x*
⟨*proof*⟩

lemma *below-sinrD*: *p* ⊑ *sinr*·*x* $\Longrightarrow$ ∃ *y*. *p* = *sinr*·*y* ∧ *y* ⊑ *x*
⟨*proof*⟩

15.5 Case analysis combinator

definition *sscase* :: (*'a*::*pcpo* → *'c*::*pcpo*) → (*'b*::*pcpo* → *'c*) → (*'a* ++ *'b*) → *'c*
where *sscase* = (Λ *f g s*. (λ(*t*, *x*, *y*). *If t then f*·*x else g*·*y*) (*Rep-ssum s*))

translations
case s of XCONST sinl·*x* ⇒ *t1* | *XCONST sinr*·*y* ⇒ *t2* ⇌ *CONST sscase*·(Λ *x*. *t1*)·(Λ *y*. *t2*)·*s*

case s of (XCONST sinl :: 'a)·x ⇒ t1 | XCONST sinr·y ⇒ t2 ⇀ *CONST sscase·(Λ x. t1)·(Λ y. t2)·s*

translations
Λ(*XCONST sinl·x*). *t* ⇌ *CONST sscase·(Λ x. t)·⊥*
Λ(*XCONST sinr·y*). *t* ⇌ *CONST sscase·⊥·(Λ y. t)*

lemma *beta-sscase*: *sscase·f·g·s* = (λ(*t*, *x*, *y*). *If t then f·x else g·y*) (*Rep-ssum s*)
⟨*proof*⟩

lemma *sscase1* [*simp*]: *sscase·f·g·⊥* = ⊥
⟨*proof*⟩

lemma *sscase2* [*simp*]: *x* ≠ ⊥ ⟹ *sscase·f·g·(sinl·x)* = *f·x*
⟨*proof*⟩

lemma *sscase3* [*simp*]: *y* ≠ ⊥ ⟹ *sscase·f·g·(sinr·y)* = *g·y*
⟨*proof*⟩

lemma *sscase4* [*simp*]: *sscase·sinl·sinr·z* = *z*
⟨*proof*⟩

15.6 Strict sum preserves flatness

instance *ssum* :: (*flat*, *flat*) *flat*
⟨*proof*⟩

end

16 The Strict Function Type

theory *Sfun*
imports *Cfun*
begin

pcpodef (*'a::pcpo*, *'b::pcpo*) *sfun* (**infixr** ‹→!› *0*) = {*f* :: *'a* → *'b*. *f·⊥* = ⊥}
⟨*proof*⟩

type-notation (*ASCII*)
sfun (**infixr** ‹−>!› *0*)

TODO: Define nice syntax for abstraction, application.

definition *sfun-abs* :: (*'a::pcpo* → *'b::pcpo*) → (*'a* →! *'b*)
where *sfun-abs* = (Λ *f*. *Abs-sfun* (*strictify·f*))

definition *sfun-rep* :: (*'a::pcpo* →! *'b::pcpo*) → *'a* → *'b*
where *sfun-rep* = (Λ *f*. *Rep-sfun f*)

lemma *sfun-rep-beta*: *sfun-rep·f* = *Rep-sfun f*

⟨*proof*⟩

lemma *sfun-rep-strict1* [*simp*]: *sfun-rep*·⊥ = ⊥
⟨*proof*⟩

lemma *sfun-rep-strict2* [*simp*]: *sfun-rep*·*f*·⊥ = ⊥
⟨*proof*⟩

lemma *strictify-cancel*: *f*·⊥ = ⊥ ⟹ *strictify*·*f* = *f*
⟨*proof*⟩

lemma *sfun-abs-sfun-rep* [*simp*]: *sfun-abs*·(*sfun-rep*·*f*) = *f*
⟨*proof*⟩

lemma *sfun-rep-sfun-abs* [*simp*]: *sfun-rep*·(*sfun-abs*·*f*) = *strictify*·*f*
⟨*proof*⟩

lemma *sfun-eq-iff*: *f* = *g* ⟷ *sfun-rep*·*f* = *sfun-rep*·*g*
⟨*proof*⟩

lemma *sfun-below-iff*: *f* ⊑ *g* ⟷ *sfun-rep*·*f* ⊑ *sfun-rep*·*g*
⟨*proof*⟩

end

17 Map functions for various types

theory *Map-Functions*
imports *Deflation Sprod Ssum Sfun Up*
begin

17.1 Map operator for continuous function space

definition *cfun-map* :: ('*b* → '*a*) → ('*c* → '*d*) → ('*a* → '*c*) → ('*b* → '*d*)
where *cfun-map* = (Λ *a b f x*. *b*·(*f*·(*a*·*x*)))

lemma *cfun-map-beta* [*simp*]: *cfun-map*·*a*·*b*·*f*·*x* = *b*·(*f*·(*a*·*x*))
⟨*proof*⟩

lemma *cfun-map-ID*: *cfun-map*·*ID*·*ID* = *ID*
⟨*proof*⟩

lemma *cfun-map-map*: *cfun-map*·*f1*·*g1*·(*cfun-map*·*f2*·*g2*·*p*) = *cfun-map*·(Λ *x*. *f2*·(*f1*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
⟨*proof*⟩

lemma *ep-pair-cfun-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*cfun-map*·*p1*·*e2*) (*cfun-map*·*e1*·*p2*)

⟨*proof*⟩

lemma *deflation-cfun-map*:
  assumes *deflation d1* **and** *deflation d2*
  shows *deflation* (*cfun-map*·*d1*·*d2*)
⟨*proof*⟩

lemma *finite-range-cfun-map*:
  assumes *a*: *finite* (*range* ($\lambda x.$ *a*·*x*))
  assumes *b*: *finite* (*range* ($\lambda y.$ *b*·*y*))
  shows *finite* (*range* ($\lambda f.$ *cfun-map*·*a*·*b*·*f*)) (**is** *finite* (*range ?h*))
⟨*proof*⟩

lemma *finite-deflation-cfun-map*:
  assumes *finite-deflation d1* **and** *finite-deflation d2*
  shows *finite-deflation* (*cfun-map*·*d1*·*d2*)
⟨*proof*⟩

Finite deflations are compact elements of the function space

lemma *finite-deflation-imp-compact*: *finite-deflation d* $\Longrightarrow$ *compact d*
  ⟨*proof*⟩

17.2 Map operator for product type

definition *prod-map* :: ($'a \to 'b$) $\to$ ($'c \to 'd$) $\to$ $'a \times 'c \to 'b \times 'd$
  where *prod-map* = (Λ *f g p*. (*f*·(*fst p*), *g*·(*snd p*)))

lemma *prod-map-Pair* [*simp*]: *prod-map*·*f*·*g*·(*x*, *y*) = (*f*·*x*, *g*·*y*)
  ⟨*proof*⟩

lemma *prod-map-ID*: *prod-map*·*ID*·*ID* = *ID*
  ⟨*proof*⟩

lemma *prod-map-map*: *prod-map*·*f1*·*g1*·(*prod-map*·*f2*·*g2*·*p*) = *prod-map*·(Λ *x*. *f1*·(*f2*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
  ⟨*proof*⟩

lemma *ep-pair-prod-map*:
  assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
  shows *ep-pair* (*prod-map*·*e1*·*e2*) (*prod-map*·*p1*·*p2*)
⟨*proof*⟩

lemma *deflation-prod-map*:
  assumes *deflation d1* **and** *deflation d2*
  shows *deflation* (*prod-map*·*d1*·*d2*)
⟨*proof*⟩

lemma *finite-deflation-prod-map*:
  assumes *finite-deflation d1* **and** *finite-deflation d2*

shows *finite-deflation* $(prod\text{-}map \cdot d1 \cdot d2)$
⟨*proof*⟩

17.3 Map function for lifted cpo

definition *u-map* $:: ('a \to 'b) \to 'a\ u \to 'b\ u$
where $u\text{-}map = (\Lambda\ f.\ fup \cdot (up\ oo\ f))$

lemma *u-map-strict* [*simp*]: $u\text{-}map \cdot f \cdot \bot = \bot$
⟨*proof*⟩

lemma *u-map-up* [*simp*]: $u\text{-}map \cdot f \cdot (up \cdot x) = up \cdot (f \cdot x)$
⟨*proof*⟩

lemma *u-map-ID*: $u\text{-}map \cdot ID = ID$
⟨*proof*⟩

lemma *u-map-map*: $u\text{-}map \cdot f \cdot (u\text{-}map \cdot g \cdot p) = u\text{-}map \cdot (\Lambda\ x.\ f \cdot (g \cdot x)) \cdot p$
⟨*proof*⟩

lemma *u-map-oo*: $u\text{-}map \cdot (f\ oo\ g) = u\text{-}map \cdot f\ oo\ u\text{-}map \cdot g$
⟨*proof*⟩

lemma *ep-pair-u-map*: $ep\text{-}pair\ e\ p \Longrightarrow ep\text{-}pair\ (u\text{-}map \cdot e)\ (u\text{-}map \cdot p)$
⟨*proof*⟩

lemma *deflation-u-map*: $deflation\ d \Longrightarrow deflation\ (u\text{-}map \cdot d)$
⟨*proof*⟩

lemma *finite-deflation-u-map*:
assumes *finite-deflation d*
shows *finite-deflation* $(u\text{-}map \cdot d)$
⟨*proof*⟩

17.4 Map function for strict products

definition *sprod-map* $:: ('a::pcpo \to 'b::pcpo) \to ('c::pcpo \to 'd::pcpo) \to 'a \otimes 'c \to 'b \otimes 'd$
where $sprod\text{-}map = (\Lambda\ f\ g.\ ssplit \cdot (\Lambda\ x\ y.\ (:f \cdot x,\ g \cdot y:)))$

lemma *sprod-map-strict* [*simp*]: $sprod\text{-}map \cdot a \cdot b \cdot \bot = \bot$
⟨*proof*⟩

lemma *sprod-map-spair* [*simp*]: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow sprod\text{-}map \cdot f \cdot g \cdot (:x,\ y:) = (:f \cdot x,\ g \cdot y:)$
⟨*proof*⟩

lemma *sprod-map-spair'*: $f \cdot \bot = \bot \Longrightarrow g \cdot \bot = \bot \Longrightarrow sprod\text{-}map \cdot f \cdot g \cdot (:x,\ y:) = (:f \cdot x,\ g \cdot y:)$
⟨*proof*⟩

lemma *sprod-map-ID*: $sprod\text{-}map \cdot ID \cdot ID = ID$
⟨*proof*⟩

lemma *sprod-map-map*:
$[\![f1 \cdot \bot = \bot;\ g1 \cdot \bot = \bot]\!] \Longrightarrow$
$sprod\text{-}map \cdot f1 \cdot g1 \cdot (sprod\text{-}map \cdot f2 \cdot g2 \cdot p) =$
$sprod\text{-}map \cdot (\Lambda\ x.\ f1 \cdot (f2 \cdot x)) \cdot (\Lambda\ x.\ g1 \cdot (g2 \cdot x)) \cdot p$
⟨*proof*⟩

lemma *ep-pair-sprod-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* $(sprod\text{-}map \cdot e1 \cdot e2)$ $(sprod\text{-}map \cdot p1 \cdot p2)$
⟨*proof*⟩

lemma *deflation-sprod-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* $(sprod\text{-}map \cdot d1 \cdot d2)$
⟨*proof*⟩

lemma *finite-deflation-sprod-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* $(sprod\text{-}map \cdot d1 \cdot d2)$
⟨*proof*⟩

17.5 Map function for strict sums

definition *ssum-map* :: $('a::pcpo \to 'b::pcpo) \to ('c::pcpo \to 'd::pcpo) \to 'a \oplus 'c \to 'b \oplus 'd$
where $ssum\text{-}map = (\Lambda\ f\ g.\ sscase \cdot (sinl\ oo\ f) \cdot (sinr\ oo\ g))$

lemma *ssum-map-strict* [*simp*]: $ssum\text{-}map \cdot f \cdot g \cdot \bot = \bot$
⟨*proof*⟩

lemma *ssum-map-sinl* [*simp*]: $x \neq \bot \Longrightarrow ssum\text{-}map \cdot f \cdot g \cdot (sinl \cdot x) = sinl \cdot (f \cdot x)$
⟨*proof*⟩

lemma *ssum-map-sinr* [*simp*]: $x \neq \bot \Longrightarrow ssum\text{-}map \cdot f \cdot g \cdot (sinr \cdot x) = sinr \cdot (g \cdot x)$
⟨*proof*⟩

lemma *ssum-map-sinl'*: $f \cdot \bot = \bot \Longrightarrow ssum\text{-}map \cdot f \cdot g \cdot (sinl \cdot x) = sinl \cdot (f \cdot x)$
⟨*proof*⟩

lemma *ssum-map-sinr'*: $g \cdot \bot = \bot \Longrightarrow ssum\text{-}map \cdot f \cdot g \cdot (sinr \cdot x) = sinr \cdot (g \cdot x)$
⟨*proof*⟩

lemma *ssum-map-ID*: $ssum\text{-}map \cdot ID \cdot ID = ID$
⟨*proof*⟩

lemma *ssum-map-map*:
$[\![f1\cdot\bot = \bot;\ g1\cdot\bot = \bot]\!] \Longrightarrow$
$ssum\text{-}map\cdot f1\cdot g1\cdot(ssum\text{-}map\cdot f2\cdot g2\cdot p) =$
$ssum\text{-}map\cdot(\Lambda\ x.\ f1\cdot(f2\cdot x))\cdot(\Lambda\ x.\ g1\cdot(g2\cdot x))\cdot p$
⟨*proof*⟩

lemma *ep-pair-ssum-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* $(ssum\text{-}map\cdot e1\cdot e2)$ $(ssum\text{-}map\cdot p1\cdot p2)$
⟨*proof*⟩

lemma *deflation-ssum-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* $(ssum\text{-}map\cdot d1\cdot d2)$
⟨*proof*⟩

lemma *finite-deflation-ssum-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* $(ssum\text{-}map\cdot d1\cdot d2)$
⟨*proof*⟩

17.6 Map operator for strict function space

definition *sfun-map* :: $('b::pcpo \to 'a::pcpo) \to ('c::pcpo \to 'd::pcpo) \to ('a \to!\ 'c) \to ('b \to!\ 'd)$
where $sfun\text{-}map = (\Lambda\ a\ b.\ sfun\text{-}abs\ oo\ cfun\text{-}map\cdot a\cdot b\ oo\ sfun\text{-}rep)$

lemma *sfun-map-ID*: $sfun\text{-}map\cdot ID\cdot ID = ID$
⟨*proof*⟩

lemma *sfun-map-map*:
assumes $f2\cdot\bot = \bot$ **and** $g2\cdot\bot = \bot$
shows $sfun\text{-}map\cdot f1\cdot g1\cdot(sfun\text{-}map\cdot f2\cdot g2\cdot p) =$
$sfun\text{-}map\cdot(\Lambda\ x.\ f2\cdot(f1\cdot x))\cdot(\Lambda\ x.\ g1\cdot(g2\cdot x))\cdot p$
⟨*proof*⟩

lemma *ep-pair-sfun-map*:
assumes *1*: *ep-pair e1 p1*
assumes *2*: *ep-pair e2 p2*
shows *ep-pair* $(sfun\text{-}map\cdot p1\cdot e2)$ $(sfun\text{-}map\cdot e1\cdot p2)$
⟨*proof*⟩

lemma *deflation-sfun-map*:
assumes *1*: *deflation d1*
assumes *2*: *deflation d2*
shows *deflation* $(sfun\text{-}map\cdot d1\cdot d2)$
⟨*proof*⟩

lemma *finite-deflation-sfun-map*:

assumes *finite-deflation d1*
and *finite-deflation d2*
shows *finite-deflation* (*sfun-map*·*d1*·*d2*)
⟨*proof*⟩

end

18 The cpo of cartesian products

theory *Cprod*
imports *Cfun*
begin

18.1 Continuous case function for unit type

definition *unit-when* :: $'a \to unit \to 'a$
where *unit-when* = (Λ *a* -. *a*)

translations
Λ(). *t* ⇌ *CONST unit-when*·*t*

lemma *unit-when* [*simp*]: *unit-when*·*a*·*u* = *a*
⟨*proof*⟩

18.2 Continuous version of split function

definition *csplit* :: $('a \to 'b \to 'c) \to ('a \times 'b) \to 'c$
where *csplit* = (Λ *f p*. *f*·(*fst p*)·(*snd p*))

translations
Λ(*CONST Pair x y*). *t* ⇌ *CONST csplit*·(Λ *x y*. *t*)

abbreviation *cfst* :: $'a \times 'b \to 'a$
where *cfst* ≡ *Abs-cfun fst*

abbreviation *csnd* :: $'a \times 'b \to 'b$
where *csnd* ≡ *Abs-cfun snd*

18.3 Convert all lemmas to the continuous versions

lemma *csplit1* [*simp*]: *csplit*·*f*·⊥ = *f*·⊥·⊥
⟨*proof*⟩

lemma *csplit-Pair* [*simp*]: *csplit*·*f*·(*x*, *y*) = *f*·*x*·*y*
⟨*proof*⟩

end

19 Profinite and bifinite cpos

theory *Bifinite*
imports *Map-Functions Cprod Sprod Sfun Up HOL−Library.Countable*
begin

19.1 Chains of finite deflations

locale *approx-chain* =
fixes *approx* :: $nat \Rightarrow 'a \rightarrow 'a$
assumes *chain-approx* [*simp*]: *chain* ($\lambda i.$ *approx i*)
assumes *lub-approx* [*simp*]: ($\bigsqcup i.$ *approx i*) = *ID*
assumes *finite-deflation-approx* [*simp*]: $\bigwedge i.$ *finite-deflation* (*approx i*)
begin

lemma *deflation-approx*: *deflation* (*approx i*)
⟨*proof*⟩

lemma *approx-idem*: *approx i·*(*approx i·x*) = *approx i·x*
⟨*proof*⟩

lemma *approx-below*: *approx i·x* $\sqsubseteq$ *x*
⟨*proof*⟩

lemma *finite-range-approx*: *finite* (*range* ($\lambda x.$ *approx i·x*))
⟨*proof*⟩

lemma *compact-approx* [*simp*]: *compact* (*approx n·x*)
⟨*proof*⟩

lemma *compact-eq-approx*: *compact x* $\Longrightarrow \exists i.$ *approx i·x* = *x*
⟨*proof*⟩

end

19.2 Omega-profinite and bifinite domains

class *bifinite* = *pcpo* +
assumes *bifinite*: $\exists (a::nat \Rightarrow 'a \rightarrow 'a).$ *approx-chain a*

class *profinite* = *cpo* +
assumes *profinite*: $\exists (a::nat \Rightarrow 'a_\bot \rightarrow 'a_\bot).$ *approx-chain a*

19.3 Building approx chains

lemma *approx-chain-iso*:
assumes *a*: *approx-chain a*
assumes [*simp*]: $\bigwedge x.$ *f·*(*g·x*) = *x*
assumes [*simp*]: $\bigwedge y.$ *g·*(*f·y*) = *y*
shows *approx-chain* ($\lambda i.$ *f oo a i oo g*)

⟨*proof*⟩

lemma *approx-chain-u-map*:
assumes *approx-chain a*
shows *approx-chain* (λ*i*. *u-map*·(*a i*))
⟨*proof*⟩

lemma *approx-chain-sfun-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* (λ*i*. *sfun-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-sprod-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* (λ*i*. *sprod-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-ssum-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* (λ*i*. *ssum-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-cfun-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* (λ*i*. *cfun-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-prod-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* (λ*i*. *prod-map*·(*a i*)·(*b i*))
⟨*proof*⟩

Approx chains for countable discrete types.

definition *discr-approx* :: *nat* ⇒ *'a::countable discr u* → *'a discr u*
where *discr-approx* = (λ*i*. Λ(*up*·*x*). *if to-nat* (*undiscr x*) < *i then up*·*x else* ⊥)

lemma *chain-discr-approx* [*simp*]: *chain discr-approx*
⟨*proof*⟩

lemma *lub-discr-approx* [*simp*]: (⨆ *i*. *discr-approx i*) = *ID*
⟨*proof*⟩

lemma *inj-on-undiscr* [*simp*]: *inj-on undiscr A*
⟨*proof*⟩

lemma *finite-deflation-discr-approx*: *finite-deflation* (*discr-approx i*)
⟨*proof*⟩

lemma *discr-approx*: *approx-chain discr-approx*

⟨*proof*⟩

19.4 Class instance proofs

instance *bifinite* ⊆ *profinite*
⟨*proof*⟩

instance *u* :: (*profinite*) *bifinite*
⟨*proof*⟩

Types $'a \rightarrow 'b$ and $'a_{\perp} \rightarrow! \ 'b$ are isomorphic.

definition $encode\text{-}cfun = (\Lambda\ f.\ sfun\text{-}abs\cdot(fup\cdot f))$

definition $decode\text{-}cfun = (\Lambda\ g\ x.\ sfun\text{-}rep\cdot g\cdot(up\cdot x))$

lemma *decode-encode-cfun* [*simp*]: $decode\text{-}cfun\cdot(encode\text{-}cfun\cdot x) = x$
⟨*proof*⟩

lemma *encode-decode-cfun* [*simp*]: $encode\text{-}cfun\cdot(decode\text{-}cfun\cdot y) = y$
⟨*proof*⟩

instance *cfun* :: (*profinite*, *bifinite*) *bifinite*
⟨*proof*⟩

Types $('a \times 'b)_{\perp}$ and $'a_{\perp} \otimes 'b_{\perp}$ are isomorphic.

definition $encode\text{-}prod\text{-}u = (\Lambda(up\cdot(x, y)).\ (:up\cdot x,\ up\cdot y:))$

definition $decode\text{-}prod\text{-}u = (\Lambda(:up\cdot x,\ up\cdot y:).\ up\cdot(x, y))$

lemma *decode-encode-prod-u* [*simp*]: $decode\text{-}prod\text{-}u\cdot(encode\text{-}prod\text{-}u\cdot x) = x$
⟨*proof*⟩

lemma *encode-decode-prod-u* [*simp*]: $encode\text{-}prod\text{-}u\cdot(decode\text{-}prod\text{-}u\cdot y) = y$
⟨*proof*⟩

instance *prod* :: (*profinite*, *profinite*) *profinite*
⟨*proof*⟩

instance *prod* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

instance *sfun* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

instance *sprod* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

instance *ssum* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

lemma *approx-chain-unit*: *approx-chain* (⊥ :: *nat* ⇒ *unit* → *unit*)
⟨*proof*⟩

instance *unit* :: *bifinite*
⟨*proof*⟩

instance *discr* :: (*countable*) *profinite*
⟨*proof*⟩

instance *lift* :: (*countable*) *bifinite*
⟨*proof*⟩

end

20 Defining algebraic domains by ideal completion

theory *Completion*
imports *Cfun*
begin

20.1 Ideals over a preorder

locale *preorder* =
fixes r :: *'a::type* ⇒ *'a* ⇒ *bool* (**infix** ‹⪯› *50*)
assumes *r-refl*: $x \preceq x$
assumes *r-trans*: $⟦x \preceq y;\ y \preceq z⟧ \Longrightarrow x \preceq z$
begin

definition
ideal :: *'a set* ⇒ *bool* **where**
ideal $A = ((\exists x.\ x \in A) \wedge (\forall x{\in}A.\ \forall y{\in}A.\ \exists z{\in}A.\ x \preceq z \wedge y \preceq z) \wedge$
$(\forall x\ y.\ x \preceq y \longrightarrow y \in A \longrightarrow x \in A))$

lemma *idealI*:
assumes $\exists x.\ x \in A$
assumes $\bigwedge x\ y.\ ⟦x \in A;\ y \in A⟧ \Longrightarrow \exists z{\in}A.\ x \preceq z \wedge y \preceq z$
assumes $\bigwedge x\ y.\ ⟦x \preceq y;\ y \in A⟧ \Longrightarrow x \in A$
shows *ideal* A
⟨*proof*⟩

lemma *idealD1*:
ideal $A \Longrightarrow \exists x.\ x \in A$
⟨*proof*⟩

lemma *idealD2*:
⟦*ideal* $A;\ x \in A;\ y \in A⟧ \Longrightarrow \exists z{\in}A.\ x \preceq z \wedge y \preceq z$
⟨*proof*⟩

lemma *idealD3*:
$\llbracket ideal\ A;\ x \preceq y;\ y \in A \rrbracket \Longrightarrow x \in A$
⟨*proof*⟩

lemma *ideal-principal*: $ideal\ \{x.\ x \preceq z\}$
⟨*proof*⟩

lemma *ex-ideal*: $\exists A.\ A \in \{A.\ ideal\ A\}$
⟨*proof*⟩

The set of ideals is a cpo

lemma *ideal-UN*:
fixes $A :: nat \Rightarrow 'a\ set$
assumes *ideal-A*: $\bigwedge i.\ ideal\ (A\ i)$
assumes *chain-A*: $\bigwedge i\ j.\ i \le j \Longrightarrow A\ i \subseteq A\ j$
shows $ideal\ (\bigcup i.\ A\ i)$
⟨*proof*⟩

lemma *typedef-ideal-po*:
fixes $Abs :: 'a\ set \Rightarrow 'b::below$
assumes *type*: *type-definition Rep Abs* $\{S.\ ideal\ S\}$
assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
shows $OFCLASS('b,\ po\text{-}class)$
⟨*proof*⟩

lemma
fixes $Abs :: 'a\ set \Rightarrow 'b::po$
assumes *type*: *type-definition Rep Abs* $\{S.\ ideal\ S\}$
assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
assumes S: $chain\ S$
shows *typedef-ideal-lub*: $range\ S <<|\ Abs\ (\bigcup i.\ Rep\ (S\ i))$
and *typedef-ideal-rep-lub*: $Rep\ (\bigsqcup i.\ S\ i) = (\bigcup i.\ Rep\ (S\ i))$
⟨*proof*⟩

lemma *typedef-ideal-cpo*:
fixes $Abs :: 'a\ set \Rightarrow 'b::po$
assumes *type*: *type-definition Rep Abs* $\{S.\ ideal\ S\}$
assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
shows $OFCLASS('b,\ cpo\text{-}class)$
⟨*proof*⟩

end

interpretation *below*: *preorder below* $:: 'a::po \Rightarrow 'a \Rightarrow bool$
⟨*proof*⟩

20.2 Lemmas about least upper bounds

lemma *is-ub-thelub-ex*: $\llbracket \exists u.\ S <<|\ u;\ x \in S \rrbracket \Longrightarrow x \sqsubseteq lub\ S$

⟨*proof*⟩

lemma *is-lub-thelub-ex*: $[\![\exists\, u.\ S <<|\ u;\ S <|\ x]\!] \Longrightarrow lub\ S \sqsubseteq x$
⟨*proof*⟩

20.3 Locale for ideal completion

hide-const (**open**) *Filter.principal*

locale *ideal-completion* = *preorder* +
  fixes $principal :: 'a::type \Rightarrow 'b$
  fixes $rep :: 'b \Rightarrow 'a::type\ set$
  assumes *ideal-rep*: $\bigwedge x.\ ideal\ (rep\ x)$
  assumes *rep-lub*: $\bigwedge Y.\ chain\ Y \Longrightarrow rep\ (\bigsqcup i.\ Y\ i) = (\bigcup i.\ rep\ (Y\ i))$
  assumes *rep-principal*: $\bigwedge a.\ rep\ (principal\ a) = \{b.\ b \preceq a\}$
  assumes *belowI*: $\bigwedge x\ y.\ rep\ x \subseteq rep\ y \Longrightarrow x \sqsubseteq y$
  assumes *countable*: $\exists f::'a \Rightarrow nat.\ inj\ f$
begin

lemma *rep-mono*: $x \sqsubseteq y \Longrightarrow rep\ x \subseteq rep\ y$
⟨*proof*⟩

lemma *below-def*: $x \sqsubseteq y \longleftrightarrow rep\ x \subseteq rep\ y$
⟨*proof*⟩

lemma *principal-below-iff-mem-rep*: $principal\ a \sqsubseteq x \longleftrightarrow a \in rep\ x$
⟨*proof*⟩

lemma *principal-below-iff* [*simp*]: $principal\ a \sqsubseteq principal\ b \longleftrightarrow a \preceq b$
⟨*proof*⟩

lemma *principal-eq-iff*: $principal\ a = principal\ b \longleftrightarrow a \preceq b \wedge b \preceq a$
⟨*proof*⟩

lemma *eq-iff*: $x = y \longleftrightarrow rep\ x = rep\ y$
⟨*proof*⟩

lemma *principal-mono*: $a \preceq b \Longrightarrow principal\ a \sqsubseteq principal\ b$
⟨*proof*⟩

lemma *ch2ch-principal* [*simp*]:
  $\forall\, i.\ Y\ i \preceq Y\ (Suc\ i) \Longrightarrow chain\ (\lambda i.\ principal\ (Y\ i))$
⟨*proof*⟩

20.3.1 Principal ideals approximate all elements

lemma *compact-principal* [*simp*]: *compact* (*principal a*)
⟨*proof*⟩

Construct a chain whose lub is the same as a given ideal

lemma *obtain-principal-chain*:
obtains Y **where** $\forall i.\ Y\ i \preceq Y\ (Suc\ i)$ **and** $x = (\bigsqcup i.\ principal\ (Y\ i))$
⟨*proof*⟩

lemma *principal-induct*:
assumes *adm*: $adm\ P$
assumes P: $\bigwedge a.\ P\ (principal\ a)$
shows $P\ x$
⟨*proof*⟩

lemma *compact-imp-principal*: $compact\ x \Longrightarrow \exists\, a.\ x = principal\ a$
⟨*proof*⟩

20.4 Defining functions in terms of basis elements

definition
$extension :: ('a::type \Rightarrow 'c) \Rightarrow 'b \rightarrow 'c$ **where**
$extension = (\lambda f.\ (\Lambda\ x.\ lub\ (f\ `\ rep\ x)))$

lemma *extension-lemma*:
fixes $f :: 'a::type \Rightarrow 'c$
assumes *f-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow f\ a \sqsubseteq f\ b$
shows $\exists\, u.\ f\ `\ rep\ x <<|\ u$
⟨*proof*⟩

lemma *extension-beta*:
fixes $f :: 'a::type \Rightarrow 'c$
assumes *f-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow f\ a \sqsubseteq f\ b$
shows $extension\ f \cdot x = lub\ (f\ `\ rep\ x)$
⟨*proof*⟩

lemma *extension-principal*:
fixes $f :: 'a::type \Rightarrow 'c$
assumes *f-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow f\ a \sqsubseteq f\ b$
shows $extension\ f \cdot (principal\ a) = f\ a$
⟨*proof*⟩

lemma *extension-mono*:
assumes *f-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow f\ a \sqsubseteq f\ b$
assumes *g-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow g\ a \sqsubseteq g\ b$
assumes *below*: $\bigwedge a.\ f\ a \sqsubseteq g\ a$
shows $extension\ f \sqsubseteq extension\ g$
⟨*proof*⟩

lemma *cont-extension*:
assumes *f-mono*: $\bigwedge a\ b\ x.\ a \preceq b \Longrightarrow f\ x\ a \sqsubseteq f\ x\ b$
assumes *f-cont*: $\bigwedge a.\ cont\ (\lambda x.\ f\ x\ a)$
shows $cont\ (\lambda x.\ extension\ (\lambda a.\ f\ x\ a))$
⟨*proof*⟩

end

lemma (**in** *preorder*) *typedef-ideal-completion*:
  fixes *Abs* :: *'a set* ⇒ *'b*
  assumes *type*: *type-definition Rep Abs* {*S*. *ideal S*}
  assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
  assumes *principal*: $\bigwedge a.\ principal\ a = Abs\ \{b.\ b \preceq a\}$
  assumes *countable*: $\exists f{::}'a \Rightarrow nat.\ inj\ f$
  shows *ideal-completion r principal Rep*
⟨*proof*⟩

end

21 A universal bifinite domain

theory *Universal*
imports *Bifinite Completion HOL−Library.Nat-Bijection*
begin

unbundle *no binomial-syntax*

21.1 Basis for universal domain

21.1.1 Basis datatype

type-synonym *ubasis* = *nat*

definition
  node :: *nat* ⇒ *ubasis* ⇒ *ubasis set* ⇒ *ubasis*
where
  node i a S = *Suc* (*prod-encode* (*i*, *prod-encode* (*a*, *set-encode S*)))

lemma *node-not-0* [*simp*]: $node\ i\ a\ S \neq 0$
⟨*proof*⟩

lemma *node-gt-0* [*simp*]: $0 < node\ i\ a\ S$
⟨*proof*⟩

lemma *node-inject* [*simp*]:
  ⟦*finite S*; *finite T*⟧
    $\Longrightarrow node\ i\ a\ S = node\ j\ b\ T \longleftrightarrow i = j \wedge a = b \wedge S = T$
⟨*proof*⟩

lemma *node-gt0*: $i < node\ i\ a\ S$
⟨*proof*⟩

lemma *node-gt1*: $a < node\ i\ a\ S$
⟨*proof*⟩

lemma *nat-less-power2*: $n < 2\hat{\ }n$
⟨*proof*⟩

lemma *node-gt2*: $[\![$*finite* S; $b \in S]\!] \Longrightarrow b <$ *node* i a S
⟨*proof*⟩

lemma *eq-prod-encode-pairI*:
$[\![$*fst* (*prod-decode* x) $= a$; *snd* (*prod-decode* x) $= b]\!] \Longrightarrow x =$ *prod-encode* (a, b)
⟨*proof*⟩

lemma *node-cases*:
assumes *1*: $x = 0 \Longrightarrow P$
assumes *2*: $\bigwedge i\ a\ S.\ [\![$*finite* S; $x =$ *node* i a $S]\!] \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *node-induct*:
assumes *1*: $P\ 0$
assumes *2*: $\bigwedge i\ a\ S.\ [\![P\ a$; *finite* S; $\forall b \in S.\ P\ b]\!] \Longrightarrow P$ (*node* i a S)
shows $P\ x$
⟨*proof*⟩

21.1.2 Basis ordering

inductive
ubasis-le :: *nat* ⇒ *nat* ⇒ *bool*
where
ubasis-le-refl: *ubasis-le* a a
| *ubasis-le-trans*:
$[\![$*ubasis-le* a b; *ubasis-le* b $c]\!] \Longrightarrow$ *ubasis-le* a c
| *ubasis-le-lower*:
finite $S \Longrightarrow$ *ubasis-le* a (*node* i a S)
| *ubasis-le-upper*:
$[\![$*finite* S; $b \in S$; *ubasis-le* a $b]\!] \Longrightarrow$ *ubasis-le* (*node* i a S) b

lemma *ubasis-le-minimal*: *ubasis-le* 0 x
⟨*proof*⟩

interpretation *udom*: *preorder ubasis-le*
⟨*proof*⟩

21.1.3 Generic take function

function
ubasis-until :: (*ubasis* ⇒ *bool*) ⇒ *ubasis* ⇒ *ubasis*
where
ubasis-until P $0 = 0$
| *finite* $S \Longrightarrow$ *ubasis-until* P (*node* i a S) $=$
(*if* P (*node* i a S) *then node* i a S *else ubasis-until* P a)

⟨*proof*⟩

termination *ubasis-until*
⟨*proof*⟩

lemma *ubasis-until*: $P\ 0 \Longrightarrow P\ (ubasis\text{-}until\ P\ x)$
⟨*proof*⟩

lemma *ubasis-until′*: $0 < ubasis\text{-}until\ P\ x \Longrightarrow P\ (ubasis\text{-}until\ P\ x)$
⟨*proof*⟩

lemma *ubasis-until-same*: $P\ x \Longrightarrow ubasis\text{-}until\ P\ x = x$
⟨*proof*⟩

lemma *ubasis-until-idem*:
$P\ 0 \Longrightarrow ubasis\text{-}until\ P\ (ubasis\text{-}until\ P\ x) = ubasis\text{-}until\ P\ x$
⟨*proof*⟩

lemma *ubasis-until-0*:
$\forall x.\ x \neq 0 \longrightarrow \neg\ P\ x \Longrightarrow ubasis\text{-}until\ P\ x = 0$
⟨*proof*⟩

lemma *ubasis-until-less*: $ubasis\text{-}le\ (ubasis\text{-}until\ P\ x)\ x$
⟨*proof*⟩

lemma *ubasis-until-chain*:
assumes *PQ*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
shows $ubasis\text{-}le\ (ubasis\text{-}until\ P\ x)\ (ubasis\text{-}until\ Q\ x)$
⟨*proof*⟩

lemma *ubasis-until-mono*:
assumes $\bigwedge i\ a\ S\ b.\ [\![finite\ S;\ P\ (node\ i\ a\ S);\ b \in S;\ ubasis\text{-}le\ a\ b]\!] \Longrightarrow P\ b$
shows $ubasis\text{-}le\ a\ b \Longrightarrow ubasis\text{-}le\ (ubasis\text{-}until\ P\ a)\ (ubasis\text{-}until\ P\ b)$
⟨*proof*⟩

lemma *finite-range-ubasis-until*:
$finite\ \{x.\ P\ x\} \Longrightarrow finite\ (range\ (ubasis\text{-}until\ P))$
⟨*proof*⟩

21.2 Defining the universal domain by ideal completion

typedef $udom = \{S.\ udom.ideal\ S\}$
⟨*proof*⟩

instantiation *udom* :: *below*
begin

definition
$x \sqsubseteq y \longleftrightarrow Rep\text{-}udom\ x \subseteq Rep\text{-}udom\ y$

instance ⟨*proof*⟩
end

instance *udom* :: *po*
⟨*proof*⟩

instance *udom* :: *cpo*
⟨*proof*⟩

definition
udom-principal :: *nat* ⇒ *udom* **where**
udom-principal t = *Abs-udom* {*u. ubasis-le u t*}

lemma *ubasis-countable*: ∃*f*::*ubasis* ⇒ *nat. inj f*
⟨*proof*⟩

interpretation *udom*:
ideal-completion ubasis-le udom-principal Rep-udom
⟨*proof*⟩

Universal domain is pointed

lemma *udom-minimal*: *udom-principal 0* ⊑ *x*
⟨*proof*⟩

instance *udom* :: *pcpo*
⟨*proof*⟩

lemma *inst-udom-pcpo*: ⊥ = *udom-principal 0*
⟨*proof*⟩

21.3 Compact bases of domains

typedef *'a compact-basis* = {*x*::*'a*::*pcpo. compact x*}
⟨*proof*⟩

lemma *Rep-compact-basis'* [*simp*]: *compact* (*Rep-compact-basis a*)
⟨*proof*⟩

lemma *Abs-compact-basis-inverse'* [*simp*]:
compact x ⟹ *Rep-compact-basis* (*Abs-compact-basis x*) = *x*
⟨*proof*⟩

instantiation *compact-basis* :: (*pcpo*) *below*
begin

definition
compact-le-def:
(⊑) ≡ (λ*x y. Rep-compact-basis x* ⊑ *Rep-compact-basis y*)

instance ⟨*proof*⟩
end

instance *compact-basis* :: (*pcpo*) *po*
⟨*proof*⟩

definition
approximants :: *'a::pcpo* ⇒ *'a compact-basis set* **where**
approximants = (λ*x*. {*a*. *Rep-compact-basis a* ⊑ *x*})

definition
compact-bot :: *'a::pcpo compact-basis* **where**
compact-bot = *Abs-compact-basis* ⊥

lemma *Rep-compact-bot* [*simp*]: *Rep-compact-basis compact-bot* = ⊥
⟨*proof*⟩

lemma *compact-bot-minimal* [*simp*]: *compact-bot* ⊑ *a*
⟨*proof*⟩

21.4 Universality of *udom*

We use a locale to parameterize the construction over a chain of approx functions on the type to be embedded.

locale *bifinite-approx-chain* =
approx-chain approx **for** *approx* :: *nat* ⇒ *'a::bifinite* → *'a*
begin

21.4.1 Choosing a maximal element from a finite set

lemma *finite-has-maximal*:
fixes *A* :: *'a compact-basis set*
shows ⟦*finite A*; *A* ≠ {}⟧ ⟹ ∃ *x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*
⟨*proof*⟩

definition
choose :: *'a compact-basis set* ⇒ *'a compact-basis*
where
choose A = (*SOME x*. *x* ∈ {*x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*})

lemma *choose-lemma*:
⟦*finite A*; *A* ≠ {}⟧ ⟹ *choose A* ∈ {*x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*}
⟨*proof*⟩

lemma *maximal-choose*:
⟦*finite A*; *y* ∈ *A*; *choose A* ⊑ *y*⟧ ⟹ *choose A* = *y*
⟨*proof*⟩

lemma *choose-in*: $\llbracket$*finite A*; $A \neq \{\}\rrbracket \Longrightarrow$ *choose* $A \in A$
⟨*proof*⟩

function
choose-pos :: *'a compact-basis set* ⇒ *'a compact-basis* ⇒ *nat*
where
choose-pos A x =
(*if finite A* ∧ $x \in A \wedge x \neq$ *choose A*
then Suc (*choose-pos* (*A* − {*choose A*}) *x*) *else 0*)
⟨*proof*⟩

termination *choose-pos*
⟨*proof*⟩

declare *choose-pos.simps* [*simp del*]

lemma *choose-pos-choose*: *finite A* ⟹ *choose-pos A* (*choose A*) = *0*
⟨*proof*⟩

lemma *inj-on-choose-pos* [*OF refl*]:
⟦*card A* = *n*; *finite A*⟧ ⟹ *inj-on* (*choose-pos A*) *A*
⟨*proof*⟩

lemma *choose-pos-bounded* [*OF refl*]:
⟦*card A* = *n*; *finite A*; $x \in A$⟧ ⟹ *choose-pos A x* < *n*
⟨*proof*⟩

lemma *choose-pos-lessD*:
⟦*choose-pos A x* < *choose-pos A y*; *finite A*; $x \in A$; $y \in A$⟧ ⟹ $x \not\sqsubseteq y$
⟨*proof*⟩

21.4.2 Compact basis take function

primrec
cb-take :: *nat* ⇒ *'a compact-basis* ⇒ *'a compact-basis* **where**
cb-take 0 = (λ*x*. *compact-bot*)
| *cb-take* (*Suc n*) = (λ*a*. *Abs-compact-basis* (*approx n*·(*Rep-compact-basis a*)))

declare *cb-take.simps* [*simp del*]

lemma *cb-take-zero* [*simp*]: *cb-take 0 a* = *compact-bot*
⟨*proof*⟩

lemma *Rep-cb-take*:
Rep-compact-basis (*cb-take* (*Suc n*) *a*) = *approx n*·(*Rep-compact-basis a*)
⟨*proof*⟩

lemmas *approx-Rep-compact-basis* = *Rep-cb-take* [*symmetric*]

lemma *cb-take-covers*: $\exists\, n.\ cb\text{-}take\ n\ x = x$
⟨*proof*⟩

lemma *cb-take-less*: $cb\text{-}take\ n\ x \sqsubseteq x$
⟨*proof*⟩

lemma *cb-take-idem*: $cb\text{-}take\ n\ (cb\text{-}take\ n\ x) = cb\text{-}take\ n\ x$
⟨*proof*⟩

lemma *cb-take-mono*: $x \sqsubseteq y \Longrightarrow cb\text{-}take\ n\ x \sqsubseteq cb\text{-}take\ n\ y$
⟨*proof*⟩

lemma *cb-take-chain-le*: $m \leq n \Longrightarrow cb\text{-}take\ m\ x \sqsubseteq cb\text{-}take\ n\ x$
⟨*proof*⟩

lemma *finite-range-cb-take*: $finite\ (range\ (cb\text{-}take\ n))$
⟨*proof*⟩

21.4.3 Rank of basis elements

definition
  $rank :: 'a\ compact\text{-}basis \Rightarrow nat$
where
  $rank\ x = (LEAST\ n.\ cb\text{-}take\ n\ x = x)$

lemma *compact-approx-rank*: $cb\text{-}take\ (rank\ x)\ x = x$
⟨*proof*⟩

lemma *rank-leD*: $rank\ x \leq n \Longrightarrow cb\text{-}take\ n\ x = x$
⟨*proof*⟩

lemma *rank-leI*: $cb\text{-}take\ n\ x = x \Longrightarrow rank\ x \leq n$
⟨*proof*⟩

lemma *rank-le-iff*: $rank\ x \leq n \longleftrightarrow cb\text{-}take\ n\ x = x$
⟨*proof*⟩

lemma *rank-compact-bot* [*simp*]: $rank\ compact\text{-}bot = 0$
⟨*proof*⟩

lemma *rank-eq-0-iff* [*simp*]: $rank\ x = 0 \longleftrightarrow x = compact\text{-}bot$
⟨*proof*⟩

definition
  $rank\text{-}le :: 'a\ compact\text{-}basis \Rightarrow 'a\ compact\text{-}basis\ set$
where
  $rank\text{-}le\ x = \{y.\ rank\ y \leq rank\ x\}$

definition

rank-lt :: *'a compact-basis* ⇒ *'a compact-basis set*
where
rank-lt x = {*y*. *rank y* < *rank x*}

definition
rank-eq :: *'a compact-basis* ⇒ *'a compact-basis set*
where
rank-eq x = {*y*. *rank y* = *rank x*}

lemma *rank-eq-cong*: *rank x* = *rank y* ⟹ *rank-eq x* = *rank-eq y*
⟨*proof*⟩

lemma *rank-lt-cong*: *rank x* = *rank y* ⟹ *rank-lt x* = *rank-lt y*
⟨*proof*⟩

lemma *rank-eq-subset*: *rank-eq x* ⊆ *rank-le x*
⟨*proof*⟩

lemma *rank-lt-subset*: *rank-lt x* ⊆ *rank-le x*
⟨*proof*⟩

lemma *finite-rank-le*: *finite* (*rank-le x*)
⟨*proof*⟩

lemma *finite-rank-eq*: *finite* (*rank-eq x*)
⟨*proof*⟩

lemma *finite-rank-lt*: *finite* (*rank-lt x*)
⟨*proof*⟩

lemma *rank-lt-Int-rank-eq*: *rank-lt x* ∩ *rank-eq x* = {}
⟨*proof*⟩

lemma *rank-lt-Un-rank-eq*: *rank-lt x* ∪ *rank-eq x* = *rank-le x*
⟨*proof*⟩

21.4.4 Sequencing basis elements

definition
place :: *'a compact-basis* ⇒ *nat*
where
place x = *card* (*rank-lt x*) + *choose-pos* (*rank-eq x*) *x*

lemma *place-bounded*: *place x* < *card* (*rank-le x*)
⟨*proof*⟩

lemma *place-ge*: *card* (*rank-lt x*) ≤ *place x*
⟨*proof*⟩

lemma *place-rank-mono*:
fixes $x\ y$ :: *'a compact-basis*
shows $rank\ x < rank\ y \Longrightarrow place\ x < place\ y$
⟨*proof*⟩

lemma *place-eqD*: $place\ x = place\ y \Longrightarrow x = y$
⟨*proof*⟩

lemma *inj-place*: *inj place*
⟨*proof*⟩

21.4.5 Embedding and projection on basis elements

definition
sub :: *'a compact-basis* ⇒ *'a compact-basis*
where
$sub\ x = (case\ rank\ x\ of\ 0 \Rightarrow compact\text{-}bot \mid Suc\ k \Rightarrow cb\text{-}take\ k\ x)$

lemma *rank-sub-less*: $x \neq compact\text{-}bot \Longrightarrow rank\ (sub\ x) < rank\ x$
⟨*proof*⟩

lemma *place-sub-less*: $x \neq compact\text{-}bot \Longrightarrow place\ (sub\ x) < place\ x$
⟨*proof*⟩

lemma *sub-below*: $sub\ x \sqsubseteq x$
⟨*proof*⟩

lemma *rank-less-imp-below-sub*: $[\![x \sqsubseteq y; rank\ x < rank\ y]\!] \Longrightarrow x \sqsubseteq sub\ y$
⟨*proof*⟩

function *basis-emb* :: *'a compact-basis* ⇒ *ubasis*
where $basis\text{-}emb\ x = (if\ x = compact\text{-}bot\ then\ 0\ else$
$node\ (place\ x)\ (basis\text{-}emb\ (sub\ x))$
$(basis\text{-}emb\ `\ \{y.\ place\ y < place\ x \wedge x \sqsubseteq y\}))$
⟨*proof*⟩

termination *basis-emb*
⟨*proof*⟩

declare *basis-emb.simps* [*simp del*]

lemma *basis-emb-compact-bot* [*simp*]:
$basis\text{-}emb\ compact\text{-}bot = 0$
⟨*proof*⟩

lemma *basis-emb-rec*:
$basis\text{-}emb\ x = node\ (place\ x)\ (basis\text{-}emb\ (sub\ x))\ (basis\text{-}emb\ `\ \{y.\ place\ y < place\ x \wedge x \sqsubseteq y\})$
if $x \neq compact\text{-}bot$

⟨*proof*⟩

lemma *basis-emb-eq-0-iff* [*simp*]:
$basis\text{-}emb\ x = 0 \longleftrightarrow x = compact\text{-}bot$
⟨*proof*⟩

lemma *fin1*: $finite\ \{y.\ place\ y < place\ x \wedge x \sqsubseteq y\}$
⟨*proof*⟩

lemma *fin2*: $finite\ (basis\text{-}emb\ `\ \{y.\ place\ y < place\ x \wedge x \sqsubseteq y\})$
⟨*proof*⟩

lemma *rank-place-mono*:
$[\![place\ x < place\ y;\ x \sqsubseteq y]\!] \Longrightarrow rank\ x < rank\ y$
⟨*proof*⟩

lemma *basis-emb-mono*:
$x \sqsubseteq y \Longrightarrow ubasis\text{-}le\ (basis\text{-}emb\ x)\ (basis\text{-}emb\ y)$
⟨*proof*⟩

lemma *inj-basis-emb*: *inj basis-emb*
⟨*proof*⟩

definition
$basis\text{-}prj :: ubasis \Rightarrow 'a\ compact\text{-}basis$
where
$basis\text{-}prj\ x = inv\ basis\text{-}emb$
$(ubasis\text{-}until\ (\lambda x.\ x \in range\ (basis\text{-}emb :: 'a\ compact\text{-}basis \Rightarrow ubasis))\ x)$

lemma *basis-prj-basis-emb*: $\bigwedge x.\ basis\text{-}prj\ (basis\text{-}emb\ x) = x$
⟨*proof*⟩

lemma *basis-prj-node*:
$[\![finite\ S;\ node\ i\ a\ S \notin range\ (basis\text{-}emb :: 'a\ compact\text{-}basis \Rightarrow nat)]\!]$
$\Longrightarrow basis\text{-}prj\ (node\ i\ a\ S) = (basis\text{-}prj\ a :: 'a\ compact\text{-}basis)$
⟨*proof*⟩

lemma *basis-prj-0*: $basis\text{-}prj\ 0 = compact\text{-}bot$
⟨*proof*⟩

lemma *node-eq-basis-emb-iff*:
$finite\ S \Longrightarrow node\ i\ a\ S = basis\text{-}emb\ x \longleftrightarrow$
$x \neq compact\text{-}bot \wedge i = place\ x \wedge a = basis\text{-}emb\ (sub\ x) \wedge$
$S = basis\text{-}emb\ `\ \{y.\ place\ y < place\ x \wedge x \sqsubseteq y\}$
⟨*proof*⟩

lemma *basis-prj-mono*: $ubasis\text{-}le\ a\ b \Longrightarrow basis\text{-}prj\ a \sqsubseteq basis\text{-}prj\ b$
⟨*proof*⟩

lemma *basis-emb-prj-less*: *ubasis-le* (*basis-emb* (*basis-prj x*)) *x*
⟨*proof*⟩

lemma *ideal-completion*:
ideal-completion below Rep-compact-basis (*approximants* :: ′*a* ⇒ -)
⟨*proof*⟩

end

interpretation *compact-basis*:
ideal-completion below Rep-compact-basis
approximants :: ′*a*::*bifinite* ⇒ ′*a compact-basis set*
⟨*proof*⟩

21.4.6 EP-pair from any bifinite domain into *udom*

context *bifinite-approx-chain* **begin**

definition
udom-emb :: ′*a* → *udom*
where
udom-emb = *compact-basis.extension* (λ*x*. *udom-principal* (*basis-emb x*))

definition
udom-prj :: *udom* → ′*a*
where
udom-prj = *udom.extension* (λ*x*. *Rep-compact-basis* (*basis-prj x*))

lemma *udom-emb-principal*:
udom-emb·(*Rep-compact-basis x*) = *udom-principal* (*basis-emb x*)
⟨*proof*⟩

lemma *udom-prj-principal*:
udom-prj·(*udom-principal x*) = *Rep-compact-basis* (*basis-prj x*)
⟨*proof*⟩

lemma *ep-pair-udom*: *ep-pair udom-emb udom-prj*
⟨*proof*⟩

end

abbreviation *udom-emb* ≡ *bifinite-approx-chain.udom-emb*
abbreviation *udom-prj* ≡ *bifinite-approx-chain.udom-prj*

lemmas *ep-pair-udom* =
bifinite-approx-chain.ep-pair-udom [*unfolded bifinite-approx-chain-def*]

21.5 Chain of approx functions for type *udom*

definition

udom-approx :: *nat* ⇒ *udom* → *udom*
where
udom-approx i =
udom.extension (λ*x*. *udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*))

lemma *udom-approx-mono*:
ubasis-le a b ⟹
udom-principal (*ubasis-until* (λ*y*. *y* ≤ *i*) *a*) ⊑
udom-principal (*ubasis-until* (λ*y*. *y* ≤ *i*) *b*)
⟨*proof*⟩

lemma *adm-mem-finite*: ⟦*cont f*; *finite S*⟧ ⟹ *adm* (λ*x*. *f x* ∈ *S*)
⟨*proof*⟩

lemma *udom-approx-principal*:
udom-approx i·(*udom-principal x*) =
udom-principal (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*)
⟨*proof*⟩

lemma *finite-deflation-udom-approx*: *finite-deflation* (*udom-approx i*)
⟨*proof*⟩

interpretation *udom-approx*: *finite-deflation udom-approx i*
⟨*proof*⟩

lemma *chain-udom-approx* [*simp*]: *chain* (λ*i*. *udom-approx i*)
⟨*proof*⟩

lemma *lub-udom-approx* [*simp*]: (⨆ *i*. *udom-approx i*) = *ID*
⟨*proof*⟩

lemma *udom-approx* [*simp*]: *approx-chain udom-approx*
⟨*proof*⟩

instance *udom* :: *bifinite*
⟨*proof*⟩

hide-const (**open**) *node*

unbundle *binomial-syntax*

end

22 Algebraic deflations

theory *Algebraic*
imports *Universal Map-Functions*
begin

22.1 Type constructor for finite deflations

typedef *'a::bifinite fin-defl* = {*d*::*'a* → *'a*. *finite-deflation d*}
⟨*proof*⟩

instantiation *fin-defl* :: (*bifinite*) *below*
begin

definition *below-fin-defl-def*:
below ≡ λ*x y*. *Rep-fin-defl x* ⊑ *Rep-fin-defl y*

instance ⟨*proof*⟩
end

instance *fin-defl* :: (*bifinite*) *po*
⟨*proof*⟩

lemma *finite-deflation-Rep-fin-defl*: *finite-deflation* (*Rep-fin-defl d*)
⟨*proof*⟩

lemma *deflation-Rep-fin-defl*: *deflation* (*Rep-fin-defl d*)
⟨*proof*⟩

interpretation *Rep-fin-defl*: *finite-deflation Rep-fin-defl d*
⟨*proof*⟩

lemma *fin-defl-belowI*:
(⋀*x*. *Rep-fin-defl a*·*x* = *x* ⟹ *Rep-fin-defl b*·*x* = *x*) ⟹ *a* ⊑ *b*
⟨*proof*⟩

lemma *fin-defl-belowD*:
⟦*a* ⊑ *b*; *Rep-fin-defl a*·*x* = *x*⟧ ⟹ *Rep-fin-defl b*·*x* = *x*
⟨*proof*⟩

lemma *fin-defl-eqI*:
a = *b* **if** (⋀*x*. *Rep-fin-defl a*·*x* = *x* ⟷ *Rep-fin-defl b*·*x* = *x*)
⟨*proof*⟩

lemma *Rep-fin-defl-mono*: *a* ⊑ *b* ⟹ *Rep-fin-defl a* ⊑ *Rep-fin-defl b*
⟨*proof*⟩

lemma *Abs-fin-defl-mono*:
⟦*finite-deflation a*; *finite-deflation b*; *a* ⊑ *b*⟧
⟹ *Abs-fin-defl a* ⊑ *Abs-fin-defl b*
⟨*proof*⟩

lemma (**in** *finite-deflation*) *compact-belowI*:
d ⊑ *f* **if** ⋀*x*. *compact x* ⟹ *d*·*x* = *x* ⟹ *f*·*x* = *x*
⟨*proof*⟩

lemma *compact-Rep-fin-defl* [*simp*]: *compact* (*Rep-fin-defl a*)
⟨*proof*⟩

22.2 Defining algebraic deflations by ideal completion

typedef *'a::bifinite defl* = {*S::'a fin-defl set. below.ideal S*}
⟨*proof*⟩

instantiation *defl* :: (*bifinite*) *below*
begin

definition $x \sqsubseteq y \longleftrightarrow$ *Rep-defl x* $\subseteq$ *Rep-defl y*

instance ⟨*proof*⟩

end

instance *defl* :: (*bifinite*) *po*
⟨*proof*⟩

instance *defl* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition *defl-principal* :: *'a::bifinite fin-defl* ⇒ *'a defl*
where *defl-principal t* = *Abs-defl* {*u. u* ⊑ *t*}

lemma *fin-defl-countable*: ∃*f::'a::bifinite fin-defl* ⇒ *nat. inj f*
⟨*proof*⟩

interpretation *defl*: *ideal-completion below defl-principal Rep-defl*
⟨*proof*⟩

Algebraic deflations are pointed

lemma *defl-minimal*: *defl-principal* (*Abs-fin-defl* ⊥) ⊑ *x*
⟨*proof*⟩

instance *defl* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-defl-pcpo*: ⊥ = *defl-principal* (*Abs-fin-defl* ⊥)
⟨*proof*⟩

22.3 Applying algebraic deflations

definition *cast* :: *'a::bifinite defl* → *'a* → *'a*
where *cast* = *defl.extension Rep-fin-defl*

lemma *cast-defl-principal*: *cast*·(*defl-principal a*) = *Rep-fin-defl a*
⟨*proof*⟩

lemma *deflation-cast*: *deflation* (*cast*·*d*)
⟨*proof*⟩

lemma *finite-deflation-cast*: *compact d* ⟹ *finite-deflation* (*cast*·*d*)
⟨*proof*⟩

interpretation *cast*: *deflation cast*·*d*
⟨*proof*⟩

declare *cast.idem* [*simp*]

lemma *compact-cast* [*simp*]: *compact* (*cast*·*d*) **if** *compact d*
⟨*proof*⟩

lemma *cast-below-cast*: *cast*·*A* ⊑ *cast*·*B* ⟷ *A* ⊑ *B*
⟨*proof*⟩

lemma *compact-cast-iff*: *compact* (*cast*·*d*) ⟷ *compact d*
⟨*proof*⟩

lemma *cast-below-imp-below*: *cast*·*A* ⊑ *cast*·*B* ⟹ *A* ⊑ *B*
⟨*proof*⟩

lemma *cast-eq-imp-eq*: *cast*·*A* = *cast*·*B* ⟹ *A* = *B*
⟨*proof*⟩

lemma *cast-strict1* [*simp*]: *cast*·⊥ = ⊥
⟨*proof*⟩

lemma *cast-strict2* [*simp*]: *cast*·*A*·⊥ = ⊥
⟨*proof*⟩

22.4 Deflation combinators

definition
  defl-fun1 e p f =
    defl.extension (λ*a*.
      defl-principal (*Abs-fin-defl*
        (*e oo f*·(*Rep-fin-defl a*) *oo p*)))

definition
  defl-fun2 e p f =
    defl.extension (λ*a*.
      defl.extension (λ*b*.
        defl-principal (*Abs-fin-defl*
          (*e oo f*·(*Rep-fin-defl a*)·(*Rep-fin-defl b*) *oo p*))))

lemma *cast-defl-fun1*:

assumes *ep*: *ep-pair e p*
assumes *f*: $\bigwedge a$. *finite-deflation a* $\Longrightarrow$ *finite-deflation* $(f \cdot a)$
shows *cast*$\cdot$(*defl-fun1 e p f*$\cdot A$) = *e oo f*$\cdot$(*cast*$\cdot A$) *oo p*
⟨*proof*⟩

lemma *cast-defl-fun2*:
assumes *ep*: *ep-pair e p*
assumes *f*: $\bigwedge a\ b$. *finite-deflation a* $\Longrightarrow$ *finite-deflation b* $\Longrightarrow$
finite-deflation $(f \cdot a \cdot b)$
shows *cast*$\cdot$(*defl-fun2 e p f*$\cdot A \cdot B$) = *e oo f*$\cdot$(*cast*$\cdot A$)$\cdot$(*cast*$\cdot B$) *oo p*
⟨*proof*⟩

end

23 Representable domains

theory *Representable*
imports *Algebraic Map-Functions HOL−Library.Countable*
begin

23.1 Class of representable domains

We define a "domain" as a pcpo that is isomorphic to some algebraic deflation over the universal domain; this is equivalent to being omega-bifinite.

A predomain is a cpo that, when lifted, becomes a domain. Predomains are represented by deflations over a lifted universal domain type.

class *predomain-syn* = *cpo* +
fixes *liftemb* :: $'a_{\bot} \rightarrow udom_{\bot}$
fixes *liftprj* :: $udom_{\bot} \rightarrow 'a_{\bot}$
fixes *liftdefl* :: *'a itself* ⇒ *udom u defl*

class *predomain* = *predomain-syn* +
assumes *predomain-ep*: *ep-pair liftemb liftprj*
assumes *cast-liftdefl*: *cast*$\cdot$(*liftdefl TYPE('a)*) = *liftemb oo liftprj*

syntax *-LIFTDEFL* :: *type* ⇒ *logic* (‹(1LIFTDEFL/(1'(-')))›)
syntax-consts *-LIFTDEFL* ⇌ *liftdefl*
translations *LIFTDEFL('t)* ⇌ *CONST liftdefl TYPE('t)*

definition *liftdefl-of* :: *udom defl* → *udom u defl*
where *liftdefl-of* = *defl-fun1 ID ID u-map*

lemma *cast-liftdefl-of*: *cast*$\cdot$(*liftdefl-of*$\cdot t$) = *u-map*$\cdot$(*cast*$\cdot t$)
⟨*proof*⟩

class *domain* = *predomain-syn* + *pcpo* +
fixes *emb* :: *'a* → *udom*
fixes *prj* :: *udom* → *'a*

fixes *defl* :: *'a itself* ⇒ *udom defl*
assumes *ep-pair-emb-prj*: *ep-pair emb prj*
assumes *cast-DEFL*: *cast·(defl TYPE('a)) = emb oo prj*
assumes *liftemb-eq*: *liftemb = u-map·emb*
assumes *liftprj-eq*: *liftprj = u-map·prj*
assumes *liftdefl-eq*: *liftdefl TYPE('a) = liftdefl-of·(defl TYPE('a))*

syntax *-DEFL* :: *type* ⇒ *logic* (‹(1DEFL/(1'(-')))›)
syntax-consts *-DEFL* ⇌ *defl*
translations *DEFL('t)* ⇌ *CONST defl TYPE('t)*

instance *domain* ⊆ *predomain*
⟨*proof*⟩

Constants *liftemb* and *liftprj* imply class predomain.

⟨*ML*⟩

interpretation *predomain*: *pcpo-ep-pair liftemb liftprj*
⟨*proof*⟩

interpretation *domain*: *pcpo-ep-pair emb prj*
⟨*proof*⟩

lemmas *emb-inverse = domain.e-inverse*
lemmas *emb-prj-below = domain.e-p-below*
lemmas *emb-eq-iff = domain.e-eq-iff*
lemmas *emb-strict = domain.e-strict*
lemmas *prj-strict = domain.p-strict*

23.2 Domains are bifinite

lemma *approx-chain-ep-cast*:
assumes *ep*: *ep-pair (e::'a::pcpo → 'b::bifinite) (p::'b → 'a)*
assumes *cast-t*: *cast·t = e oo p*
shows ∃(*a::nat ⇒ 'a::pcpo → 'a*). *approx-chain a*
⟨*proof*⟩

instance *domain* ⊆ *bifinite*
⟨*proof*⟩

instance *predomain* ⊆ *profinite*
⟨*proof*⟩

23.3 Universal domain ep-pairs

definition *u-emb = udom-emb (λi. u-map·(udom-approx i))*
definition *u-prj = udom-prj (λi. u-map·(udom-approx i))*

definition *prod-emb = udom-emb (λi. prod-map·(udom-approx i)·(udom-approx i))*

definition *prod-prj* = *udom-prj* (λ*i*. *prod-map*·(*udom-approx i*)·(*udom-approx i*))

definition *sprod-emb* = *udom-emb* (λ*i*. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))
definition *sprod-prj* = *udom-prj* (λ*i*. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))

definition *ssum-emb* = *udom-emb* (λ*i*. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))
definition *ssum-prj* = *udom-prj* (λ*i*. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))

definition *sfun-emb* = *udom-emb* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))
definition *sfun-prj* = *udom-prj* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))

lemma *ep-pair-u*: *ep-pair u-emb u-prj*
⟨*proof*⟩

lemma *ep-pair-prod*: *ep-pair prod-emb prod-prj*
⟨*proof*⟩

lemma *ep-pair-sprod*: *ep-pair sprod-emb sprod-prj*
⟨*proof*⟩

lemma *ep-pair-ssum*: *ep-pair ssum-emb ssum-prj*
⟨*proof*⟩

lemma *ep-pair-sfun*: *ep-pair sfun-emb sfun-prj*
⟨*proof*⟩

23.4 Type combinators

definition *u-defl* :: *udom defl* → *udom defl*
where *u-defl* = *defl-fun1 u-emb u-prj u-map*

definition *prod-defl* :: *udom defl* → *udom defl* → *udom defl*
where *prod-defl* = *defl-fun2 prod-emb prod-prj prod-map*

definition *sprod-defl* :: *udom defl* → *udom defl* → *udom defl*
where *sprod-defl* = *defl-fun2 sprod-emb sprod-prj sprod-map*

definition *ssum-defl* :: *udom defl* → *udom defl* → *udom defl*
where *ssum-defl* = *defl-fun2 ssum-emb ssum-prj ssum-map*

definition *sfun-defl* :: *udom defl* → *udom defl* → *udom defl*
where *sfun-defl* = *defl-fun2 sfun-emb sfun-prj sfun-map*

lemma *cast-u-defl*:
cast·(*u-defl*·*A*) = *u-emb oo u-map*·(*cast*·*A*) *oo u-prj*
⟨*proof*⟩

lemma *cast-prod-defl*:
cast·(*prod-defl*·*A*·*B*) =
prod-emb oo prod-map·(*cast*·*A*)·(*cast*·*B*) *oo prod-prj*
⟨*proof*⟩

lemma *cast-sprod-defl*:
cast·(*sprod-defl*·*A*·*B*) =
sprod-emb oo sprod-map·(*cast*·*A*)·(*cast*·*B*) *oo sprod-prj*
⟨*proof*⟩

lemma *cast-ssum-defl*:
cast·(*ssum-defl*·*A*·*B*) =
ssum-emb oo ssum-map·(*cast*·*A*)·(*cast*·*B*) *oo ssum-prj*
⟨*proof*⟩

lemma *cast-sfun-defl*:
cast·(*sfun-defl*·*A*·*B*) =
sfun-emb oo sfun-map·(*cast*·*A*)·(*cast*·*B*) *oo sfun-prj*
⟨*proof*⟩

Special deflation combinator for unpointed types.

definition *u-liftdefl* :: *udom u defl* → *udom defl*
where *u-liftdefl* = *defl-fun1 u-emb u-prj ID*

lemma *cast-u-liftdefl*:
cast·(*u-liftdefl*·*A*) = *u-emb oo cast*·*A oo u-prj*
⟨*proof*⟩

lemma *u-liftdefl-liftdefl-of*:
u-liftdefl·(*liftdefl-of*·*A*) = *u-defl*·*A*
⟨*proof*⟩

23.5 Class instance proofs

23.5.1 Universal domain

instantiation *udom* :: *domain*
begin

definition [*simp*]:
emb = (*ID* :: *udom* → *udom*)

definition [*simp*]:
prj = (*ID* :: *udom* → *udom*)

definition
defl (*t*::*udom itself*) = ($\bigsqcup$ *i*. *defl-principal* (*Abs-fin-defl* (*udom-approx i*)))

definition
(*liftemb* :: *udom u* → *udom u*) = *u-map*·*emb*

definition
 (*liftprj* :: *udom u* → *udom u*) = *u-map·prj*

definition
 liftdefl (*t*::*udom itself*) = *liftdefl-of·DEFL*(*udom*)

instance ⟨*proof*⟩

end

23.5.2 Lifted cpo

instantiation *u* :: (*predomain*) *domain*
begin

definition
 emb = *u-emb oo liftemb*

definition
 prj = *liftprj oo u-prj*

definition
 defl (*t*::′*a u itself*) = *u-liftdefl·LIFTDEFL*(′*a*)

definition
 (*liftemb* :: ′*a u u* → *udom u*) = *u-map·emb*

definition
 (*liftprj* :: *udom u* → ′*a u u*) = *u-map·prj*

definition
 liftdefl (*t*::′*a u itself*) = *liftdefl-of·DEFL*(′*a u*)

instance ⟨*proof*⟩

end

lemma *DEFL-u*: *DEFL*(′*a*::*predomain u*) = *u-liftdefl·LIFTDEFL*(′*a*)
⟨*proof*⟩

23.5.3 Strict function space

instantiation *sfun* :: (*domain*, *domain*) *domain*
begin

definition
 emb = *sfun-emb oo sfun-map·prj·emb*

definition

prj = *sfun-map·emb·prj oo sfun-prj*

definition
defl (*t*::('*a* →! '*b*) *itself*) = *sfun-defl·DEFL*('*a*)·*DEFL*('*b*)

definition
(*liftemb* :: ('*a* →! '*b*) *u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → ('*a* →! '*b*) *u*) = *u-map·prj*

definition
liftdefl (*t*::('*a* →! '*b*) *itself*) = *liftdefl-of·DEFL*('*a* →! '*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-sfun*:
DEFL('*a*::*domain* →! '*b*::*domain*) = *sfun-defl·DEFL*('*a*)·*DEFL*('*b*)
⟨*proof*⟩

23.5.4 Continuous function space

instantiation *cfun* :: (*predomain*, *domain*) *domain*
begin

definition
emb = *emb oo encode-cfun*

definition
prj = *decode-cfun oo prj*

definition
defl (*t*::('*a* → '*b*) *itself*) = *DEFL*('*a u* →! '*b*)

definition
(*liftemb* :: ('*a* → '*b*) *u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → ('*a* → '*b*) *u*) = *u-map·prj*

definition
liftdefl (*t*::('*a* → '*b*) *itself*) = *liftdefl-of·DEFL*('*a* → '*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-cfun*:
$DEFL('a::predomain \rightarrow 'b::domain) = DEFL('a\ u \rightarrow!\ 'b)$
⟨*proof*⟩

23.5.5 Strict product

instantiation *sprod* :: (*domain*, *domain*) *domain*
begin

definition
$emb = sprod\text{-}emb\ oo\ sprod\text{-}map \cdot emb \cdot emb$

definition
$prj = sprod\text{-}map \cdot prj \cdot prj\ oo\ sprod\text{-}prj$

definition
$defl\ (t::('a \otimes 'b)\ itself) = sprod\text{-}defl \cdot DEFL('a) \cdot DEFL('b)$

definition
$(liftemb :: ('a \otimes 'b)\ u \rightarrow udom\ u) = u\text{-}map \cdot emb$

definition
$(liftprj :: udom\ u \rightarrow ('a \otimes 'b)\ u) = u\text{-}map \cdot prj$

definition
$liftdefl\ (t::('a \otimes 'b)\ itself) = liftdefl\text{-}of \cdot DEFL('a \otimes 'b)$

instance ⟨*proof*⟩

end

lemma *DEFL-sprod*:
$DEFL('a::domain \otimes 'b::domain) = sprod\text{-}defl \cdot DEFL('a) \cdot DEFL('b)$
⟨*proof*⟩

23.5.6 Cartesian product

definition $prod\text{-}liftdefl :: udom\ u\ defl \rightarrow udom\ u\ defl \rightarrow udom\ u\ defl$
where $prod\text{-}liftdefl = defl\text{-}fun2\ (u\text{-}map \cdot prod\text{-}emb\ oo\ decode\text{-}prod\text{-}u)$
$(encode\text{-}prod\text{-}u\ oo\ u\text{-}map \cdot prod\text{-}prj)\ sprod\text{-}map$

lemma *cast-prod-liftdefl*:
$cast \cdot (prod\text{-}liftdefl \cdot a \cdot b) =$
$(u\text{-}map \cdot prod\text{-}emb\ oo\ decode\text{-}prod\text{-}u)\ oo\ sprod\text{-}map \cdot (cast \cdot a) \cdot (cast \cdot b)\ oo$
$(encode\text{-}prod\text{-}u\ oo\ u\text{-}map \cdot prod\text{-}prj)$
⟨*proof*⟩

instantiation *prod* :: (*predomain*, *predomain*) *predomain*
begin

definition
liftemb = (*u-map·prod-emb oo decode-prod-u*) *oo*
(*sprod-map·liftemb·liftemb oo encode-prod-u*)

definition
liftprj = (*decode-prod-u oo sprod-map·liftprj·liftprj*) *oo*
(*encode-prod-u oo u-map·prod-prj*)

definition
liftdefl (*t*::(′*a* × ′*b*) *itself*) = *prod-liftdefl·LIFTDEFL*(′*a*)·*LIFTDEFL*(′*b*)

instance ⟨*proof*⟩

end

instantiation *prod* :: (*domain*, *domain*) *domain*
begin

definition
emb = *prod-emb oo prod-map·emb·emb*

definition
prj = *prod-map·prj·prj oo prod-prj*

definition
defl (*t*::(′*a* × ′*b*) *itself*) = *prod-defl·DEFL*(′*a*)·*DEFL*(′*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-prod*:
DEFL(′*a*::*domain* × ′*b*::*domain*) = *prod-defl·DEFL*(′*a*)·*DEFL*(′*b*)
⟨*proof*⟩

lemma *LIFTDEFL-prod*:
LIFTDEFL(′*a*::*predomain* × ′*b*::*predomain*) =
prod-liftdefl·LIFTDEFL(′*a*)·*LIFTDEFL*(′*b*)
⟨*proof*⟩

23.5.7 Unit type

instantiation *unit* :: *domain*
begin

definition
emb = (⊥ :: *unit* → *udom*)

definition

prj = (⊥ :: *udom* → *unit*)

definition
defl (*t*::*unit itself*) = ⊥

definition
(*liftemb* :: *unit u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → *unit u*) = *u-map*·*prj*

definition
liftdefl (*t*::*unit itself*) = *liftdefl-of*·*DEFL*(*unit*)

instance ⟨*proof*⟩

end

23.5.8 Discrete cpo

instantiation *discr* :: (*countable*) *predomain*
begin

definition
(*liftemb* :: *'a discr u* → *udom u*) = *strictify*·*up oo udom-emb discr-approx*

definition
(*liftprj* :: *udom u* → *'a discr u*) = *udom-prj discr-approx oo fup*·*ID*

definition
liftdefl (*t*::*'a discr itself*) =
(⨆ *i*. *defl-principal* (*Abs-fin-defl* (*liftemb oo discr-approx i oo* (*liftprj*::*udom u* → *'a discr u*))))

instance ⟨*proof*⟩

end

23.5.9 Strict sum

instantiation *ssum* :: (*domain*, *domain*) *domain*
begin

definition
emb = *ssum-emb oo ssum-map*·*emb*·*emb*

definition
prj = *ssum-map*·*prj*·*prj oo ssum-prj*

definition

defl (*t*::(′*a* ⊕ ′*b*) *itself*) = *ssum-defl*·*DEFL*(′*a*)·*DEFL*(′*b*)

definition
(*liftemb* :: (′*a* ⊕ ′*b*) *u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → (′*a* ⊕ ′*b*) *u*) = *u-map*·*prj*

definition
liftdefl (*t*::(′*a* ⊕ ′*b*) *itself*) = *liftdefl-of*·*DEFL*(′*a* ⊕ ′*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-ssum*:
DEFL(′*a*::*domain* ⊕ ′*b*::*domain*) = *ssum-defl*·*DEFL*(′*a*)·*DEFL*(′*b*)
⟨*proof*⟩

23.5.10 Lifted HOL type

instantiation *lift* :: (*countable*) *domain*
begin

definition
emb = *emb oo* (Λ *x*. *Rep-lift x*)

definition
prj = (Λ *y*. *Abs-lift y*) *oo prj*

definition
defl (*t*::′*a lift itself*) = *DEFL*(′*a discr u*)

definition
(*liftemb* :: ′*a lift u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → ′*a lift u*) = *u-map*·*prj*

definition
liftdefl (*t*::′*a lift itself*) = *liftdefl-of*·*DEFL*(′*a lift*)

instance ⟨*proof*⟩

end

end

24 The unit domain

theory *One*
imports *Lift*
begin

type-synonym *one* = *unit lift*

translations
(*type*) *one* ↽ (*type*) *unit lift*

definition *ONE* :: *one*
where $ONE \equiv Def\ ()$

Exhaustion and Elimination for type *one*

lemma *Exh-one*: $t = \bot \vee t = ONE$
⟨*proof*⟩

lemma *oneE* [*case-names bottom ONE*]: $[\![p = \bot \Longrightarrow Q; p = ONE \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *one-induct* [*case-names bottom ONE*]: $P\ \bot \Longrightarrow P\ ONE \Longrightarrow P\ x$
⟨*proof*⟩

lemma *dist-below-one* [*simp*]: $ONE \not\sqsubseteq \bot$
⟨*proof*⟩

lemma *below-ONE* [*simp*]: $x \sqsubseteq ONE$
⟨*proof*⟩

lemma *ONE-below-iff* [*simp*]: $ONE \sqsubseteq x \longleftrightarrow x = ONE$
⟨*proof*⟩

lemma *ONE-defined* [*simp*]: $ONE \neq \bot$
⟨*proof*⟩

lemma *one-neq-iffs* [*simp*]:
$x \neq ONE \longleftrightarrow x = \bot$
$ONE \neq x \longleftrightarrow x = \bot$
$x \neq \bot \longleftrightarrow x = ONE$
$\bot \neq x \longleftrightarrow x = ONE$
⟨*proof*⟩

lemma *compact-ONE*: *compact ONE*
⟨*proof*⟩

Case analysis function for type *one*

definition *one-case* :: $'a::pcpo \rightarrow one \rightarrow 'a$
where $one\text{-}case = (\Lambda\ a\ x.\ seq \cdot x \cdot a)$

translations
case x of XCONST ONE $\Rightarrow$ *t* $\rightleftharpoons$ *CONST one-case·t·x*
case x of XCONST ONE :: *'a* $\Rightarrow$ *t* $\rightharpoonup$ *CONST one-case·t·x*
Λ (*XCONST ONE*). *t* $\rightleftharpoons$ *CONST one-case·t*

lemma *one-case1* [*simp*]: (*case* $\perp$ *of ONE* $\Rightarrow$ *t*) = $\perp$
⟨*proof*⟩

lemma *one-case2* [*simp*]: (*case ONE of ONE* $\Rightarrow$ *t*) = *t*
⟨*proof*⟩

lemma *one-case3* [*simp*]: (*case x of ONE* $\Rightarrow$ *ONE*) = *x*
⟨*proof*⟩

end

theory *Fixrec*
imports *Cprod Sprod Ssum Up One Tr Cfun*
keywords *fixrec* :: *thy-defn*
begin

25 Fixed point operator and admissibility

25.1 Iteration

primrec *iterate* :: *nat* $\Rightarrow$ (*'a* $\rightarrow$ *'a*) $\rightarrow$ (*'a* $\rightarrow$ *'a*)
where
iterate 0 = (Λ *F x*. *x*)
| *iterate* (*Suc n*) = (Λ *F x*. *F*·(*iterate n*·*F*·*x*))

Derive inductive properties of iterate from primitive recursion

lemma *iterate-0* [*simp*]: *iterate 0*·*F*·*x* = *x*
⟨*proof*⟩

lemma *iterate-Suc* [*simp*]: *iterate* (*Suc n*)·*F*·*x* = *F*·(*iterate n*·*F*·*x*)
⟨*proof*⟩

declare *iterate.simps* [*simp del*]

lemma *iterate-Suc2*: *iterate* (*Suc n*)·*F*·*x* = *iterate n*·*F*·(*F*·*x*)
⟨*proof*⟩

lemma *iterate-iterate*: *iterate m*·*F*·(*iterate n*·*F*·*x*) = *iterate* (*m* + *n*)·*F*·*x*
⟨*proof*⟩

The sequence of function iterations is a chain.

lemma *chain-iterate* [*simp*]: *chain* (λ*i*. *iterate i*·*F*·$\perp$)

⟨*proof*⟩

25.2 Least fixed point operator

definition *fix* :: $('a::pcpo \to 'a) \to 'a$
where $fix = (\Lambda\ F.\ \bigsqcup i.\ iterate\ i \cdot F \cdot \bot)$

Binder syntax for *fix*

abbreviation *fix-syn* :: $('a::pcpo \Rightarrow 'a) \Rightarrow 'a$ (**binder** ‹μ › *10*)
where *fix-syn* $(\lambda x.\ f\ x) \equiv fix \cdot (\Lambda\ x.\ f\ x)$

notation (*ASCII*)
fix-syn (**binder** ‹*FIX* › *10*)

Properties of *fix*

direct connection between *fix* and iteration

lemma *fix-def2*: $fix \cdot F = (\bigsqcup i.\ iterate\ i \cdot F \cdot \bot)$
⟨*proof*⟩

lemma *iterate-below-fix*: $iterate\ n \cdot f \cdot \bot \sqsubseteq fix \cdot f$
⟨*proof*⟩

Kleene's fixed point theorems for continuous functions in pointed omega cpo's

lemma *fix-eq*: $fix \cdot F = F \cdot (fix \cdot F)$
⟨*proof*⟩

lemma *fix-least-below*: $F \cdot x \sqsubseteq x \Longrightarrow fix \cdot F \sqsubseteq x$
⟨*proof*⟩

lemma *fix-least*: $F \cdot x = x \Longrightarrow fix \cdot F \sqsubseteq x$
⟨*proof*⟩

lemma *fix-eqI*:
assumes *fixed*: $F \cdot x = x$
and *least*: $\bigwedge z.\ F \cdot z = z \Longrightarrow x \sqsubseteq z$
shows $fix \cdot F = x$
⟨*proof*⟩

lemma *fix-eq2*: $f \equiv fix \cdot F \Longrightarrow f = F \cdot f$
⟨*proof*⟩

lemma *fix-eq3*: $f \equiv fix \cdot F \Longrightarrow f \cdot x = F \cdot f \cdot x$
⟨*proof*⟩

lemma *fix-eq4*: $f = fix \cdot F \Longrightarrow f = F \cdot f$
⟨*proof*⟩

lemma *fix-eq5*: $f = fix \cdot F \Longrightarrow f \cdot x = F \cdot f \cdot x$
⟨*proof*⟩

strictness of *fix*

lemma *fix-bottom-iff*: $fix \cdot F = \bot \longleftrightarrow F \cdot \bot = \bot$
⟨*proof*⟩

lemma *fix-strict*: $F \cdot \bot = \bot \Longrightarrow fix \cdot F = \bot$
⟨*proof*⟩

lemma *fix-defined*: $F \cdot \bot \neq \bot \Longrightarrow fix \cdot F \neq \bot$
⟨*proof*⟩

fix applied to identity and constant functions

lemma *fix-id*: $(\mu\ x.\ x) = \bot$
⟨*proof*⟩

lemma *fix-const*: $(\mu\ x.\ c) = c$
⟨*proof*⟩

25.3 Fixed point induction

lemma *fix-ind*: $adm\ P \Longrightarrow P\ \bot \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow P\ (F \cdot x)) \Longrightarrow P\ (fix \cdot F)$
⟨*proof*⟩

lemma *cont-fix-ind*: $cont\ F \Longrightarrow adm\ P \Longrightarrow P\ \bot \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow P\ (F\ x)) \Longrightarrow P\ (fix \cdot (Abs\text{-}cfun\ F))$
⟨*proof*⟩

lemma *def-fix-ind*: $[\![f \equiv fix \cdot F;\ adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F \cdot x)]\!] \Longrightarrow P\ f$
⟨*proof*⟩

lemma *fix-ind2*:
assumes *adm*: $adm\ P$
assumes *0*: $P\ \bot$ **and** *1*: $P\ (F \cdot \bot)$
assumes *step*: $\bigwedge x.\ [\![P\ x;\ P\ (F \cdot x)]\!] \Longrightarrow P\ (F \cdot (F \cdot x))$
shows $P\ (fix \cdot F)$
⟨*proof*⟩

lemma *parallel-fix-ind*:
assumes *adm*: $adm\ (\lambda x.\ P\ (fst\ x)\ (snd\ x))$
assumes *base*: $P\ \bot\ \bot$
assumes *step*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow P\ (F \cdot x)\ (G \cdot y)$
shows $P\ (fix \cdot F)\ (fix \cdot G)$
⟨*proof*⟩

lemma *cont-parallel-fix-ind*:
assumes $cont\ F$ **and** $cont\ G$
assumes $adm\ (\lambda x.\ P\ (fst\ x)\ (snd\ x))$

assumes $P \perp \perp$
assumes $\bigwedge x\ y.\ P\ x\ y \Longrightarrow P\ (F\ x)\ (G\ y)$
shows $P\ (\mathit{fix}\cdot(\mathit{Abs\text{-}cfun}\ F))\ (\mathit{fix}\cdot(\mathit{Abs\text{-}cfun}\ G))$
$\langle \mathit{proof} \rangle$

25.4 Fixed-points on product types

Bekic's Theorem: Simultaneous fixed points over pairs can be written in terms of separate fixed points.

lemma *fix-cprod*:
fixes $F :: {'a}{::}\mathit{pcpo} \times {'b}{::}\mathit{pcpo} \to {'a} \times {'b}$
shows
$\mathit{fix}\cdot F =$
$(\mu\ x.\ \mathit{fst}\ (F\cdot(x,\ \mu\ y.\ \mathit{snd}\ (F\cdot(x,\ y)))),$
$\mu\ y.\ \mathit{snd}\ (F\cdot(\mu\ x.\ \mathit{fst}\ (F\cdot(x,\ \mu\ y.\ \mathit{snd}\ (F\cdot(x,\ y)))),\ y)))$
(**is** $\mathit{fix}\cdot F = (?x,\ ?y)$)
$\langle \mathit{proof} \rangle$

26 Package for defining recursive functions in HOLCF

26.1 Pattern-match monad

pcpodef $'a\ \mathit{match} = \mathit{UNIV}{::}(\mathit{one} \mathbin{++} {'a}\ u)\ \mathit{set}$
$\langle \mathit{proof} \rangle$

definition
$\mathit{fail} :: {'a}\ \mathit{match}$ **where**
$\mathit{fail} = \mathit{Abs\text{-}match}\ (\mathit{sinl}\cdot\mathit{ONE})$

definition
$\mathit{succeed} :: {'a} \to {'a}\ \mathit{match}$ **where**
$\mathit{succeed} = (\Lambda\ x.\ \mathit{Abs\text{-}match}\ (\mathit{sinr}\cdot(\mathit{up}\cdot x)))$

lemma *matchE* [*case-names bottom fail succeed*, *cases type*: *match*]:
$[\![p = \perp \Longrightarrow Q;\ p = \mathit{fail} \Longrightarrow Q;\ \bigwedge x.\ p = \mathit{succeed}\cdot x \Longrightarrow Q]\!] \Longrightarrow Q$
$\langle \mathit{proof} \rangle$

lemma *succeed-defined* [*simp*]: $\mathit{succeed}\cdot x \neq \perp$
$\langle \mathit{proof} \rangle$

lemma *fail-defined* [*simp*]: $\mathit{fail} \neq \perp$
$\langle \mathit{proof} \rangle$

lemma *succeed-eq* [*simp*]: $(\mathit{succeed}\cdot x = \mathit{succeed}\cdot y) = (x = y)$
$\langle \mathit{proof} \rangle$

lemma *succeed-neq-fail* [*simp*]:
$\mathit{succeed}\cdot x \neq \mathit{fail}\ \mathit{fail} \neq \mathit{succeed}\cdot x$

⟨*proof*⟩

26.1.1 Run operator

definition
run :: *'a match* → *'a::pcpo* **where**
run = (Λ *m. sscase*·⊥·(*fup*·*ID*)·(*Rep-match m*))

rewrite rules for run

lemma *run-strict* [*simp*]: *run*·⊥ = ⊥
⟨*proof*⟩

lemma *run-fail* [*simp*]: *run*·*fail* = ⊥
⟨*proof*⟩

lemma *run-succeed* [*simp*]: *run*·(*succeed*·*x*) = *x*
⟨*proof*⟩

26.1.2 Monad plus operator

definition
mplus :: *'a match* → *'a match* → *'a match* **where**
mplus = (Λ *m1 m2. sscase*·(Λ -. *m2*)·(Λ -. *m1*)·(*Rep-match m1*))

abbreviation
mplus-syn :: [*'a match*, *'a match*] ⇒ *'a match* (**infixr** ‹+++› *65*) **where**
m1 +++ *m2* == *mplus*·*m1*·*m2*

rewrite rules for mplus

lemma *mplus-strict* [*simp*]: ⊥ +++ *m* = ⊥
⟨*proof*⟩

lemma *mplus-fail* [*simp*]: *fail* +++ *m* = *m*
⟨*proof*⟩

lemma *mplus-succeed* [*simp*]: *succeed*·*x* +++ *m* = *succeed*·*x*
⟨*proof*⟩

lemma *mplus-fail2* [*simp*]: *m* +++ *fail* = *m*
⟨*proof*⟩

lemma *mplus-assoc*: (*x* +++ *y*) +++ *z* = *x* +++ (*y* +++ *z*)
⟨*proof*⟩

26.2 Match functions for built-in types

definition
match-bottom :: *'a::pcpo* → *'c match* → *'c match*
where

$match\text{-}bottom = (\Lambda\ x\ k.\ seq \cdot x \cdot fail)$

definition
$match\text{-}Pair :: 'a \times 'b \rightarrow ('a \rightarrow 'b \rightarrow 'c\ match) \rightarrow 'c\ match$
where
$match\text{-}Pair = (\Lambda\ x\ k.\ csplit \cdot k \cdot x)$

definition
$match\text{-}spair :: 'a::pcpo \otimes 'b::pcpo \rightarrow ('a \rightarrow 'b \rightarrow 'c\ match) \rightarrow 'c::pcpo\ match$
where
$match\text{-}spair = (\Lambda\ x\ k.\ ssplit \cdot k \cdot x)$

definition
$match\text{-}sinl :: 'a::pcpo \oplus 'b::pcpo \rightarrow ('a \rightarrow 'c::pcpo\ match) \rightarrow 'c\ match$
where
$match\text{-}sinl = (\Lambda\ x\ k.\ sscase \cdot k \cdot (\Lambda\ b.\ fail) \cdot x)$

definition
$match\text{-}sinr :: 'a::pcpo \oplus 'b::pcpo \rightarrow ('b \rightarrow 'c::pcpo\ match) \rightarrow 'c\ match$
where
$match\text{-}sinr = (\Lambda\ x\ k.\ sscase \cdot (\Lambda\ a.\ fail) \cdot k \cdot x)$

definition
$match\text{-}up :: 'a\ u \rightarrow ('a \rightarrow 'c::pcpo\ match) \rightarrow 'c\ match$
where
$match\text{-}up = (\Lambda\ x\ k.\ fup \cdot k \cdot x)$

definition
$match\text{-}ONE :: one \rightarrow 'c::pcpo\ match \rightarrow 'c\ match$
where
$match\text{-}ONE = (\Lambda\ ONE\ k.\ k)$

definition
$match\text{-}TT :: tr \rightarrow 'c::pcpo\ match \rightarrow 'c\ match$
where
$match\text{-}TT = (\Lambda\ x\ k.\ If\ x\ then\ k\ else\ fail)$

definition
$match\text{-}FF :: tr \rightarrow 'c::pcpo\ match \rightarrow 'c\ match$
where
$match\text{-}FF = (\Lambda\ x\ k.\ If\ x\ then\ fail\ else\ k)$

lemma *match-bottom-simps* [*simp*]:
$match\text{-}bottom \cdot x \cdot k = (if\ x = \bot\ then\ \bot\ else\ fail)$
⟨*proof*⟩

lemma *match-Pair-simps* [*simp*]:
$match\text{-}Pair \cdot (x,\ y) \cdot k = k \cdot x \cdot y$
⟨*proof*⟩

lemma *match-spair-simps* [*simp*]:
$[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow \textit{match-spair}\cdot(:x,\ y:)\cdot k = k\cdot x\cdot y$
$\textit{match-spair}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-sinl-simps* [*simp*]:
$x \neq \bot \Longrightarrow \textit{match-sinl}\cdot(\textit{sinl}\cdot x)\cdot k = k\cdot x$
$y \neq \bot \Longrightarrow \textit{match-sinl}\cdot(\textit{sinr}\cdot y)\cdot k = \textit{fail}$
$\textit{match-sinl}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-sinr-simps* [*simp*]:
$x \neq \bot \Longrightarrow \textit{match-sinr}\cdot(\textit{sinl}\cdot x)\cdot k = \textit{fail}$
$y \neq \bot \Longrightarrow \textit{match-sinr}\cdot(\textit{sinr}\cdot y)\cdot k = k\cdot y$
$\textit{match-sinr}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-up-simps* [*simp*]:
$\textit{match-up}\cdot(\textit{up}\cdot x)\cdot k = k\cdot x$
$\textit{match-up}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-ONE-simps* [*simp*]:
$\textit{match-ONE}\cdot \textit{ONE}\cdot k = k$
$\textit{match-ONE}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-TT-simps* [*simp*]:
$\textit{match-TT}\cdot \textit{TT}\cdot k = k$
$\textit{match-TT}\cdot \textit{FF}\cdot k = \textit{fail}$
$\textit{match-TT}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-FF-simps* [*simp*]:
$\textit{match-FF}\cdot \textit{FF}\cdot k = k$
$\textit{match-FF}\cdot \textit{TT}\cdot k = \textit{fail}$
$\textit{match-FF}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

26.3 Mutual recursion

The following rules are used to prove unfolding theorems from fixed-point definitions of mutually recursive functions.

lemma *Pair-equalI*: $[\![x \equiv \textit{fst}\ p;\ y \equiv \textit{snd}\ p]\!] \Longrightarrow (x,\ y) \equiv p$
⟨*proof*⟩

lemma *Pair-eqD1*: $(x,\ y) = (x',\ y') \Longrightarrow x = x'$
⟨*proof*⟩

lemma *Pair-eqD2*: $(x, y) = (x', y') \Longrightarrow y = y'$
⟨*proof*⟩

lemma *def-cont-fix-eq*:
$[\![f \equiv fix\cdot(Abs\text{-}cfun\ F);\ cont\ F]\!] \Longrightarrow f = F\ f$
⟨*proof*⟩

lemma *def-cont-fix-ind*:
$[\![f \equiv fix\cdot(Abs\text{-}cfun\ F);\ cont\ F;\ adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F\ x)]\!] \Longrightarrow P\ f$
⟨*proof*⟩

lemma for proving rewrite rules

lemma *ssubst-lhs*: $[\![t = s;\ P\ s = Q]\!] \Longrightarrow P\ t = Q$
⟨*proof*⟩

26.4 Initializing the fixrec package

⟨*ML*⟩

hide-const (**open**) *succeed fail run*

end

27 Domain package

theory *Domain*
imports *Representable Map-Functions Fixrec*
keywords
lazy unsafe **and**
domaindef domain :: *thy-defn* **and**
domain-isomorphism :: *thy-decl*
begin

27.1 Continuous isomorphisms

A locale for continuous isomorphisms

locale *iso* =
fixes $abs :: 'a::pcpo \to 'b::pcpo$
fixes $rep :: 'b \to 'a$
assumes *abs-iso* [*simp*]: $rep\cdot(abs\cdot x) = x$
assumes *rep-iso* [*simp*]: $abs\cdot(rep\cdot y) = y$
begin

lemma *swap*: *iso rep abs*
⟨*proof*⟩

lemma *abs-below*: $(abs\cdot x \sqsubseteq abs\cdot y) = (x \sqsubseteq y)$

⟨*proof*⟩

lemma *rep-below*: $(rep \cdot x \sqsubseteq rep \cdot y) = (x \sqsubseteq y)$
⟨*proof*⟩

lemma *abs-eq*: $(abs \cdot x = abs \cdot y) = (x = y)$
⟨*proof*⟩

lemma *rep-eq*: $(rep \cdot x = rep \cdot y) = (x = y)$
⟨*proof*⟩

lemma *abs-strict*: $abs \cdot \bot = \bot$
⟨*proof*⟩

lemma *rep-strict*: $rep \cdot \bot = \bot$
⟨*proof*⟩

lemma *abs-defin'*: $abs \cdot x = \bot \Longrightarrow x = \bot$
⟨*proof*⟩

lemma *rep-defin'*: $rep \cdot z = \bot \Longrightarrow z = \bot$
⟨*proof*⟩

lemma *abs-defined*: $z \neq \bot \Longrightarrow abs \cdot z \neq \bot$
⟨*proof*⟩

lemma *rep-defined*: $z \neq \bot \Longrightarrow rep \cdot z \neq \bot$
⟨*proof*⟩

lemma *abs-bottom-iff*: $(abs \cdot x = \bot) = (x = \bot)$
⟨*proof*⟩

lemma *rep-bottom-iff*: $(rep \cdot x = \bot) = (x = \bot)$
⟨*proof*⟩

lemma *casedist-rule*: $rep \cdot x = \bot \vee P \Longrightarrow x = \bot \vee P$
⟨*proof*⟩

lemma *compact-abs-rev*: $compact\ (abs \cdot x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-rep-rev*: $compact\ (rep \cdot x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-abs*: $compact\ x \Longrightarrow compact\ (abs \cdot x)$
⟨*proof*⟩

lemma *compact-rep*: $compact\ x \Longrightarrow compact\ (rep \cdot x)$
⟨*proof*⟩

lemma *iso-swap*: $(x = abs \cdot y) = (rep \cdot x = y)$
⟨*proof*⟩

end

27.2 Proofs about take functions

This section contains lemmas that are used in a module that supports the domain isomorphism package; the module contains proofs related to take functions and the finiteness predicate.

lemma *deflation-abs-rep*:
  fixes *abs* **and** *rep* **and** *d*
  assumes *abs-iso*: $\bigwedge x.\ rep \cdot (abs \cdot x) = x$
  assumes *rep-iso*: $\bigwedge y.\ abs \cdot (rep \cdot y) = y$
  shows $deflation\ d \Longrightarrow deflation\ (abs\ oo\ d\ oo\ rep)$
⟨*proof*⟩

lemma *deflation-chain-min*:
  assumes *chain*: *chain d*
  assumes *defl*: $\bigwedge n.\ deflation\ (d\ n)$
  shows $d\ m \cdot (d\ n \cdot x) = d\ (min\ m\ n) \cdot x$
⟨*proof*⟩

lemma *lub-ID-take-lemma*:
  assumes *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
  assumes $\bigwedge n.\ t\ n \cdot x = t\ n \cdot y$ **shows** $x = y$
⟨*proof*⟩

lemma *lub-ID-reach*:
  assumes *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
  shows $(\bigsqcup n.\ t\ n \cdot x) = x$
⟨*proof*⟩

lemma *lub-ID-take-induct*:
  assumes *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
  assumes *adm P* **and** $\bigwedge n.\ P\ (t\ n \cdot x)$ **shows** *P x*
⟨*proof*⟩

27.3 Finiteness

Let a "decisive" function be a deflation that maps every input to either itself or bottom. Then if a domain's take functions are all decisive, then all values in the domain are finite.

definition
  $decisive :: ('a::pcpo \rightarrow 'a) \Rightarrow bool$
where
  $decisive\ d \longleftrightarrow (\forall x.\ d \cdot x = x \lor d \cdot x = \bot)$

lemma *decisiveI*: $(\bigwedge x.\ d{\cdot}x = x \vee d{\cdot}x = \bot) \Longrightarrow$ *decisive d*
⟨*proof*⟩

lemma *decisive-cases*:
assumes *decisive d* **obtains** $d{\cdot}x = x \mid d{\cdot}x = \bot$
⟨*proof*⟩

lemma *decisive-bottom*: *decisive* $\bot$
⟨*proof*⟩

lemma *decisive-ID*: *decisive ID*
⟨*proof*⟩

lemma *decisive-ssum-map*:
assumes *f*: *decisive f*
assumes *g*: *decisive g*
shows *decisive* (*ssum-map*$\cdot f\cdot g$)
⟨*proof*⟩

lemma *decisive-sprod-map*:
assumes *f*: *decisive f*
assumes *g*: *decisive g*
shows *decisive* (*sprod-map*$\cdot f\cdot g$)
⟨*proof*⟩

lemma *decisive-abs-rep*:
fixes *abs rep*
assumes *iso*: *iso abs rep*
assumes *d*: *decisive d*
shows *decisive* (*abs oo d oo rep*)
⟨*proof*⟩

lemma *lub-ID-finite*:
assumes *chain*: *chain d*
assumes *lub*: $(\bigsqcup n.\ d\ n) = ID$
assumes *decisive*: $\bigwedge n.$ *decisive* $(d\ n)$
shows $\exists\, n.\ d\ n{\cdot}x = x$
⟨*proof*⟩

lemma *lub-ID-finite-take-induct*:
assumes *chain d* **and** $(\bigsqcup n.\ d\ n) = ID$ **and** $\bigwedge n.$ *decisive* $(d\ n)$
shows $(\bigwedge n.\ P\ (d\ n{\cdot}x)) \Longrightarrow P\ x$
⟨*proof*⟩

27.4 Proofs about constructor functions

Lemmas for proving nchotomy rule:

lemma *ex-one-bottom-iff*:

$(\exists x.\ P\ x \wedge x \neq \bot) = P\ ONE$
⟨*proof*⟩

lemma *ex-up-bottom-iff*:
$(\exists x.\ P\ x \wedge x \neq \bot) = (\exists x.\ P\ (up{\cdot}x))$
⟨*proof*⟩

lemma *ex-sprod-bottom-iff*:
$(\exists y.\ P\ y \wedge y \neq \bot) =$
$(\exists x\ y.\ (P\ (:x,\ y:) \wedge x \neq \bot) \wedge y \neq \bot)$
⟨*proof*⟩

lemma *ex-sprod-up-bottom-iff*:
$(\exists y.\ P\ y \wedge y \neq \bot) =$
$(\exists x\ y.\ P\ (:up{\cdot}x,\ y:) \wedge y \neq \bot)$
⟨*proof*⟩

lemma *ex-ssum-bottom-iff*:
$(\exists x.\ P\ x \wedge x \neq \bot) =$
$((\exists x.\ P\ (sinl{\cdot}x) \wedge x \neq \bot) \vee$
$(\exists x.\ P\ (sinr{\cdot}x) \wedge x \neq \bot))$
⟨*proof*⟩

lemma *exh-start*: $p = \bot \vee (\exists x.\ p = x \wedge x \neq \bot)$
⟨*proof*⟩

lemmas *ex-bottom-iffs* =
ex-ssum-bottom-iff
ex-sprod-up-bottom-iff
ex-sprod-bottom-iff
ex-up-bottom-iff
ex-one-bottom-iff

Rules for turning nchotomy into exhaust:

lemma *exh-casedist0*: $[\![R;\ R \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *exh-casedist1*: $((P \vee Q \Longrightarrow R) \Longrightarrow S) \equiv ([\![P \Longrightarrow R;\ Q \Longrightarrow R]\!] \Longrightarrow S)$
⟨*proof*⟩

lemma *exh-casedist2*: $(\exists x.\ P\ x \Longrightarrow Q) \equiv (\bigwedge x.\ P\ x \Longrightarrow Q)$
⟨*proof*⟩

lemma *exh-casedist3*: $(P \wedge Q \Longrightarrow R) \equiv (P \Longrightarrow Q \Longrightarrow R)$
⟨*proof*⟩

lemmas *exh-casedists* = *exh-casedist1 exh-casedist2 exh-casedist3*

Rules for proving constructor properties

lemmas *con-strict-rules* =
sinl-strict sinr-strict spair-strict1 spair-strict2

lemmas *con-bottom-iff-rules* =
sinl-bottom-iff sinr-bottom-iff spair-bottom-iff up-defined ONE-defined

lemmas *con-below-iff-rules* =
sinl-below sinr-below sinl-below-sinr sinr-below-sinl con-bottom-iff-rules

lemmas *con-eq-iff-rules* =
sinl-eq sinr-eq sinl-eq-sinr sinr-eq-sinl con-bottom-iff-rules

lemmas *sel-strict-rules* =
cfcomp2 sscase1 sfst-strict ssnd-strict fup1

lemma *sel-app-extra-rules*:
$sscase \cdot ID \cdot \bot \cdot (sinr \cdot x) = \bot$
$sscase \cdot ID \cdot \bot \cdot (sinl \cdot x) = x$
$sscase \cdot \bot \cdot ID \cdot (sinl \cdot x) = \bot$
$sscase \cdot \bot \cdot ID \cdot (sinr \cdot x) = x$
$fup \cdot ID \cdot (up \cdot x) = x$
⟨*proof*⟩

lemmas *sel-app-rules* =
sel-strict-rules sel-app-extra-rules
ssnd-spair sfst-spair up-defined spair-defined

lemmas *sel-bottom-iff-rules* =
cfcomp2 sfst-bottom-iff ssnd-bottom-iff

lemmas *take-con-rules* =
ssum-map-sinl′ ssum-map-sinr′ sprod-map-spair′ u-map-up
deflation-strict deflation-ID ID1 cfcomp2

27.5 ML setup

named-theorems *domain-deflation theorems like deflation a ==> deflation (foo-map$a)*
and *domain-map-ID theorems like foo-map$ID = ID*

⟨*ML*⟩

27.6 Representations of types

lemma *emb-prj*: $emb \cdot ((prj \cdot x) :: 'a :: domain) = cast \cdot DEFL('a) \cdot x$
⟨*proof*⟩

lemma *emb-prj-emb*:
fixes $x :: 'a :: domain$
assumes $DEFL('a) \sqsubseteq DEFL('b)$
shows $emb \cdot (prj \cdot (emb \cdot x) :: 'b :: domain) = emb \cdot x$

⟨*proof*⟩

lemma *prj-emb-prj*:
assumes $DEFL('a::domain) \sqsubseteq DEFL('b::domain)$
shows $prj\cdot(emb\cdot(prj\cdot x :: 'b)) = (prj\cdot x :: 'a)$
⟨*proof*⟩

Isomorphism lemmas used internally by the domain package:

lemma *domain-abs-iso*:
fixes *abs* **and** *rep*
assumes *DEFL*: $DEFL('b::domain) = DEFL('a::domain)$
assumes *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
assumes *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
shows $rep\cdot(abs\cdot x) = x$
⟨*proof*⟩

lemma *domain-rep-iso*:
fixes *abs* **and** *rep*
assumes *DEFL*: $DEFL('b::domain) = DEFL('a::domain)$
assumes *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
assumes *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
shows $abs\cdot(rep\cdot x) = x$
⟨*proof*⟩

27.7 Deflations as sets

definition *defl-set* :: $'a::bifinite\ defl \Rightarrow 'a\ set$
where *defl-set* $A = \{x.\ cast\cdot A\cdot x = x\}$

lemma *adm-defl-set*: $adm\ (\lambda x.\ x \in defl\text{-}set\ A)$
⟨*proof*⟩

lemma *defl-set-bottom*: $\bot \in defl\text{-}set\ A$
⟨*proof*⟩

lemma *defl-set-cast* [*simp*]: $cast\cdot A\cdot x \in defl\text{-}set\ A$
⟨*proof*⟩

lemma *defl-set-subset-iff*: $defl\text{-}set\ A \subseteq defl\text{-}set\ B \longleftrightarrow A \sqsubseteq B$
⟨*proof*⟩

27.8 Proving a subtype is representable

Temporarily relax type constraints.

⟨*ML*⟩

lemma *typedef-domain-class*:
fixes $Rep :: 'a::pcpo \Rightarrow udom$
fixes $Abs :: udom \Rightarrow 'a::pcpo$

fixes *t* :: *udom defl*
assumes *type*: *type-definition Rep Abs* (*defl-set t*)
assumes *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
assumes *emb*: $emb \equiv (\Lambda\ x.\ Rep\ x)$
assumes *prj*: $prj \equiv (\Lambda\ x.\ Abs\ (cast \cdot t \cdot x))$
assumes *defl*: $defl \equiv (\lambda\ a{::}'a\ itself.\ t)$
assumes *liftemb*: $(liftemb :: 'a\ u \rightarrow udom\ u) \equiv u\text{-}map \cdot emb$
assumes *liftprj*: $(liftprj :: udom\ u \rightarrow 'a\ u) \equiv u\text{-}map \cdot prj$
assumes *liftdefl*: $(liftdefl :: 'a\ itself \Rightarrow \text{-}) \equiv (\lambda t.\ liftdefl\text{-}of \cdot DEFL('a))$
shows $OFCLASS('a,\ domain\text{-}class)$
⟨*proof*⟩

lemma *typedef-DEFL*:
assumes $defl \equiv (\lambda a{::}'a{::}pcpo\ itself.\ t)$
shows $DEFL('a{::}pcpo) = t$
⟨*proof*⟩

Restore original typing constraints.

⟨*ML*⟩

27.9 Isomorphic deflations

definition *isodefl* :: $('a{::}domain \rightarrow 'a) \Rightarrow udom\ defl \Rightarrow bool$
where $isodefl\ d\ t \longleftrightarrow cast \cdot t = emb\ oo\ d\ oo\ prj$

definition *isodefl'* :: $('a{::}predomain \rightarrow 'a) \Rightarrow udom\ u\ defl \Rightarrow bool$
where $isodefl'\ d\ t \longleftrightarrow cast \cdot t = liftemb\ oo\ u\text{-}map \cdot d\ oo\ liftprj$

lemma *isodeflI*: $(\bigwedge x.\ cast \cdot t \cdot x = emb \cdot (d \cdot (prj \cdot x))) \Longrightarrow isodefl\ d\ t$
⟨*proof*⟩

lemma *cast-isodefl*: $isodefl\ d\ t \Longrightarrow cast \cdot t = (\Lambda\ x.\ emb \cdot (d \cdot (prj \cdot x)))$
⟨*proof*⟩

lemma *isodefl-strict*: $isodefl\ d\ t \Longrightarrow d \cdot \bot = \bot$
⟨*proof*⟩

lemma *isodefl-imp-deflation*:
fixes $d :: 'a{::}domain \rightarrow 'a$
assumes *isodefl d t* **shows** *deflation d*
⟨*proof*⟩

lemma *isodefl-ID-DEFL*: $isodefl\ (ID :: 'a \rightarrow 'a)\ DEFL('a{::}domain)$
⟨*proof*⟩

lemma *isodefl-LIFTDEFL*:
$isodefl'\ (ID :: 'a \rightarrow 'a)\ LIFTDEFL('a{::}predomain)$
⟨*proof*⟩

lemma *isodefl-DEFL-imp-ID*: *isodefl* $(d :: 'a \to 'a)$ *DEFL*$('a::domain) \Longrightarrow d =$ *ID*
⟨*proof*⟩

lemma *isodefl-bottom*: *isodefl* $\perp \perp$
⟨*proof*⟩

lemma *adm-isodefl*:
 cont f $\Longrightarrow$ *cont g* $\Longrightarrow$ *adm* $(\lambda x.$ *isodefl* $(f\ x)\ (g\ x))$
⟨*proof*⟩

lemma *isodefl-lub*:
 assumes *chain d* **and** *chain t*
 assumes $\bigwedge i.$ *isodefl* $(d\ i)\ (t\ i)$
 shows *isodefl* $(\bigsqcup i.\ d\ i)\ (\bigsqcup i.\ t\ i)$
⟨*proof*⟩

lemma *isodefl-fix*:
 assumes $\bigwedge d\ t.$ *isodefl d t* $\Longrightarrow$ *isodefl* $(f \cdot d)\ (g \cdot t)$
 shows *isodefl* $(fix \cdot f)\ (fix \cdot g)$
⟨*proof*⟩

lemma *isodefl-abs-rep*:
 fixes *abs* **and** *rep* **and** *d*
 assumes *DEFL*: *DEFL*$('b::domain) =$ *DEFL*$('a::domain)$
 assumes *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
 assumes *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
 shows *isodefl d t* $\Longrightarrow$ *isodefl* $(abs\ oo\ d\ oo\ rep)\ t$
⟨*proof*⟩

lemma *isodefl'-liftdefl-of*: *isodefl d t* $\Longrightarrow$ *isodefl'* d $(liftdefl\text{-}of \cdot t)$
⟨*proof*⟩

lemma *isodefl-sfun*:
 isodefl d1 t1 $\Longrightarrow$ *isodefl d2 t2* $\Longrightarrow$
  isodefl $(sfun\text{-}map \cdot d1 \cdot d2)\ (sfun\text{-}defl \cdot t1 \cdot t2)$
⟨*proof*⟩

lemma *isodefl-ssum*:
 isodefl d1 t1 $\Longrightarrow$ *isodefl d2 t2* $\Longrightarrow$
  isodefl $(ssum\text{-}map \cdot d1 \cdot d2)\ (ssum\text{-}defl \cdot t1 \cdot t2)$
⟨*proof*⟩

lemma *isodefl-sprod*:
 isodefl d1 t1 $\Longrightarrow$ *isodefl d2 t2* $\Longrightarrow$
  isodefl $(sprod\text{-}map \cdot d1 \cdot d2)\ (sprod\text{-}defl \cdot t1 \cdot t2)$
⟨*proof*⟩

lemma *isodefl-prod*:

isodefl d1 t1 $\Longrightarrow$ *isodefl d2 t2* $\Longrightarrow$
isodefl (*prod-map*·*d1*·*d2*) (*prod-defl*·*t1*·*t2*)
⟨*proof*⟩

lemma *isodefl-u*:
isodefl d t $\Longrightarrow$ *isodefl* (*u-map*·*d*) (*u-defl*·*t*)
⟨*proof*⟩

lemma *isodefl-u-liftdefl*:
isodefl′ d t $\Longrightarrow$ *isodefl* (*u-map*·*d*) (*u-liftdefl*·*t*)
⟨*proof*⟩

lemma *encode-prod-u-map*:
encode-prod-u·(*u-map*·(*prod-map*·*f*·*g*)·(*decode-prod-u*·*x*))
= *sprod-map*·(*u-map*·*f*)·(*u-map*·*g*)·*x*
⟨*proof*⟩

lemma *isodefl-prod-u*:
assumes *isodefl′ d1 t1* **and** *isodefl′ d2 t2*
shows *isodefl′* (*prod-map*·*d1*·*d2*) (*prod-liftdefl*·*t1*·*t2*)
⟨*proof*⟩

lemma *encode-cfun-map*:
encode-cfun·(*cfun-map*·*f*·*g*·(*decode-cfun*·*x*))
= *sfun-map*·(*u-map*·*f*)·*g*·*x*
⟨*proof*⟩

lemma *isodefl-cfun*:
assumes *isodefl* (*u-map*·*d1*) *t1* **and** *isodefl d2 t2*
shows *isodefl* (*cfun-map*·*d1*·*d2*) (*sfun-defl*·*t1*·*t2*)
⟨*proof*⟩

27.10 Setting up the domain package

named-theorems *domain-defl-simps theorems like DEFL('a t) = t-defl$DEFL('a)*
and *domain-isodefl theorems like isodefl d t ==> isodefl (foo-map$d) (foo-defl$t)*

⟨*ML*⟩

lemmas [*domain-defl-simps*] =
DEFL-cfun DEFL-sfun DEFL-ssum DEFL-sprod DEFL-prod DEFL-u
liftdefl-eq LIFTDEFL-prod u-liftdefl-liftdefl-of

lemmas [*domain-map-ID*] =
cfun-map-ID sfun-map-ID ssum-map-ID sprod-map-ID prod-map-ID u-map-ID

lemmas [*domain-isodefl*] =
isodefl-u isodefl-sfun isodefl-ssum isodefl-sprod
isodefl-cfun isodefl-prod isodefl-prod-u isodefl′-liftdefl-of

isodefl-u-liftdefl

lemmas [*domain-deflation*] =
deflation-cfun-map deflation-sfun-map deflation-ssum-map
deflation-sprod-map deflation-prod-map deflation-u-map

⟨*ML*⟩

end

28 A compact basis for powerdomains

theory *Compact-Basis*
imports *Universal*
begin

28.1 A compact basis for powerdomains

definition *pd-basis* = {*S*::′*a*::*bifinite compact-basis set*. *finite S* ∧ *S* ≠ {}}

typedef ′*a*::*bifinite pd-basis* = *pd-basis* :: ′*a compact-basis set set*
⟨*proof*⟩

lemma *finite-Rep-pd-basis* [*simp*]: *finite* (*Rep-pd-basis u*)
⟨*proof*⟩

lemma *Rep-pd-basis-nonempty* [*simp*]: *Rep-pd-basis u* ≠ {}
⟨*proof*⟩

The powerdomain basis type is countable.

lemma *pd-basis-countable*: ∃*f*::′*a*::*bifinite pd-basis* ⇒ *nat*. *inj f* (**is** *Ex ?P*)
⟨*proof*⟩

28.2 Unit and plus constructors

definition
PDUnit :: ′*a*::*bifinite compact-basis* ⇒ ′*a pd-basis* **where**
PDUnit = (λ*x*. *Abs-pd-basis* {*x*})

definition
PDPlus :: ′*a*::*bifinite pd-basis* ⇒ ′*a pd-basis* ⇒ ′*a pd-basis* **where**
PDPlus t u = *Abs-pd-basis* (*Rep-pd-basis t* ∪ *Rep-pd-basis u*)

lemma *Rep-PDUnit*:
Rep-pd-basis (*PDUnit x*) = {*x*}
⟨*proof*⟩

lemma *Rep-PDPlus*:
Rep-pd-basis (*PDPlus u v*) = *Rep-pd-basis u* ∪ *Rep-pd-basis v*

⟨*proof*⟩

lemma *PDUnit-inject* [*simp*]: (*PDUnit a* = *PDUnit b*) = (*a* = *b*)
⟨*proof*⟩

lemma *PDPlus-assoc*: *PDPlus* (*PDPlus t u*) *v* = *PDPlus t* (*PDPlus u v*)
⟨*proof*⟩

lemma *PDPlus-commute*: *PDPlus t u* = *PDPlus u t*
⟨*proof*⟩

lemma *PDPlus-absorb*: *PDPlus t t* = *t*
⟨*proof*⟩

lemma *pd-basis-induct1* [*case-names PDUnit PDPlus*]:
assumes *PDUnit*: ⋀*a*. *P* (*PDUnit a*)
assumes *PDPlus*: ⋀*a t*. *P t* ⟹ *P* (*PDPlus* (*PDUnit a*) *t*)
shows *P x*
⟨*proof*⟩

lemma *pd-basis-induct* [*case-names PDUnit PDPlus*]:
assumes *PDUnit*: ⋀*a*. *P* (*PDUnit a*)
assumes *PDPlus*: ⋀*t u*. ⟦*P t*; *P u*⟧ ⟹ *P* (*PDPlus t u*)
shows *P x*
⟨*proof*⟩

28.3 Fold operator

definition
fold-pd ::
('*a*::*bifinite compact-basis* ⇒ '*b*::*type*) ⇒ ('*b* ⇒ '*b* ⇒ '*b*) ⇒ '*a pd-basis* ⇒ '*b*
where *fold-pd g f t* = *semilattice-set.F f* (*g* ' *Rep-pd-basis t*)

lemma *fold-pd-PDUnit*:
assumes *semilattice f*
shows *fold-pd g f* (*PDUnit x*) = *g x*
⟨*proof*⟩

lemma *fold-pd-PDPlus*:
assumes *semilattice f*
shows *fold-pd g f* (*PDPlus t u*) = *f* (*fold-pd g f t*) (*fold-pd g f u*)
⟨*proof*⟩

end

29 Upper powerdomain

theory *UpperPD*
imports *Compact-Basis*

begin

29.1 Basis preorder

definition
upper-le :: *'a*::*bifinite pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** ‹≤♯› *50*) **where**
upper-le = (λ*u v*. ∀ *y*∈*Rep-pd-basis v*. ∃ *x*∈*Rep-pd-basis u*. *x* ⊑ *y*)

lemma *upper-le-refl* [*simp*]: *t* ≤♯ *t*
⟨*proof*⟩

lemma *upper-le-trans*: ⟦*t* ≤♯ *u*; *u* ≤♯ *v*⟧ ⟹ *t* ≤♯ *v*
⟨*proof*⟩

interpretation *upper-le*: *preorder upper-le*
⟨*proof*⟩

lemma *upper-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♯ *t*
⟨*proof*⟩

lemma *PDUnit-upper-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♯ *PDUnit y*
⟨*proof*⟩

lemma *PDPlus-upper-mono*: ⟦*s* ≤♯ *t*; *u* ≤♯ *v*⟧ ⟹ *PDPlus s u* ≤♯ *PDPlus t v*
⟨*proof*⟩

lemma *PDPlus-upper-le*: *PDPlus t u* ≤♯ *t*
⟨*proof*⟩

lemma *upper-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* ≤♯ *PDUnit b*) = (*a* ⊑ *b*)
⟨*proof*⟩

lemma *upper-le-PDPlus-PDUnit-iff*:
(*PDPlus t u* ≤♯ *PDUnit a*) = (*t* ≤♯ *PDUnit a* ∨ *u* ≤♯ *PDUnit a*)
⟨*proof*⟩

lemma *upper-le-PDPlus-iff*: (*t* ≤♯ *PDPlus u v*) = (*t* ≤♯ *u* ∧ *t* ≤♯ *v*)
⟨*proof*⟩

lemma *upper-le-induct* [*induct set*: *upper-le*]:
assumes *le*: *t* ≤♯ *u*
assumes *1*: ⋀*a b*. *a* ⊑ *b* ⟹ *P* (*PDUnit a*) (*PDUnit b*)
assumes *2*: ⋀*t u a*. *P t* (*PDUnit a*) ⟹ *P* (*PDPlus t u*) (*PDUnit a*)
assumes *3*: ⋀*t u v*. ⟦*P t u*; *P t v*⟧ ⟹ *P t* (*PDPlus u v*)
shows *P t u*
⟨*proof*⟩

29.2 Type definition

typedef *′a::bifinite upper-pd* (‹(‹*notation*=‹*postfix upper-pd*›› ′(-′)♯)›) =
{*S*::′*a pd-basis set. upper-le.ideal S*}
⟨*proof*⟩

instantiation *upper-pd* :: (*bifinite*) *below*
begin

definition
$x \sqsubseteq y \longleftrightarrow$ *Rep-upper-pd x* ⊆ *Rep-upper-pd y*

instance ⟨*proof*⟩
end

instance *upper-pd* :: (*bifinite*) *po*
⟨*proof*⟩

instance *upper-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
upper-principal :: ′*a*::*bifinite pd-basis* ⇒ ′*a upper-pd* **where**
upper-principal t = *Abs-upper-pd* {*u. u* ≤♯ *t*}

interpretation *upper-pd*:
ideal-completion upper-le upper-principal Rep-upper-pd
⟨*proof*⟩

Upper powerdomain is pointed

lemma *upper-pd-minimal*: *upper-principal* (*PDUnit compact-bot*) ⊑ *ys*
⟨*proof*⟩

instance *upper-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-upper-pd-pcpo*: ⊥ = *upper-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

29.3 Monadic unit and plus

definition
upper-unit :: ′*a*::*bifinite* → ′*a upper-pd* **where**
upper-unit = *compact-basis.extension* (λ*a. upper-principal* (*PDUnit a*))

definition
upper-plus :: ′*a*::*bifinite upper-pd* → ′*a upper-pd* → ′*a upper-pd* **where**
upper-plus = *upper-pd.extension* (λ*t. upper-pd.extension* (λ*u.*
upper-principal (*PDPlus t u*)))

abbreviation
upper-add :: *'a::bifinite upper-pd* ⇒ *'a upper-pd* ⇒ *'a upper-pd*
(**infixl** ‹∪♯› *65*) **where**
xs ∪♯ *ys* == *upper-plus·xs·ys*

syntax
-upper-pd :: *args* ⇒ *logic* (‹(‹*indent=1 notation=*‹*mixfix upper-pd enumeration*››{-}♯)›)
translations
{*x*,*xs*}♯ == {*x*}♯ ∪♯ {*xs*}♯
{*x*}♯ == *CONST upper-unit·x*

lemma *upper-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♯ = *upper-principal* (*PDUnit a*)
⟨*proof*⟩

lemma *upper-plus-principal* [*simp*]:
upper-principal t ∪♯ *upper-principal u* = *upper-principal* (*PDPlus t u*)
⟨*proof*⟩

interpretation *upper-add*: *semilattice upper-add* ⟨*proof*⟩

lemmas *upper-plus-assoc* = *upper-add.assoc*
lemmas *upper-plus-commute* = *upper-add.commute*
lemmas *upper-plus-absorb* = *upper-add.idem*
lemmas *upper-plus-left-commute* = *upper-add.left-commute*
lemmas *upper-plus-left-absorb* = *upper-add.left-idem*

Useful for *simp add*: *upper-plus-ac*

lemmas *upper-plus-ac* =
upper-plus-assoc upper-plus-commute upper-plus-left-commute

Useful for *simp only*: *upper-plus-aci*

lemmas *upper-plus-aci* =
upper-plus-ac upper-plus-absorb upper-plus-left-absorb

lemma *upper-plus-below1*: *xs* ∪♯ *ys* ⊑ *xs*
⟨*proof*⟩

lemma *upper-plus-below2*: *xs* ∪♯ *ys* ⊑ *ys*
⟨*proof*⟩

lemma *upper-plus-greatest*: ⟦*xs* ⊑ *ys*; *xs* ⊑ *zs*⟧ ⟹ *xs* ⊑ *ys* ∪♯ *zs*
⟨*proof*⟩

lemma *upper-below-plus-iff* [*simp*]:
xs ⊑ *ys* ∪♯ *zs* ⟷ *xs* ⊑ *ys* ∧ *xs* ⊑ *zs*
⟨*proof*⟩

lemma *upper-plus-below-unit-iff* [*simp*]:
$xs \cup\sharp\ ys \sqsubseteq \{z\}\sharp \longleftrightarrow xs \sqsubseteq \{z\}\sharp \vee ys \sqsubseteq \{z\}\sharp$
⟨*proof*⟩

lemma *upper-unit-below-iff* [*simp*]: $\{x\}\sharp \sqsubseteq \{y\}\sharp \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemmas *upper-pd-below-simps* =
upper-unit-below-iff
upper-below-plus-iff
upper-plus-below-unit-iff

lemma *upper-unit-eq-iff* [*simp*]: $\{x\}\sharp = \{y\}\sharp \longleftrightarrow x = y$
⟨*proof*⟩

lemma *upper-unit-strict* [*simp*]: $\{\bot\}\sharp = \bot$
⟨*proof*⟩

lemma *upper-plus-strict1* [*simp*]: $\bot \cup\sharp\ ys = \bot$
⟨*proof*⟩

lemma *upper-plus-strict2* [*simp*]: $xs \cup\sharp\ \bot = \bot$
⟨*proof*⟩

lemma *upper-unit-bottom-iff* [*simp*]: $\{x\}\sharp = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *upper-plus-bottom-iff* [*simp*]:
$xs \cup\sharp\ ys = \bot \longleftrightarrow xs = \bot \vee ys = \bot$
⟨*proof*⟩

lemma *compact-upper-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\sharp$
⟨*proof*⟩

lemma *compact-upper-unit-iff* [*simp*]: *compact* $\{x\}\sharp \longleftrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-upper-plus* [*simp*]:
⟦*compact xs*; *compact ys*⟧ $\Longrightarrow$ *compact* $(xs \cup\sharp\ ys)$
⟨*proof*⟩

29.4 Induction rules

lemma *upper-pd-induct1*:
assumes *P*: *adm P*
assumes *unit*: $\bigwedge x.\ P\ \{x\}\sharp$
assumes *insert*: $\bigwedge x\ ys.$ ⟦$P\ \{x\}\sharp$; $P\ ys$⟧ $\Longrightarrow P\ (\{x\}\sharp \cup\sharp\ ys)$
shows *P* (*xs*::'*a*::*bifinite upper-pd*)
⟨*proof*⟩

lemma *upper-pd-induct* [*case-names adm upper-unit upper-plus*, *induct type*: *upper-pd*]:
assumes *P*: *adm P*
assumes *unit*: ⋀*x*. *P* {*x*}♯
assumes *plus*: ⋀*xs ys*. ⟦*P xs*; *P ys*⟧ ⟹ *P* (*xs* ∪♯ *ys*)
shows *P* (*xs*::′*a*::*bifinite upper-pd*)
⟨*proof*⟩

29.5 Monadic bind

definition
upper-bind-basis ::
′*a*::*bifinite pd-basis* ⇒ (′*a* → ′*b upper-pd*) → ′*b*::*bifinite upper-pd* **where**
upper-bind-basis = *fold-pd*
(λ*a*. Λ *f*. *f*·(*Rep-compact-basis a*))
(λ*x y*. Λ *f*. *x*·*f* ∪♯ *y*·*f*)

lemma *ACI-upper-bind*:
semilattice (λ*x y*. Λ *f*. *x*·*f* ∪♯ *y*·*f*)
⟨*proof*⟩

lemma *upper-bind-basis-simps* [*simp*]:
upper-bind-basis (*PDUnit a*) =
(Λ *f*. *f*·(*Rep-compact-basis a*))
upper-bind-basis (*PDPlus t u*) =
(Λ *f*. *upper-bind-basis t*·*f* ∪♯ *upper-bind-basis u*·*f*)
⟨*proof*⟩

lemma *upper-bind-basis-mono*:
t ≤♯ *u* ⟹ *upper-bind-basis t* ⊑ *upper-bind-basis u*
⟨*proof*⟩

definition
upper-bind :: ′*a*::*bifinite upper-pd* → (′*a* → ′*b upper-pd*) → ′*b*::*bifinite upper-pd*
where
upper-bind = *upper-pd.extension upper-bind-basis*

syntax
-upper-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
(‹(‹*indent=3 notation=*‹*binder upper-bind*››⋃♯-∈-./ -)› [*0*, *0*, *10*] *10*)

translations
⋃♯*x*∈*xs*. *e* == *CONST upper-bind*·*xs*·(Λ *x*. *e*)

lemma *upper-bind-principal* [*simp*]:
upper-bind·(*upper-principal t*) = *upper-bind-basis t*
⟨*proof*⟩

lemma *upper-bind-unit* [*simp*]:
$upper\text{-}bind \cdot \{x\}\sharp \cdot f = f \cdot x$
⟨*proof*⟩

lemma *upper-bind-plus* [*simp*]:
$upper\text{-}bind \cdot (xs \cup\sharp\ ys) \cdot f = upper\text{-}bind \cdot xs \cdot f \cup\sharp\ upper\text{-}bind \cdot ys \cdot f$
⟨*proof*⟩

lemma *upper-bind-strict* [*simp*]: $upper\text{-}bind \cdot \bot \cdot f = f \cdot \bot$
⟨*proof*⟩

lemma *upper-bind-bind*:
$upper\text{-}bind \cdot (upper\text{-}bind \cdot xs \cdot f) \cdot g = upper\text{-}bind \cdot xs \cdot (\Lambda\ x.\ upper\text{-}bind \cdot (f \cdot x) \cdot g)$
⟨*proof*⟩

29.6 Map

definition
$upper\text{-}map :: ('a::bifinite \rightarrow 'b::bifinite) \rightarrow 'a\ upper\text{-}pd \rightarrow 'b\ upper\text{-}pd$ **where**
$upper\text{-}map = (\Lambda\ f\ xs.\ upper\text{-}bind \cdot xs \cdot (\Lambda\ x.\ \{f \cdot x\}\sharp))$

lemma *upper-map-unit* [*simp*]:
$upper\text{-}map \cdot f \cdot \{x\}\sharp = \{f \cdot x\}\sharp$
⟨*proof*⟩

lemma *upper-map-plus* [*simp*]:
$upper\text{-}map \cdot f \cdot (xs \cup\sharp\ ys) = upper\text{-}map \cdot f \cdot xs \cup\sharp\ upper\text{-}map \cdot f \cdot ys$
⟨*proof*⟩

lemma *upper-map-bottom* [*simp*]: $upper\text{-}map \cdot f \cdot \bot = \{f \cdot \bot\}\sharp$
⟨*proof*⟩

lemma *upper-map-ident*: $upper\text{-}map \cdot (\Lambda\ x.\ x) \cdot xs = xs$
⟨*proof*⟩

lemma *upper-map-ID*: $upper\text{-}map \cdot ID = ID$
⟨*proof*⟩

lemma *upper-map-map*:
$upper\text{-}map \cdot f \cdot (upper\text{-}map \cdot g \cdot xs) = upper\text{-}map \cdot (\Lambda\ x.\ f \cdot (g \cdot x)) \cdot xs$
⟨*proof*⟩

lemma *upper-bind-map*:
$upper\text{-}bind \cdot (upper\text{-}map \cdot f \cdot xs) \cdot g = upper\text{-}bind \cdot xs \cdot (\Lambda\ x.\ g \cdot (f \cdot x))$
⟨*proof*⟩

lemma *upper-map-bind*:
$upper\text{-}map \cdot f \cdot (upper\text{-}bind \cdot xs \cdot g) = upper\text{-}bind \cdot xs \cdot (\Lambda\ x.\ upper\text{-}map \cdot f \cdot (g \cdot x))$
⟨*proof*⟩

lemma *ep-pair-upper-map*: *ep-pair e p* $\Longrightarrow$ *ep-pair* (*upper-map*·*e*) (*upper-map*·*p*)
⟨*proof*⟩

lemma *deflation-upper-map*: *deflation d* $\Longrightarrow$ *deflation* (*upper-map*·*d*)
⟨*proof*⟩

lemma *finite-deflation-upper-map*:
assumes *finite-deflation d* **shows** *finite-deflation* (*upper-map*·*d*)
⟨*proof*⟩

29.7 Upper powerdomain is bifinite

lemma *approx-chain-upper-map*:
assumes *approx-chain a*
shows *approx-chain* (λi. *upper-map*·(*a i*))
⟨*proof*⟩

instance *upper-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

29.8 Join

definition
upper-join :: *'a*::*bifinite upper-pd upper-pd* $\to$ *'a upper-pd* **where**
upper-join = (Λ *xss*. *upper-bind*·*xss*·(Λ *xs*. *xs*))

lemma *upper-join-unit* [*simp*]:
upper-join·{*xs*}$\sharp$ = *xs*
⟨*proof*⟩

lemma *upper-join-plus* [*simp*]:
upper-join·(*xss* $\cup\sharp$ *yss*) = *upper-join*·*xss* $\cup\sharp$ *upper-join*·*yss*
⟨*proof*⟩

lemma *upper-join-bottom* [*simp*]: *upper-join*·$\bot$ = $\bot$
⟨*proof*⟩

lemma *upper-join-map-unit*:
upper-join·(*upper-map*·*upper-unit*·*xs*) = *xs*
⟨*proof*⟩

lemma *upper-join-map-join*:
upper-join·(*upper-map*·*upper-join*·*xsss*) = *upper-join*·(*upper-join*·*xsss*)
⟨*proof*⟩

lemma *upper-join-map-map*:
upper-join·(*upper-map*·(*upper-map*·*f*)·*xss*) =
upper-map·*f*·(*upper-join*·*xss*)

⟨*proof*⟩

end

30 Lower powerdomain

theory *LowerPD*
imports *Compact-Basis*
begin

30.1 Basis preorder

definition
lower-le :: *'a::bifinite pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** ‹≤♭› *50*) **where**
lower-le = (*λu v*. ∀ *x*∈*Rep-pd-basis u*. ∃ *y*∈*Rep-pd-basis v*. *x* ⊑ *y*)

lemma *lower-le-refl* [*simp*]: *t* ≤♭ *t*
⟨*proof*⟩

lemma *lower-le-trans*: ⟦*t* ≤♭ *u*; *u* ≤♭ *v*⟧ ⟹ *t* ≤♭ *v*
⟨*proof*⟩

interpretation *lower-le*: *preorder lower-le*
⟨*proof*⟩

lemma *lower-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♭ *t*
⟨*proof*⟩

lemma *PDUnit-lower-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♭ *PDUnit y*
⟨*proof*⟩

lemma *PDPlus-lower-mono*: ⟦*s* ≤♭ *t*; *u* ≤♭ *v*⟧ ⟹ *PDPlus s u* ≤♭ *PDPlus t v*
⟨*proof*⟩

lemma *PDPlus-lower-le*: *t* ≤♭ *PDPlus t u*
⟨*proof*⟩

lemma *lower-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* ≤♭ *PDUnit b*) = (*a* ⊑ *b*)
⟨*proof*⟩

lemma *lower-le-PDUnit-PDPlus-iff*:
(*PDUnit a* ≤♭ *PDPlus t u*) = (*PDUnit a* ≤♭ *t* ∨ *PDUnit a* ≤♭ *u*)
⟨*proof*⟩

lemma *lower-le-PDPlus-iff*: (*PDPlus t u* ≤♭ *v*) = (*t* ≤♭ *v* ∧ *u* ≤♭ *v*)
⟨*proof*⟩

lemma *lower-le-induct* [*induct set*: *lower-le*]:

assumes *le*: $t \leq\flat\ u$
assumes *1*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P\ (PDUnit\ a)\ (PDUnit\ b)$
assumes *2*: $\bigwedge t\ u\ a.\ P\ (PDUnit\ a)\ t \Longrightarrow P\ (PDUnit\ a)\ (PDPlus\ t\ u)$
assumes *3*: $\bigwedge t\ u\ v.\ [\![P\ t\ v;\ P\ u\ v]\!] \Longrightarrow P\ (PDPlus\ t\ u)\ v$
shows $P\ t\ u$
⟨*proof*⟩

30.2 Type definition

typedef *'a::bifinite lower-pd* (‹(‹notation=‹postfix lower-pd››'(-')♭)›) =
{*S*::*'a pd-basis set. lower-le.ideal S*}
⟨*proof*⟩

instantiation *lower-pd* :: (*bifinite*) *below*
begin

definition
$x \sqsubseteq y \longleftrightarrow$ *Rep-lower-pd x* $\subseteq$ *Rep-lower-pd y*

instance ⟨*proof*⟩
end

instance *lower-pd* :: (*bifinite*) *po*
⟨*proof*⟩

instance *lower-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
lower-principal :: *'a::bifinite pd-basis* ⇒ *'a lower-pd* **where**
lower-principal t = *Abs-lower-pd* {*u*. $u \leq\flat\ t$}

interpretation *lower-pd*:
ideal-completion lower-le lower-principal Rep-lower-pd
⟨*proof*⟩

Lower powerdomain is pointed

lemma *lower-pd-minimal*: *lower-principal* (*PDUnit compact-bot*) ⊑ *ys*
⟨*proof*⟩

instance *lower-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-lower-pd-pcpo*: ⊥ = *lower-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

30.3 Monadic unit and plus

definition

lower-unit :: *'a::bifinite* → *'a lower-pd* **where**
lower-unit = *compact-basis.extension* (*λa. lower-principal* (*PDUnit a*))

definition
lower-plus :: *'a::bifinite lower-pd* → *'a lower-pd* → *'a lower-pd* **where**
lower-plus = *lower-pd.extension* (*λt. lower-pd.extension* (*λu.*
lower-principal (*PDPlus t u*)))

abbreviation
lower-add :: *'a::bifinite lower-pd* ⇒ *'a lower-pd* ⇒ *'a lower-pd*
(**infixl** ‹∪♭› *65*) **where**
xs ∪♭ *ys* == *lower-plus·xs·ys*

syntax
-lower-pd :: *args* ⇒ *logic* (‹(‹*indent=1 notation=*‹*mixfix lower-pd enumeration*››{-}♭)›)
translations
{*x*,*xs*}♭ == {*x*}♭ ∪♭ {*xs*}♭
{*x*}♭ == *CONST lower-unit·x*

lemma *lower-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♭ = *lower-principal* (*PDUnit a*)
⟨*proof*⟩

lemma *lower-plus-principal* [*simp*]:
lower-principal t ∪♭ *lower-principal u* = *lower-principal* (*PDPlus t u*)
⟨*proof*⟩

interpretation *lower-add*: *semilattice lower-add* ⟨*proof*⟩

lemmas *lower-plus-assoc* = *lower-add.assoc*
lemmas *lower-plus-commute* = *lower-add.commute*
lemmas *lower-plus-absorb* = *lower-add.idem*
lemmas *lower-plus-left-commute* = *lower-add.left-commute*
lemmas *lower-plus-left-absorb* = *lower-add.left-idem*

Useful for *simp add*: *lower-plus-ac*

lemmas *lower-plus-ac* =
lower-plus-assoc lower-plus-commute lower-plus-left-commute

Useful for *simp only*: *lower-plus-aci*

lemmas *lower-plus-aci* =
lower-plus-ac lower-plus-absorb lower-plus-left-absorb

lemma *lower-plus-below1*: *xs* ⊑ *xs* ∪♭ *ys*
⟨*proof*⟩

lemma *lower-plus-below2*: *ys* ⊑ *xs* ∪♭ *ys*
⟨*proof*⟩

lemma *lower-plus-least*: $[\![xs \sqsubseteq zs;\ ys \sqsubseteq zs]\!] \Longrightarrow xs \cup\flat\ ys \sqsubseteq zs$
⟨*proof*⟩

lemma *lower-plus-below-iff* [*simp*]:
$xs \cup\flat\ ys \sqsubseteq zs \longleftrightarrow xs \sqsubseteq zs \wedge ys \sqsubseteq zs$
⟨*proof*⟩

lemma *lower-unit-below-plus-iff* [*simp*]:
$\{x\}\flat \sqsubseteq ys \cup\flat\ zs \longleftrightarrow \{x\}\flat \sqsubseteq ys \vee \{x\}\flat \sqsubseteq zs$
⟨*proof*⟩

lemma *lower-unit-below-iff* [*simp*]: $\{x\}\flat \sqsubseteq \{y\}\flat \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemmas *lower-pd-below-simps* =
lower-unit-below-iff
lower-plus-below-iff
lower-unit-below-plus-iff

lemma *lower-unit-eq-iff* [*simp*]: $\{x\}\flat = \{y\}\flat \longleftrightarrow x = y$
⟨*proof*⟩

lemma *lower-unit-strict* [*simp*]: $\{\bot\}\flat = \bot$
⟨*proof*⟩

lemma *lower-unit-bottom-iff* [*simp*]: $\{x\}\flat = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *lower-plus-bottom-iff* [*simp*]:
$xs \cup\flat\ ys = \bot \longleftrightarrow xs = \bot \wedge ys = \bot$
⟨*proof*⟩

lemma *lower-plus-strict1* [*simp*]: $\bot \cup\flat\ ys = ys$
⟨*proof*⟩

lemma *lower-plus-strict2* [*simp*]: $xs \cup\flat\ \bot = xs$
⟨*proof*⟩

lemma *compact-lower-unit*: $compact\ x \Longrightarrow compact\ \{x\}\flat$
⟨*proof*⟩

lemma *compact-lower-unit-iff* [*simp*]: $compact\ \{x\}\flat \longleftrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-lower-plus* [*simp*]:
$[\![compact\ xs;\ compact\ ys]\!] \Longrightarrow compact\ (xs \cup\flat\ ys)$
⟨*proof*⟩

30.4 Induction rules

lemma *lower-pd-induct1*:
assumes *P*: *adm P*
assumes *unit*: $\bigwedge x.\ P\ \{x\}\flat$
assumes *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\flat;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\flat \cup\flat\ ys)$
shows *P* (*xs*::*'a*::*bifinite lower-pd*)
⟨*proof*⟩

lemma *lower-pd-induct* [*case-names adm lower-unit lower-plus*, *induct type*: *lower-pd*]:
assumes *P*: *adm P*
assumes *unit*: $\bigwedge x.\ P\ \{x\}\flat$
assumes *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\flat\ ys)$
shows *P* (*xs*::*'a*::*bifinite lower-pd*)
⟨*proof*⟩

30.5 Monadic bind

definition
lower-bind-basis ::
'a::*bifinite pd-basis* ⇒ (*'a* → *'b lower-pd*) → *'b*::*bifinite lower-pd* **where**
lower-bind-basis = *fold-pd*
(λ*a*. Λ *f*. *f*·(*Rep-compact-basis a*))
(λ*x y*. Λ *f*. *x*·*f* ∪♭ *y*·*f*)

lemma *ACI-lower-bind*:
semilattice (λ*x y*. Λ *f*. *x*·*f* ∪♭ *y*·*f*)
⟨*proof*⟩

lemma *lower-bind-basis-simps* [*simp*]:
lower-bind-basis (*PDUnit a*) =
(Λ *f*. *f*·(*Rep-compact-basis a*))
lower-bind-basis (*PDPlus t u*) =
(Λ *f*. *lower-bind-basis t*·*f* ∪♭ *lower-bind-basis u*·*f*)
⟨*proof*⟩

lemma *lower-bind-basis-mono*:
$t \le\flat\ u \Longrightarrow$ *lower-bind-basis t* ⊑ *lower-bind-basis u*
⟨*proof*⟩

definition
lower-bind :: *'a*::*bifinite lower-pd* → (*'a* → *'b lower-pd*) → *'b*::*bifinite lower-pd*
where
lower-bind = *lower-pd.extension lower-bind-basis*

syntax
-lower-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
(‹(‹*indent=3 notation=*‹*binder lower-bind*››⋃♭-∈-./ -)› [*0*, *0*, *10*] *10*)

translations

$\bigcup\flat x{\in}xs.\ e == CONST\ lower\text{-}bind{\cdot}xs{\cdot}(\Lambda\ x.\ e)$

lemma *lower-bind-principal* [*simp*]:
$lower\text{-}bind{\cdot}(lower\text{-}principal\ t) = lower\text{-}bind\text{-}basis\ t$
⟨*proof*⟩

lemma *lower-bind-unit* [*simp*]:
$lower\text{-}bind{\cdot}\{x\}\flat{\cdot}f = f{\cdot}x$
⟨*proof*⟩

lemma *lower-bind-plus* [*simp*]:
$lower\text{-}bind{\cdot}(xs \cup\flat\ ys){\cdot}f = lower\text{-}bind{\cdot}xs{\cdot}f \cup\flat\ lower\text{-}bind{\cdot}ys{\cdot}f$
⟨*proof*⟩

lemma *lower-bind-strict* [*simp*]: $lower\text{-}bind{\cdot}\bot{\cdot}f = f{\cdot}\bot$
⟨*proof*⟩

lemma *lower-bind-bind*:
$lower\text{-}bind{\cdot}(lower\text{-}bind{\cdot}xs{\cdot}f){\cdot}g = lower\text{-}bind{\cdot}xs{\cdot}(\Lambda\ x.\ lower\text{-}bind{\cdot}(f{\cdot}x){\cdot}g)$
⟨*proof*⟩

30.6 Map

definition
$lower\text{-}map :: ('a::bifinite \rightarrow 'b::bifinite) \rightarrow 'a\ lower\text{-}pd \rightarrow 'b\ lower\text{-}pd$ **where**
$lower\text{-}map = (\Lambda\ f\ xs.\ lower\text{-}bind{\cdot}xs{\cdot}(\Lambda\ x.\ \{f{\cdot}x\}\flat))$

lemma *lower-map-unit* [*simp*]:
$lower\text{-}map{\cdot}f{\cdot}\{x\}\flat = \{f{\cdot}x\}\flat$
⟨*proof*⟩

lemma *lower-map-plus* [*simp*]:
$lower\text{-}map{\cdot}f{\cdot}(xs \cup\flat\ ys) = lower\text{-}map{\cdot}f{\cdot}xs \cup\flat\ lower\text{-}map{\cdot}f{\cdot}ys$
⟨*proof*⟩

lemma *lower-map-bottom* [*simp*]: $lower\text{-}map{\cdot}f{\cdot}\bot = \{f{\cdot}\bot\}\flat$
⟨*proof*⟩

lemma *lower-map-ident*: $lower\text{-}map{\cdot}(\Lambda\ x.\ x){\cdot}xs = xs$
⟨*proof*⟩

lemma *lower-map-ID*: $lower\text{-}map{\cdot}ID = ID$
⟨*proof*⟩

lemma *lower-map-map*:
$lower\text{-}map{\cdot}f{\cdot}(lower\text{-}map{\cdot}g{\cdot}xs) = lower\text{-}map{\cdot}(\Lambda\ x.\ f{\cdot}(g{\cdot}x)){\cdot}xs$
⟨*proof*⟩

lemma *lower-bind-map*:

$lower\text{-}bind\cdot(lower\text{-}map\cdot f\cdot xs)\cdot g = lower\text{-}bind\cdot xs\cdot(\Lambda\ x.\ g\cdot(f\cdot x))$
⟨proof⟩

lemma *lower-map-bind*:
$lower\text{-}map\cdot f\cdot(lower\text{-}bind\cdot xs\cdot g) = lower\text{-}bind\cdot xs\cdot(\Lambda\ x.\ lower\text{-}map\cdot f\cdot(g\cdot x))$
⟨proof⟩

lemma *ep-pair-lower-map*: $ep\text{-}pair\ e\ p \Longrightarrow ep\text{-}pair\ (lower\text{-}map\cdot e)\ (lower\text{-}map\cdot p)$
⟨proof⟩

lemma *deflation-lower-map*: $deflation\ d \Longrightarrow deflation\ (lower\text{-}map\cdot d)$
⟨proof⟩

lemma *finite-deflation-lower-map*:
assumes *finite-deflation d* **shows** *finite-deflation* $(lower\text{-}map\cdot d)$
⟨proof⟩

30.7 Lower powerdomain is bifinite

lemma *approx-chain-lower-map*:
assumes *approx-chain a*
shows *approx-chain* $(\lambda i.\ lower\text{-}map\cdot(a\ i))$
⟨proof⟩

instance *lower-pd* :: (*bifinite*) *bifinite*
⟨proof⟩

30.8 Join

definition
$lower\text{-}join :: 'a::bifinite\ lower\text{-}pd\ lower\text{-}pd \rightarrow 'a\ lower\text{-}pd$ **where**
$lower\text{-}join = (\Lambda\ xss.\ lower\text{-}bind\cdot xss\cdot(\Lambda\ xs.\ xs))$

lemma *lower-join-unit* [*simp*]:
$lower\text{-}join\cdot\{xs\}\flat = xs$
⟨proof⟩

lemma *lower-join-plus* [*simp*]:
$lower\text{-}join\cdot(xss \cup\flat\ yss) = lower\text{-}join\cdot xss \cup\flat\ lower\text{-}join\cdot yss$
⟨proof⟩

lemma *lower-join-bottom* [*simp*]: $lower\text{-}join\cdot\bot = \bot$
⟨proof⟩

lemma *lower-join-map-unit*:
$lower\text{-}join\cdot(lower\text{-}map\cdot lower\text{-}unit\cdot xs) = xs$
⟨proof⟩

lemma *lower-join-map-join*:

lower-join·(*lower-map*·*lower-join*·*xsss*) = *lower-join*·(*lower-join*·*xsss*)
⟨*proof*⟩

lemma *lower-join-map-map*:
lower-join·(*lower-map*·(*lower-map*·*f*)·*xss*) =
lower-map·*f*·(*lower-join*·*xss*)
⟨*proof*⟩

end

31 Convex powerdomain

theory *ConvexPD*
imports *UpperPD LowerPD*
begin

31.1 Basis preorder

definition
convex-le :: *'a::bifinite pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** ‹≤♮› *50*) **where**
convex-le = (λ*u v*. *u* ≤♯ *v* ∧ *u* ≤♭ *v*)

lemma *convex-le-refl* [*simp*]: *t* ≤♮ *t*
⟨*proof*⟩

lemma *convex-le-trans*: ⟦*t* ≤♮ *u*; *u* ≤♮ *v*⟧ ⟹ *t* ≤♮ *v*
⟨*proof*⟩

interpretation *convex-le*: *preorder convex-le*
⟨*proof*⟩

lemma *upper-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♮ *t*
⟨*proof*⟩

lemma *PDUnit-convex-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♮ *PDUnit y*
⟨*proof*⟩

lemma *PDPlus-convex-mono*: ⟦*s* ≤♮ *t*; *u* ≤♮ *v*⟧ ⟹ *PDPlus s u* ≤♮ *PDPlus t v*
⟨*proof*⟩

lemma *convex-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* ≤♮ *PDUnit b*) = (*a* ⊑ *b*)
⟨*proof*⟩

lemma *convex-le-PDUnit-lemma1*:
(*PDUnit a* ≤♮ *t*) = (∀ *b*∈*Rep-pd-basis t*. *a* ⊑ *b*)
⟨*proof*⟩

lemma *convex-le-PDUnit-PDPlus-iff* [*simp*]:

$(PDUnit\ a \le\natural\ PDPlus\ t\ u) = (PDUnit\ a \le\natural\ t \wedge PDUnit\ a \le\natural\ u)$
⟨*proof*⟩

lemma *convex-le-PDUnit-lemma2*:
$(t \le\natural\ PDUnit\ b) = (\forall\ a \in Rep\text{-}pd\text{-}basis\ t.\ a \sqsubseteq b)$
⟨*proof*⟩

lemma *convex-le-PDPlus-PDUnit-iff* [*simp*]:
$(PDPlus\ t\ u \le\natural\ PDUnit\ a) = (t \le\natural\ PDUnit\ a \wedge u \le\natural\ PDUnit\ a)$
⟨*proof*⟩

lemma *convex-le-PDPlus-lemma*:
assumes *z*: $PDPlus\ t\ u \le\natural\ z$
shows $\exists\ v\ w.\ z = PDPlus\ v\ w \wedge t \le\natural\ v \wedge u \le\natural\ w$
⟨*proof*⟩

lemma *convex-le-induct* [*induct set*: *convex-le*]:
assumes *le*: $t \le\natural\ u$
assumes *2*: $\bigwedge t\ u\ v.\ [\![P\ t\ u;\ P\ u\ v]\!] \Longrightarrow P\ t\ v$
assumes *3*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P\ (PDUnit\ a)\ (PDUnit\ b)$
assumes *4*: $\bigwedge t\ u\ v\ w.\ [\![P\ t\ v;\ P\ u\ w]\!] \Longrightarrow P\ (PDPlus\ t\ u)\ (PDPlus\ v\ w)$
shows $P\ t\ u$
⟨*proof*⟩

31.2 Type definition

typedef *'a::bifinite convex-pd* (‹(‹notation=‹postfix convex-pd››'(-')♮)›) =
{*S*::*'a pd-basis set. convex-le.ideal S*}
⟨*proof*⟩

instantiation *convex-pd* :: (*bifinite*) *below*
begin

definition
$x \sqsubseteq y \longleftrightarrow Rep\text{-}convex\text{-}pd\ x \subseteq Rep\text{-}convex\text{-}pd\ y$

instance ⟨*proof*⟩
end

instance *convex-pd* :: (*bifinite*) *po*
⟨*proof*⟩

instance *convex-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
convex-principal :: *'a::bifinite pd-basis* ⇒ *'a convex-pd* **where**
$convex\text{-}principal\ t = Abs\text{-}convex\text{-}pd\ \{u.\ u \le\natural\ t\}$

interpretation *convex-pd*:
ideal-completion convex-le convex-principal Rep-convex-pd
⟨*proof*⟩

Convex powerdomain is pointed

lemma *convex-pd-minimal*: *convex-principal* (*PDUnit compact-bot*) ⊑ *ys*
⟨*proof*⟩

instance *convex-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-convex-pd-pcpo*: ⊥ = *convex-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

31.3 Monadic unit and plus

definition
convex-unit :: *'a::bifinite* → *'a convex-pd* **where**
convex-unit = *compact-basis.extension* (λ*a*. *convex-principal* (*PDUnit a*))

definition
convex-plus :: *'a::bifinite convex-pd* → *'a convex-pd* → *'a convex-pd* **where**
convex-plus = *convex-pd.extension* (λ*t*. *convex-pd.extension* (λ*u*.
convex-principal (*PDPlus t u*)))

abbreviation
convex-add :: *'a::bifinite convex-pd* ⇒ *'a convex-pd* ⇒ *'a convex-pd*
(**infixl** ‹∪♮› *65*) **where**
xs ∪♮ *ys* == *convex-plus·xs·ys*

syntax
-convex-pd :: *args* ⇒ *logic* (‹(‹*indent=1 notation=*‹*mixfix convex-pd enumeration*››{-}♮)›)
translations
{*x*,*xs*}♮ == {*x*}♮ ∪♮ {*xs*}♮
{*x*}♮ == *CONST convex-unit·x*

lemma *convex-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♮ = *convex-principal* (*PDUnit a*)
⟨*proof*⟩

lemma *convex-plus-principal* [*simp*]:
convex-principal t ∪♮ *convex-principal u* = *convex-principal* (*PDPlus t u*)
⟨*proof*⟩

interpretation *convex-add*: *semilattice convex-add* ⟨*proof*⟩

lemmas *convex-plus-assoc* = *convex-add.assoc*
lemmas *convex-plus-commute* = *convex-add.commute*

lemmas *convex-plus-absorb* = *convex-add.idem*
lemmas *convex-plus-left-commute* = *convex-add.left-commute*
lemmas *convex-plus-left-absorb* = *convex-add.left-idem*

Useful for *simp add*: *convex-plus-ac*

lemmas *convex-plus-ac* =
convex-plus-assoc convex-plus-commute convex-plus-left-commute

Useful for *simp only*: *convex-plus-aci*

lemmas *convex-plus-aci* =
convex-plus-ac convex-plus-absorb convex-plus-left-absorb

lemma *convex-unit-below-plus-iff* [*simp*]:
$\{x\}\natural \sqsubseteq ys \cup\natural\ zs \longleftrightarrow \{x\}\natural \sqsubseteq ys \wedge \{x\}\natural \sqsubseteq zs$
⟨*proof*⟩

lemma *convex-plus-below-unit-iff* [*simp*]:
$xs \cup\natural\ ys \sqsubseteq \{z\}\natural \longleftrightarrow xs \sqsubseteq \{z\}\natural \wedge ys \sqsubseteq \{z\}\natural$
⟨*proof*⟩

lemma *convex-unit-below-iff* [*simp*]: $\{x\}\natural \sqsubseteq \{y\}\natural \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *convex-unit-eq-iff* [*simp*]: $\{x\}\natural = \{y\}\natural \longleftrightarrow x = y$
⟨*proof*⟩

lemma *convex-unit-strict* [*simp*]: $\{\bot\}\natural = \bot$
⟨*proof*⟩

lemma *convex-unit-bottom-iff* [*simp*]: $\{x\}\natural = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *compact-convex-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\natural$
⟨*proof*⟩

lemma *compact-convex-unit-iff* [*simp*]: *compact* $\{x\}\natural \longleftrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-convex-plus* [*simp*]:
⟦*compact xs*; *compact ys*⟧ $\Longrightarrow$ *compact* ($xs \cup\natural\ ys$)
⟨*proof*⟩

31.4 Induction rules

lemma *convex-pd-induct1*:
assumes *P*: *adm P*
assumes *unit*: $\bigwedge x.\ P\ \{x\}\natural$
assumes *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\natural;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\natural \cup\natural\ ys)$
shows *P* (*xs*::'*a*::*bifinite convex-pd*)

⟨*proof*⟩

lemma *convex-pd-induct* [*case-names adm convex-unit convex-plus*, *induct type*: *convex-pd*]:
  assumes *P*: *adm P*
  assumes *unit*: ⋀*x*. *P* {*x*}♮
  assumes *plus*: ⋀*xs ys*. ⟦*P xs*; *P ys*⟧ ⟹ *P* (*xs* ∪♮ *ys*)
  shows *P* (*xs*::′*a*::*bifinite convex-pd*)
⟨*proof*⟩

31.5 Monadic bind

definition
  convex-bind-basis ::
  ′*a*::*bifinite pd-basis* ⇒ (′*a* → ′*b convex-pd*) → ′*b*::*bifinite convex-pd* **where**
  convex-bind-basis = *fold-pd*
    (λ*a*. Λ *f*. *f*·(*Rep-compact-basis a*))
    (λ*x y*. Λ *f*. *x*·*f* ∪♮ *y*·*f*)

lemma *ACI-convex-bind*:
  semilattice (λ*x y*. Λ *f*. *x*·*f* ∪♮ *y*·*f*)
⟨*proof*⟩

lemma *convex-bind-basis-simps* [*simp*]:
  convex-bind-basis (*PDUnit a*) =
    (Λ *f*. *f*·(*Rep-compact-basis a*))
  convex-bind-basis (*PDPlus t u*) =
    (Λ *f*. *convex-bind-basis t*·*f* ∪♮ *convex-bind-basis u*·*f*)
⟨*proof*⟩

lemma *convex-bind-basis-mono*:
  t ≤♮ *u* ⟹ *convex-bind-basis t* ⊑ *convex-bind-basis u*
⟨*proof*⟩

definition
  convex-bind :: ′*a*::*bifinite convex-pd* → (′*a* → ′*b convex-pd*) → ′*b*::*bifinite convex-pd*
where
  convex-bind = *convex-pd.extension convex-bind-basis*

syntax
  -convex-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
    (‹(‹*indent=3 notation=*‹*binder convex-bind*››⋃♮-∈-./ -)› [*0*, *0*, *10*] *10*)

translations
  ⋃♮*x*∈*xs*. *e* == *CONST convex-bind*·*xs*·(Λ *x*. *e*)

lemma *convex-bind-principal* [*simp*]:
  convex-bind·(*convex-principal t*) = *convex-bind-basis t*
⟨*proof*⟩

lemma *convex-bind-unit* [*simp*]:
$convex\text{-}bind \cdot \{x\}\natural \cdot f = f \cdot x$
⟨*proof*⟩

lemma *convex-bind-plus* [*simp*]:
$convex\text{-}bind \cdot (xs \cup\natural\ ys) \cdot f = convex\text{-}bind \cdot xs \cdot f \cup\natural\ convex\text{-}bind \cdot ys \cdot f$
⟨*proof*⟩

lemma *convex-bind-strict* [*simp*]: $convex\text{-}bind \cdot \bot \cdot f = f \cdot \bot$
⟨*proof*⟩

lemma *convex-bind-bind*:
$convex\text{-}bind \cdot (convex\text{-}bind \cdot xs \cdot f) \cdot g =$
$convex\text{-}bind \cdot xs \cdot (\Lambda\ x.\ convex\text{-}bind \cdot (f \cdot x) \cdot g)$
⟨*proof*⟩

31.6 Map

definition
$convex\text{-}map :: ('a::bifinite \to 'b) \to 'a\ convex\text{-}pd \to 'b::bifinite\ convex\text{-}pd$ **where**
$convex\text{-}map = (\Lambda\ f\ xs.\ convex\text{-}bind \cdot xs \cdot (\Lambda\ x.\ \{f \cdot x\}\natural))$

lemma *convex-map-unit* [*simp*]:
$convex\text{-}map \cdot f \cdot \{x\}\natural = \{f \cdot x\}\natural$
⟨*proof*⟩

lemma *convex-map-plus* [*simp*]:
$convex\text{-}map \cdot f \cdot (xs \cup\natural\ ys) = convex\text{-}map \cdot f \cdot xs \cup\natural\ convex\text{-}map \cdot f \cdot ys$
⟨*proof*⟩

lemma *convex-map-bottom* [*simp*]: $convex\text{-}map \cdot f \cdot \bot = \{f \cdot \bot\}\natural$
⟨*proof*⟩

lemma *convex-map-ident*: $convex\text{-}map \cdot (\Lambda\ x.\ x) \cdot xs = xs$
⟨*proof*⟩

lemma *convex-map-ID*: $convex\text{-}map \cdot ID = ID$
⟨*proof*⟩

lemma *convex-map-map*:
$convex\text{-}map \cdot f \cdot (convex\text{-}map \cdot g \cdot xs) = convex\text{-}map \cdot (\Lambda\ x.\ f \cdot (g \cdot x)) \cdot xs$
⟨*proof*⟩

lemma *convex-bind-map*:
$convex\text{-}bind \cdot (convex\text{-}map \cdot f \cdot xs) \cdot g = convex\text{-}bind \cdot xs \cdot (\Lambda\ x.\ g \cdot (f \cdot x))$
⟨*proof*⟩

lemma *convex-map-bind*:

convex-map·f·(convex-bind·xs·g) = convex-bind·xs·(Λ x. convex-map·f·(g·x))
⟨*proof*⟩

lemma *ep-pair-convex-map*: *ep-pair e p* ⟹ *ep-pair* (*convex-map·e*) (*convex-map·p*)
⟨*proof*⟩

lemma *deflation-convex-map*: *deflation d* ⟹ *deflation* (*convex-map·d*)
⟨*proof*⟩

lemma *finite-deflation-convex-map*:
assumes *finite-deflation d* **shows** *finite-deflation* (*convex-map·d*)
⟨*proof*⟩

31.7 Convex powerdomain is bifinite

lemma *approx-chain-convex-map*:
assumes *approx-chain a*
shows *approx-chain* (*λi. convex-map·*(*a i*))
⟨*proof*⟩

instance *convex-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

31.8 Join

definition
convex-join :: *'a::bifinite convex-pd convex-pd* → *'a convex-pd* **where**
convex-join = (Λ *xss. convex-bind·xss·*(Λ *xs. xs*))

lemma *convex-join-unit* [*simp*]:
convex-join·{xs}♮ = xs
⟨*proof*⟩

lemma *convex-join-plus* [*simp*]:
convex-join·(xss ∪♮ yss) = convex-join·xss ∪♮ convex-join·yss
⟨*proof*⟩

lemma *convex-join-bottom* [*simp*]: *convex-join·⊥ = ⊥*
⟨*proof*⟩

lemma *convex-join-map-unit*:
convex-join·(convex-map·convex-unit·xs) = xs
⟨*proof*⟩

lemma *convex-join-map-join*:
convex-join·(convex-map·convex-join·xsss) = convex-join·(convex-join·xsss)
⟨*proof*⟩

lemma *convex-join-map-map*:

convex-join·(*convex-map*·(*convex-map*·*f*)·*xss*) =
convex-map·*f*·(*convex-join*·*xss*)
⟨*proof*⟩

31.9 Conversions to other powerdomains

Convex to upper

lemma *convex-le-imp-upper-le*: $t \leq\natural\ u \Longrightarrow t \leq\sharp\ u$
⟨*proof*⟩

definition
convex-to-upper :: *′a::bifinite convex-pd* → *′a upper-pd* **where**
convex-to-upper = *convex-pd.extension upper-principal*

lemma *convex-to-upper-principal* [*simp*]:
convex-to-upper·(*convex-principal t*) = *upper-principal t*
⟨*proof*⟩

lemma *convex-to-upper-unit* [*simp*]:
convex-to-upper·{*x*}♮ = {*x*}♯
⟨*proof*⟩

lemma *convex-to-upper-plus* [*simp*]:
convex-to-upper·(*xs* ∪♮ *ys*) = *convex-to-upper*·*xs* ∪♯ *convex-to-upper*·*ys*
⟨*proof*⟩

lemma *convex-to-upper-bind* [*simp*]:
convex-to-upper·(*convex-bind*·*xs*·*f*) =
upper-bind·(*convex-to-upper*·*xs*)·(*convex-to-upper oo f*)
⟨*proof*⟩

lemma *convex-to-upper-map* [*simp*]:
convex-to-upper·(*convex-map*·*f*·*xs*) = *upper-map*·*f*·(*convex-to-upper*·*xs*)
⟨*proof*⟩

lemma *convex-to-upper-join* [*simp*]:
convex-to-upper·(*convex-join*·*xss*) =
upper-bind·(*convex-to-upper*·*xss*)·*convex-to-upper*
⟨*proof*⟩

Convex to lower

lemma *convex-le-imp-lower-le*: $t \leq\natural\ u \Longrightarrow t \leq\flat\ u$
⟨*proof*⟩

definition
convex-to-lower :: *′a::bifinite convex-pd* → *′a lower-pd* **where**
convex-to-lower = *convex-pd.extension lower-principal*

lemma *convex-to-lower-principal* [*simp*]:

convex-to-lower·(*convex-principal t*) = *lower-principal t*
⟨*proof*⟩

lemma *convex-to-lower-unit* [*simp*]:
convex-to-lower·{*x*}♮ = {*x*}♭
⟨*proof*⟩

lemma *convex-to-lower-plus* [*simp*]:
convex-to-lower·(*xs* ∪♮ *ys*) = *convex-to-lower*·*xs* ∪♭ *convex-to-lower*·*ys*
⟨*proof*⟩

lemma *convex-to-lower-bind* [*simp*]:
convex-to-lower·(*convex-bind*·*xs*·*f*) =
lower-bind·(*convex-to-lower*·*xs*)·(*convex-to-lower oo f*)
⟨*proof*⟩

lemma *convex-to-lower-map* [*simp*]:
convex-to-lower·(*convex-map*·*f*·*xs*) = *lower-map*·*f*·(*convex-to-lower*·*xs*)
⟨*proof*⟩

lemma *convex-to-lower-join* [*simp*]:
convex-to-lower·(*convex-join*·*xss*) =
lower-bind·(*convex-to-lower*·*xss*)·*convex-to-lower*
⟨*proof*⟩

Ordering property

lemma *convex-pd-below-iff*:
(*xs* ⊑ *ys*) =
(*convex-to-upper*·*xs* ⊑ *convex-to-upper*·*ys* ∧
convex-to-lower·*xs* ⊑ *convex-to-lower*·*ys*)
⟨*proof*⟩

lemmas *convex-plus-below-plus-iff* =
convex-pd-below-iff [**where** *xs*=*xs* ∪♮ *ys* **and** *ys*=*zs* ∪♮ *ws*]
for *xs ys zs ws*

lemmas *convex-pd-below-simps* =
convex-unit-below-plus-iff
convex-plus-below-unit-iff
convex-plus-below-plus-iff
convex-unit-below-iff
convex-to-upper-unit
convex-to-upper-plus
convex-to-lower-unit
convex-to-lower-plus
upper-pd-below-simps
lower-pd-below-simps

end

32 Powerdomains

theory *Powerdomains*
imports *ConvexPD Domain*
begin

32.1 Universal domain embeddings

definition *upper-emb* = *udom-emb* (*λi. upper-map*·(*udom-approx i*))
definition *upper-prj* = *udom-prj* (*λi. upper-map*·(*udom-approx i*))

definition *lower-emb* = *udom-emb* (*λi. lower-map*·(*udom-approx i*))
definition *lower-prj* = *udom-prj* (*λi. lower-map*·(*udom-approx i*))

definition *convex-emb* = *udom-emb* (*λi. convex-map*·(*udom-approx i*))
definition *convex-prj* = *udom-prj* (*λi. convex-map*·(*udom-approx i*))

lemma *ep-pair-upper*: *ep-pair upper-emb upper-prj*
⟨*proof*⟩

lemma *ep-pair-lower*: *ep-pair lower-emb lower-prj*
⟨*proof*⟩

lemma *ep-pair-convex*: *ep-pair convex-emb convex-prj*
⟨*proof*⟩

32.2 Deflation combinators

definition *upper-defl* :: *udom defl* → *udom defl*
where *upper-defl* = *defl-fun1 upper-emb upper-prj upper-map*

definition *lower-defl* :: *udom defl* → *udom defl*
where *lower-defl* = *defl-fun1 lower-emb lower-prj lower-map*

definition *convex-defl* :: *udom defl* → *udom defl*
where *convex-defl* = *defl-fun1 convex-emb convex-prj convex-map*

lemma *cast-upper-defl*:
cast·(*upper-defl*·*A*) = *upper-emb oo upper-map*·(*cast*·*A*) *oo upper-prj*
⟨*proof*⟩

lemma *cast-lower-defl*:
cast·(*lower-defl*·*A*) = *lower-emb oo lower-map*·(*cast*·*A*) *oo lower-prj*
⟨*proof*⟩

lemma *cast-convex-defl*:
cast·(*convex-defl*·*A*) = *convex-emb oo convex-map*·(*cast*·*A*) *oo convex-prj*
⟨*proof*⟩

32.3 Domain class instances

instantiation *upper-pd* :: (*domain*) *domain*
begin

definition
emb = *upper-emb oo upper-map·emb*

definition
prj = *upper-map·prj oo upper-prj*

definition
defl (*t*::′*a upper-pd itself*) = *upper-defl·DEFL*(′*a*)

definition
(*liftemb* :: ′*a upper-pd u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → ′*a upper-pd u*) = *u-map·prj*

definition
liftdefl (*t*::′*a upper-pd itself*) = *liftdefl-of·DEFL*(′*a upper-pd*)

instance ⟨*proof*⟩

end

instantiation *lower-pd* :: (*domain*) *domain*
begin

definition
emb = *lower-emb oo lower-map·emb*

definition
prj = *lower-map·prj oo lower-prj*

definition
defl (*t*::′*a lower-pd itself*) = *lower-defl·DEFL*(′*a*)

definition
(*liftemb* :: ′*a lower-pd u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → ′*a lower-pd u*) = *u-map·prj*

definition
liftdefl (*t*::′*a lower-pd itself*) = *liftdefl-of·DEFL*(′*a lower-pd*)

instance ⟨*proof*⟩

end

instantiation *convex-pd* :: (*domain*) *domain*
begin

definition
emb = *convex-emb oo convex-map·emb*

definition
prj = *convex-map·prj oo convex-prj*

definition
defl (*t*::'*a convex-pd itself*) = *convex-defl·DEFL*('*a*)

definition
(*liftemb* :: '*a convex-pd u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → '*a convex-pd u*) = *u-map·prj*

definition
liftdefl (*t*::'*a convex-pd itself*) = *liftdefl-of·DEFL*('*a convex-pd*)

instance ⟨*proof*⟩

end

lemma *DEFL-upper*: *DEFL*('*a*::*domain upper-pd*) = *upper-defl·DEFL*('*a*)
⟨*proof*⟩

lemma *DEFL-lower*: *DEFL*('*a*::*domain lower-pd*) = *lower-defl·DEFL*('*a*)
⟨*proof*⟩

lemma *DEFL-convex*: *DEFL*('*a*::*domain convex-pd*) = *convex-defl·DEFL*('*a*)
⟨*proof*⟩

32.4 Isomorphic deflations

lemma *isodefl-upper*:
isodefl d t ⟹ *isodefl* (*upper-map·d*) (*upper-defl·t*)
⟨*proof*⟩

lemma *isodefl-lower*:
isodefl d t ⟹ *isodefl* (*lower-map·d*) (*lower-defl·t*)
⟨*proof*⟩

lemma *isodefl-convex*:
isodefl d t ⟹ *isodefl* (*convex-map·d*) (*convex-defl·t*)
⟨*proof*⟩

32.5 Domain package setup for powerdomains

lemmas [*domain-defl-simps*] = *DEFL-upper DEFL-lower DEFL-convex*
lemmas [*domain-map-ID*] = *upper-map-ID lower-map-ID convex-map-ID*
lemmas [*domain-isodefl*] = *isodefl-upper isodefl-lower isodefl-convex*

lemmas [*domain-deflation*] =
deflation-upper-map deflation-lower-map deflation-convex-map

⟨*ML*⟩

end

theory *HOLCF*
imports
Main
Domain
Powerdomains
begin

default-sort *domain*

end